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ON A GENERALIZED FUNDAMENTAL EQUATION OF INFORMATION

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1. Introduction. The object of this paper is to determine the general solution of the functional equation

$$\text{(FE)} \qquad f(x) + \alpha(1 - x)g\left(\frac{y}{1 - x}\right) = h(y) + \alpha(1 - y)k\left(\frac{x}{1 - y}\right)$$
$$\text{(for all } x, y \in [0, 1[\text{ with } x + y \in [0, 1]\text{)},$$

where α is multiplicative. It turns out that non-trivial embeddings of the reals in the complex generate some interesting solutions.

In many applications, various special cases of (FE) have occurred ([**1**, **3**, **4**, **6**, **10**, **11**, **14**]). The special case where $f = g = h = k$ and $\alpha =$ the identity map is known as the fundamental equation of information, and has been extensively investigated by many authors ([**5**]). The case where $f = g = h = k$ and α is multiplicative was treated in [**13**, **14**]. The general solution of (FE) when $\alpha(1 - x) = (1 - x)^{\beta}$ has been obtained in [**9**], except when $\beta = 2$. The gap has been covered in [**1**, **2**, **11**, **12**]. With these in mind it is desirable to know the general solution of (FE). Due to some technical difficulty, the case where α is multiplicative but not a power remained open. In this paper we resolve this by using embeddings of the reals in the complex, giving rise to new solutions. The method employed also fills a gap left in [**7**].

2. General solution of the equation (FE).

THEOREM 2.1. *Let* f, h: $[0, 1[\to \mathbf{R}$, g, k: $[0, 1] \to \mathbf{R}$ *and* α: $[0, 1] \to \mathbf{R}$ *be functions satisfying*

$$\text{(FE)} \qquad f(x) + \alpha(1 - x)g\left(\frac{y}{1 - x}\right) = h(y) + \alpha(1 - y)k\left(\frac{x}{1 - y}\right)$$
$$\textit{(for all } x, y \in [0, 1[\textit{ with } x + y \in [0, 1]\text{)},$$

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where α satisfies

$$\text{(M)} \quad \begin{cases} \alpha(xy) = \alpha(x)\alpha(y) \text{ for all } x, y \in \,]0, 1[\\ \alpha(0) = 0 \text{ and } \alpha(1) = 1. \end{cases}$$

Then they are given by

$$\text{(2.1)} \quad \begin{array}{ll} f(x) = \Phi(1 - x) + a_1\alpha(x) + b_1\alpha(1 - x) + c, & \text{on } [0, 1[\\ g(x) = \Phi(x) + a_2\alpha(x) + b_2\alpha(1 - x) - b_1, & \text{on } [0, 1] \\ h(x) = \Phi(x) + a_2\alpha(x) + b_3\alpha(1 - x) + c, & \text{on } [0, 1] \\ k(x) = \Phi(1 - x) + a_1\alpha(x) + b_2\alpha(1 - x) - b_3, & \text{on } [0, 1] \end{array}$$

where

$$\text{(2.2)} \quad \begin{array}{ll} \text{either} & \alpha(x) = x^2 \text{ and } \Phi(x) = D(x), \\ \text{or} & \alpha(x) = |\phi(x)|^2 \text{ and } \Phi(x) = a\,\mathrm{Im}\,\phi(x), \\ \text{or} & \alpha(x) = x \text{ and } \Phi(x) = xL(x) + (1 - x)L(1 - x), \\ \text{or} & \alpha \text{ is an arbitrary map satisfying (M) and } \Phi = 0 \end{array}$$

with constants a, a_i, b_i, c and functions D:$\mathbf{R} \to \mathbf{R}$, ϕ: $\mathbf{R} \to \mathbf{C}$ (complex numbers), and L: $]0, \infty[\to \mathbf{R}$ satisfying

$$\text{(2.3)} \quad \begin{array}{l} D(x + y) = D(x) + D(y),\ D(xy) = xD(y) + yD(x) \\ \qquad\qquad\qquad - \text{a derivation } [\mathbf{16}], \\ \phi(x + y) = \phi(x) + \phi(y),\ \phi(xy) = \phi(x)\phi(y),\ \phi \text{ injective} \\ \qquad\qquad\qquad - \text{an embedding}, \\ L(xy) = L(x) + L(y) - \text{logarithmic}. \end{array}$$

The converse is also true.

Proof. Suppose f, g, h, k and α satisfy (FE) and (M). By putting $x = 0$ in (FE) we get

$$\text{(2.4)} \quad h(y) = g(y) - k(0)\alpha(1 - y) + f(0) \text{ on } [0, 1[.$$

Similarly by specializing $y = 0$, $y = 1 - x$ respectively in (FE) and using (2.4), we get

$$\text{(2.5)} \quad f(x) = g(1 - x) + [k(1) - k(0)]\alpha(x) - g(1)\alpha(1 - x) + f(0) \text{ on }]0, 1[.$$

$$\text{(2.6)} \quad k(x) = g(1 - x) + [k(1) - k(0)]\alpha(x) - [g(1) - g(0)]\alpha(1 - x) + [f(0) - h(0)] \text{ on }]0, 1[.$$

These functional dependence and (FE) imply

$$\text{(2.7)} \quad g(1 - x) + \alpha(1 - x)g\left(\frac{y}{1 - x}\right) = g(y)$$

$$+ \alpha(1-y)g\left(1-\frac{x}{1-y}\right)$$
$$- [g(1) - g(0)]\alpha(1-x-y)$$
$$+ g(1)\alpha(1-x) - g(0)\alpha(1-y)$$
$$(x \in [0, 1[, y \in [0, 1[, 0 < x + y < 1).$$

By defining

$$\beta(x) = g(x) - [g(1) - g(0)]\alpha(x) - g(0) \text{ on } [0, 1] \tag{2.8}$$

it reduces to

$$\beta(1-x) + \alpha(1-x)\beta\left(\frac{y}{1-x}\right) = \beta(y) \tag{2.9}$$
$$+ \alpha(1-y)\beta\left(1-\frac{x}{1-y}\right) \quad (x \in]0, 1[, y \in [0, 1[,$$
$$0 < x + y < 1).$$

Interchanging x and y in (2.9) and combining the result with (2.9) we obtain

$$m(x) + m(y) = \alpha(1-x)m\left(\frac{y}{1-x}\right) + \alpha(1-y)m\left(\frac{x}{1-y}\right) \tag{2.10}$$

$$n(x) + \alpha(1-x)n\left(\frac{y}{1-x}\right) = n(y) + \alpha(1-y)n\left(\frac{x}{1-y}\right) \tag{2.11}$$
$$(x, y, x + y \in]0, 1[)$$

where

$$m(x) = \frac{1}{2}\beta(x) - \frac{1}{2}\beta(1-x) \text{ and } n(x) = \frac{1}{2}\beta(x) + \frac{1}{2}\beta(1-x) \tag{2.12}$$

are the skew-symmetric and symmetric parts of β.

The general (skew-symmetric) solution of (2.10) on]0, 1[will be derived in the next section as

$$\begin{cases} \alpha(x) = x^2 \text{ and } m(x) = D(x), \\ \text{or } \alpha(x) = |\phi(x)|^2 \text{ and } m(x) = a \operatorname{Im} \phi(x), \\ \text{or } \alpha \text{ is arbitrary and } m = 0, \end{cases} \tag{2.13}$$

where D and ϕ are described in (2.3), and a is a constant. The general (symmetric) solution of (2.11) on]0, 1[is given by ([**5**, **13**, **14**]),

$$(2.14)\qquad \begin{cases} \alpha(x) = x \text{ and } n(x) = xL(x) + (1-x)L(1-x) \quad \text{or} \\ \alpha \text{ is arbitrary and } n(x) = A[\alpha(x) + \alpha(1-x) - 1], \end{cases}$$

where A is a constant and L is described in (2.3). This in turn gives the form β on]0, 1[, and so, from (2.8),

$$(2.15)\qquad \begin{aligned} g(x) &= D(x) + a_2\alpha(x) + b_2\alpha(1-x) \\ &\qquad - b_1 \text{ while } \alpha(x) = x^2, \quad \text{or} \\ g(x) &= a \operatorname{Im}\phi(x) + a_2\alpha(x) + b_2\alpha(1-x) - b_1 \\ &\qquad \text{while } \alpha(x) = |\phi(x)|^2 \quad \text{or} \\ g(x) &= xL(x) + (1-x)L(1-x) + a_2\alpha(x) + b_2\alpha(1-x) \\ &\qquad - b_1 \text{ while } \alpha(x) = x \quad \text{or} \\ g(x) &= a_2\alpha(x) + b_2\alpha(1-x) - b_1 \text{ while } \alpha \text{ is arbitrary} \end{aligned}$$

on]0, 1[, where a, a_2, b_1, b_2 are constants. These general forms of g on]0, 1[indeed satisfy (2.7) if, and only if, $g(0) = b_2 - b_1$ and $g(1) = a_2 - b_1$; we can thus claim that (2.15) indeed represents g on [0, 1] as asserted in the theorem with the convention $0L(0) = 0$.

From the form of g given by (2.15) and (2.4), (2.5) and (2.6) we obtain the forms of f, h, and k on]0, 1[as asserted in (2.1). These functions indeed satisfy (FE) if, and only if, their boundary values are also given by (2.1).

The converse is straightforward, and this proves the theorem.

3. General skew-symmetric solution of the equation (2.10). We present the general non-trivial skew-symmetric solution of (2.10) through the following proposition.

PROPOSITION 3.1. *Let* $m(\not\equiv 0)$, α:]0, 1[$\to \mathbf{R}$ *be mappings satisfying*

$$(3.1)\qquad m(x) + m(y) = \alpha(1-x)m\left(\frac{y}{1-x}\right) + \alpha(1-y)m\left(\frac{x}{1-y}\right),$$

$$(3.2)\qquad m(x) + m(1-x) = 0,$$

and

$$(3.3)\qquad \alpha(uv) = \alpha(u)\alpha(v),$$

for all $x, y, u\ v \in$]0, 1[*with* $x + y \in$]0, 1[. *Let*

$$(3.4)\qquad \beta(x) := \frac{1}{2}(\alpha(x) - \alpha(1-x) + 1) \quad \text{on }]0, 1[.$$

Then for some constant c the following equations also hold:

$$(3.5)\qquad \alpha(1-x)m\left(\frac{y}{1-x}\right) = \beta(1-x)m(y) - \beta(y)m(1-x)$$

(3.6) $\quad m(x + y) = m(x) + m(y)$

(3.7) $\quad \beta(x + y) = \beta(x) + \beta(y)$

(3.8) $\quad m(uv) = \beta(u)m(v) + \beta(v)m(u)$

(3.9) $\quad \beta(uv) = \beta(u)\beta(v) + cm(u)m(v),$

in particular

$$\beta(u^2) = \beta^2(u) + cm^2(u)$$

(3.10) $\quad \alpha(u) = \beta^2(u) - cm^2(u)$

for all $x, y, u, v \in \,]0, 1[$ *with* $x + y \in \,]0, 1[$.

Proof. Consider the following computations:

$$m\left(\frac{x}{1-y}\right) - m(x) = -\left[m\left(1 - \frac{x}{1-y}\right) + m(x)\right] \quad \text{(by 3.2)}$$

$$= -\left[\alpha\left(\frac{x}{1-y}\right)m\left(\frac{x}{x/(1-y)}\right)\right.$$

$$\left. + \alpha(1-x)\, m\left(\frac{1 - (x/(1-y))}{1-x}\right)\right]$$

(by (3.1), noting that $1 - \dfrac{x}{1-y}, x, \left(1 - \dfrac{x}{1-y}\right) + x \in \,]0, 1[$)

$$= -\alpha\left(\frac{x}{1-y}\right)m(1-y) - \alpha(1-x)m\left(\frac{1-x-y}{(1-x)(1-y)}\right)$$

$$= \alpha\left(\frac{x}{1-y}\right)m(y) - \alpha(1-x)m\left(\frac{1-x-y}{(1-x)(1-y)}\right) \quad \text{(by 3.2).}$$

We multiply both sides by $\alpha(1 - y)$ while using (3.3) and rearranging terms to get

$$\alpha(1-y)m\left(\frac{x}{1-y}\right) - \alpha(1-y)m(x) - \alpha(x)m(y)$$

$$= -\alpha((1-x)(1-y))m\left(\frac{1-x-y}{(1-x)(1-y)}\right).$$

The symmetry of the right hand side in x and y implies that of the left, which can be arranged in the form

$$(3.11)\quad [\alpha(1-y)-\alpha(y)]m(x)+[\alpha(x)-\alpha(1-x)]m(y) = \alpha(1-y)m\left(\frac{x}{1-y}\right)-\alpha(1-x)m\left(\frac{y}{1-x}\right).$$

The significance of this relation is that it allows us to compute the difference of the two terms on the right side of (3.1), and therefore it leads to a separation of the two terms. To carry this out we subtract (3.11) from (3.1) and use (3.2) to obtain (3.5). Equation (3.5) is more informative than (3.1), since with (3.2) we can indeed simplify the right side of (3.1) to that of its left side.

From (3.5) we get

$$\alpha(x+y)m(x/x+y)=\beta(x+y)m(x)-\beta(x)m(x+y) \quad \text{and}$$

$$\alpha(x+y)m(y/x+y)=\beta(x+y)m(y)-\beta(y)m(x+y).$$

On the other hand, because of (3.2),

$$\alpha(x+y)m(x/x+y)+\alpha(x+y)m(y/x+y)=0;$$

hence we obtain

$$(3.12)\quad \beta(x+y)[m(x)+m(y)]-[\beta(x)+\beta(y)]m(x+y)=0.$$

Noting from its definition (3.4) that β has the property

$$(3.13)\quad \beta(u)+\beta(1-u)=1 \quad \text{for all } u\in\,]0,1[.$$

We replace in (3.12) first x by $1-x-y$, then y by $1-x-y$, and add the two resulting equations to (3.12) while using (3.2) to obtain (3.6).

To each x_0 with $m(1-x_0)\neq 0$, the additivity of m in (3.5) implies that $y\rightarrow\beta(y)$ is additive on $]0,1-x_0[$. Since we assume the additive m is non-trivial, we can choose such $1-x_0$ as close to 1 as we wish. This proves the additivity (3.7) of β on $]0,1[$.

Setting $u=y/1-x$ and $v=1-x$, we rewrite (3.5) as

$$(3.14)\quad \alpha(v)m(u)=\beta(v)m(uv)-\beta(uv)m(v).$$

We replace v by $1-v$ and substract the resulting equation from it while using (3.4), (3.6), (3.7) and (3.13). This leads to (3.8).

Applying (3.8) twice to $m(uvw)$ which is first conceived as $m(u(vw))$ and then as $m((uv)w)$, we get

$$m(uvw)=\beta(u)\beta(v)m(w)+\beta(u)\beta(w)m(v)+\beta(vw)m(u)$$

and also

$$m(uvw)=\beta(uv)m(w)+\beta(v)\beta(w)m(u)+\beta(u)\beta(w)m(v).$$

Comparison leads to

$$[\beta(u)\beta(v) - \beta(uv)]m(w) = [\beta(v)\beta(w) - \beta(vw)]m(u),$$

and by fixing $w = w_0$ with $m(w_0) \neq 0$ we get

$$(3.15) \qquad \beta(uv) = \beta(u)\beta(v) - \gamma(v)m(u),$$

where

$$\gamma(v) := m(w_0)^{-1}[\beta(v)\beta(w_0) - \beta(vw_0)].$$

Using the same idea we can compute $\beta(uvw)$ in two ways using (3.15) along with (3.8) and compare the results to obtain

$$m(u)[\gamma(v)\beta(w) + \gamma(w)\beta(v) - \gamma(vw)] = 0.$$

Since $m \neq 0$, we have

$$(3.16) \qquad \gamma(vw) = \gamma(v)\beta(w) + \gamma(w)\beta(v).$$

If γ happens to be equal to zero, (3.15) already gives (3.9) with $c = 0$. Else (3.16) is just like (3.8) where $m \neq 0$, and so just as (3.15) has been obtained from (3.8), we get

$$(3.17) \qquad \beta(uv) = \beta(u)\beta(v) - \delta(v)\gamma(u),$$

for some appropriate function δ. Now compare (3.15) with (3.17) to get $\gamma = -cm$ for some constant $c \neq 0$ and with it (3.15) can be written symmetrically as (3.9). This ends the proof of (3.9).

In (3.14) we replace $m(uv)$ and $\beta(uv)$ by the right sides of (3.8) and (3.9) to obtain (3.10). This proves the proposition.

Extension of functions in Proposition 3.1. The additivity of m and β, given by (3.6) and (3.7) respectively, enables us to extend m and β uniquely to additive $\bar{m}$ and $\bar{\beta}$ on $\mathbf{R}$. For instance, $\bar{m}$ is defined by $\bar{m}(rx) = rm(x)$ for arbitrary rational r and $x \in]0, 1[$. The extended $\bar{m}$ and $\bar{\beta}$ satisfy (3.2), (3.8) and (3.9) on $\mathbf{R}$. We can also extend α to a multiplicative $\bar{\alpha}$ on $\mathbf{R}$ through (3.10). In other words

$$\bar{\alpha}(u) := \bar{\beta}^2(u) - c\bar{m}^2(u) \text{ for } u \in \mathbf{R}.$$

Further, $\bar{m}(1) = 0$ (by 3.2)) and $\bar{\beta}(1) = 1$ (by 3.8)) which help to show that the extended $\bar{\beta}$ and $\bar{\alpha}$ satisfy (3.4) on $\mathbf{R}$. Now, it is easy to check that the remaining two equations (3.5) and (3.1) also hold for the extended functions on $\mathbf{R}$. In summary all equations in Proposition 3.1 can be conceived to hold for all real arguments.

THEOREM 3.2. *Let* m, α: $]0, 1[\to \mathbf{R}$ *be functions satisfying* (3.1) [(2.10)], (3.2) *and* (3.3). *Then they are given by* (2.13). *The converse also holds.*

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Proof. Suppose m and α satisfy (3.1) to (3.3).

If $m \equiv 0$, then α can be arbitrary as claimed in (2.13).

So let us assume $m \not\equiv 0$. Then by Proposition 3.1 and the subsequent extensions (3.1) to (3.10) hold on **R**. Without ambiguity, we will omit the bars.

Let us first treat the case when $c < 0$ say $c = -\sigma^2$ $(\sigma \neq 0)$. The map

$$\phi := \beta + i\sigma m$$

from **R** into **C** is additive (which follows from that of β and m) and multiplicative (which is equivalent to (3.8) and (3.9)); that is, ϕ is a ring homomorphism of the reals into the complex field. Since $m \neq 0$, ϕ is a non-trivial embedding of **R** into **C** [**8**].

Now (3.10) is translated into $\alpha = |\phi|^2 (= \beta^2 + \sigma^2 m^2)$ and indeed

$$m = \frac{1}{\sigma} \operatorname{Im} \phi$$

as claimed in (2.13).

We now treat the case when $c \geqq 0$. Equation (3.9) implies $\beta(u^2) \geqq 0$ and so $\beta \geqq 0$ on $[0, \infty[$. Since β is additive, $\beta(u) = au$. Since $\beta(1) = 1$, $\beta(u) = u$. This in turn in (3.9) implies $c = 0$. Thus (3.10) and (3.8) give $\alpha(u) = u^2$ and $m(uv) = um(v) + vm(u)$ which proves the first line in (2.13).

The converse is straightfoward. This completes the proof of Theorem 3.2.

4. Remarks on the system of functional equations (3.6) **and** (3.8). Consider the system of equations

(4.1) $$m(x + y) = m(x) + m(y)$$

(4.2) $$m(xy) = \beta(x)m(y) + \beta(y)m(x).$$

In [7] the domain of m and β is a commutative integral domain with identity 1, and the range is a field of characteristic different from 2. The solutions of the system when $\beta(1) \neq 1$ were obtained, whereas the solutions covering the case $\beta(1) = 1$ were not adequately treated. The method employed in the previous section sheds some light on how solutions can be obtained including the case $\beta(1) = 1$. We illustrate this by means of a proposition, accompanied by examples. In [**15**], (4.2) has been solved on abelian groups using the method of determinants.

Let S be a set on which two binary operations $+$ and $\cdot$ are defined, and let F be a field. Let m and β be maps from S into F satisfying (4.1) and (4.2). Since when $m = 0$, β can be arbitrary, from now on $m \neq 0$ is always assumed. Our reasonings in the previous section establishing (3.9) from (3.8) with $m \neq 0$ is based on the associativity of multiplication $(\cdot)$. Thus if $(S, \cdot)$ is a semigroup and $m(\neq 0)$, β satisfy (4.2) for all $x, y \in S$, then they satisfy also

$$\beta(xy) = \beta(x)\beta(y) + cm(x)m(y) \tag{4.3}$$

for some constant $c \in F$. When multiplication is distributive over addition, and m, β satisfy (4.1) and (4.2), then β is additive as well, i.e.,

$$\beta(x + y) = \beta(x) + \beta(y). \tag{4.4}$$

Thus if $(S, +, \cdot)$ is a ring, we can make use of (4.3) and (4.4) if necessary to solve the system (4.1) and (4.2). We summarize it as a proposition.

PROPOSITION. *Let S be a ring and F be a field. Let $m \neq 0$ and β be maps from S to F satisfying* (4.1) *and* (4.2). *Then* (4.3) *and* (4.4) *also hold for some constant $c \in F$. Consider $k = \sqrt{c}$ which may or may not be in F, and the field $F(k)$ which may be F itself or an extension of F accordingly. Then the maps $\phi = \beta + km$ and $\psi = \beta - km$ from S into $F(k)$ are ring homomorphisms. When* char $(F) \neq 2$, *we get three sets of solutions depending on $k = \sqrt{c}$ as follows*:

Case (i). *When $k = c = 0$, then β is a ring homomorphism and m is conceived as a generalized derivation in the sense that it satisfies* (4.1) *and* (4.2) *along with a ring homomorphism β. The general construction of such generalized derivation requires further investigation. Consider the special case when S is a subfield of F; then $\beta: S \to F$ is an embedding and $m = D \circ \beta$ where $D: \beta(S) \to F$ is a nontrivial derivation* [**16**, *Chapter* II, *Section* 17].

Case (ii). *When $k \neq 0$. Suppose* char $(F) \neq 2$. *Then we can express β and m in terms of ϕ and ψ. There are two subcases*:

(ii a) *Suppose $c = k^2$ with $k \in F$. Then the general solution is given by*

$$\beta = \frac{1}{2}(\phi + \psi) \quad \text{and} \quad m = \frac{1}{2k}(\phi - \psi)$$

where ϕ and ψ are distinct ring homomorphisms.

(ii b) *Suppose $c = k^2$, $k \notin F$. We can define on the field extension $F(k)$ the conjugate operation $\overline{a + bk} = a - bk$ for $a, b \in F$, which is a natural automorphism on $F(k)$. Then $\psi = \bar{\phi}$ and the general solutions are given by*

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$$\beta = \frac{1}{2}(\phi + \bar{\phi}) \quad \textit{and} \quad m = \frac{1}{2k}(\phi - \bar{\phi}),$$

where ϕ is a ring homomorphism of S into $F(k)$ such that $\phi \neq \bar{\phi}$.

Example 1. Let $S = \mathbf{Z}_n$. If char (F) is not a divisor of n, there will be no nontrivial additive map $m:S \to F$. Else additive maps m, β are of the form

$$m(x) = xa \quad \text{and} \quad \beta(x) = xb \quad (x \in \mathbf{Z}_n),$$

where $a \neq 0$ and b are constants. They satisfy the equation (4.2) if and only if $2b = 1$. In this example, $\beta(1) = 1$ is impossible (else $m = 0$). Similar arguments apply when $S = \mathbf{Z}$.

Example 2. Let $S = \mathbf{Z}_{p^2} = \mathbf{Z}_p(t)$ where t is a root of the irreducible polynomial $z^2 + \lambda z + \mu$ over $\mathbf{Z}_p$, and let F be a field of characteristic p. Additive maps $m, \beta:S \to F$ are of the form

$$m(r + ts) = rm(1) + sm(t) = ra + sa' \quad \text{and}$$

$$\beta(r + ts) = rb + sb'$$

where $a, a', b, b' \in F$ are constants. They satisfy (4.2) if and only if these constants satisfy

$$-\lambda a' - \mu a = 2a'b', \ a' = a'b + ab' \quad \text{and} \quad a = 2ab.$$

Thus the general solution of the system is

$$\begin{cases} m(r + ts) = ra + sa' \\ \beta(r + ts) = rb + sb' \end{cases} \quad r, s, \in \mathbf{Z}_p$$

where the constants fulfill either ($a = 0$, $a' \neq 0$, $b = 1$, $2b' = -\lambda$) or ($a \neq 0$, $a' = 2ab'$, $2b = 1$, $4b'^2 + 2\lambda b' + \mu = 0$).

In this example, there is no nontrivial m when $p = 2$, and there is a nontrivial m along with $\beta(1) = 1$ when $p \neq 2$.

From this example, we can find the general solution when the ring S is finitely generated under addition, although the determination of the conditions on the constants involved could be very tedious.

Example 3. Let $S = F = \mathbf{R}$.

Case (i). When $k = c = 0$, the only nontrivial homomorphism $\beta:\mathbf{R} \to \mathbf{R}$ is the identity map, and m is the usual derivation (2.3). In Case (ii a), $\beta = \frac{1}{2}I$ where I is the identity map and $m = (\pm)\frac{1}{2k}I$. In case (ii b),

$$F(k) = \mathbf{C},\ m = \frac{1}{2k}(\phi - \bar{\phi}),\ \text{and}\ \beta = \frac{1}{2}(\phi + \bar{\phi}),$$

where ϕ is a nontrivial embedding of $\mathbf{R}$ into $\mathbf{C}$ and k is a purely imaginary constant in $\mathbf{C}$. In case (i) and (ii b), $\beta(1) = 1$ (which was not treated in [7]), while in case (ii a), $\beta(1) = \frac{1}{2}$.

Let $S = F = \mathbf{C}$ in the proposition. If $c = 0$, the general solution is either as given in case (i), or else it is governed by (ii a).

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