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On $\mathcal{CR}$-Epic Embeddings and Absolute $\mathcal{CR}$-Epic Spaces

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Abstract. We study Tychonoff spaces X with the property that, for all topological embeddings $X \to Y$, the induced map $C(Y) \to C(X)$ is an epimorphism of rings. Such spaces are called absolute $\mathcal{CR}$-epic. The simplest examples of absolute $\mathcal{CR}$-epic spaces are σ-compact locally compact spaces and Lindelöf P-spaces. We show that absolute $\mathcal{CR}$-epic first countable spaces must be locally compact.

However, a "bad" class of absolute $\mathcal{CR}$-epic spaces is exhibited whose pathology settles, in the negative, a number of open questions. Spaces which are not absolute $\mathcal{CR}$-epic abound, and some are presented.

1 Introduction

If X is a topological space the set $C(X)$ of all continuous functions from X to $\mathbb{R}$ can be considered as an object in a number of categories. It is a group, a commutative ring, a reduced ring (one with no non-zero nilpotents), an Archimedean f-ring, and others. In this paper we are considering it exclusively as a commutative ring. We denote by $C^*(X)$ the subring of $C(X)$ consisting of the bounded functions from X to $\mathbb{R}$.

Barr, Burgess and Raphael [3] considered the following question (and some related ones): If X is a topological space and Y is a subspace, when is it the case that the homomorphism $C(X) \to C(Y)$ induced by restriction is an epimorphism? In this paper, we are concerned mainly with the question of when every embedding of X into some larger space Y has the property that $C(Y) \to C(X)$ is epic. It is easy to reduce this question to the case in which Y is compact and X is dense in Y. Such a space Y is called a *compactification* of X.

Hager and Martinez [12] used the phrase "C-epic" to describe an embedding $g\colon X \to Y$ for which the induced map $C(Y) \to C(X)$, given by $f \mapsto f \circ g$, is epic in a particular category. However the notion of epimorphism depends on the category and we wished to find a term in which the name of the category appears as a parameter. Accordingly we describe an embedding $X \to Y$ as an $\mathcal{X}$-*epic embedding* if $C(Y) \to C(X)$ is epic in the category $\mathcal{X}$. We say that an object X is an *absolute* $\mathcal{X}$-*epic* if every embedding $X \to Y$ is $\mathcal{X}$-epic.

In this paper we will be concerned exclusively with the case that $\mathcal{X} = \mathcal{CR}$, the category of commutative rings.

Definition 1.1 An inclusion $X \to Y$ is called a $\mathcal{CR}$-*epic embedding* if the map $C(Y) \to C(X)$ induced by restriction is an epimorphism in the category of com-

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mutative rings. The space X is called *absolute* $\mathcal{CR}$*-epic* if every embedding of X is $\mathcal{CR}$-epic.

1.1 Notation

All spaces considered in this paper are assumed to be Tychonoff (completely regular Hausdorff) and all functions, unless explicitly stated otherwise, are assumed continuous. As usual, βX denotes the Stone-Čech compactification of the space X. It is the unique compact space in which X is dense and C^*-embedded. See [11, Ch. 6] for details. We denote by υX the Hewitt realcompactification of a Tychonoff space X; see [11, Ch. 8] or [22, 5.5(c), 5.10]. A space X is called *realcompact* if $X = \upsilon X$. A space is called *almost Lindelöf* if, of any two disjoint zero-sets, at least one is Lindelöf. It is shown in [18, 5.4] that a space X is almost Lindelöf if and only if it differs from υX by at most one point and υX is Lindelöf.

1.2 Examples

Here are some examples of absolute $\mathcal{CR}$-epic spaces, mostly developed below.

1. Locally compact Lindelöf spaces (Theorem 2.14).
2. Lindelöf *P*-spaces (Theorem 5.2).
3. Almost compact spaces (defined in § 1.3; see [11, 6J(5)]).
4. The sum (that is, disjoint union) of an absolute $\mathcal{CR}$-epic Lindelöf space and an almost compact space (Theorem 3.1).
5. The sum of countably many absolute $\mathcal{CR}$-epic Lindelöf spaces (Proposition 2.16).

Here are some examples of spaces that are not absolute $\mathcal{CR}$-epic.

6. A non-open dense countable intersection of cozero-sets in a locally connected space (this includes $\mathbb{R} \setminus \mathbb{Q}$) [10, Theorem 3.10(2)].
7. A non-open dense co-σ boundary of an arbitrary space [19, 2.5].
8. A proper dense subspace of $\beta\mathbb{N} \setminus \mathbb{N}$ (assuming CH), (Corollary 2.5).
9. Any space X for which $|\upsilon X \setminus X| > 1$ (Theorem 2.9); for example, any pseudocompact space that is not almost compact, or the sum of two non-realcompact spaces.
10. A dense subset of $\beta\mathbb{Q} \setminus \mathbb{Q}$. This includes certain countable extremally disconnected spaces (Corollary 4.4).
11. An uncountable sum of spaces (Corollary 2.10); in particular, any locally compact, paracompact space that is not σ-compact, [8, 5.1.27].
12. A first countable space that is not locally compact (Corollary 2.22), for example the rational numbers.

1.3 Properties

There are a number of properties of topological spaces we will be using in this article that we summarize here. Most of them are from [11].

A space X is called *almost compact* if $|\beta X \setminus X| \leq 1$. A space is called *pseudocompact* if every real-valued function is bounded and therefore extends to βX. An almost compact space is pseudocompact [11, Problem 6J], but the converse is not true. One characterization of almost compact spaces is that given two disjoint zero-sets, at least one is compact (compare the definition of almost Lindelöf).

A space X is *realcompact* if for every point $p \in \beta X \setminus X$, there is a function in $C(X)$ that does not extend to p.

Any countable union of cozero-sets is a cozero-set [22, 1.4(i)(2)].

The following theorem, which is central to the discussion in the present paper, is essentially Theorem 2.6 of [3].

Theorem 1.2 *Suppose the dense embedding $X \subseteq Y$ is $\mathcal{CR}$-epic. Then every function in $C(X)$ extends to an open set of Y that contains X.*

2 Absolute $\mathcal{CR}$-Epic Spaces

The following two results are found in [12, 1.1 and 1.6, resp.], (although the category is different, the same easy proofs work).

Proposition 2.1 *For a space X to be absolute $\mathcal{CR}$-epic, it is sufficient that its embedding in every compactification be $\mathcal{CR}$-epic.*

Proposition 2.2 *Suppose that we have a commutative diagram*

$$\begin{tikzcd}
 & X \arrow[dl, "q'"'] \arrow[dr, "q"] & \\
K' \arrow[rr, "p"'] & & K
\end{tikzcd}$$

such that q and q' are embeddings. Then if q is $\mathcal{CR}$-epic, so is q'. In particular, if X is locally compact, it is absolute $\mathcal{CR}$-epic if and only if the embedding into its one-point compactification is $\mathcal{CR}$-epic.

Proposition 2.3 *If a sum of two spaces is absolute $\mathcal{CR}$-epic, so is each summand.*

Proof If X is dense in K, then $X + Y$ is dense in $K + \beta Y$. Assuming $X + Y$ is absolute $\mathcal{CR}$-epic, the result can be read from this diagram

$$\begin{tikzcd}
C(X) & C(K) \arrow[l] \\
C(X+Y) \arrow[u, two heads] & C(K+\beta Y) \arrow[l, two heads] \arrow[u]
\end{tikzcd}$$

■

Proposition 2.4 *If X is dense in Y and $C(Y) \to C(X)$ is epic, then $C(X)$ has the same cardinality as $C(Y)$.*

Proof This follows from the computation of the size of $C(X)$ in [15], in conjunction with the fact shown in [14, 1.5] that an epimorphism between infinite algebras does not increase cardinality. ■

Corollary 2.5 (CH) *No proper dense subspace of $\beta\mathbb{N} \setminus \mathbb{N}$ is absolute $\mathcal{CR}$-epic.*

Proof Using CH, it follows from [26, Theorem 2.2] that a subset $X \subseteq \beta\mathbb{N} \setminus \mathbb{N}$ is C^*-embedded in $\beta\mathbb{N} \setminus \mathbb{N}$ if and only if $|C(X)| = c$. Thus if $X \subseteq \beta\mathbb{N} \setminus \mathbb{N}$ is epic, X must be C^*-embedded in $\beta\mathbb{N} \setminus \mathbb{N}$. But [9, Theorem 4.6a] shows that in the presence of CH, proper dense subsets of $\beta\mathbb{N} \setminus \mathbb{N}$ cannot be C^*-embedded. ■

Epimorphisms in the category of rings are characterized by the following theorem which is found in [20] and in [25, p. 73].

Proposition 2.6 *A homomorphism $w\colon R \to S$ in the category of rings is an epimorphism if and only if for each $f \in S$ there exist matrices G, A, H of sizes $1 \times n$, $n \times n$ and $n \times 1$, respectively, such that*

(i) *$f = GAH$,*
(ii) *G and H have entries in S,*
(iii) *the entries of A, GA, and AH are in the image of w.*

Such a decomposition of f is called an $n \times n$ zig-zag *for f with respect to w.*

In most, but not all cases, w will be an injective ring homomorphism (induced by the inclusion of a dense subspace) and we will treat R as a subring of S. In that case we will simply say that A, GA, and AH are matrices over R.

The name "zig-zag" is not especially appropriate. It appears to go back to a theorem of Isbell's that characterizes epics in the category of semigroups, [14].

A useful observation, whose proof we leave to the reader follows.

Proposition 2.7 *Suppose*

$$\begin{tikzcd} P \arrow[r, two heads] \arrow[d] & R \arrow[d] \\ S \arrow[r] & T \end{tikzcd}$$

is a commutative diagram of rings such that $P \to R$ is surjective. If an element $f \in T$ has an $n \times n$ zig-zag with respect to R, then it also has an $n \times n$ zig-zag with respect to S.

Corollary 2.8 *Suppose that for every compactification $X \subseteq K$, every element of $C(X)$ has a 1×1 zig-zag with respect to $C(K)$. Then the same is true with $X \to K$ replaced by any embedding $X \to Y$.*

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2.1 Results from Hager and Martinez

Hager and Martinez [12] study epimorphisms between function rings, not in the category of commutative rings, but rather in the category of Archimedean l-groups; see their paper for details. The crucial fact appears on p. 116, where it is shown that if $C(Y) \to C(X)$ is epic in the category of commmutative rings, it is epic in their category. Thus, any result of theirs that concludes that an embedding induced by an inclusion of spaces is not epic has, as a corollary, that the embedding is not epic in the category of rings and ring homomorphisms. The following theorem states three consequences of their results that are pertinent for us here, with the references cited to their paper.

Theorem 2.9 *An absolute $\mathcal{CR}$-epic space*

(i) *is almost Lindelöf* [12, Theorem 9.1]*;*
(ii) *is almost compact if it is pseudocompact* [12, Corollary 2.9 a]*;*
(iii) *does not have a basically disconnected compactification except possibly its Stone-Čech compactification* [12, Example 3.4].

Corollary 2.10

(i) *The categorical sum of uncountably many spaces cannot be absolute $\mathcal{CR}$-epic;*
(ii) *the categorical sum of an absolute $\mathcal{CR}$-epic space with itself can fail to be absolute $\mathcal{CR}$-epic;*
(iii) *the product of an absolute $\mathcal{CR}$-epic space with itself can fail to be absolute $\mathcal{CR}$-epic;*
(iv) *the preimage of an absolute $\mathcal{CR}$-epic space under a perfect irreducible map can fail to be absolute $\mathcal{CR}$-epic.*

Proof

(i) By splitting the index set into two uncountable subsets, we can write the space as $X = X_1 + X_2$, where neither summand is Lindelöf. Clearly X_1 and X_2 are zero-sets so that X is not almost Lindelöf (*cf.* Definition 1.1).

The next three parts use the space W of all countable ordinals with the order topology. This space is locally compact and $\beta W = \upsilon W = W^*$ can be thought of as consisting of all ordinals up to and including the first uncountable ordinal ω_1 (see [11, 6J]).

(ii) The space $W + W$ is pseudocompact but not absolute $\mathcal{CR}$-epic because $|\upsilon(W \times W) \setminus (W \times W)| = 2$ (see [11, 8M(3)]).

(iii) The space $X = W \times W$ is pseudocompact and therefore $\beta X = \upsilon X$. It follows that $\upsilon X \setminus X = (W \times (\omega)_1) \cup ((\omega)_1 \times W)$, so X is not almost compact. See [11, 8L and 8M] for all details.

(iv) Let D denote the set of all isolated points (non-limit ordinals) in W^*. Then W^* is a compactification of the uncountable discrete space D and, by [11, 6.4], there is a continuous surjection $f\colon \beta D \to W^*$ that fixes D pointwise. Then $f^{-1}(W) = \aleph_0(D)$, the subspace of βD consisting of D and the limit points of countable subsets of D. Although W is absolute $\mathcal{CR}$-epic, $\aleph_0 D$ is pseudocompact, but not almost compact and therefore not absolute $\mathcal{CR}$-epic. But the map $f|\aleph_0 D\colon \aleph_0 D \to W$ is a perfect irreducible continuous surjection (see [22, Chapter 6]). ■

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The following theorem appears as [3, Proposition 2.1(ii)], but the proof given here is slightly easier.

Proposition 2.11 *Suppose that $X \subseteq Y$ and $f \in C(X)$. If f extends to a cozero-set of Y, then f has a 1×1 zig-zag with respect to $C(Y)$.*

Proof Suppose that f extends to $\operatorname{coz}(u)$. We may suppose that $u \geq 0$. Define $a\colon Y \to R$ by $a|\operatorname{coz}(u) = (f/(1+f^2))u$ and $a|Z(u) = 0$. Then a is easily verified to be in $C(Y)$. Define $g\colon X \to R$ by $g = \sqrt{(1+f^2)/(u|X)}$. Clearly g is in $C(X)$. One verifies that $f = gag$, and that $g(a|X)$ extends continuously to Y. Thus gag is a 1×1 zig-zag for f. ∎

The following well-known characterization of Lindelöf spaces (among completely regular spaces) is due to Smirnov [24]. We give an easy proof of the direction we use. See [8, 3.12.25] for the converse.

Proposition 2.12 *A space X is Lindelöf if and only if, whenever X is embedded in a space Y, every open subset of Y containing X, contains a cozero-set containing X.*

Proof Let $X \subseteq U \subseteq Y$ with U open. Any open subset in the completely regular space Y is a union of cozero-sets. But the space is Lindelöf and hence a countable union of these cozero-sets covers X. But a countable union of cozero-sets is a cozero-set. ∎

Corollary 2.13 *Let X be a Lindelöf space densely embedded in a space Y. Then the embedding is $\mathcal{CR}$-epic if and only if every function in $C(X)$ extends to an open subset of Y that contains X.*

Proof Theorem 1.2 gives one direction and the two immediately preceding propositions give the other. ∎

Theorem 2.14 *A locally compact Lindelöf space is absolute $\mathcal{CR}$-epic. (This includes* [3, 2.15(ii)]*).*

Proof A locally compact space is open in any compactification. Thus if the locally compact space X is dense in a compact space K, it is open. If it is Lindelöf, it is a cozero-set in K and the conclusion follows from Proposition 2.11. ∎

Lemma 2.15 *Let X be an absolute $\mathcal{CR}$-epic Lindelöf space. Then whenever X is embedded in a space Y any function in $C(X)$ can be extended to a cozero-set in Y and has a 1×1 zig-zag.*

Proof By Theorem 1.2, every $f \in C(X)$ extends to an open set $U \supseteq X$. Thus f extends to a cozero-set and, by Proposition 2.11, has a 1×1 zig-zag. ∎

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Proposition 2.16 *A sum of countably many absolute $\mathcal{CR}$-epic Lindelöf spaces is absolute $\mathcal{CR}$-epic.*

Proof Let $L = \sum L_i$ be densely embedded in the compact space K and suppose that each L_i is absolute $\mathcal{CR}$-epic Lindelöf. Let $f \in C(L)$. Since each L_i is open in L, choose an open set U_i of K whose intersection with L is L_i. If $i \neq j$, the set $U_i \cap U_j \cap L = \varnothing$. Since L is dense in K, this means that $U_i \cap U_j = \varnothing$. Since L_i is absolute $\mathcal{CR}$-epic, there is, by Proposition 2.12, a function $u_i \in C(K)$ such that $L_i \subseteq \operatorname{coz}(u_i) \subseteq U_i$ and such that $f|L_i$ extends to $g_i \in C(\operatorname{coz}(u_i))$. Let $V = \bigcup \operatorname{coz}(u_i)$ and let $g = \bigcup g_i$. Then $V \in \operatorname{coz} K$, $g \in C(V)$, and $g|X = f$. Thus each function in $C(L)$ extends to a cozero-set in K, whence $C(K) \to C(L)$ is epic. ■

The following is an immediate consequence of [17, Theorem 3.2]. To apply it, one has to understand that $Y \cap \beta X = X$ "multiply" means that for each $y \in Y \setminus X$ there is at least one $f \in C^*(X)$ that cannot be extended to $X \cup \{y\}$ while $Y \cap \beta X = X$ "singly" means that there is a single $f \in C^*(X)$ that cannot be extended to any point of $Y \setminus X$.

Lemma 2.17 *Let X be a dense subspace of the space Y and let S be a countable subset of $Y \setminus X$. Suppose that for each $s \in S$ there is a function in $C^*(X)$ that does not extend continuously to $X \cup \{s\}$. Then there is a single function in $C^*(X)$ that does not extend continuously to $X \cup \{s\}$ for any $s \in S$.*

Theorem 2.18 *Suppose that X is dense in Y and that S is a countable subset of $Y \setminus X$ for which $X \cap \operatorname{cl} S \neq \varnothing$. If for each $s \in S$, X is not C^*-embedded in $X \cup \{s\}$, then the embedding $X \subseteq Y$ is not $\mathcal{CR}$-epic.*

Proof By the preceding lemma there is a single $f \in C^*(X)$ that does not extend continuously to $X \cup \{s\}$ for any $s \in S$. If V is open in Y and $X \subseteq V \subseteq Y$, then $V \cap S \neq \varnothing$ as $X \cap \operatorname{cl} S \neq \varnothing$. Thus f cannot extend continuously to V. By Theorem 1.2 the embedding $X \subseteq Y$ is not $\mathcal{CR}$-epic. ■

Corollary 2.19 *If X is absolute $\mathcal{CR}$-epic then υX is absolute $\mathcal{CR}$-epic. The converse is false.*

Proof Any pseudocompact space that is not almost compact establishes the second claim. For the first claim, if K is a compactification of υX, then it is also a compactification of X and one has the induced maps $C(K) \to C(\upsilon X) \to C(X)$. Since the second map is an isomorphism and the composite is epic, so is the first map. ■

Theorem 2.20 *Let E be an equivalence relation on the compact space K. Let K/E denote the set of equivalence classes of E and $f\colon K \to K/E$ be the natural map sending each $x \in K$ to its equivalence class $[x]$. Give K/E the quotient topology induced by f. Then*

(i) *Suppose that if A is any closed subset of K, then $f^{-1}(f[A])$ is closed in K. Then K/E is a Hausdorff space.*
(ii) *Suppose that K is a compactification of a space X, E is an equivalence relation on K as in 1, and if $x \in X$ then $[x] = \{x\}$. Then K/E is a compactification of X.*

Proof Part (i) follows directly by combining [8, 2.4.9, 2.4.10 and 3.2.11]. By the hypotheses in (ii), $f|X$ maps X homeomorphically onto the dense subspace $f[X]$ of the space K/E, which is compact as f is a continuous surjection and Hausdorff by 1.

This argument also shows that the quotient map $q\colon K \to K/E$ is closed and we claim that its restriction to X is also closed. In fact, if $A \subseteq X$ is closed in X, let $A = B \cap X$, where B is closed in K. By hypothesis, no point of X is identified by E to any other point of K, either in or out of X. From this, it is immediate that $q(A) = q(X) \cap q(K)$ and hence that $q(A)$ is closed in $q(X)$. Thus $q|X$ is closed and injective, and thus maps X homeomorphically on $q(X)$. ■

Theorem 2.21 *Suppose X is a dense subspace of the compact space K. Suppose there is a countable discrete subset $S \subseteq K \setminus X$ such that $\mathrm{cl}_K S = S \cup \{p\}$ where $p \in X$. Then X is not absolute $\mathcal{CR}$-epic.*

Proof Decompose S as a union of two disjoint infinite sets, say $S = A \cup B$ where $A = \{a_1, a_2, \dots\}$ and $B = \{b_1, b_2, \dots\}$. Let E be the equivalence relation generated by letting $a_n E b_n$ for all $n \in \mathbb{N}$. Suppose A is closed in K. It is straightforward to verify that if $A \cap S$ is finite, then $f^{-1}(f[A]) = A \cup F$, where F is a finite set, and hence is closed in K. If $A \cap S$ is infinite, then as A is closed, it must contain p (as each neighbourhood of p meets a cofinite subset of S). Thus $f^{-1}(f[A]) = A \cup (\{p\} \cup f^{-1}(f[A \cap S]))$. But $\{p\} \cup f^{-1}(f[A \cap S])$ is homeomorphic to the one-point compactification of the discrete space $f^{-1}(f[A \cap S])$, so $f^{-1}(f[A])$ is the union of two closed sets and hence is closed in K. Hence by Theorem 2.20, K/E is a compact Hausdorff space and X is dense in it. For each n there is a function on $f_n \in C^*(K)$ for which $f_n(a_n) = 0$ and $f_n(b_n) = 1$. The restriction of f_n to X obviously cannot be extended to the point represented by a_n. From Theorem 2.18, it follows that $X \to K/E$ is not $\mathcal{CR}$-epic. ■

Corollary 2.22 *If X is an absolute $\mathcal{CR}$-epic space and is first countable at a point p, then X is locally compact at p. In particular, a first countable absolute $\mathcal{CR}$-epic space must be locally compact.*

Proof It is sufficient to observe that if p is a first countable point of X, then it is also first countable in any ambient space in which it is dense. For if $X \subseteq Y$ and $\{U_n\}$ is a countable neighbourhood base at p, then any collection $\{V_n\}$ of neighbourhoods of p in Y for which $V_n \cap X = U_n$ is a neighbourhood base for p. If X is not locally compact at p, then no neighbourhood of p lies entirely inside X and then there is a sequence of points outside X that converges to p. ■

Lemma 2.23 *Suppose K is a compactification of X and T is a closed discrete subset of $K \setminus X$ such that there is some point $p \in \mathrm{cl}_K T$ that is a G_δ point of X. Then there is a sequence $s_1, s_2, \dots$ of points of T that converges to p.*

Proof Since p is a G_δ point of X, there is a nested sequence $\{U_n\}$ of closed neighbourhoods of p in K such that $\bigcap(X \cap U_n) = \{p\}$. Let s_n be any point of $U_n \cap T$ and let $S = \{s_1, s_2, \dots\}$. First we claim that $\operatorname{cl}_{K\setminus X} S$ does not contain any point of K other than p. For if $q \notin X$, then q has some neighbourhood in K whose intersection with $K \setminus X$ contains no point of S (except possibly q) and since $S \subseteq K \setminus X$ this neighbourhood contains no point of S. If $q \in X$, but $q \neq p$, then $q \notin \bigcap U_n$ so there is an n with $q \notin U_n$. Since U_n is closed, $K \setminus U_n$ is a neighbourhood of q that contains at most finitely many points of S, which may be deleted to find a neighbourhood of q that does not meet S. Finally, we see that S, an infinite discrete set in a compact space, cannot be closed, so it has some limit point and the only possibility is p. ∎

Corollary 2.24 *If the points of X are G_δ and there is a compactification K of X such that $K \setminus X$ contains an infinite closed discrete set, then X is not absolute $\mathcal{CR}$-epic.*

Proof It suffices to observe that $K \setminus X$ cannot be closed in K and hence has limit points in X. Then the preceding lemma applies to give us the hypotheses of Theorem 2.21. ∎

Corollary 2.25 *Let X be a space whose points are all G_δ. If X is absolute $\mathcal{CR}$-epic, then for every compactification K of X, the space $K \setminus X$ is countably compact. In particular, if $\beta X \setminus X$ is not countably compact, then X is not absolute $\mathcal{CR}$-epic.*

Proof A space is countably compact if and only if it has no countably infinite closed discrete subset (see [8, 3.10.3]). Now apply the previous corollary. ∎

Corollary 2.26 *A first countable Lindelöf space is absolute $\mathcal{CR}$-epic if and only if it is locally compact. (Note that the space Ψ to be described in §3.3 is first countable, not Lindelöf, and can be absolute $\mathcal{CR}$-epic).*

Lemma 2.27 *Let X be absolute $\mathcal{CR}$-epic and dense in the realcompact space K. Then the following are equivalent:*

(i) *X is z-embedded in K (this means that every zero-set of X extends to a zero-set of K);*
(ii) *every $f \in C(X)$ satisfies a 1×1 zig-zag over $C(K)$.*

Proof (i) $\Rightarrow$ (ii). Let W be the intersection of the realcompact spaces between X and K. Then W is realcompact by [11, 8.9]. By [4, 2.4] X is C-embedded in W. Thus W is a copy of υX and W is absolute $\mathcal{CR}$-epic by Corollary 2.19. Therefore W is Lindelöf by Theorem iii.1. Now obtain the 1×1 zig-zag as follows. Take f on X. It extends to F on W and since W is Lindelöf and absolute $\mathcal{CR}$-epic, by Corollary 2.13, F extends to G on a cozero-set of K between W and K and Proposition 2.11 applies.

(ii) $\Rightarrow$ (i). Let $f \in C(X)$ with zero-set Z, and take a 1×1 zig-zag $f = gah$, with a, ga, and ah in $C(K)$. Then $fa = gaha \in C(K)$. We claim that $Z(fa) \cap X = Z(f)$. In fact, if $x \in X$ and $f(x)a(x) = 0$, then either $f(x) = 0$, in which case $x \in Z(f)$,

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or $a(x) = 0$, in which case $f(x) = g(x)a(x)h(x) = 0$ as well, so that $x \in Z(f)$. The opposite inclusion is obvious. Thus X is z-embedded in K. ∎

Theorem 2.28 (Taxonomy) *An absolute $\mathcal{CR}$-epic space X differs from its Hewitt realcompactification by at most one point, and fits into one of the following three classes:*

Almost compact

In this case $C(X) = C(\beta X)$ and for any embedding $X \to Y$, the induced $C(Y) \to C(X)$ is surjective, thus no zig-zags are needed;

Realcompact

In this case the space is Lindelöf and all zig-zags can be taken to be 1×1;

All others

Any such space has a compactification K and a function $f \in C(X)$ that does not have a 1×1 zig-zag; equivalently X is not z-embedded in K.

Proof As noted in Theorem 2.9, an absolute $\mathcal{CR}$-epic space is almost Lindelöf. Suppose that $|\beta X \setminus X| = 1$ (that is, X is almost compact but not compact). By [11, 8A.4, and 9D.3], it follows that $\upsilon X = \beta X$. But $C(X) = C(\upsilon X)$ so that $C(X) = C(\beta X)$. If X is embedded in Y, then the conclusion follows from the fact that βX is closed in the normal space βY. Next suppose X is absolute $\mathcal{CR}$-epic and realcompact. Then it is also Lindelöf and the result follows from Lemma 2.15.

Finally, suppose that X is absolute $\mathcal{CR}$-epic, but neither realcompact nor almost compact. Since Lindelöf spaces are realcompact, it is not Lindelöf. According to [4, Theorem 4.1], it is not z-embedded in some compactification K. The rest follows by Lemma 2.27. ∎

Theorem 2.29 *Let X be absolute $\mathcal{CR}$-epic. Then the following are equivalent:*

(i) *X is Lindelöf or almost compact;*
(ii) *in every compactification K of X every function extends to a cozero-set of K;*
(iii) *in every compactification K of X every function extends to a realcompact subset of K;*
(iv) *X is z-embedded in each compactification.*

Proof That (i) $\Rightarrow$ (ii) is clear in the almost compact case from [11, 6J5] and in the Lindelöf case from Proposition 2.12 and Corollary 2.13. That (ii) $\Rightarrow$ (iii) follows from [11, 8.14]. The proof that (iii) $\Rightarrow$ (iv) can be done by an argument similar to that of Lemma 2.27. Finally it is shown in [4, 4.1] that (iv) $\Rightarrow$ (i) (actually that they are equivalent). ∎

3 An Absolute $\mathcal{CR}$-Epic Space With Bad Properties

In this section, we will show that the third possibility of the Taxonomy Theorem (2.28) actually arises. This will allow us to produce a class of absolute $\mathcal{CR}$-epic spaces which have a compactification K such that not every function in $C(X)$ can be extended to a cozero-set of K; thus 1×1 zig-zags are not sufficient for the embedding $C(K) \to C(X)$, and hence X is not z-embedded in K.

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3.1 Some (Previously) Open Questions

The existence of this class of spaces answers several questions that have been raised in the literature and elsewhere:

(i) Schwartz [23, following Proposition 4.2] asked whether $\mathcal{CR}$-epic embeddings have to be z-embeddings (see Lemma 5.1 below for a case in which the answer is positive). This gives an example to the contrary.

(ii) [3, Section 3.10] asked whether a G-embedded space need be z-embedded. This gives a negative answer.

(iii) S. Watson [private communication] has asked whether an epimorphism of rings of continuous functions is necessarily an epimorphism of the underlying semigroups. Again the examples here provide a negative answer.

3.2 A Construction

Let L be any non-compact Lindelöf absolute $\mathcal{CR}$-epic space and A be any almost compact space that is not compact. Let K be a compactification of $X = L + A$ (topological sum). If $\mathrm{cl}_K(L) \cap \mathrm{cl}_K(A) = \varnothing$, any function on X has the property that both its restriction to L and to A have 1×1 zig-zags with respect to $\mathrm{cl}_K(L)$ and $\mathrm{cl}_K(A)$, resp. Since $K = \mathrm{cl}_K(L) + \mathrm{cl}_K(A)$, these can be extended to a 1×1 zig-zag for f.

However, we can always find a compactification K of X in which $\mathrm{cl}_K(L) \cap \mathrm{cl}_K(A) \neq \varnothing$. Let $\beta A \backslash A = \{s\}$ and let p be an arbitrary point of $\beta L \backslash L$. The equivalence relation on $\beta(A + L) = \beta A + \beta L$ that identifies s with p is obviously compact and identifies no point of X with any other point of βX so Theorem 2.20 applies.

So suppose X is densely embedded in K and $\mathrm{cl}_K(L) \cap \mathrm{cl}_K(A) \neq \varnothing$. Since A is almost compact and not compact, $\mathrm{cl}_K(A) \setminus A$ consists of exactly one point, which we call p and then $\mathrm{cl}_K(L) \cap \mathrm{cl}_K(A) = \{p\}$. We claim that $\upsilon X = L + A^*$. To see this, first recall that $\beta X = \beta L + \beta A = \beta L + A^*$. Then since L is Lindelöf, it is realcompact and so is $L + A^*$. If $f \in C(X)$, then $f|A$ extends continuously to A^*. Clearly X is dense and C-embedded in $L + A^*$ and it follows that $\upsilon X = L + A^*$.

Theorem 3.1 *A sum of a Lindelöf absolute $\mathcal{CR}$-epic space and an almost compact space is absolute $\mathcal{CR}$-epic*

Proof Let $X = L + A$ with L Lindelöf absolute $\mathcal{CR}$-epic and A almost compact and suppose K is a compactification of X. If $f \in C(X)$, the function $f|A$ extends to a function on A^* and from there to a function g defined on all of K. The function $h = f - g|X$ vanishes on A and, if we can exhibit a zig-zag for h, we can add it to g to get one for f. The fact that L is absolute $\mathcal{CR}$-epic implies that there is a function u defined on $\mathrm{cl}_K(L)$ such that h extends to $\mathrm{coz}(u)$. We can extend u to all of K by normality. Let v be the function defined as $(1 + h^2)/u$ when $u \neq 0$ and 0 otherwise. We claim that

$$h = v \frac{hu^2}{(1 + h^2)^2} v$$

is a zig-zag for h. In fact, on $L \cap \mathrm{coz}(u)$, this reduces directly to h, while it is 0 on A.

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Finally, the product

$$v\frac{hu^2}{(1+h^2)^2} = \frac{1+h^2}{u}\frac{hu^2}{(1+h^2)^2} = \frac{hu}{1+h^2}$$

is defined and continuous on $\mathrm{coz}(u)$. Moreover, it is the product of a function that is bounded everywhere and one that vanishes wherever the the first factor is not continuous, and such a function has a continuous extension everywhere. ■

3.3 Example

Aside from the more obvious examples of this result, here is a naturally occurring one using the space Ψ, which was created by J. Isbell and independently by S. Mrowka; see [11, 5I] for the details and all undefined notation. The construction of Ψ depends on the choice of a maximal family of infinite subsets of $\mathbb{N}$ with the property that the intersection of any two is finite (such a family is called a *maximal almost disjoint* or *mad* family). It turns out that for some such maximal families, Ψ is absolute $\mathcal{CR}$-epic and for others it is not, according to whether Ψ is or is not almost compact.

Now choose a mad family for the construction of Ψ that yields an almost compact space. Let M be a member of the mad family whose complement is infinite. Take $d \in D$ the point at infinity in the one point compactification of M. Let $Y = \Psi \setminus \{d\}$. Then M is clopen in Y and $Y \setminus M$ is the result of the Ψ construction on the set $\mathbb{N} \setminus M$ using the mad family $\{A \setminus M \mid A \in \mathcal{E}\}$. Furthermore $Y \setminus M$ is almost compact by the following argument: $\Psi \setminus (M \cup \{d\})$ is C^*-embedded in Ψ since it is clopen in Ψ; therefore $\Psi \setminus (M \cup \{d\})$ is C^*-embedded in $\beta\Psi$ and hence is C^*-embedded in $\beta\Psi \setminus (M \cup \{d\})$ and is dense as well. Thus the passage to the compact space $\beta\Psi \setminus (M \cup \{d\})$ adds only one point and that makes $Y \setminus M$ almost compact. So Y is the topological union of the countable discrete space M and the almost compact non-compact space $Y \setminus M$ and the preceding theorem applies.

Incidentally, the spaces Ψ, for different choices of a mad family, show that open subsets, zero-sets, and C-embedded subspaces of absolute $\mathcal{CR}$-epic spaces need not be absolute $\mathcal{CR}$-epic. For an open example, take Ψ not almost compact, embedded in its one-point compactification. For C-embedded subsets, take Ψ not almost compact in $\beta\Psi$. For zero-sets, take Ψ almost compact (hence absolute $\mathcal{CR}$-epic) and take as a subspace the space called D in [11, 5I].

By the way, we have not been able to determine whether closed C-embedded subspaces of absolute $\mathcal{CR}$-epic spaces must be absolute $\mathcal{CR}$-epic, nor whether cozero-sets in absolute $\mathcal{CR}$-epic spaces must be absolute $\mathcal{CR}$-epic. But the following resulted from an attempt to resolve the latter question in the almost compact case.

Proposition 3.2 *The following are equivalent for a space X,*

(i) $|\upsilon X \setminus X| \leq 1$ *and υX is locally compact and σ-compact;*
(ii) *X is a cozero-set of an almost compact space A.*

Proof (ii) $\Rightarrow$ (i). If X is a cozero-set in a compact space A, then X is realcompact, locally compact and σ-compact, so $\upsilon X = X$ and has the requisite properties.

Suppose that A is not compact. Let $\beta A = A \cup \{p\}$ and $X = \operatorname{coz}(f)$ for some $f \in C(A)$. Since $\beta A = \upsilon A$ there is a $u \in C(\beta A)$ so that $u|A = f$, and $\operatorname{coz}(u) \cap A = \operatorname{coz}(f) = X$. By [11, 8G1], X is C-embedded and dense in $\operatorname{coz}(u)$, and $\operatorname{coz}(u)$ is realcompact since it is a cozero-set of βA. Therefore $\operatorname{coz}(u) = \upsilon X$ and $|\upsilon X \setminus X| \le 1$ (0 if $p \notin \operatorname{coz}(u)$, 1 otherwise), and υX, as a cozero-set of a compact set, has the required properties.

(i) $\Rightarrow$ (ii). If $\upsilon X = X$ then X is locally compact and σ-compact, so it is open in βX and therefore a cozero-set in the compact space βX, and hence of A.

In the other case, $\upsilon X \setminus X = \{p\}$ and we let $A = \beta X \setminus \{p\}$. Since $X \subseteq A \subsetneqq \beta X$, A is almost compact. Now υX is locally compact and σ-compact by hypothesis, so $\upsilon X \in \operatorname{coz}(\beta X)$. Thus $X = \upsilon X \cap A \in \operatorname{coz}(A)$. ■

This raises the related question : if $|\upsilon X \setminus X| = 1$, and υX is locally compact and σ-compact, is X absolute $\mathcal{CR}$-epic?

4 Spaces That Are Not Absolute $\mathcal{CR}$-Epic

Theorem 4.1 *Let K be a compactification of a space X and suppose that there is a sequence $\{a_n \mid n \in \mathbb{N}\} \subseteq K \setminus X$ so that:*

(i) *for all $n \in \mathbb{N}$, $K \setminus \{a_n\}$ is not almost compact,*

(ii) *$X \cap \operatorname{cl}_K\{a_n \mid n \in \mathbb{N}\} \neq \varnothing$.*

Then there is an $f \in C(X)$ that does not extend continuously to any open set of K that contains X. Thus the embedding of X into K is not $\mathcal{CR}$-epic and hence X is not absolute $\mathcal{CR}$-epic.

Proof By (i) there exists, for each n, a function $g_n \colon K \setminus \{a_n\} \to [0,1]$ that does not extend continuously to K. Thus for each $n \in \mathbb{N}$, $f_n = g_n|X$ does not extend to $X \cup \{a_n\}$ (see [11, 6H]). Define $f = \sum_{n=1}^{\infty} 2^{-n} f_n$. Let $p \in X \cap \operatorname{cl}_K\{a_n\}$. If f extended continuously to an open V such that $X \subseteq V \subseteq K$, then $p \in V$ and $a_n \in V$ for some n. But then f extends to $X \cup \{a_n\}$ which is false. ■

In what follows we will utilize several different "cardinal functions" defined on a topological space. The best reference for them is the survey article by Hodel. Recall from [13, Chapter 3] that a space X has countable π-weight at a point p if there is a countable family (a "π-base") $\{V_n \mid n \in \mathbb{N}\}$ of non-empty open subsets of X such that if U is open in X and $p \in U$, then there is an n_u such that $V_{n_u} \subseteq U$. We write "$\pi\mathrm{w}(X, p) = \aleph_0$". We will use the following well-known result that is a special case of a local version of [13, 7.1].

Theorem 4.2 *Let K be compact and let $p \in K$. If $\{p\}$ is a G_δ in K, then K is first countable at p.*

For a space X, define $L(X)$ to be the set of points of X that have a locally compact neighbourhood. Thus $L(X)$ is the set of points at which X is locally compact.

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Theorem 4.3 *Let X be a space and suppose that either*

(i) *There exists a point $p \in X \setminus \mathrm{cl}_X L(X)$ such that $\pi\mathrm{w}(X, p) = \aleph_0$, or*
(ii) *there exists a point $p \in X \setminus L(X)$ at which X is first countable.*

Suppose further that X has a compactification K that is first countable at a dense set of points of $K \setminus X$. Then the embedding of X into K is not $\mathcal{CR}$-epic.

Proof If (i) holds, let $\{V(n)\}$, $n \in \mathbb{N}$, be a countable π-base at $p \in X$. By replacing each $V(n)$ by $V(n) \setminus \mathrm{cl}_X L(X)$ and discarding empty sets if necessary, we may assume without loss of generality that $V(n) \cap L(X)$ is empty for each n. Let $W(n) = K \setminus \mathrm{cl}_K(X \setminus V(n))$. Clearly $V(n) = W(n) \cap X$. As X is not locally compact at any point of $V(n)$, it follows that $W(n) \setminus X$ is nonempty.

If (ii) holds, let $\{V(n)\}$, $n \in \mathbb{N}$, be a countable neighbourhood base at $p \in X$, and again choose $W(n)$ open in K as above. If $W(n) \subseteq X$, then X would be locally compact at p, in contradiction to our choice of p, so again we must have $W(n) \setminus X$ nonempty.

In either case, by hypothesis we can find, for each $n \in \mathbb{N}$, a point $a_n \in W(n) \setminus X$ such that K is first countable at a_n. We claim that $p \in \mathrm{cl}_K(\{a_n \mid n \in \mathbb{N}\})$. To see this, let S be open in K and contain p, and by the regularity of K choose T open in K such that $p \in T$ and $\mathrm{cl}_K(T) \subseteq S$. As $T \cap X$ is an X-neighbourhood of p, by hypothesis there is an integer k such that $V(k) \subseteq T \cap X$. We will show that S contains $W(k)$ and hence a_k. Assume not, and suppose that $W(k) \setminus S \neq \varnothing$. Then $W(k) \setminus \mathrm{cl}_K(T)$ is nonempty and open in K, so it has nonempty intersection with X. Thus $(K \setminus \mathrm{cl}_K(X \setminus V(k)) \cap (K \setminus \mathrm{cl}_K(T)) \cap X \neq \varnothing$. Now $X \cap \mathrm{cl}_K(X \setminus V(k)) = X \setminus V(k)$ and $X \cap \mathrm{cl}_K(T) = \mathrm{cl}_X(T \cap X)$. So we deduce that $V(k) \cap (X \setminus \mathrm{cl}_X(T \cap X)$ is nonempty. But $V(k)$ was chosen to be a subset of $T \cap X$, so we have a contradiction. Since a_n is a point of first countability of K, it follows that for every n, $K \setminus \{a_n\}$ is not almost compact. The result now follows from Theorem 4.1. ■

Corollary 4.4 *No dense subset of $\beta\mathbb{Q} \setminus \mathbb{Q}$ is absolute $\mathcal{CR}$-epic. In particular, there is a countable extremally disconnected space that is not absolute $\mathcal{CR}$-epic (there is a countable dense set of remote points in $\beta\mathbb{Q}$; see* [5, Section 12]*).*

Proof Observe that $\beta\mathbb{Q}$ has countable π-weight and has a dense extremally disconnected subspace E. A countable dense subset T of E will be extremally disconnected nowhere locally compact. Now apply Theorem 4.3(i) to the space T and its compactification $\beta\mathbb{Q}$. The corollary follows. ■

Corollary 4.5 *Assume that either*

(i) *There exists a point of $X \setminus \mathrm{cl}_X L(X)$ of countable π-weight in X, or*
(ii) *there exists a point of $X \setminus L(X)$ at which X is first countable.*

Assume further that X is σ-compact and has a compactification K such that $K \setminus X$ has a dense set of G_δ-points. Then the embedding of X into K is not $\mathcal{CR}$-epic.

Proof As X is σ-compact, $K \setminus X$ is a G_δ set of K. Thus a G_δ-point of $K \setminus X$ will be a G_δ-point of K, and hence a point of first countability of K by Theorem 4.2. The hypotheses of Theorem 4.3 are thus satisfied, and the result in our two cases follows from the corresponding cases of Theorem 4.3. ■

A space is called *Čech-complete* (or an *absolute G_δ*) if it is a G_δ set in a compact space. It is an immediate consequence of Theorem 4.2 that if p is a G_δ point of a Čech-complete space Y, then Y is first countable at p.

The special case of the following theorem in which Y is compact appears in [8, Problem 3.12.11(a)] and [13, 7.19]. The more general case is also mentioned in [8, Problem 3.12.11(b)].

Theorem 4.6 *If Y is a Čech-complete space with no points of first countability then $|Y| \geq 2^{\aleph_1}$.*

If $2^{\aleph_1} = 2^{\aleph_0}$ (as happens in some models of set theory), this tells us nothing helpful. So we will assume Lusin's hypothesis ($2^{\aleph_1} > 2^{\aleph_0}$), which is a weakening of the continuum hypothesis, in some of what follows.

Theorem 4.7 *Assume Lusin's hypothesis. Let Y be Čech-complete and assume that $|Y| \leq 2^{\aleph_0}$. Then Y is first countable at a dense set of points.*

Proof Let $S = \{y \in Y \mid Y \text{ is first countable at } y\}$ and let $\varnothing \neq T = Y \setminus \operatorname{cl} S$. Then T is Čech-complete because it is an open set in Y that is Čech-complete. If q were a point of first countability of T, then $\{q\}$ would be a G_δ of T and hence one of Y. By Theorem 4.2, q is a point of first countability of Y contradicting the definition of S. Thus T is Čech-complete with no points of first countability, so by Theorem 4.6, $|T| \geq 2^{\aleph_1} > 2^{\aleph_0}$. This contradicts the assumption made on the cardinality of Y. ■

Theorem 4.8 *Assume Lusin's hypothesis. Let X be a σ-compact space such that either*

(i) *There exists a point of $X \setminus \operatorname{cl}_X L(X)$ of countable π-weight in X, or*
(ii) *there exists a point of $X \setminus L(X)$ at which X is first countable.*

If K is a compactification of X of cardinality $\leq 2^{\aleph_0}$ then the embedding of X into K is not $\mathcal{CR}$-epic.

Proof As X is σ-compact, $K \setminus X$ is Čech-complete. Hence by Theorem 4.7, $K \setminus X$ is first countable at a dense set of points. The result now follows from Corollary 4.5. ■

Corollary 4.9 *Let X be a countable absolute $\mathcal{CR}$-epic space without isolated points. Assume Lusin's hypothesis. Then either X has uncountable π-weight at each of its points, or each compactification of X has cardinality greater than $2^{\aleph_0}$.*

Proof Countable spaces without isolated points are σ-compact nowhere locally compact, so the result follows from the contrapositive of Theorem 4.7. ■

5 P-Spaces and Almost-P-Spaces

Recall that X is called a P-space if each zero-set is open; it is called an almost-P-space if each zero-set has a non-empty interior; and it is called an F-space if every cozero-set is C^*-embedded.

Lemma 5.1 *Let X be a P-space and assume that X is $\mathcal{CR}$-epic in K. Then X is z-embedded in K.*

Proof By working with βK one sees easily that it suffices to consider the case when K is compact. By [12, 8.2], X is z-embedded in K if and only if $C(K)$ is relatively uniformly dense (rud) in $C(X)$; see [12, p. 129] for the definition of rud.

Since X is absolute $\mathcal{CR}$-epic in K, $C(K) \to C(X)$ is $\mathcal{W}$-epic, for $\mathcal{W}$ the category of Archimedean ℓ-groups with unit, [12, p. 114]. Since X is a P-space, $C(X)$ is epicomplete (*cf.* [1, Theorem 2.1 and section 5]) so that $C(K) \to C(X)$ is an epicompletion of $C(K)$ in W. Now by [2, Theorem 16], $C(K)$ is relatively uniformly dense in $C(X)$. ■

Thus the answer to Schwartz's question (see the start of §3) is "yes" for P-spaces. Note that results Theorem 2.29 and Theorem 3.1 show that the answer to the general question is "no".

Theorem 5.2 *A P-space is absolute $\mathcal{CR}$-epic if and only if it is Lindelöf.*

Proof A theorem due originally to Jerison shows that a Lindelöf space is z-embedded in any compactification; see [4, 4.1(c)]. If it is also a P-space, then the range of every real-valued function is countable and, as noted in the proof of [3, 4.1(ii)], each real-valued function on X extends to a cozero-set of K containing X. Thus X is absolute $\mathcal{CR}$-epic by Proposition 2.11.

Conversely, suppose that X is absolute $\mathcal{CR}$-epic. By the preceding lemma, X is z-embedded in each compactification K, and by [4, 4.1] X is either almost compact or Lindelöf. But almost compact P-spaces are realcompact (pseudocompact P-spaces are finite [11, 4K(2)]). Thus X is Lindelöf. ■

Note that the countable space of Corollary 4.4 shows that a Lindelöf F space need not be absolute $\mathcal{CR}$-epic.

It has been shown by [7] that there is a model of set theory in which $\beta\mathbb{N} \setminus \mathbb{N}$ has a proper dense C^*-embedded subset, say X. In this model, there exists an almost compact (therefore absolute $\mathcal{CR}$-epic), almost-P, non-compact space Y as follows: let $p \in (\beta\mathbb{N} \setminus \mathbb{N}) \setminus X$ and let $Y = (\beta\mathbb{N} \setminus \mathbb{N}) - \{p\})$. Since X is C^*-embedded in the compact space $\beta\mathbb{N} \setminus \mathbb{N}$, Y is as well, so Y is almost compact. Since Y is dense in the almost-P-space $\beta\mathbb{N} \setminus \mathbb{N}$, it is also an almost-$P$-space [16]. Clearly, Y is not compact. Thus a non-compact, almost-P-space can be absolute $\mathcal{CR}$-epic.

This leaves open the question of whether one can show in ZFC that there exists an almost-P, almost compact, non-compact space. In any case, in the model used above, this absolute $\mathcal{CR}$-epic space exists, and it is therefore possible to do the construction of Section 3 using exclusively almost-P-spaces.

6 Some Open Questions

1. If an absolute $\mathcal{CR}$-epic space X is embedded in some space Y, can every function in $C(X)$ be written as a symmetric diagonal zig-zag, meaning one of the form GDG^T, where D is a diagonal matrix whose entries, along with those of GD (and hence DG^T) lie in $C(Y)$?
2. Can there be uniform bound on the size of zig-zags of $\mathcal{CR}$-epic embeddings?
3. Is the sum of an absolute $\mathcal{CR}$-epic Lindelöf space and an arbitrary absolute $\mathcal{CR}$-epic space, absolute $\mathcal{CR}$-epic? It is easy to give an example of a countable metric space that is not locally compact, but is the union of a countable set of isolated points and a single point, (*e.g.*, $A_0 \cup A_2$ from [3, 3.12]). This space is not absolute $\mathcal{CR}$-epic by Corollary 2.22, so being the union of a (dense) open absolute $\mathcal{CR}$-epic Lindelöf subspace and a compact subspace does not suffice to make a space absolute $\mathcal{CR}$-epic. As well, being scattered does not suffice to make a countable space absolute $\mathcal{CR}$-epic.
4. If X is countable and nowhere locally compact, can it be absolute $\mathcal{CR}$-epic? For example, the space $[0,1]^c$ has a countable dense subspace that is of uncountable π-weight; is that subspace absolute $\mathcal{CR}$-epic? If so, then countable absolute $\mathcal{CR}$-epic spaces need not have isolated points. Is the space of van Douwen and Pryzmusinski, found in [6], absolute $\mathcal{CR}$-epic?

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