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THE EFFECTIVE VERSION OF BROOKS' THEOREM

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One of the fundamental results on graph coloring is the following classical theorem of Brooks.

BROOKS' THEOREM. *Suppose that $k \geqq 3$ and that G is a k-regular graph which does not induce a $(k+1)$-clique. Then G is k-colorable.*

Brooks proved his theorem in [**1**]; several more recent proofs have appeared in [**3**], [**4**] and [**5**]. All the proofs of this theorem have the common feature of applying only to finite graphs; the transition to infinite graphs can be accomplished by a very standard implementation of the Compactness Theorem (or some other equally noneffective device such as the theorem of deBruijn and Erdös [**2**] asserting that a graph is k-colorable if and only if each of its finite subgraphs is). Thus, it is not immediately apparent that an effective version of Brooks' Theorem exists. It is our purpose to show, however, that the effective analogue of Brooks' Theorem is indeed true.

THEOREM. *Suppose that $k \geqq 3$ and that G is a recursive k-regular graph which does not induce a $(k+1)$-clique. Then G is recursively k-colorable.*

It behooves us at this point to make precise the notions in the theorem. A *graph* G is a pair (V, E), where V is a set of *vertices* and E, the set of *edges*, is a set of 2-subsets of V. If V is a subset of ω, the set of natural numbers, and both V and E are recursive (in the sense of recursive function theory [**6**]), then G is *recursive*. If $X \subseteq V$, then set

$$N(X) = \{y \in V : \{x, y\} \in E \text{ for some } x \in X\},$$

and $N(x) = N(\{x\})$ for $x \in V$. Then G is *locally finite* if $N(x)$ is finite for each vertex x. The *degree* of x, denoted by $\deg x$, is $|N(x)|$, so that G is *k-regular* if and only if $\deg x = k$ for each vertex x. A *k-coloring* of G is a function $\chi : V \rightarrow \{0, 1, \ldots, k-1\}$ such that whenever x, y are vertices and $x \in N(y)$, then $\chi(x) \neq \chi(y)$. A graph is *k-colorable* if there is a *k-coloring* of it, and it is *recursively k-colorable* if there is a recursive k-coloring. The graph G is a *$(k+1)$-clique* if it is k-regular and has exactly $k+1$ vertices.

Throughout this paper, all subgraphs considered will be induced subgraphs, so we will unambiguously identify a subgraph with its set of

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vertices. For a graph G, if $X \subseteq G$, then we define $N_i(X)$ inductively on i by

$$N_0(X) = X \text{ and } N_{i+1}(X) = N_i(X) \cup N(N_i(X)).$$

The theorem will be proved by an induction on k, the most difficult portion being the basis step $k = 3$. We will begin with the proof of the inductive step, which is essentially the content of the following lemma. Recall that a subset X of a graph G is *independent* if and only if $X \cap N(X) = \emptyset$.

LEMMA 1. *Suppose that $k > 3$ and that G is a recursive k-regular graph which does not induce a $(k + 1)$-clique. Then there is a recursive, independent $X \subseteq G$ such that:*
(1) *whenever $x \in G - X$, then $N(x) \cap X \neq \emptyset$;*
(2) *if $C \subseteq G$ is a k-clique, then $C \cap X \neq \emptyset$.*

Proof. It is easy to see that if $C \subseteq G$ is a k-clique, then there is at most one k-clique $D \neq C$ such that $D \cap C \neq \emptyset$. Let

$$X_1 = \{x \in G : x = \min (C \cap D) \text{ for distinct } k\text{-cliques } C, D \subseteq G\}.$$

Clearly, X_1 is recursive and independent.

Now let $\mathscr{C}$ be the set of k-cliques $C \subseteq G$ such that whenever $D \neq C$ is a k-clique, then $D \cap C = \emptyset$. We will obtain a recursive independent $X_2 \subseteq G$ such that

(3) $\quad |X_2 \cap C| = 1$ for each $C \in \mathscr{C}$;

(4) $\quad X_2 \subseteq \bigcup \mathscr{C}$.

To do so, let $\mathscr{Y}$ be the set of all finite independent $Y \subseteq \bigcup \mathscr{C}$ which satisfy:

(5) whenever $x, y \in Y$ and $C \in \mathscr{C}$ are such that $x \neq y$ and $C \cap N(x) \neq \emptyset \neq C \cap N(y)$, then $C \cap Y \neq \emptyset$.

Notice that $\mathscr{Y} \neq \emptyset$ since $\emptyset \in \mathscr{Y}$. Thus, to show the existence of X_2 it suffices to verify the claim: whenever $Y \in \mathscr{Y}$ and $C \in \mathscr{C}$, then there is $Z \in \mathscr{Y}$ such that $Y \subseteq Z$ and $Z \cap C \neq \emptyset$.

To verify the claim, suppose that $Y \cap C = \emptyset$ where $Y \in \mathscr{Y}$ and $C \in \mathscr{C}$. Because of (5), there is $a \in C$ such that $N(a) \cap Y = \emptyset$. Let $Y_0 = Y \cup \{a\}$. If Y_0 fails to be in $\mathscr{Y}$, it is because (5) fails and there is exactly one pair $x, y \in Y_0$ for which (5) fails. Now let $\mathscr{Y}_0$ be the set of all finite independent $Y_1 \subseteq \bigcup \mathscr{C}$ such that $Y_0 \subseteq Y_1$ and Y_1 fails to be in $\mathscr{Y}$ because (5) fails for exactly one pair x, y. For $Y_1 \in \mathscr{Y}_0$, let

$$n(Y_1) = |\{y \in Y_1 : \text{there is } C \in \mathscr{C} \text{ such that } Y_1 \cap C = \emptyset \text{ yet } N(Y_1) \cap C \neq \emptyset\}|.$$

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Choose $Y_1 \in \mathscr{Y}_0$ which minimizes $n(Y_1)$, and let $x, y \in Y_1$ be the unique pair of elements in Y_1 for which (5) fails. Let $C_1 \in \mathscr{C}$ be the k-clique demonstrating that failure; that is,

$$C_1 \cap N(x) \neq \emptyset \neq C_1 \cap N(y) \text{ and } C_1 \cap Y_1 = \emptyset.$$

Choose $z \in C_1 - N(Y_1)$, and let $Z = Y_1 \cup \{z\}$. Clearly, if $Z \in \mathscr{Y}_0$, then $n(Z) < n(Y_1)$, contradicting minimality. Hence $Z \notin \mathscr{Y}_0$, and therefore $Z \in \mathscr{Y}$. This proves the claim and shows the existence of X_2.

It is clear that $X_1 \cup X_2$ is independent and recursive. Let $X \subseteq G$ be a recursive, maximal independent set containing $X_1 \cup X_2$. Then X has the desired properties.

It is clear that Lemma 1 gives us the inductive step of the proof of the theorem. For, if G is a k-regular graph which does not induce a $(k+1)$-clique, then obtain X as in Lemma 1, use the inductive hypothesis to obtain a recursive $(k-1)$-coloring $\psi:(G - X) \rightarrow k - 1$, and then set

$$\chi = \psi \cup (X \times \{k - 1\}).$$

Having proved the inductive step, we need only prove the theorem in the case that $k = 3$. The basic strategy to be used to achieve this is the following lemma, the proof of which constitutes the bulk of the proof of the theorem.

LEMMA 2. *Suppose that G is a recursive 3-regular graph which does not induce a 4-clique. Then with each finite $X \subseteq G$ there is effectively associated some r such that if $\chi:N_r(X) \rightarrow 3$ is 3-coloring, then $\chi|X$ can be extended to a 3-coloring of G.*

To see how Lemma 2 implies the theorem (in the case $k = 3$), let $\{x_n:n < \omega\}$ be some effective enumeration of the vertices of G, and let $X_n = \{x_0, x_1, \dots, x_{n-1}\}$. Lemma 2 easily implies the existence of a recursive $g:\omega \rightarrow \omega$ such that whenever $n < \omega$ and $\psi:N_{g(n)}(X_n) \rightarrow 3$ is a 3-coloring, then $\psi|X_n$ can be extended to a 3-coloring of G. Inductively define 3-colorings $\psi_n:N_{g(n)}(X_n) \rightarrow 3$ so that for any $n < \omega$, ψ_{n+1} extends $\psi_n|X_n$. By Brooks' Theorem ψ_0 exists. For $n > 0$ ψ_n exists by Lemma 2, and ψ_n can be chosen by some effective method. Thus, let $\chi:G \rightarrow 3$ be such that $\chi(x_n) = \psi_{n+1}(x_n)$. Clearly, χ is a recursive 3-coloring of G.

Thus, all that remains is to prove Lemma 2.

For a graph G, we let

$$\Delta(G) = \max(\{\deg x:x \in G\});$$

and we let

$$\delta(G) = \{x \in G:\deg x \leqq 2\}.$$

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If $P \subseteq G$, then P is a *path* from x to y if P is connected, $\Delta(P) \leqq 2$ and $\deg_P x = \deg_P y = 1$.

There are two definitions which will play important roles.

Definition 3. A finite graph H has *property A* if $\Delta(H) = 3$ and any 3-coloring $\psi:\delta(H) \to 3$ can be extended to a 3-coloring $\chi:H \to 3$.

Definition 4. A finite graph H has *property B* if there are distinct vertices $x, y \in H$ such that $x \notin N(y)$ and there are paths P_1, P_2, P_3 from x to y such that $P_1 \cap P_2 = P_2 \cap P_3 = P_3 \cap P_1 = \{x, y\}$.

The utility of these definitions with regard to Brooks' Theorem is demonstrated in the following lemma.

LEMMA 5. *Let H be a finite graph such that $\Delta(H) \leqq 3$, and let $X \subseteq H$ be such that $H - X$ is connected and that one of the following properties holds:*

(1) *$H - X$ has an induced subgraph with property* A;

(2) *$H - X$ has property* B;

(3) $(H - X) \cap \delta(H) \neq \emptyset$.

Then any 3-coloring $\psi:X \to 3$ can be extended to a 3-coloring $\chi:H \to 3$.

Proof. If $H - X$ is a component of H, then each of (1)–(3) implies that $H - X$ is not a 4-clique, so that by Brooks' Theorem there is a 3-coloring $\psi':H - X \to 3$. Let $\chi = \psi \cup \psi'$. Thus assume that $H - X$ is not a component of H.

Now let $a_0, a_1, \ldots, a_n$ be a list of elements of $H - X$ arranged so that for $1 \leqq i \leqq n$, a_i is connected by an edge to some a_j $(j < i)$. Notice that we can assign 3 colors to $a_n, a_{n-1}, \ldots, a_1$ in that order so that each a_j $(n \geqq j \geqq 1)$ is assigned a color different from any color previously assigned to a point to which it is connected by an edge. Thus, there is a 3-coloring $\psi_0:H - \{a_0\} \to 3$ extending ψ.

The choice of a_0 was so far arbitrary. We now impose some conditions on a_0 according to which of the conditions (1)–(3) is satisfied.

Suppose (1). Let H_0 be the induced subgraph of $H - X$ with property A. Choose $a_0 \in H_0 - \delta(H_0)$. Let $\phi:H_0 \to 3$ be a 3-coloring of H_0 extending $\psi_0|\delta(H_0)$. Now define $\chi:H \to 3$ by

$$\chi(x) = \begin{cases} \phi(x) \text{ if } x \in H_0 - \delta(H_0); \\ \psi_0(x) \text{ otherwise.} \end{cases}$$

Clearly, $\chi:H \to 3$ extends ψ. To see that χ is a 3-coloring, consider $x, y \in H$ such that $x \in N(y)$. The only possibility for a problem occurs when, for example, $x \in H_0 - \delta(H_0)$. But then $y \in H_0$ so that $\chi(x) = \phi(x) \neq \phi(y) = \chi(y)$.

Suppose (2) (but not (1) or (3)). Since $H - X$ has property B there are $x, y \in H - X$ and paths $P_1, P_2, P_3 \subseteq H - X$ as in Definition 4. Let

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$a_0 = x$. Since $\deg y = 3$, there are distinct vertices $b, c, \in N(y)$ such that $\psi_0(b) = \psi_0(c)$. Then consider (say) P_1 where $b, c \in P_1$. Then, as in Brooks' proof [**1**] of his theorem, there is a 3-coloring $\chi : H \to 3$ such that

$$\psi_0|(H - P_1) = \chi|(H - P_1).$$

Clearly, χ extends ψ.

Finally suppose (3) (but not (1)). Let $a_0 \in (H - X) \cap \delta(H)$. Let $j < 3$ be a color not assigned by ψ_0 to any point in $N(a_0)$. Then let

$$\chi = \psi_0 \cup \{\langle a_0, j\rangle\}.$$

Clearly, χ is a 3-coloring of H extending ψ.

In the next two lemmas some graphs with property A are presented.

LEMMA 6. *Let H be a finite graph such that $\Delta(H) \leqq 3$, and suppose that $Z \subseteq H$ is such that:*

$$\delta(H) \neq \emptyset$$

$$N_2(Z) = H,$$

$$N_1(Z) \cap \delta(H) = \emptyset,$$

$$|N_1(Z) - Z| \leqq 2.$$

Then H has property A.

Proof. Let $\psi : \delta(H) \to 3$ be a 3-coloring. By (3) of Lemma 5 there is a 3-coloring $\psi_0 : H - Z \to 3$ extending ψ. If $|N_1(Z) - Z| = 1$, then let $\psi_1 : N_1(Z) \to 3$ be a 3-coloring such that $\psi_1(a) = \psi_0(a)$, where $a \in N_1(Z) - Z$. Then set $\chi = \psi_0 \cup \psi$. So suppose $N_1(Z) - Z = \{a, b\}$, where $a \neq b$. If each ψ_0 as before is such that $\psi_0(a) = \psi_0(b)$, then each of a and b is joined to 2 points of $N_2(Z) - N_1(Z)$. Thus there are $a', b' \in Z$ which are the only points of Z in $N(a)$ and $N(b)$ respectively. Then let $\psi_2 : Z \to 3$ be any 3-coloring such that $\psi_2(a') \neq \psi_0(a) \neq \psi(b')$, and let $\chi = \psi_0 \cup \psi_2$. Finally, if each ψ_0 as before is such that $\psi_0(a) \neq \psi_0(b)$, then form the graph H' by adjoining to the graph $N_1(Z)$ an edge between a and b. Clearly, H' has maximal degree $\leqq 3$ and does not induce a 4-clique, so by Brooks' Theorem there is a 3-coloring $\psi_3 : H' \to 3$. Since $\psi_3(a) \neq \psi_3(b)$ we can assume that $\psi_3(a) = \psi_0(a)$ and $\psi_3(b) = \psi_0(b)$. Then set $\chi = \psi_0 \cup \psi$.

LEMMA 7. *Let H be a finite graph such that $\Delta(H) \leqq 3$, let $z_0 \subseteq H$ and let $Z_{i+1} = N_{i+1}(Z_0) - N_i(Z_0)$. Suppose the following hold:*

$$H = N_h(Z_0);$$

$$|\{x \in Z_i : N(x) \cap Z_{i+1} \neq \emptyset| \leqq 2 \text{ for } i < h;$$

$$1 \leqq |\delta(H) \cap Z_h| \leqq 2;$$

$$Z_0 \subseteq \delta(H) \subseteq Z_0 \cap Z_h.$$

Then, if h is sufficiently large, H has property A.

Proof. Choose h large enough so that the following proof works. (It appears that the least value of h for which the lemma is true is 6.) It is quite easy to see, using Lemma 5(3), that any 3-coloring of $Z_0 \cup \ldots \cup Z_i$ can be extended to a 3-coloring of $Z_0 \cup \ldots \cup Z_{i+1}$. Similarly, any 3-coloring of $Z_{i+1} \cup \ldots \cup Z_h$ can be extended to a 3-coloring of $Z_i \cup \ldots \cup Z_h$. Thus, it suffices to show that there are i, j $(0 \leqq i < j \leqq h)$ such that $Z_i \cup \ldots \cup Z_j$ has property A.

For each $i < h$, let $a_i, b_i \in Z_i$ be such that

$$\{a_i, b_i\} = \{z \in Z_i : N(z) \cap Z_{i+1} \neq \emptyset\},$$

and let

$$\{a_h, b_h\} = \delta(H) \cap Z_h.$$

Note that it is possible that $a_i = b_i$. It is very easy to check that whenever $0 \leqq i < h - 1$ and $a_{i+1} = b_{i+1}$, then $a_i = b_i$ if and only if $a_{i+2} \neq b_{i+2}$. If $i < j < h$ are such that $a_i = b_i$, $a_{i+1} = b_{i+1}$, $a_j = b_j$ and $a_{j+1} = b_{j+1}$, then clearly $Z_i \cup \ldots \cup Z_{j+1}$ has property A. Thus, without loss of generality, we can assume that $a_i \neq b_i$ for each $i \leqq h$.

If a_i and b_i are connected by an edge for $j \leqq i \leqq j + 3$, then $Z_j \cup \ldots \cup Z_{j+3}$ is uniquely determined and is easily seen to have property A. Thus we can assume that a_i and b_i are not connected by an edge for sufficiently many i.

Notice that if a_{i+1} and b_{i+1} are not connected by an edge, then any 3-coloring $Z_0 \cup \ldots \cup Z_i$ can be extended to a 3-coloring of $Z_0 \cup \ldots \cup Z_{i+1}$ which assigns the same color to a_{i+1} and b_{i+1}. Similarly, if $j < h - 1$ and a_j and b_j are not connected by an edge, then any 3-coloring of $Z_{j+2} \cup \ldots \cup Z_h$ can be extended to a 3-coloring of $Z_j \cup \ldots \cup Z_h$ which assigns the same color to a_j and b_j. Furthermore, notice that if a_i and b_i are connected by an edge, but a_{i+1} and b_{i+1} are not, then neither are a_{i+2} and b_{i+2}. Thus, there are i, j such that $i + 1 < j < h - 1$ and a_i, a_{i+1}, a_j are not connected by edges to b_i, b_{i+1}, b_j respectively. Now let $\psi : \delta(H) \to 3$ be a 3-coloring, and let

$$\chi_0 : (Z_0 \cup \ldots \cup Z_i) \cup (Z_j \cup \ldots \cup Z_h) \to 3$$

be a 3-coloring extending ψ such that $\chi_0(a_i) = \chi_0(b_i)$ and $\chi_0(a_j) = \chi_0(b_j)$. Clearly there is a 3-coloring χ_1 of $Z_i \cup \ldots \cup Z_j$ such that

$$\chi_1(a_i) = \chi_1(b_i) = \chi_0(a_i) \text{ and}$$
$$\chi_1(a_j) = \chi_1(b_j) = \chi_0(a_j).$$

Then $\chi = \chi_1 \cup \chi_2$ is the desired coloring.

LEMMA 8. *Let G be a graph such that $\Delta(G) \leqq 3$, let $X \subseteq G$ be finite, and let $m < \omega$. Then there is r $(2 \leqq r < \omega)$ such that for each component Y of $N_r(X) - X$ one of the following holds:*

(1) *Y has an induced subgraph with property* A;

(2) *Y has property* B;

(3) $Y \cap \delta(G) \neq \emptyset$;

(4) $Y \subseteq N_{r-1}(X)$;

(5) $|Y \cap (N_r(X) - N_{r-1}(X))| \geqq m$, *and whenever distinct* $x, y \in Y \cap (N_{r-1}(X) - N_{r-2}(X))$, *then there is a path* P *from* x *to* y *such that*

$$P \subseteq (Y \cap N_{r-2}(X)) \cup \{x, y\}.$$

Proof. Let $H = \cup \{N_s(X) : s < \omega\}$. Choose $s \geqq 3$ large enough so that each component W of $H - X$ satisfies each of the following conditions:

(A0) $W \cap N_s(X)$ is a component of $N_s(X) - X$.

(A1) if W has an induced subgraph with property A, then $W \cap N_s(X)$ has an induced subgraph with property A;

(A2) if W has property B, then $W \cap N_s(X)$ has property B;

(A3) if $W \cap \delta(G) \neq \emptyset$, then $W \cap N_s(X) \cap \delta(G) \neq \emptyset$;

(A4) if W is finite, then $W \subseteq N_s(X)$.

Clearly such an s exists since there are only finitely many components of $H - X$.

Now let W be a component of $H - X$ such that (A1)–(A4) hold vacuously; that is:

(B1) W has no induced subgraph with property A;

(B2) W does not have property B;

(B3) $W \cap \delta(G) = \emptyset$;

(B4) W is infinite.

We will find $u < \omega$ such that if $r \geqq u$ and $Y = W \cap N_r(X)$, then (5) is satisfied. Clearly, this will suffice to prove the lemma.

Let $t \geqq s + 2$ be such that whenever $x, y \in W \cap (N_s(X) - N_{s-1}(X))$ and x, y are in the same component of $W - N_{s-1}(X)$, then they are already in the same component of $W \cap (N_{t-2}(X) - N_{s-1}(X))$.

We claim that if $r \geqq t$ and $x, y \in W \cap (N_{r-1}(X) - N_{r-2}(X))$ are distinct, then there is a path P from x to y such that

$$P \subseteq (W \cap N_{r-2}(X)) \cup \{x, y\}.$$

To see this let $x_1, y_1 \in W \cap (N_s(X) - N_{s-1}(X))$ be such that there are paths P_1, Q_1 from x_1, y_1 to x_1', y_1' respectively so that

$$P_1 \subseteq (N_{r-2}(X) - N_{s-1}(X)) \cup \{x\} \text{ and}$$

$$Q_1 \subseteq (N_{r-2}(X) - N_{s-1}(X)) \cup \{y\}.$$

By the condition on t there is a component R of $W \cup (N_{t-2}(X) - N_s(X))$ which contains x_1 and y_1. But then $P_1 \cup Q_1 \cup R$ is connected and

$$(P_1 \cup Q_1 \cup R) \cap N_{r-1}(X) = \{x, y\}.$$

Then there is a path P from x to y such that

$$P \subseteq (W \cap N_{r-2}(X)) \cup \{x, y\}.$$

Thus, to complete the proof of the lemma it suffices to find $u < \omega$ such that whenever $r \geqq u$, then

$$|W \cap (N_r(X) - N_{r-1}(X))| \geqq m.$$

We now claim that if $n \geqq t$ and Z is a component of $W \cap (N_n(X) - N_{n-1}(X))$, then $N_1(Z) - N_n(X) \neq \phi$. For, suppose not. Then $N_1(Z) \subseteq N_n(X)$. If $|N_1(Z) \cap N_{n-1}(X)| \leqq 2$, then by Lemma 6 $N_2(Z)$ has property A. On the other hand, if $|N_1(Z) \cap N_{n-1}(X)| \geqq 3$, then by the condition on t it follows that W has property B, and this contradicts (B2). This proves the claim.

Thus, if $C(n)$ is the number of components of $W - N_{n-1}(X)$, the above claim shows that $t \leqq n \leqq i$ implies $C(n) \leqq C(i)$.

Suppose that $\lim_n C(n) < \infty$. Choose $n \geqq t$ so large that if $i > n$ then $C(i) = C(n)$. If Z is a component of $W - N_{i-1}(X)$ for some $i \geqq n$, then $Z \cap N_i(X)$ has at most two elements which are connected by an edge to some element in $W - N_i(X)$, as otherwise W would have property B. So let Z be a component of $W - N_{n-1}(X)$, and let

$$Z_0 = \{x \in Z \cap N_n(X) : N(x) \cap (W - N_n(X)) \neq \phi\}.$$

Choosing h as in Lemma 7, let $Z_{i+1} = N_1(Z_i) - N_{n+1}(X)$ for $i \leqq h$. Then by Lemma 7 $Z_0 \cup \ldots \cup Z_h$ has property A, contradicting (B1).

Thus, $\lim_n C(n) = \infty$. Clearly, if $u \geqq n$ is such that $C(u) = m$, then $C(r) \geqq m$ for $r \geqq u$, so that u has the desired property.

LEMMA 9. *Let G be a graph such that $\Delta(G) \leqq 3$. Let $X \subseteq G$ be finite and let $m = 24|X|$. Let r be as in Lemma 8. Suppose $s < \omega$ and that $\psi : N_r(X) \to 3$ is a 3-coloring. Then there is a 3-coloring $\chi : N_s(X) \to 3$ extending $\psi|X$.*

Proof. Let $Z_0, Z_1, \ldots, Z_t$ be the components of $N_s(X) - X$. It suffices to find for each $j \leqq t$ a 3-coloring $\chi_j : X \cap Z_j \to 3$ such that $\chi_j|X = \psi|X$. For, then just set $\chi = \chi_0 \cup \ldots \cup \chi_t$.

So suppose Z is a component of $N_s(X) - X$. We will show that there is a 3-coloring $\chi : X \cup Z \to 3$ such that $\chi|X = \psi|X$. If $Z \subseteq N_r(X)$, then the lemma follows trivially by setting $\chi = \psi|(X \cup Z)$. Thus letting $D = Z - N_r(X)$, we can assume $D \neq \emptyset$ and, in particular, $s > r$.

From Lemma 5 we can make the following assumptions:

(1) $Z - X$ has no induced subgraph with property A;

(2) $Z - X$ does not have property B;

(3) $(Z - X) \cap \delta(G) = \emptyset$;

We need a fact about the components of $Z \cap (N_r(X) - N_{r-1}(X))$.

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(4) If W is a component of $Z \cap (N_r(X) - N_{r-1}(X))$, then

$$N_1(W) - N_r(X) \neq \emptyset.$$

To see this suppose $N_1(W) - N_r(X) = \emptyset$. Then $N_1(W) \subseteq N_r(X)$. If $|N_1(W) \cap N_{r-1}(X)| \leqq 2$, then by Lemma 6 $N_2(W)$ has property A, contradicting (1). On the other hand, if $|N_1(W) \cap N_{r-1}(X)| \geqq 3$, then from (5) of Lemma 8 it follows that $Z - X$ has property B, contradicting (2). Thus (4) is true. For each $z \in D$, let

$$p(z) = \{y \in Z \cap (N_r(X) - N_{r-1}(X)) : \text{there is a path } P \text{ from } y \text{ to } z \text{ such that } P \subseteq (N_s(X) - N_r(X)) \cup \{y\}\}.$$

Notice that $p(z) \neq \emptyset$ for each $z \in D$.

Let $Y_0, Y_1, \ldots, Y_c$ be a list of the components of $Z \cap (N_r(X) - X)$. A consequence of (4) is that for each $i \leqq c$ there is $z \in D$ such that $p(z) \cap Y_i \neq \emptyset$. We now improve upon (4).

(5) Suppose $i \leqq c$ and W is a component of $Y_i \cap (N_r(X) - N_{r-1}(X))$. Then there is $j \neq i$ and $z \in D$ such that

$$p(z) \cap W \neq \emptyset \neq p(z) \cap Y_j.$$

For, suppose not. Let $D_0 = \{z \in D : p(z) \cap W \neq \emptyset\}$. Let

$$T = \{t : r < t \leqq s \text{ and } p(z) \cap W \neq \emptyset \text{ for some } z \in N_t(X) - N_{t-1}(X)\}.$$

By (4), $T \neq \emptyset$ so let $t = \max T$, and let $z \in N_t(X) - N_{t-1}(X)$ be such that $p(z) \cap W \neq \emptyset$. Let W_0 be the component of $N_t(X) - N_{t-1}(X)$ to which z belongs. By an argument like the one verifying (4) we see that there are $j \neq i$ and $x \in N_1(W_0)$ such that $p(x) \cap Y_j \neq \emptyset$. Thus (5) holds.

Notice that the number of components of $N_r(X) - X$ is at most $3|X|$. Thus,

(6) $\quad c < 3|X|$.

Also, notice that

(7) $\quad$ for each component W of $Z \cap (N_r(X) - N_{r-1}(X))$, $|W| \leqq 4$.

For, if $|W| > 4$, then there necessarily would be $x_1, x_2, x_3 \in W$ and distinct $y_1, y_2, y_3 \in N_{r-1}(X) - N_{r-2}(X)$ such that $y_i \in N(x_i)$. Then by (5) of Lemma 8, $Z - X$ would have property B, contradicting (2).

Now consider Y_c, noting that

$$|Y_c \cap (N_r(X) - N_{r-1}(X))| \geqq m = 24|X|.$$

Thus, from (6),

$$|Y_c \cap (N_r(X) - N_{r-1}(X))| > 8c.$$

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Then from (7), it follows that the number of components of $Y_c \cap (N_r(X) - N_{r-1}(X))$ is $>2c$. Therefore, from (5), it follows that there is $i < c$ and there are distinct $x_1, x_2, x_3 \in Y_c$ and there are $y_1, y_2, y_3 \in Y_i$ and $z_1, z_2, z_3 \in D$ such that

$$\{x_1, y_1\} \subseteq p(z_1),\ \{x_2, y_2\} \subseteq p(z_2) \text{ and } \{x_3, y_3\} \subseteq p(z_3).$$

Then there are paths P_1, P_2, P_3 from x_1, x_2, x_3 to y_1, y_2, y_3 respectively such that

$$P_j \subseteq (N_s(X) - N_r(X)) \cup \{x_j, y_j\}$$

for $j = 1, 2, 3$. But then there are paths Q_1, Q_2, Q_3 from x_1, x_2, x_3 to y_1 such that

$$Q_j \subseteq (N_s(X) - N_r(X)) \cup \{x_j\} \cup Y_i$$

for $j = 1, 2, 3$. It easily follows that Z has property B, contradicting (2).

The proof of Lemma 2, and hence also of the theorem, is now quite clear. For, given a recursive 3-regular graph G and a finite subset $X \subseteq G$, let $m = 24|X|$, and then obtain r as in Lemma 8. Clearly, since such an r exists, there is an effective way to obtain it. By Lemma 9, this is the required r, so the theorem is proved.

Brooks' Theorem is usually stated so as to refer to graphs G with maximal degree $\Delta(G) \leqq k$ rather than to k-regular G. These two ways of stating Brooks' Theorem are equivalent; however, a little extra care must be exercised in getting the effective version of the other form of Brooks' Theorem. It is very easy to see that every recursive graph G for which $\Delta(G) \leqq k$ is recursively $(k + 1)$-colorable. However, for each $k \geqq 2$, there even are examples of recursive trees G such that $\Delta(G) = k$ yet G is not recursively k-colorable. A graph G is *highly recursive* if it is recursive, locally finite, and the function deg is recursive. Notice that a recursive graph G with $\Delta(G) \leqq k$ is highly recursive if and only if it is an induced subgraph of a recursive k-regular graph. Thus we get the following corollary to the theorem.

COROLLARY 10. *Suppose that $k \geqq 3$ and that G is a highly recursive graph with $\Delta(G) \leqq k$ and G does not induce a $(k + 1)$-clique. Then G is recursively k-colorable.*

We conclude with another corollary, which is a strengthening of the theorem, but which also indicates that the theorem has little to do with recursion theory. It asserts the existence of a function, which also happens to be recursive, but whose existence is much more interesting than the unsurprising fact of its recursiveness.

From the statement of Lemma 8, we easily see that the r whose existence is claimed there can be made to depend only on $|X|$ and m, and not on X or G. Thus, the r of Lemma 2 need depend only on $|X|$. Also, by an inspection of the proof of Lemma 1, we see that such an r can be chosen to work even in the case of k-regular graphs for arbitrary k. This results in the following corollary which is new even without the requirement of recursiveness.

COROLLARY 11. *There is a recursive function* $f:\omega \rightarrow \omega$ *such that whenever* $3 \leqq k < \omega$, G *is a graph such that* $\Delta(G) \leqq k$ *and* G *does not induce a* $(k + 1)$*-clique,* $X \subseteq G$ *is such that* $|X| = n < \omega$, *and* $\chi: N_{f(n)}(X) \rightarrow k$ *is a* k*-coloring, then* $\chi|X$ *can be extended to a* k*-coloring of* G.

For other results on recursive colorings of graphs we refer the reader to [**7**].

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