ON THE NUMBER OF LIMIT CYCLES IN PERTURBATIONS
OF A QUADRATIC REVERSIBLE CENTER

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Abstract

This paper is concerned with the bifurcation of limit cycles from a quadratic reversible system under polynomial perturbations. It is proved that the cyclicity of the period annulus is two, and also a linear estimate of the number of zeros of the Abelian integral for the system under polynomial perturbations of arbitrary degree n is given.

1. Introduction

The weak version of Hilbert' 16th problem, asking what is the number of limit cycles for the polynomial perturbation of planar systems with centers, is still open even for the quadratic case (see [1]). Zoladek [14] classified planar quadratic systems with at least one center into four types, Q^H, Q_3^R, Q_3^{LV} and Q_4, and conjectured that the cyclicity of the periodic annulus of quadratic systems is two. The conjecture has been proved completely for the cases Q^H and Q_4 (see [2, 5, 6]). Because of the rich dynamical behaviors, mathematicians focus on the case Q_3^R. For $Q_3^R \setminus Q^H$, since the integrating factor is no longer a constant, the problem is much more difficult. After an affine transformation, any system of $Q_3^R \setminus Q^H$ can be written as

$$\begin{cases} \dot{x} = ax^2 + by^2 - 2(b-1)y + b - 2, \\ \dot{y} = -2xy, \end{cases} \tag{1.1}$$

where $a, b \in R$. Excellent results on limit cycles of system (1.1) under quadratic perturbations have been obtained in recent years. In [3], the authors considered the case $a = -1/2$ and $\{b : -1 < b < 1\}$, and they presented the upper bound of the limit

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cycles and the corresponding bifurcation diagrams. The case where $a = -3/2$ has been investigated in [9, 13]: by using some complex methods, C. Liu (unpublished) proved that four limit cycles could be found for $\{b : 0 < b < 2\}$ with $(3, 1)$-configuration. Liang [9] and Zhao [13] considered the cases where $\{b : b < 0, b = 2\}$ and $\{b : b > 2\}$, and proved that the least upper bound of the number of limit cycles is two.

In this paper, we will study the case where $a = -3/4$ and $b = 0$ in system (1.1), which after an affine transformation becomes

$$\begin{cases} \dot{x} = xy, \\ \dot{y} = \frac{3}{8}y^2 + \frac{9}{2^{10}} (1 - x). \end{cases} \tag{1.2}$$

Our main results are the following theorems.

THEOREM 1.1. *Under quadratic perturbations, the exact upper bound on the number of limit cycles bifurcating from the period annulus of integrable system (1.2) is two, that is, the cyclicity of the periodic annulus of system (1.2) is two.*

THEOREM 1.2. *Under polynomial perturbations of arbitrary degree n, the upper bound of the number (taking multiplicities into account) of limit cycles arising from the period annulus of integrable system (1.2) does not exceed* $30n - 37$.

As a complement to [3, 9, 13], Theorem 1.1 partly verifies the conjecture given in [14] for system (1.2), while Theorem 1.2 improves the linear estimate $30n + 20$ of the zeros of the associated Abelian integrals in [11].

2. The Abelian integral and Picard–Fuchs equation

As a planar integrable system, there is a C^1 function which is constant along any solution curve of the system, and the integrable system (1.2) has first integral

$$H(x, y) = x^{-3/4}(\tfrac{1}{2}y^2 + \tfrac{9}{2^8}x + \tfrac{3}{2^8}) = h \tag{2.1}$$

with the integrating factor $M(x, y) = x^{-7/4}$.

We denote the closed connected component of the real curve $H(x, y) = h$ by Γ_h, defined on the maximal open interval $S = (3/2^6, +\infty)$. The phase portrait of system (1.2) in the Poincaré disk is shown in Figure 1.

LEMMA 2.1. *Under quadratic perturbations, the Abelian integral $I(h)$ related to system (1.2) can be expressed as*

$$I(h) = \alpha I_0(h) + \beta I_1(h) + \gamma I_{-1}(h),$$

where α, β and γ are constants, and $I_i(h) = \oint_{\Gamma_h} x^{i-7/4}y \, dx$ with $i = -1, 0$ and 1.

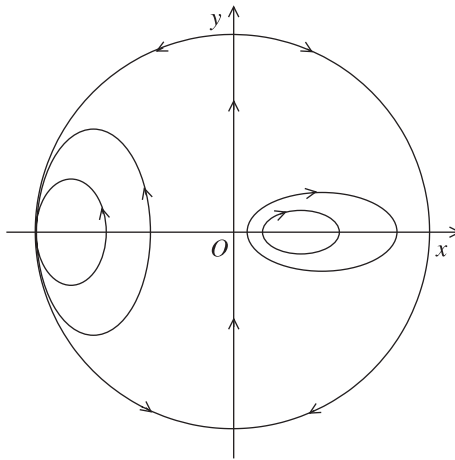PROOF. The conclusion follows directly from [8, Theorem 2]. □

FIGURE 1. Phase portrait of system (1.2) in the Poincaré disk.

By the standard method in [2], we get the Picard–Fuchs equation in the following lemma.

LEMMA 2.2. *For system (1.2), the vector function $J(h) = \mathrm{col}(I_0(h), I_1(h), I_{-1}(h),$ $I_{1/4}(h))$ satisfies the Picard–Fuchs equation as follows:*

$$J(h) = M_1(h)J'(h), \tag{2.2}$$

where

$$M_1(h) = \begin{pmatrix} h & 0 & 0 & -\dfrac{3}{2^6} \\[2mm] 0 & \dfrac{1}{3}h & 0 & -\dfrac{1}{2^6} \\[2mm] -\dfrac{6}{7}h & 0 & \dfrac{3}{7}h & \dfrac{9}{7 \times 2^6} \\[2mm] \dfrac{3^5}{5 \times 2^{20}h^2} & -\dfrac{3^5}{2^{20}h^2} & \dfrac{2 \times 3^4}{5 \times 2^{20}h^2} & \dfrac{3}{2}h \end{pmatrix}.$$

LEMMA 2.3. *The ratio $\varphi(h)/h$ satisfies the equation*

$$\left(\frac{\varphi(h)}{h}\right)' = \left(m + \frac{2^8}{3}k - \frac{6k\mathcal{H}}{h^2}\right)I_0''(h) + \left(n - \frac{2k\mathcal{H}}{3h^2}\right)I_{-1}''(h), \tag{2.3}$$

where $\varphi(h) = I(h) - hI'(h)/3$, $\mathcal{H} = 3^5 \times 2^{-20}h^{-2}/5$, m, n and k are constants relevant to α, β and γ.

PROOF. Differentiating (2.2) with respect to h yields

$$M_2(h)J'(h) = M_1(h)J''(h), \tag{2.4}$$

where

$$M_2(h) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \dfrac{2}{3} & 0 & 0 \\ \dfrac{6}{7} & 0 & \dfrac{4}{7} & 0 \\ \dfrac{3^5}{5 \times 2^{19} h^3} & -\dfrac{3^5}{2^{19} h^3} & \dfrac{2 \times 3^4}{5 \times 2^{19} h^3} & -\dfrac{1}{2} \end{pmatrix}.$$

The first and second rows of (2.4) give

$$I''_{1/4}(h) = \tfrac{2^6}{3} h I''_0(h), \tag{2.5}$$

$$I'_1(h) = \tfrac{1}{2} h I''_1(h) - \tfrac{3}{2^7} I''_{1/4}(h). \tag{2.6}$$

Substituting (2.5) and (2.6) into the third row of (2.4),

$$\tfrac{3}{2} h I''_{-1}(h) - \tfrac{3}{2} h I''_0(h) = 3 I'_0(h) + 2 I'_{-1}(h). \tag{2.7}$$

Similarly, the first row of (2.4) combined with (2.5)–(2.7) yields

$$I'_{1/4}(h) = 6 \mathcal{H} I''_0(h) + \tfrac{2}{3} \mathcal{H} I''_{-1}(h) - 2^6 h^2 I''_0(h). \tag{2.8}$$

By using the Picard–Fuchs function, we get

$$\begin{aligned} I(h) &= (\alpha - \tfrac{6}{7}\gamma) h I'_0(h) + \tfrac{1}{3}\beta h I'_1(h) + \tfrac{3}{7}\gamma h I'_{-1}(h) \\ &\quad + (-\tfrac{3}{2^6}\alpha - \tfrac{1}{2^6}\beta + \tfrac{9}{2^6 \times 7}\gamma) I'_{1/4}(h). \end{aligned} \tag{2.9}$$

Meanwhile, differentiating both sides of (1.2) with respect to h directly yields

$$I'(h) = \alpha I'_0(h) + \beta I'_1(h) + \gamma I'_{-1}(h). \tag{2.10}$$

From (2.9) and (2.10), we get

$$\varphi(h) = (\tfrac{2}{3}\alpha - \tfrac{6}{7}\gamma) h I'_0(h) + \tfrac{2}{21}\gamma h I'_{-1}(h) + (-\tfrac{3}{2^6}\alpha - \tfrac{1}{2^6}\beta + \tfrac{9}{2^6 \times 7}\gamma) I'_{1/4}(h),$$

which implies that

$$\left(\frac{\varphi(h)}{h}\right)' = m I''_0(h) + n I''_{-1}(h) + \frac{k}{h} I''_{1/4}(h) - \frac{k}{h^2} I'_{1/4}(h), \tag{2.11}$$

where

$$m = 2\alpha/3 - 6\gamma/7, \quad n = 2\gamma/21, \quad k = -3 \times 2^{-6}\alpha - 2^{-6}\beta + 9 \times 2^{-6}\gamma/7.$$

Substituting (2.5) and (2.8) into (2.11), we get (2.3).   □

REMARK 2.4. From equation (2.4), we obtain

$$G(h)J''(h) = M_3(h)J'(h), \tag{2.12}$$

where $G(h) = h(16777216h^4 - 81)$ and

$$M_3(h) = \begin{pmatrix} \frac{81}{5} & 0 & \frac{54}{5} & -262144h^3 \\ \frac{81}{5} & 33554432h^4 - 16 & \frac{54}{5} & -262144h^3 \\ 33554432h^4 - \frac{729}{5} & 0 & \frac{67108864}{3}h^4 - \frac{486}{5} & -262144h^3 \\ \frac{1728}{5}h & 0 & \frac{1152}{5}h & -\frac{16777216}{3}h^4. \end{pmatrix}.$$

The first array of (2.12) yields

$$G(h)I_0''(h) = \tfrac{1}{5}f(h),$$

where

$$f(h) = 81I_0'(h) + 54I_{-1}'(h) - 1310720h^3 I_{1/4}'(h).$$

By direct calculation, we get

$$G(h)f'(h) = -f^2(h) + \frac{G(h)}{h}f(h) + g(h).$$

where

$$g(h) = 1310720h^2(67108864h^5/3 - 108h - 3)I_{1/4}'(h)/I_0'(h).$$

Obviously $f(3/2^6) = 0$, then $f'(3/2^6) = (g(h)/G(h))|_{h=3/2^6}$. If $f(h)$ has another zero point for $h \in S$, without loss of generality, we suppose that $h \in S$ is adjacent to $3/2^6$ such that $f(h_0) = 0$, then $f'(h_0)f'(3/2^6) < 0$, which contradicts $g(h) > 0$ for $h \in [3/2^6, +\infty)$.

Hence $f(h)$ has no zero point for $h \in S$, which implies that $I_0''(h) \neq 0$.

LEMMA 2.5. *The ratio $v(h) = I_{-1}''(h)/I_0''(h)$ satisfies the Riccati equation as follows:*

$$\tilde{G}(h)v'(h) = -a_{12}v^2(h) + (a_{22} - a_{11})v(h) + a_{21}, \tag{2.13}$$

where $\tilde{G}(h) = 6h(16777216h^4 - 81)$ and

$$(a_{ij})_{2\times2} = \begin{pmatrix} -729 - 234881024h^4 & -81 \\ -2187 + 67108864h^4 & -243 + 33554432h^4 \end{pmatrix}.$$

PROOF. Differentiating (2.7) and (2.8) with respect to h, we get

$$-I_0'''(h) + I_{-1}'''(h) = 3h^{-1}I_0''(h) + \tfrac{1}{3}h^{-1}I_{-1}''(h), \tag{2.14a}$$

$$(6\mathcal{H} - 2^6h^2)I_0'''(h) + \tfrac{2}{3}\mathcal{H}I_{-1}'''(h) = (\tfrac{7}{3} \times 2^6h - 6\mathcal{H}'(h))I_0''(h) + \tfrac{2}{3}\mathcal{H}'(h)I_{-1}''(h). \tag{2.14b}$$

The definition of $v(h)$ implies that

$$v'(h) = \frac{I_{-1}'''(h)}{I_0''(h)} - v(h)\frac{I_0'''(h)}{I_0''(h)}. \tag{2.15}$$

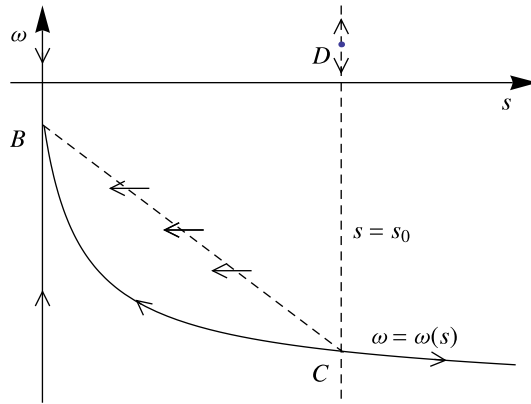Substituting (2.14a) and (2.14b) into (2.15), we get Lemma 2.5. □

FIGURE 2. Behaviors of system (3.1).

3. Global behaviors of the auxiliary curve and centroid curve

Let $s = h^4$, $\omega(s) = v(h)$, the Riccati function (2.13) is equivalent to the two-dimensional system

$$\begin{cases} \dot{s} = 24s(16777216s - 81), \\ \dot{\omega} = 81\omega^2 + (486 + 268435456s)\omega + 67108864s - 2187, \end{cases} \tag{3.1}$$

which has four singularities: two saddles $A(0, 3)$ and $C(81/16777216, -23)$, a stable node $B(0, -9)$ and an unstable node $D(81/16777216, 1)$; see Figure 2. We denote the auxiliary curve on the (s, ω)-plane by C_ω:

$$C_\omega = \{(s, \omega(s)) : \omega(s) = v(h), s \in (81/16777216, +\infty)\}.$$

LEMMA 3.1. C_ω is located on the unstable manifold of the saddle C.

PROOF. Let $P(h) = I_1(h)/I_0(h)$, $Q(h) = I_{-1}(h)/I_0(h)$, $R(h) = I_{1/4}(h)/I_0(h)$. Since the level curve Γ_h shrinks to the center $(1, 0)$ as h tends to $3/2^6$, by the mean value theorem, $P(h)$, $Q(h)$ and $R(h)$ converge to 1 as h tends to $3/2^6$.

Differentiating (2.2) with respect to h, we obtain

$$\tilde{G}(h)J''(h) = M_4(h)J(h), \tag{3.2}$$

where

$$M_4(h) = \begin{pmatrix} \dfrac{1701}{5h} & -\dfrac{729}{h} & \dfrac{1134}{5h} & -1048576h^2 \\ -\dfrac{1701}{5h} & \dfrac{9(81 + 67108864h^4)}{h} & -\dfrac{1134}{5h} & 5242880h^2 \\ \dfrac{7(-2187 + 335544320h^4)}{5h} & \dfrac{6561}{h} & \dfrac{6561}{h} & 9437184h^2 \\ \dfrac{36288}{5} & -15552 & \dfrac{24192}{5} & -\dfrac{67108864}{3}h^3 \end{pmatrix}.$$

We denote $M_4(h)$ by the form $(b_{ij})_{4 \times 4}$. From the first and third arrays of equation (3.2), we get

$$\nu(h) = \frac{b_{31} + b_{32}P(h) + b_{33}Q(h) + b_{34}R(h)}{b_{11} + b_{12}P(h) + b_{13}Q(h) + b_{14}R(h)}. \tag{3.3}$$

Substituting (2.1) into (3.3), we obtain $\omega(81/16777216) = -23$.

Therefore, we get the stable and unstable manifolds of the saddle C:

$$W_C^s = \{(s, \omega(s)) : s = \tfrac{81}{16777216}\},$$
$$W_C^u = \{(s, \omega(s)) : \omega(s) = \nu(h), s \in (\tfrac{81}{16777216}, +\infty)\}.$$

This concludes the proof. □

Moreover, we notice that there exists an isoclinic line $\omega = \tilde{\omega}(s)$ defined by $\dot{\omega} = 0$ between B and C. Comparing the slope of C_ω with $\omega = \tilde{\omega}(s)$, we can extend C_ω from C to B and determine its position (see Figure 2).

LEMMA 3.2. C_ω *is globally convex and strictly decreasing.*

PROOF. Since the curve C_ω is located on the unstable manifold of the saddle C, we can suppose that $\omega(s)$ has the following expansion near $C(81/16777216, -23)$:

$$\omega(s) = -23 + \omega_1(s - \tfrac{81}{16777216}) + \frac{\omega_2}{2!}(s - \tfrac{81}{16777216})^2 + \cdots. \tag{3.4}$$

According to system (3.1), we substitute (3.4) into the equation

$$\dot{\omega} - \tilde{G}(h)\frac{d\omega}{ds} = 0. \tag{3.5}$$

From (3.5), we obtain

$$\omega_1 = -\frac{381681664}{234}, \quad \omega_2 = \frac{74841557479456768}{531441}.$$

Hence, the curve C_ω is convex near the left endpoint.

Next, we claim that C_ω is globally convex. Otherwise, without loss of generality, we suppose that C_ω has an inflection points right to $s = 81/16777216$, then we find a straight line L on the (s, ω)-plane such that it cuts C_ω at at least three points and intersects $s = 81/16777216$ below the saddle point $(81/16777216, -23)$, leading to at least three points on L, at which the vector field (3.1) is tangent to L. This contradicts the fact that $(\dot{\omega} - a\dot{s})|_{\omega=as+b}$ is quadratic in s, where a, b are arbitrary constants. Therefore the curve C_ω is globally convex for $s \in (81/16777216, +\infty)$.

Using a horizontal line $\omega = \omega_0$ instead of the straight line L, a similar argument verifies the monotonicity of C_ω. □

We denote the centeriod curve on the (P, Q)-plane by $\Sigma = \{(P, Q)(h) : h \in S\}$.

LEMMA 3.3. *The curve Σ is increasing.*

PROOF. Using arguments as in [10, Theorem 2], we obtain $P'(h) > 0$, $Q'(h) > 0$ for $h \in S$, which implies Lemma 3.3. □

LEMMA 3.4. *The curve Σ is concave for $0 < h - 3/2^6 \ll 1$.*

PROOF. From (2.2), we obtain

$$\tilde{G}(h)J'(h) = M_5(h)J(h), \tag{3.6}$$

where

$$M_5(h) = \begin{pmatrix} -\frac{4131}{5} + 100663296h^4 & 2187 & -\frac{1134}{5} & 3145728h^3 \\ -\frac{1701}{5} & 9(81 + 33554432h^2) & -\frac{1134}{5} & 3145728h^3 \\ -\frac{6561}{5} + 201326592h^4 & 2187 & -\frac{6804}{5} + 234881024h^4 & 3145728h^3 \\ -\frac{36288}{5}h & 46656h & -\frac{24192}{5}h & 67108864h^4 \end{pmatrix}.$$

For convenience, we denote $M_5(h)$ as $(c_{ij})_{4 \times 4}$. From (3.6), we get the four-dimensional system

$$\begin{cases} \dot{h} = \tilde{G}(h), \\ \dot{P} = c_{21} + c_{22}P + c_{23}Q + c_{24}R - P(c_{11} + c_{12}P + c_{13}Q + c_{14}R), \\ \dot{Q} = c_{31} + c_{32}P + c_{33}Q + c_{34}R - Q(c_{11} + c_{12}P + c_{13}Q + c_{14}R), \\ \dot{R} = c_{41} + c_{42}P + c_{43}Q + c_{44}R - R(c_{11} + c_{12}P + c_{13}Q + c_{14}R). \end{cases} \tag{3.7}$$

The asymptotic expansions of $P(h)$ and $Q(h)$ can be expressed near the hyperbolic singular point $(3/2^6, 1, 1, 1)$ as

$$\begin{aligned} P(h) &= 1 + p_1(h - \tfrac{3}{2^6}) + p_2(h - \tfrac{3}{2^6})^2 + \cdots, \\ Q(h) &= 1 + q_1(h - \tfrac{3}{2^6}) + q_2(h - \tfrac{3}{2^6})^2 + \cdots. \end{aligned} \tag{3.8}$$

For simplicity, we take the calculation of the coefficient of $P(h)$ as an example. Notice that $P'(h)\dot{h} - \dot{P} = 0$. Substituting (3.7) and (3.8) into it, we get

$$p_1 = \tfrac{64}{3}, \qquad p_2 = \tfrac{78848}{243}.$$

Similarly,

$$q_1 = \tfrac{320}{9}, \qquad q_2 = \tfrac{148480}{729}.$$

Hence

$$\frac{d^2\tilde{Q}}{dP^2}\bigg|_{h=3/2^6} = -\frac{20}{27}.$$

Therefore, Lemma 3.4 holds. □

LEMMA 3.5. *The curve Σ is concave for $0 < 1/h \ll 1$.*

PROOF. Since Γ_h is symmetric with respect to the x-axis, we need only consider the part above the x-axis. Denoting the intersection points of Γ_h with the x-axis by $(\xi_h, 0)$ and $(\eta_h, 0)$, where $0 < \xi_h < 1 < \eta_h$, we can estimate ξ_h and η_h as h tends to positive infinity,

$$\xi_h \sim (3^{-1} \times 2^8 \times h)^{-4/3}, \quad \eta_h \sim (3^{-2} \times 2^8 \times h)^4.$$

From the definition of $I_0(h)$,

$$
\begin{aligned}
I_0(h) &= 2 \int_{\xi_h}^{\eta_h} x^{-7/4} \sqrt{2hx^{3/4} - \frac{9}{27}x - \frac{3}{27}} \, dx \\
&= 2^{3/2} h^{1/2} \int_{\xi_h}^{\eta_h} x^{-11/8} \, dx \\
&\quad + \frac{1}{2} \int_{\xi_h}^{\eta_h} x^{-11/8} \sum_{n=1}^{+\infty} \frac{\frac{1}{2}\left(\frac{1}{2} - 1\right) \cdots \left(\frac{1}{2} - n + 1\right)}{n!} \left(-\frac{9}{2^8}x^{1/4} - \frac{3}{2^8}x^{-3/4}\right)^n h^{-n} \, dx \\
&= a_0 h + a_1 h^{-1/3} + o(h^{-1})
\end{aligned}
$$

where $a_0 = 2^{15/2} 3^{-3/2} B(1/2, 3/2) > 0$, $a_1 = -2^{25/6} 3^{5/6} B(1/6, 1/2) < 0$ and $B(\alpha, \beta)$ is the beta function.

By a similar calculation,

$$
\begin{aligned}
I_1(h) &= b_0 h^3 + b_1 h^{-1/3} + o(h^{-3}), \\
I_{-1}(h) &= c_0 h^{7/3} + c_1 h^{-1/3} + o(h^{-5/3}),
\end{aligned}
$$

where $b_0 = 2^{47/2} 3^{-5} B(5/2, 3/2)$ and $c_0 = 2^{109/6} 3^{-17/6} B(11/6, 3/2)$.

The expansion series of $I_i(h)$ in h with $i = 0, 1$ and -1 show that, as h tends to positive infinity,

$$P(h) \sim \frac{b_0}{a_0} h^2, \quad Q(h) \sim \frac{c_0}{a_0} h^{4/3},$$

which implies that

$$\frac{d^2 Q}{dp^2} \sim -\frac{2a_0 c_0}{9b_0^2} h^{-8/3}.$$

Thus the statements of Lemma 3.5 holds.   □

4. Proof of Theorem 1.1

We denote the maximum number (taking multiplicities into account) of zeros of function $I(h)$ for $h \in S$ by $\#\{I(h), S\}$. Since $h \neq 0$ and $I(3/2^6) = 0$,

$$
\begin{aligned}
\#\{I(h), S\} &= \#\{h^{-3} I(h), S\} \\
&\leq \#\left\{-\frac{3}{h^4} \varphi(h), S\right\} \\
&\leq \#\left\{\left(\frac{\varphi(h)}{h}\right)', S\right\} + 1.
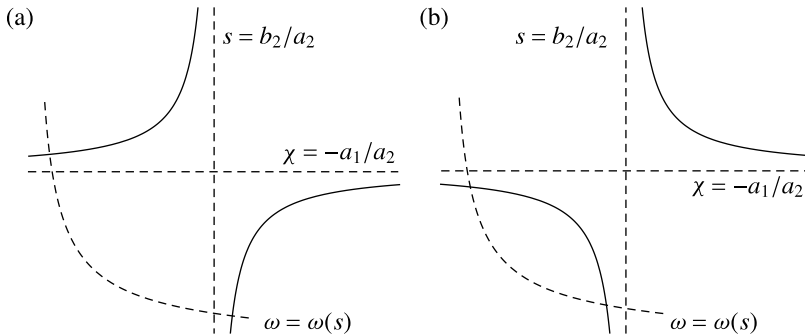\end{aligned}
\tag{4.1}
$$

FIGURE 3. Behaviors of C_ω and C_χ.

LEMMA 4.1. $\#\{I(h), S\} \leq 3$.

PROOF. From (4.1), we only need to prove that $\#\{(\varphi(h)/h)', S\} \leq 2$.

Equation (2.9) can be rewritten as follows:

$$\left(\frac{\varphi(h)}{h}\right)' = \frac{I_0''(h)}{5 \times 2^{20}h^4}\left(\left(5 \times 2^{20}\left(m + \frac{2^8}{3}k\right)h^4 - 6k\right) + \left(5 \times 2^{20}nh^4 - \frac{2}{3}k\right)\frac{I_{-1}''(h)}{I_0''(h)}\right).$$

Let

$$\Phi(s) = (a_1 s - b_1) + (a_2 s - b_2)\omega(s), \tag{4.2}$$

where $a_1 = 5 \times 2^{20}(m + 2^8 k/3)$, $b_1 = 6k$, $a_2 = 5 \times 2^{20}n$, and $b_2 = 2k/3$. Obviously, the following equality holds:

$$\#\left\{(\Phi(s), \left(\frac{81}{16777216}, +\infty\right)\right\} = \#\left\{\left(\frac{\varphi(h)}{h}\right)', S\right\}.$$

If $a_2 s - b_2 = 0$, then $\Phi(s)$ has at most one zero point.

When $a_2 s - b_2 \neq 0$, from (4.2) we can obtain

$$\Phi(s) = (a_2 s - b_2)(-\chi(s) + \omega(s))$$

where $\chi(s) = -(a_1 s - b_1)/(a_2 s - b_2)$. Hence, $\#\{(\Phi(s), (81/16777216, +\infty)\}$ is equal to the number of intersection points of C_χ and C_ω, where C_χ is the graphic of $\chi = \chi(s)$ for $s \in (81/16777216, +\infty)$.

For $a_1 b_2 - a_2 b_1 = 0$, it is easy to see that C_χ is a straight line and cuts C_ω at at most at one point. For $a_1 b_2 - a_2 b_1 \neq 0$, without loss of generality, suppose that $a_2 \neq 0$; then C_χ is a hyperbola with two asymptotes $s = b_2/a_2$ and $\chi(s) = -a_1/a_2$. If C_χ is increasing, by the monotonicity of C_ω, we conclude they have at most two intersection points (see Figure 3(a)). If C_χ is decreasing, shown in Figure 3(b), by the monotonicity of C_ω, it only intersects with one branch of the hyperbola. If it happens to the left branch, because the convexities of the two curves C_ω and C_χ are different, they have at most
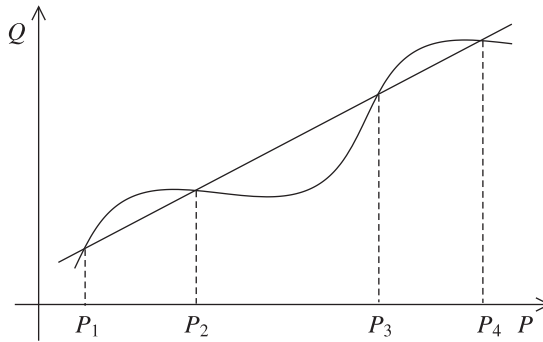
FIGURE 4. Behaviors of Σ.

two intersection points. If it happens to the right branch, since $\omega(s)$ satisfies the two-dimensional system (3.1), between any two intersection points of C_χ with C_ω, there is at least one point on C_χ at which the tangent direction of this curve coincides with the vector field (3.1). This corresponds to a zero of the function

$$\left.\left(\frac{\dot{\omega}}{\dot{s}} - \chi'(s)\right)\right|_{\omega=\chi(s)} = \frac{\alpha_1 s^2 + \beta_1 s + \gamma_1}{24(16777216s - 81)(a_2 s - b_2)^2}, \qquad (4.3)$$

where $\alpha_1, \beta_1, \gamma_1$ are constants relevant to a_1, a_2, b_1, b_2. Equation (4.3) shows that there are at most two such points on C_χ, which implies that C_χ cuts C_ω at most three times. Noticing that $\omega(0) = \chi(0) = -9$, we can conclude that they have at most two intersection points for $s \in (81/16777216, +\infty)$.

In summary, C_χ has at most two intersection points with C_ω. This completes the proof of Lemma 4.1.   □

LEMMA 4.2. *The centeriod curve Σ is globally concave without zero curvature.*

PROOF. Because of the monotonicity of Σ, we assert that the curve Σ is globally located on the right of the tangent line of Σ at the left endpoint. Otherwise, there must exist another point on Σ near the left endpoint, such that the tangent line of Σ at this point cuts two more points of Σ, that is, $I(h)$ has at least four zeros for $h > 3/2^6$, which contradicts Lemma 4.1. On the other hand, since Σ is increasing and has the same concavity at two endpoints, if it has inflection points, they must happen in pairs. Without loss of generality, we assume that there are two inflection points in $h \in S$; then we can find an increasing straight line $L_{\alpha_0,\beta_0,\gamma_0}$ such that it cuts Σ at at least three points. Suppose that their abscissas are P_1, P_2 and P_3 (see Figure 4). Then $L_{\alpha_0,\beta_0,\gamma_0}$ is below Σ when $0 < P(h) - P_3 \ll 1$. Because the slope of Σ converges to zero as h tends to positive infinity, there will be another intersection point of Σ with the increasing straight line $L_{\alpha_0,\beta_0,\gamma_0}$, that is, $I(h)$ has at least four zeros, which also leads to the same contradiction to Lemma 4.1. Hence Σ has no inflection points. Moreover, we can assert

that Σ has no quadruple tangency points, otherwise, we obtain the same contradiction as above. In this way, we complete the proof of Lemma 4.2. $\square$

PROOF OF THEOREM 1.1. Because of convexity and monotonicity of the centroid curve Σ, any straight line $L_{\alpha,\beta,\gamma} = \{(P, Q) : \alpha + \beta P + \gamma Q = 0\}$ has at most two intersection points with Σ, that is, $I(h)$ has at most two zeros. Moreover, there exist α^*, β^* and γ^*, such that $L_{\alpha^*,\beta^*,\gamma^*}$ cuts Σ at exactly two points. We conclude that two is the exact upper bound for the number of limit cycles bifurcating from system (1.2) under small quadratic polynomial perturbations. $\square$

5. Proof of Theorem 1.2

After making the transformation $x = \bar{x}^4$, $y = \bar{y}$ (which has a unique value for $x \geqslant 0$) and changing back to (x, y), the system (1.2) and its first integral (2.1) become

$$\begin{cases} \dot{x} = \frac{1}{4}xy, \\ \dot{y} = \frac{3}{8}y^2 + \frac{9}{2^{10}}(1 - x^4), \end{cases} \tag{5.1}$$
$$H(x, y) = x^{-3}(\tfrac{1}{2}y^2 + 3px^4 + p) = h,$$

with the integrating factor $M(x, y) = x^{-4}$, $p = 3/2^8$. We denote the family of closed curves surrounding the center $(1, 0)$ by $\Pi_h = \{(x, y) : H(x, y) = h\}$, where $h \in S$.

LEMMA 5.1. *Under arbitrary perturbations of degree n, the Abelian integral $V(h)$ related to system (5.1) can be expressed as*

$$I(h) = \alpha(h)J_{-1}(h) + \beta(h)J_0(h) + \gamma(h)J_1(h) + \delta(h)J_2(h), \tag{5.2}$$

where $\alpha(h), \beta(h), \gamma(h)$ and $\delta(h)$ are real polynomials in h, and $J_i(h) = \oint_{\Pi_h} x^{i-4}y\,dx$ with $i = -1, 0, 1$ and 2. If $n \geq 4$ is even,

$$\deg \alpha(h) = 2n - 4, \quad \deg \beta(h) = 2n - 5, \quad \deg \gamma(h) = 2n - 6, \quad \deg \delta(h) = 2n - 3;$$

if $n \geq 4$ is odd,

$$\deg \alpha(h) = 2n - 5, \quad \deg \beta(h) = 2n - 6, \quad \deg \gamma(h) = 2n - 7, \quad \deg \delta(h) = 2n - 4.$$

Moreover, if $n = 3$,

$$\deg \alpha(h) = 1, \quad \deg \beta(h) = \deg \gamma(h) = 0, \quad \deg \delta(h) = 2.$$

The proof of Lemma 5.1 is similar to that of Proposition 2.1 in [12], so we omit it. By similar arguments to Lemmas 2.1 and 2.2, we obtain the following lemmas.

LEMMA 5.2. *For system (5.1), $V(h) = \mathrm{col}(J_{-1}(h), J_0(h), J_1(h), J_2(h))$ satisfies the Picard–Fuchs equation*

$$V(h) = M(h)V'(h), \tag{5.3}$$

where

$$M(h) = \begin{pmatrix} \frac{3}{4}h & -3p & 0 & 0 \\ 0 & h & -4p & 0 \\ 0 & 0 & \frac{3}{2}h & -6p \\ -4p & 0 & 0 & h \end{pmatrix}.$$

LEMMA 5.3. *The ratio* $v(h) = J''_{-1}(h)/J''_0(h)$ *satisfies the Riccati equation*

$$G(h)v(h)'(h) = -\frac{16}{3}p^2 h v^2(h) + \frac{2}{3}h^3 v(h)(h) - \frac{16}{3}hp^2, \tag{5.4}$$

where $G(h) = h^4 - 256p^4$.

LEMMA 5.4. *The function* $I(h)$ *has the representation*

$$\delta_2(h)I(h) = \delta_1(h)J'(h) + \Delta(h), \tag{5.5}$$

where

$$\Delta(h) = \alpha_3(h)J'_{-1}(h) + \beta_3(h)J'_0(h) + \gamma_3(h)J'_1(h).$$

PROOF. From (5.2) and (5.3), we get

$$I(h) = \alpha_1(h)J'_{-1}(h) + \beta_1(h)J'_0(h) + \gamma_1(h)J'_1(h) + \delta_1(h)J'_2(h). \tag{5.6}$$

Differentiating (5.6) with respect to h directly yields

$$I'(h) = \alpha_2(h)J'_{-1}(h) + \beta_2(h)J'_0(h) + \gamma_2(h)J'_1(h) + \delta_2(h)J'_2(h). \tag{5.7}$$

Eliminating $J'_2(h)$ from (5.6) and (5.7), we get (5.5).   □

LEMMA 5.5. *Denote* $\Delta_1(h) = (\Delta(h)/\beta_3(h))'$, *the function* $v(h)$ *satisfies*

$$\frac{\Delta_1(h)}{J'_1(h)} = \frac{\alpha_4(h)v(h) + v_4(h)}{G(h)\beta_3^2(h)}. \tag{5.8}$$

PROOF. Differentiating (5.1) with respect to h yields

$$G(h)\begin{pmatrix} J''_{-1}(h) \\ J''_0(h) \\ J''_1(h) \\ J''_2(h) \end{pmatrix} = \begin{pmatrix} \frac{1}{3}h^3 & -\frac{16}{3}hp^2 \\ \frac{64}{3}p^3 & -\frac{4}{3}h^2 p \\ \frac{16}{3}hp^2 & -\frac{1}{3}h^3 \\ \frac{4}{3}h^2 p & -\frac{64}{3}p^3 \end{pmatrix}\begin{pmatrix} J'_{-1}(h) \\ J'_1(h) \end{pmatrix}. \tag{5.9}$$

From the definition of $\Delta(h)$ and (5.9), we get (5.8).   □

PROOF OF THEOREM 1.2. Letting $\Delta_2(h) = \alpha_4(h)v(h) + \gamma_4(h)$, and substituting it into (5.4),

$$G(h)\alpha_4(h)\Delta'_2(h) = -\frac{16}{3}ph^2\Delta_2^2(h) + R_1(h)\Delta_2 + R_2(h), \tag{5.10}$$

where

$$R_1(h) = \frac{32}{3}p^2 h^2 \gamma_4(h) + G(h)\alpha'_4(h) + \frac{2}{3}h^3 \alpha_4(h),$$

$$R_2(h) = -\frac{16}{3}p^2 h(\gamma_5^2(h) + \alpha_4^2(h)) - \frac{2}{3}h^3 \alpha_4(h)\gamma_4(h) + G(h)(\gamma'_4(h)\alpha_4(h) - \gamma_4(h)\alpha'_4(h)).$$

By the methods of [7, Lemma 4.2], from (5.5) and (5.10) we obtain

$$\#I(h) \leqslant \#\Delta(h) + \#\delta_2(h) + 1,$$
$$\#\Delta_2(h) \leqslant \#G(h) + \#\alpha_4(h) + \#R_2(h) + 1.$$

The definitions of $\Delta(h)$ and $\Delta_2(h)$ yield

$$\#\Delta(h) \leqslant \#\Delta_1(h) + 1 = \#\Delta_2(h) + 1.$$

Then we get

$$\#I(h) \leqslant \#G(h) + \#\alpha_4(h) + \#R_2(h) + \#\delta_2(h) + 2.$$

Theorem 1.2 follows from the degrees of $\alpha_4(h)$, $R_2(h)$ and $\delta_2(h)$. □

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References

[1] V. I. Arnold, 'Ten problems', *Adv. Soviet Math.* **1** (1990), 1–8.

[2] F. Chen, C. Li, J. Llibre and Z. Zhang, 'A unified proof on the weak Hilbert 16th problem for $n = 2$', *J. Differential Equations* **221** (2006), 309–342.

[3] B. Coll, C. Li and R. Prohens, 'Quadratic perturbations of a class of quadratic reversible systems with two centers', *Discrete Contin. Dyn. Syst.* **24** (2009), 1–31.

[4] S. Gautier, L. Gavrilov and I. D. Iliev, 'Perturbations of quadratic centers of genus one', arXiv:0705.1609, 2007.

[5] L. Gavrilov, 'The infinitesimal 16th Hilbert problem in the quadratic case', *Invent. Math.* **143** (2001), 449–497.

[6] E. Horozov and I. D. Iliev, 'On the number of limit cycles in perturbations of quadratic Hamiltonian systems', *Proc. Lond. Math. Soc.* **69** (1994), 198–224.

[7] E. Horozov and I. D. Iliev, 'Linear estimate for the number of zeros of Abelian integrals with cubic Hamiltonians', *Nonlinearity* **11** (1998), 1521–1537.

[8] I. D. Iliev, 'Perturbations of quadratic centers', *Bull. Sci. Math.* **122** (1998), 107–161.

[9] H. Liang and Y. Zhao, 'Quadratic perturbations of a class of quadratic reversible systems with one center', *Discrete Contin. Dyn. Syst.* **27** (2010), 325–335.

[10] C. Li and Z. Zhang, 'A criterion for determining the monotonicity of the ratio of two Abelian integrals', *J. Differential Equations* **124** (1996), 407–424.

[11] W. Li, Y. Zhao, C. Li and Z. Zhang, 'Abelian integrals for quadratic centers having almost all their orbits formed by quartics', *Nonlinearity* **15** (2002), 863–885.

[12] Y. Zhao, W. Li, C. Li and Z. Zhang, 'Linear estimate of the number of zeros for Abelian integrals for quadratic centers having almost all their orbits formed by cubics', *Sci. China (Series A)* **45** (1998), 964–974.

[13] Y. Zhao and H. Zhu, 'Bifurcation of limit cycles from a non-Hamiltonian quadratic integrable system with homoclinic loop', *Fields Inst. Commun.* to appear.

[14] H. Zoladek, 'Quadratic systems with center and their perturbations', *J. Differential Equations* **109** (1994), 223–278.

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