Can. J. Math., Vol. XXXIX, No. 1, 1987, pp. 74-99

ANALYTIC SUBALGEBRAS OF VON NEUMANN ALGEBRAS

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1. Introduction. Let M be a von Neumann algebra and let $\{\alpha_t\}_{t\in\mathbf{R}}$ be a σ-weakly continuous flow on M; i.e., suppose that $\{\alpha_t\}_{t\in\mathbf{R}}$ is a one-parameter group of *-automorphisms of M such that for each ρ in the predual, M_*, of M and for each $x \in M$, the function of t, $\rho(\alpha_t(x))$, is continuous on $\mathbf{R}$. In recent years, considerable attention has been focused on the subspace of M, $H^\infty(\alpha)$, which is defined to be

$$\{x \in M|\rho(\alpha_t(x)) \in H^\infty(\mathbf{R}), \text{ for all } \rho \in M_*\},$$

where $H^\infty(\mathbf{R})$ is the classical Hardy space consisting of the boundary values of functions bounded analytic in the upper half-plane. In Theorem 3.15 of [**8**] it is proved that in fact $H^\infty(\alpha)$ is a σ-weakly closed subalgebra of M containing the identity operator such that

$$H^\infty(\alpha) + H^\infty(\alpha)^* \quad (\equiv \{x + y^*|x, y \in H^\infty(\alpha)\})$$

is σ-weakly dense in M, and such that

$$H^\infty(\alpha) \cap H^\infty(\alpha)^* = M^\alpha \equiv \{x \in M|\alpha_t(x) = x, t \in \mathbf{R}\}$$

(see [**7**] and [**27**] also). Consequently, the elements of $H^\infty(\alpha)$ are called *analytic* with respect to $\{\alpha_t\}_{t\in\mathbf{R}}$ and $H^\infty(\alpha)$, itself, is called the *analytic subalgebra* of M determined by $\{\alpha_t\}_{t\in\mathbf{R}}$. The algebras $H^\infty(\alpha)$ provide a very interesting generalization to the noncommutative setting of certain well known classes of function algebras and, perhaps more importantly, they provide a common perspective from which one can analyze certain types of non-self-adjoint operator algebras that have received significant attention lately. Indeed, if there is a faithful family of α-invariant, normal states on M, then as is shown in [**7**] and [**8**], $H^\infty(\alpha)$ is a maximal subdiagonal algebra in M in the sense of Arveson [**1**] and, as it turns out, most subdiagonal algebras can be realized as $H^\infty(\alpha)$ for a suitable automorphism group $\{\alpha_t\}_{t\in\mathbf{R}}$. Also, each nest subalgebra of a von Neumann algebra may be realized as an $H^\infty(\alpha)$.

In this paper we contribute some partial answers to the following

Received October 8, 1984. The first author was supported in part by a grant from the National Science Foundation. The second author was supported in part by a Grant-in-Aid for Scientific Research from the Japanese Ministry of Education.

https://doi.org/10.4153/CJM-1987-005-4 Published online by Cambridge University Press

Question. When is $H^{\infty}(\alpha)$ maximal among the σ-weakly closed subalgebras of M?

The reason we are interested in this question has two components, at least. First of all, from the function algebra perspective, it is intrinsically interesting, and when we can answer it, the proofs are nontrivial. The fact that $H^{\infty}(\mathbf{R})$ and $H^{\infty}(\mathbf{T})$ are maximal weak-$*$ closed subalgebras of $L^{\infty}(\mathbf{R})$ and $L^{\infty}(\mathbf{T})$, respectively, is classical and certainly well known. The next results, in the abelian case, were obtained by the first author in [**14**], using [**13**]. Of course in the abelian case, M may be identified with $L^{\infty}(\Omega)$ for some measure space Ω, which we may take to be a standard Borel space with finite measure m if M is σ-weakly separable. In this case $\{\alpha_t\}_{t\in\mathbf{R}}$ is implemented by a measurable action of $\mathbf{R}$ on Ω leaving m quasi-invariant:

$$(\omega, t) \rightarrow T_t\omega, \quad \omega \in \Omega, t \in \mathbf{R}.$$

The space $H^{\infty}(\alpha)$, then, may be viewed as $\{\varphi \in L^{\infty}(\Omega) \mid$ for m-almost all ω, the function of t, $\varphi(T_t\omega)$, lies in $H^{\infty}(\mathbf{R})\}$. It is proved in [**14**] that if m is invariant, then $H^{\infty}(\alpha)$ is maximal among the σ-weakly closed subalgebras of $L^{\infty}(\Omega)$ if and only if m is ergodic. An examination of the proof reveals that it is not necessary to assume that m is invariant. Of course, m is ergodic if and only if M^{α} is a factor. So, in the abelian case, we conclude that $H^{\infty}(\alpha)$ is maximal among the σ-weakly closed subalgebras of M if and only if M^{α} is a factor. The first noncommutative results were obtained by the authors in joint work with M. McAsey. The strongest result of our three papers relating to the subject [**10-12**] may be expressed as follows. Suppose that N is a σ-finite von Neumann algebra and that β is a $*$-automorphism of N preserving a faithful normal state. Let M be the crossed product determined by N and β and let $\{\alpha_t\}_{t\in\mathbf{R}}$ be the (periodic) action of $\mathbf{R}$ on M that is dual, in the sense of Takesaki [**25**], to the action of $\mathbf{Z}$ on N determined by β. Then $H^{\infty}(\alpha)$, which is called the *analytic crossed product* determined by N and β, is maximal if and only if $M^{\alpha}(= N)$ is a factor. In [**10-12**], the maximality of $H^{\infty}(\alpha)$ is related to its invariant subspace structure and ideal structure. Subsequent results along these lines were obtained by the second author [**17-19**] who considered almost periodic actions of $\mathbf{R}$ (alias compact group actions) on finite von Neumann algebras and related results were obtained by Solel [**21**]. On the basis of the results of these investigations, one might be led to conjecture that $H^{\infty}(\alpha)$ is maximal among the σ-weakly closed subalgebras of M precisely when M^{α} is a factor. However, this is not the case. Indeed, if M is the algebra of 2×2 matrices, and if $\{\alpha_t\}_{t\in\mathbf{R}}$ is nontrivial, then $H^{\infty}(\alpha)$ is (isomorphic to) the algebra of upper triangular matrices, which is maximal in this case, but M^{α} is (isomorphic to) the algebra of diagonal matrices and is not a factor. In Corollary 3.12 of [**22**], Solel subsumes this example in a result that gives a necessary and sufficient condition for

$H^\infty(\alpha)$ to be a maximal σ-weakly closed subalgebra of M under the assumption that $\{\alpha_t\}_{t\in\mathbf{R}}$ is periodic. Solel's discoveries don't seem to generalize readily to the aperiodic setting and we must content ourselves with partial results. Nevertheless, it is clear that the maximality of $H^\infty(\alpha)$ is tied in some fashion to whether or not M^α is a factor.

The second reason why the maximality question intrigues us has to do with the fact that $H^\infty(\alpha)$ may always be viewed as the intersection of (some faithful, normal representation of) M with a nest algebra (see Proposition 2.4 below). If M has almost any intersection at all with the projections determining the nest, then $H^\infty(\alpha)$ is not maximal and M^α is not a factor. In some sense, which as yet we are unable to make precise, $H^\infty(\alpha)$ seems to be maximal if and only if M is highly "skewed" with respect to the projections that determine the nest. In our view, M is most highly "skewed" with respect to the projections in the nest, when M is the crossed product of another von Neumann algebra by an action of $\mathbf{R}$ or $\mathbf{Z}$. In the case of a crossed product by an $\mathbf{R}$-action, as we shall see in Theorem 5.2, $H^\infty(\alpha)$ is maximal if and only if M^α is a factor.

As mentioned above, in earlier work the maximality question is related to the invariant subspace structure of $H^\infty(\alpha)$; the same is true here. We refine the invariant subspace analysis presented in **[8]** and use it to establish our maximality theorems. In Section 2, we recall the basic facts about von Neumann algebras in standard form from the perspective of Haagerup's L^p-spaces, **[6]** and **[26]**. In Section 3, we show that when M is in standard form, there is essentially a one-to-one correspondence between invariant subspaces of $H^\infty(\alpha)$ and cocycles for $\{\alpha_t\}_{t\in\mathbf{R}}$ in M. In Section 4, we use this correspondence to show that when M is σ-finite and finite and $\{\alpha_t\}_{t\in\mathbf{R}}$ preserves a faithful, normal, finite trace, then $H^\infty(\alpha)$ is a maximal σ-weakly closed subalgebra of M whenever M^α is a factor. Besides showing, in Section 5, that $H^\infty(\alpha)$ is maximal if and only if M^α is a factor (under the assumption that M is a crossed product and $\{\alpha_t\}_{t\in\mathbf{R}}$ is a dual action), we generalize Solel's analysis in **[21]**, and identify all of the σ-weakly closed subalgebras of M containing $H^\infty(\alpha)$ when $H^\infty(\alpha)$ is not maximal.

2. Algebras and automorphisms in standard form. There are a number of ways in which one can view the standard form of a von Neumann algebra. The form we find most congenial and the one we shall use is that developed by Haagerup. So let M be a von Neumann algebra and form the noncommutative Lebesgue space $L^2(M)$ in the sense of Haagerup **[6]** (cf. **[26]** also). Recall that $L^2(M)$ is a certain space of (generally unbounded) operators. For $x \in M$, let L_x (resp. R_x) be the operator on $L^2(M)$ defined by the formula $L_xy = xy$ (resp. $R_xy = yx$), $y \in L^2(M)$. Then by Theorem 3.6 of **[26]**, L (resp. R) is a faithful normal representation (resp. antirepresentation) of M on the Hilbert space $L^2(M)$. If J is defined on

$L^2(M)$ by the formula $Jy = y^*$, $y \in L^2(M)$, then J is a conjugate linear isometric involution of $L^2(M)$. The von Neumann algebras $L(M)$ and $R(M)$ are communtants of one another (in general, for a subset S of M, we write $L(S)$ for $\{L_x\}_{x \in S}$ and $R(S)$ for $\{R_x\}_{x \in S}$), and $JL(M)J = R(M)$. In fact, the quadruple

$$\{L(M), L^2(M), J, L^2(M)_+\},$$

where $L^2(M)_+$ is the cone of positive operators in $L^2(M)$, is a standard form of M in the sense of [3], by Theorem 36 of [26]. That is, the quadruple satisfies the following assertions:

1) $JL(M)J = R(M)$;

2) $JL_cJ = L_{c^*}$ for all c in the center of M, $\mathscr{Z}(M)$;

3) $Jy = y$, for all $y \in L^2(M)_+$; and

4) $L_aJL_aJ(L^2(M)_+) \subseteq L^2(M)_+$, for all $a \in M$.

Let $\{\alpha_t\}_{t \in \mathbf{R}}$ be a σ-weakly continuous flow on M, i.e., a σ-weakly continuous, one-parameter group of $*$-automorphisms of M. Then by Theorem 3.2 of [3], there is a uniquely determined unitary group $\{U_t\}_{t \in \mathbf{R}}$ on $L^2(M)$ such that

1) $U_tJ = JU_t$,

2) $U_t(L^2(M)_+) = L^2(M)_+$, and

3) $L_{\alpha_t(x)} = U_tL_xU_t^*$

for all $x \in M$ and $t \in \mathbf{R}$.

We need Arveson's theory [2] of spectral subspaces and so we recall the definitions here. The groups $\{\alpha_t\}_{t \in \mathbf{R}}$ and $\{U_t\}_{t \in \mathbf{R}}$ may be integrated yielding representations $\alpha(\cdot)$ and $U(\cdot)$ of $L^1(\mathbf{R})$. Specifically,

$$\alpha(f)x = \int_{-\infty}^{\infty} f(t)\alpha_t(x)dt, \quad x \in M, f \in L^1(\mathbf{R}), \quad \text{and}$$

$$U(f)y = \int_{-\infty}^{\infty} f(t)U_tydt, \quad y \in L^2(M), f \in L^1(\mathbf{R}).$$

For $L^1(\mathbf{R})$, we denote by $Z(f)$ the set

$$\{t \in \mathbf{R} \mid \hat{f}(t) = 0\} \quad \text{where}$$

$$\hat{f}(t) = \int_{-\infty}^{\infty} e^{-ist}f(s)ds.$$

For $x \in M$ (resp. $y \in L^2(M)$), we define $\mathrm{Sp}_\alpha(x)$ (resp. $\mathrm{Sp}_U(y)$) to be the set

$$\cap\{Z(f) \mid f \in L^1(\mathbf{R}), \alpha(f)x = 0\}$$

(resp. $\cap\{Z(f) \mid f \in L^1(\mathbf{R}), U(f)y = 0\}$) and for any closed subset S of $\mathbf{R}$ we define the spectral subspace $M^\alpha(S)$ (resp. $L^2(M)^U(S)$) to be

$$\{x \in M \mid \mathrm{Sp}_\alpha(x) \subseteq S\}$$

(resp. $\{y \in L^2(M) | \mathrm{Sp}_U(y) \subseteq S\}$). If S is not closed $M^\alpha(S)$ and $L^2(M)^U(S)$ are defined to be the σ-weak and norm closures, respectively, of the sets

$$\{x | \mathrm{Sp}_\alpha(x) \subseteq S\} \quad \text{and} \quad \{y | \mathrm{Sp}_U(y) \subseteq S\}.$$

Finally, we define the spectrum of α, $\mathrm{Sp}(\alpha)$, to be

$$\cap\{Z(f) | \alpha(f) = 0\}$$

and the spectrum of $\{U_t\}_{t \in \mathbf{R}}$, $\mathrm{Sp}(U)$, to be

$$\cap\{Z(f) | U(f) = 0\}.$$

We refer the reader to [**2**], [**8**], and [**23**] for the basic facts about spectra.

In this paper, we write $H^\infty(\alpha)$ for $M^\alpha(\mathbf{R}_+)$ and $H_0^\infty(\alpha)$ for $M^\alpha(\mathbf{R}_{+0})$ where $\mathbf{R}_+ = [0, \infty)$ and $\mathbf{R}_{+0} = (0, \infty)$. It is not difficult to see that $H^\infty(\alpha)$ defined in this way coincides with our definition of $H^\infty(\alpha)$ in the introduction. Let E be the spectral measure of $\{U_t\}_{t \in \mathbf{R}}$ and define $H^2(\alpha)$ to be $E[0, \infty)L^2(M)$; likewise, define $H_0^2(\alpha)$ to be $E(0, \infty)L^2(M)$. Observe that in the classical setting when $M = L^\infty(\mathbf{R})$ or $L^\infty(\mathbf{T})$ and $\{\alpha_t\}_{t \in \mathbf{R}}$ is translation then $H^p(\alpha)$ coincides with the classical Hardy space $H^p(\mathbf{R})$ or $H^p(\mathbf{T})$, for $p = 2, \infty$. Of course

$$H^p(\mathbf{R}) = H_0^p(\mathbf{R})$$

while

$$H^p(\mathbf{T}) \neq H_0^p(\mathbf{T}), \quad p = 2, \infty.$$

We write $\mathfrak{L}_+$ for $L(H^\infty(\alpha))$, $\mathfrak{R}_+$ for $R(H^\infty(\alpha))$, $\mathfrak{L}[t, \infty)$ for $L(M^\alpha([t, \infty)))$, and $\mathfrak{R}[t, \infty)$ for $R(M^\alpha([t, \infty)))$. Finally, we write M^α for $M^\alpha(\{0\})$ and note that

$$M^\alpha = \{x \in M | \alpha_t(x) = x, t \in \mathbf{R}\}.$$

Definition 2.1. Let $\mathfrak{M}$ be a closed subspace of $L^2(M)$. We say that $\mathfrak{M}$ is: *left-invariant*, if $\mathfrak{L}_+\mathfrak{M} \subseteq \mathfrak{M}$; *left-reducing*, if $L(M)\mathfrak{M} \subseteq \mathfrak{M}$; *left-pure*, if $\mathfrak{M}$ contains no left-reducing subspaces; and *left-full*, if the smallest left-reducing subspace containing $\mathfrak{M}$ is all of $L^2(M)$. The right-hand versions of these concepts are defined similarly, and a closed subspace which is both left-invariant and right-invariant is called *two-sided invariant*.

The proof of the following proposition is straightforward and so will be omitted; the key fact that one needs, to fill in the details, is the relation

$$M^\alpha([t, \infty))M^\alpha([s, \infty)) \subseteq M^\alpha([s + t, \infty)).$$

PROPOSITION 2.2. *Let $\mathfrak{M}$ be a left-invariant subspace of $L^2(M)$. Then*

1) *$\mathfrak{M}$ reduces $L(M^\alpha)$;*

2) *$\bigwedge_{t>0}[\mathfrak{L}[t, \infty)\mathfrak{M}]_2$ and $\bigvee_{t<0}[\mathfrak{L}[t, \infty)\mathfrak{M}]_2$ are left-reducing subspaces of $L^2(M)$;*

https://doi.org/10.4153/CJM-1987-005-4 Published online by Cambridge University Press

3) *if* $\mathfrak{M}$ *is left-pure, then*

$$\bigwedge_{t>0}[\mathfrak{L}[t, \infty)\mathfrak{M}]_2 = \{0\};$$

and

4) $\mathfrak{M}$ *is left-full if and only if*

$$\bigvee_{t<0}[\mathfrak{L}[t, \infty)\mathfrak{M}]_2 = L^2(M),$$

where, for a subset $S \subseteq L^2(M)$, $[S]_2$ *denotes its closure in* $L^2(M)$.

If $\mathfrak{M}$ is a left-invariant subspace of $L^2(M)$, then we write $\mathfrak{M}_{(+)}$ for $\bigwedge_{t<0}[\mathfrak{L}[t, \infty)\mathfrak{M}]_2$ and $\mathfrak{M}_{(-)}$ for $\bigvee_{t>0}[\mathfrak{L}[t, \infty)\mathfrak{M}]_2$.

Definition 2.3. Let $\mathfrak{M}$ be a left-invariant subspace of $L^2(M)$. Then $\mathfrak{M}$ is said to be *left-normalized* (resp. *right-normalized*) in case $\mathfrak{M} = \mathfrak{M}_{(+)}$ (resp. $\mathfrak{M} = \mathfrak{M}_{(-)}$). If $\mathfrak{M}$ is both left- and right-normalized, then we call $\mathfrak{M}$ *completely normalized.*

The following two propositions describe some basic properties of $H^\infty(\alpha)$ and $H^2(\alpha)$ and their proofs rest ultimately on Theorem 2.9 of [**8**].

PROPOSITION 2.4. *The algebra* $H^\infty(\alpha)$ *coincides with*

$$\{x \in M | L_x E[t, \infty)L^2(M) \subseteq E[t, \infty)L^2(M), \textit{ for all } t \in \mathbf{R}\}$$

and with

$$\{x \in M | R_x E[t, \infty)L^2(M) \subseteq E[t, \infty)L^2(M), \textit{ for all } t \in \mathbf{R}\}.$$

Proof. Since

$$L_{\alpha_t(x)} = U_t L_x U_t^*, \text{ for all } x \in M \text{ and } t \in \mathbf{R},$$

the first identification is a consequence of Theorem 2.9 of [**8**]. On the other hand, since $JU_t = U_t J$ for all t and $JL_x J = R_{x^*}$, for all $x \in M$, the following calculation and Theorem 2.9 of [**8**] complete the proof:

$$\begin{aligned} R_{\alpha_t(x)} &= JL_{\alpha_t(x^*)}J = JU_t L_{x^*} U_t^* J \\ &= U_t J L_{x^*} J U_t^* = U_t R_x U_t^*. \end{aligned}$$

PROPOSITION 2.5. (1) *The space* $H^2(\alpha)$ *is a left-full, right-full, two-sided invariant subspace of* $L^2(M)$.

(2) *The space* $H_0^2(\alpha)$ *is a left-pure, right-pure, two-sided invariant subspace of* $L^2(M)$.

Proof. Proposition 2.4 shows that $H^2(\alpha)$ is a two-sided invariant and it also shows that $H_0^2(\alpha)$ is two-sided invariant once one notes that

$$E(t, \infty) = \bigvee_{s>t} E[s, \infty).$$

(1) It suffices to prove that $H^2(\alpha)$ is left-full. To this end, consider

$$\bigvee_{t \in \mathbf{R}} [\mathfrak{L}[t, \infty)H^2(\alpha)]_2 = [L(M)H^2(\alpha)]_2.$$

Since $[L(M)H^2(\alpha)]_2$ is a left-reducing subspace of $L^2(M)$, there is a projection $p \in M$ such that

$$[L(M)H^2(\alpha)]_2 = R_pL^2(M).$$

Since $H^\infty(\alpha)$ is $\{U_t\}_{t\in\mathbf{R}}$-invariant and $L(M)$ is $\{\alpha_t\}_{t\in\mathbf{R}}$-invariant, $R_pL^2(M)$ is $\{U_t\}_{t\in\mathbf{R}}$-invariant. Thus p is $\{\alpha_t\}_{t\in\mathbf{R}}$-invariant. Also, since $R_p \geqq E[0, \infty)$, we have $R_{1-p} \leqq E(-\infty, 0)$. Set

$$Q = L_{1-p}R_{1-p}.$$

Then, by Lemma 2.6 of [**3**], the quadruple

$$\{QL(M)Q, QL^2(M), QJQ, QL^2(M)_+\}$$

is a standard form for $QL(M)Q$. Since $1 - p$ is $\{\alpha_t\}_{t\in\mathbf{R}}$-invariant, Q commutes with $\{U_t\}_{t\in\mathbf{R}}$. So, if $\{\alpha_t^Q\}_{t\in\mathbf{R}}$ is the automorphism group of $QL(M)Q$ defined by

$$\alpha_t^Q(QL_xQ) = QL_{\alpha_t(x)}Q, \quad x \in M,$$

then the restriction,

$$U_t^Q = U_t|_{QL^2(M)},$$

of U_t to $QL^2(M)$ is the canonical unitary group that implements $\{\alpha_t^Q\}_{t\in\mathbf{R}}$ and preserves the standard form. On the other hand, since $R_{1-p} \leqq E(-\infty, 0)$, we find that

$$\mathrm{Sp}(U^Q) \subseteq (-\infty, 0),$$

and so

$$\mathrm{Sp}(U^Q) \subseteq (-\infty, 0) \cap (0, \infty) = \emptyset$$

since

$$U_t^QQJQ = QJQU_t^Q \quad \text{for all } t \in \mathbf{R}.$$

Thus $Q = 0$, and by Corollary 2.5 of [**3**], $p = 1$. Hence $H^2(\alpha)$ is left-full.

(2) Put

$$\mathfrak{M} = \bigwedge_{t\in\mathbf{R}} [\mathfrak{L}[t, \infty)H^2(\alpha)]_2$$

and note that by Proposition 2.2, $\mathfrak{M}$ is left-reducing. Consequently there is a projection p in M such that

$$\mathfrak{M} = R_pL^2(M).$$

Since $\mathfrak{M}$ is $\{U_t\}_{t\in\mathbf{R}}$-invariant, p is $\{\alpha_t\}_{t\in\mathbf{R}}$-invariant. Also since $E(0, \infty)$ is the projection onto

$$\bigvee_{t>0}[\mathfrak{L}[t, \infty)H^2(\alpha)\,]_2,$$

$R_p \leqq E(0, \infty)$. So, as in the proof of (1), we conclude that $p = 0$, proving that $H^2(\alpha)$ is left-pure, and completing the proof.

Remark 2.6. It follows from Proposition 2.7 of [**19**] and Lemma 3.3 of [**21**] that $H^2(\alpha)$ need not be left- or right-pure.

3. Invariant subspaces and cocycles. In this section we refine the results of [**8**] and parameterize the invariant subspaces for $H^\infty(\alpha)$ in terms of cocycles. Let $\mathfrak{M}$ be a left-invariant subspace of $L^2(M)$. By Theorem 5.2 of [**8**], $\mathfrak{M}$ has a "Wold" decomposition $\mathfrak{M}_1 \oplus \mathfrak{M}_2$, where $\mathfrak{M}_1$ is a left-pure, left-invariant subspace, and $\mathfrak{M}_2$ is a left-reducing subspace. Since it is sufficient to describe the left-pure part, $\mathfrak{M}_1$, of $\mathfrak{M}$ we assume now, without loss of generality, that $\mathfrak{M} = \mathfrak{M}_1$. Also, we shall assume that $\mathfrak{M}$ is left-normalized, the argument when $\mathfrak{M}$ is right-normalized is similar. For $t \in \mathbf{R}$, let F_t be the projection of $L^2(M)$ onto

$$\bigwedge_{s<t}[\mathfrak{L}[s, \infty)\mathfrak{M}]_2,$$

and let

$$E = \bigvee_{t\in\mathbf{R}}F_t.$$

Since

$$EL^2(M) = \bigvee_{t\in\mathbf{R}}[\mathfrak{L}[t, \infty)\mathfrak{M}]_2$$

is left-reducing by Proposition 2.2, there is a projection $p \in M$ such that $E = R_p$. Since $\mathfrak{M}$ is left-pure, by assumption,

$$\bigwedge_{t\in\mathbf{R}}[\mathfrak{L}[s, \infty)\mathfrak{M}]_2 = 0$$

by Proposition 2.2, and so

$$\bigwedge_{t\in\mathbf{R}}F_t = 0.$$

As in the proof of Theorem 5.2 of [**8**], there is a spectral measure with values in the projections on $R_pL^2(M)$ such that $F[t, \infty) = F_t$. By construction and the hypothesis that $\mathfrak{M}$ is left-normalized, we see that

$$\mathfrak{M} = F_0L^2(M) = F[0, \infty)R_pL^2(M) = F[0, \infty)L^2(M).$$

Also, by construction,

$$(\mathfrak{L}[t, \infty)\,)F[s, \infty)R_pL^2(M) \subseteq F[s + t, \infty)R_pL^2(M),$$

https://doi.org/10.4153/CJM-1987-005-4 Published online by Cambridge University Press

for all s, $t \in \mathbf{R}$. Consequently, if $\{V_t\}_{t\in\mathbf{R}}$ is the strongly continuous unitary representation of $\mathbf{R}$ on $R_pL^2(M)$ which is the Fourier-Stieltjes transform of F, then

$$L_{\alpha_t(x)}R_p = V_tL_xV_t^* \quad \text{for } x \in M \text{ and } t \in \mathbf{R},$$

by Theorem 5.2 of [**8**]. Put $A_t = V_tU_t^*$. Then since $V_t^*V_t = V_tV_t^* = R_p$, we have

$$\begin{aligned} A_tL_x &= V_tU_t^*L_x = V_tU_t^*L_xU_tU_t^* \\ &= V_tL_{\alpha_{-t}(x)}U_t^* = V_tV_t^*V_tL_{\alpha_{-t}(x)}U_t^* \\ &= V_tL_{\alpha_{-t}(x)}V_t^*V_tU_t^* = L_xR_pV_tU_t^* \\ &= L_xV_tU_t^* = L_xA_t, \end{aligned}$$

for all $x \in M$ and $t \in \mathbf{R}$. This implies that A_t lies in $L(M)' = R(M)$. Consequently, there is a strongly continuous family, $\{a_t\}_{t\in\mathbf{R}}$, of partial isometries in M such that $A_t = R_{a_t}$. Since

$$\begin{aligned} R_{a_t^*a_t} &= R_{a_t}R_{a_t^*} = V_tU_t^*U_tV_t^* \\ &= V_tV_t^* = R_p \end{aligned}$$

and

$$\begin{aligned} R_{a_ta_t^*} &= R_{a_t^*}R_{a_t} = U_tV_t^*V_tU_t^* = U_tR_pU_t^* \\ &= R_{\alpha_t(p)}, \end{aligned}$$

we find that $a_t^*a_t = p$ and $a_ta_t^* = \alpha_t(p)$. Moreover, $\{a_t\}_{t\in\mathbf{R}}$ has the cocycle property, namely, since

$$\begin{aligned} R_{\alpha_t(a_s)a_t} &= R_{a_t}U_tR_{a_s}U_t^* = V_tU_t^*U_tV_sU_s^*U_t^* \\ &= V_{s+t}U_{s+t}^* = R_{a_{t+s}}, \end{aligned}$$

$a_{t+s} = \alpha_t(a_s)a_t$ for all s, $t \in \mathbf{R}$. The discussion to this point is summarized in the first half of the following theorem; the proof of the second half is straightforward, and so will be omitted.

THEOREM 3.1. *Let $\mathfrak{M}$ be a left-pure, left-invariant subspace of $L^2(M)$ that is left-normalized (resp. right-normalized). Then there is a projection p in M, a strongly continuous unitary representation $\{V_t\}_{t\in\mathbf{R}}$ of $\mathbf{R}$ on $R_pL^2(M)$, and a strongly continuous family $\{a_t\}_{t\in\mathbf{R}}$ of partial isometries in M such that*

(1) $L_{\alpha_t(x)}R_p = V_tL_xV_t^*$, *for all $x \in M$, $t \in \mathbf{R}$;*
(2) $V_t = R_{a_t}U_t$, *for all $t \in \mathbf{R}$;*
(3) $a_t^*a_t = p$, $a_ta_t^* = \alpha_t(p)$ *and* $a_{t+s} = \alpha_t(a_s)a_t$, *for all s, $t \in \mathbf{R}$; and*
(4) $\mathfrak{M} = F[0, \infty)R_pL^2(M)$ *(resp.* $\mathfrak{M} = F(0, \infty)R_pL^2(M)$ *), where F is the spectral measure for V on $R_pL^2(M)$.*

Conversely, given a projection p in M and a strongly continuous family $\{a_t\}_{t\in\mathbf{R}}$ *of partial isometries in M which satisfies* (3), *the family* $\{V_t\}_{t\in\mathbf{R}}$, *defined by the formula*

$$V_t = R_{a_t}U_t, \quad t \in \mathbf{R},$$

is a strongly continuous unitary representation on $R_pL^2(M)$ *that satisfies* (1) *and has the property that if F is the spectral measure of* $\{V_t\}_{t\in\mathbf{R}}$ *on* $R_pL^2(M)$, *then the space* $\mathfrak{M}$ *defined by the formula(s) in* (4) *is a left-normalized* (*resp. right-normalized*), *left-pure, left-invariant subspace of* $L^2(M)$.

Definition 3.2. A strongly continuous family $\{a_t\}_{t\in\mathbf{R}}$ of partial isometries in M satisfying condition (3) in Theorem 3.1 will be called a *cocycle* and p will be called its initial projection.

Theorem 3.1 asserts that there is a one to one correspondence between cocycles and left-normalized, left-pure, left-invariant subspaces. The next proposition and its corollaries are devoted to describing the right invariant subspaces of $L^2(M)$ that are left reducing. This is the key to determining sufficient conditions for the maximality of $H^\infty(\alpha)$.

PROPOSITION 3.3. *Suppose there is a projection* $p \in M$ *such that the subspace* $R_pL^2(M)$ *is right-invariant, but not right-reducing. Then there is a projection q in* $\mathscr{Z}(M) \cap M^\alpha$ *such that* $\{\alpha_t|_{M_q}\}_{t\in\mathbf{R}}$ *is inner.*

Proof. Let $\mathfrak{M} = R_pL^2(M)$, and put

$$\mathfrak{M}_1 = \bigwedge_{t\in\mathbf{R}}[\mathfrak{R}[t, \infty)\mathfrak{M}]_2.$$

Then on the basis of Proposition 2.2, it is easy to see that $\mathfrak{M}_1$ reduces both $L(M)$ and $R(M)$, i.e.,

$$\mathfrak{M}_1 = R_qL^2(M)$$

for some central projection $q \in M$, $q \leqq p$, and

$$\mathfrak{M} \ominus \mathfrak{M}_1 = R_{(p-q)}L^2(M)$$

is right-invariant and right-pure. As a result, we may assume without loss of generality that $\mathfrak{M} = R_pL^2(M)$ is right-pure.

Form

$$\mathfrak{M}^{(+)} = \bigwedge_{t<0}[\mathfrak{R}[t, \infty)\mathfrak{M}]_2 \quad \text{and}$$

$$\mathfrak{M}^{(-)} = \bigvee_{t>0}[\mathfrak{R}[t, \infty)\mathfrak{M}]_2.$$

Since $\mathfrak{M}$ is left-reducing, so are $\mathfrak{M}^{(+)}$ and $\mathfrak{M}^{(-)}$. Hence, there are projections $p_{(+)}$ and $p_{(-)}$ in M such that

$$\mathfrak{M}^{(+)} = R_{p_{(+)}}L^2(M) \quad \text{and} \quad \mathfrak{M}^{(-)} = R_{p_{(-)}}L^2(M).$$

There are two cases to consider.

Case 1. $\mathfrak{M}^{(+)}$ and $\mathfrak{M}^{(-)}$ are right-reducing. In this case, of course, $p_{(+)}$ and $p_{(-)}$ are central, but since $\mathfrak{M}$ is left-pure, by hypothesis, and since $\mathfrak{M}^{(-)} \subseteq \mathfrak{M}$, we conclude that $p_{(-)} = 0$. But then we find that

$$(R_{p_{(+)}}L^2(M))^{(-)} = (\mathfrak{M}^{(+)})^{(-)} = \mathfrak{M}^{(-)} = \{0\}$$

by Corollary 5.6 of [**8**]. Hence

$$[\mathfrak{R}[t, \infty)R_{p_{(+)}}L^2(M)]_2 = \{0\} \quad \text{for all } t > 0;$$

i.e.,

$$p_{(+)}M^{\alpha}([t, \infty)) = \{0\} \quad \text{for all } t > 0.$$

We claim that $p_{(+)}M \subseteq M^{\alpha}$ so that $\{\alpha_t|_{Mp_{(+)}}\}_{t\in\mathbf{R}}$ is trivial and therefore inner. To see this, suppose there is an $x \in p_{(+)}M\setminus M^{\alpha}$. Then there is a nonzero $t \in \mathbf{R}$ such that $t \in \mathrm{Sp}_{\alpha}(x)$. Since $\mathrm{Sp}_{\alpha}(x^*) = -\mathrm{Sp}_{\alpha}(x)$, we may suppose that $t > 0$. We may then choose a function $f \in L^1(\mathbf{R})$ such that

$$\mathrm{supp}\ \hat{f} \subseteq [t/2, 3t/2] \quad \text{and} \quad \alpha(f)x \neq 0.$$

Of course

$$\mathrm{Sp}_{\alpha}(\alpha(f)x) \subseteq [t/2, 3t/2].$$

However, since

$$p_{(+)}M^{\alpha}([s, \infty)) = \{0\} \quad \text{for } s > 0,$$

we conclude that

$$\alpha(f)(p_{(+)}x) = \alpha(f)(x) = 0,$$

which is a contradiction. Hence, $p_{(+)}M \subseteq M^{\alpha}$ as we claimed.

Case 2. Either $\mathfrak{M}^{(+)}$ or $\mathfrak{M}^{(-)}$ is not right-reducing. We assume that $\mathfrak{M}^{(+)}$ is not right-reducing; the argument for $\mathfrak{M}^{(-)}$ is similar. By Proposition 5.5 of [**8**], $\mathfrak{M}^{(+)}$ is left-normalized with respect to $\mathfrak{R}_{+}$. By Theorem 3.1, there are a projection q in M, a strongly continuous unitary representation $\{V_t\}_{t\in\mathbf{R}}$ of $\mathbf{R}$ on $L_qL^2(M)$ and a cocycle $\{a_t\}_{t\in\mathbf{R}}$ of partial isometries in M such that:

(1) $R_{\alpha_t(x)}L_q = V_tR_xV_t^*, \quad x \in M,\ t \in \mathbf{R};$

(2) $V_t = L_{a_t}U_t, \quad t \in \mathbf{R};$ and

(3) $\mathfrak{M}^{(+)} = F[0, \infty)L_qL^2(M),$

where F is the spectral measure for $\{V_t\}_{t\in\mathbf{R}}$ on $L_qL^2(M)$. Since

$$L_qL^2(M) = \bigvee_{t\in\mathbf{R}} [\mathfrak{R}[t, \infty)\mathfrak{M}]_2$$

and $\mathfrak{M}$ is left-reducing, q is a central projection in M. Since $L_q = R_q$, (1) implies that

$$R_{\alpha_t(q)}R_q = R_{\alpha_t(q)}L_q = V_tR_qV_t^* = V_tL_qV_t^* = L_q = R_q,$$

and so $\alpha_t(q) = q$ for all $t \in \mathbf{R}$. On the other hand, since $F[t, \infty)$ is the projection of $L_qL^2(M)$ onto

$$\bigwedge_{t<s}[\mathfrak{N}[s, \infty)\mathfrak{M}]_2$$

and $\mathfrak{M}$ is left-inducing, $F[t, \infty)$ lies in $L(M)' = R(M)$. Thus $V_t \in R(M)$, and we conclude that there is a unitary representation $\{v_t\}_{t\in\mathbf{R}}$ of $\mathbf{R}$ in Mq such that $V_t = R_{v_t}$. Appealing to (1), again, we have

$$R_{\alpha_t(xq)} = R_qR_{\alpha_t(xq)} = R_{\alpha_t(xq)}L_q = V_tR_{xq}V_t^* = R_{v_t}R_{xq}R_{v_t^*} = R_{v_t^*xqv_t}.$$

Thus

$$\alpha_t(xq) = v_t^*(xq)v_t \quad \text{for all } t \in \mathbf{R},$$

and so $\{\alpha_t|_{Mq}\}_{t\in\mathbf{R}}$ is inner on Mq. This completes the proof.

COROLLARY 3.4. *Suppose that $\{\alpha_t\}_{t\in\mathbf{R}}$ is not trivial on M. If M^α is a factor, then every two-sided invariant subspace of $L^2(M)$ which is not left-reducing is left-full and left-pure.*

Proof. Let $\mathfrak{M}$ be a two-sided invariant subspace of $L^2(M)$ which is not left-reducing. Form

$$\mathfrak{M}_{+\infty} = \bigwedge_{t\in\mathbf{R}}[\mathfrak{L}[t, \infty)\mathfrak{M}]_2 \quad \text{and} \quad \mathfrak{M}_{-\infty} = \bigvee_{t\in\mathbf{R}}[\mathfrak{L}[t, \infty)\mathfrak{M}]_2.$$

Then both $\mathfrak{M}_{+\infty}$ and $\mathfrak{M}_{-\infty}$ are left-reducing and right-invariant subspaces of $L^2(M)$. If either $\mathfrak{M}_{+\infty}$ or $\mathfrak{M}_{-\infty}$ is not right-reducing, then by Proposition 3.3, there is a nonzero projection q in $\mathscr{Z}(M) \cap M^\alpha$ such that $\{\alpha_t|_{Mq}\}_{t\in\mathbf{R}}$ is inner. Since M^α is a factor, $q = 1$, and we conclude that there is a unitary group in M which implements $\{\alpha_t\}_{t\in\mathbf{R}}$. Since this unitary group is contained in the center of M^α, which is a factor, we see that $\alpha_t(x) = x$ for all $t \in \mathbf{R}$ and $x \in M$, i.e., $\{\alpha_t\}_{t\in\mathbf{R}}$ is trivial. This contradiction shows that $\mathfrak{M}_{+\infty}$ and $\mathfrak{M}_{-\infty}$ are two-sided reducing subspaces of $L^2(M)$. Hence there are central projections $p_{\pm\infty}$ such that

$$\mathfrak{M}_{\pm\infty} = R_{p_{\pm\infty}}L^2(M).$$

Form

$$\mathfrak{M}_{(+)} = \bigwedge_{t<0}[\mathfrak{L}[t, \infty)\mathfrak{M}]_2 \quad \text{and} \quad \mathfrak{M}_{(-)} = \bigvee_{t>0}[\mathfrak{L}[t, \infty)\mathfrak{M}]_2.$$

If both $\mathfrak{M}_{(+)}$ and $\mathfrak{M}_{(-)}$ are left-reducing, then $\mathfrak{M}_{(+)} = \mathfrak{M}_{-\infty}$ while $\mathfrak{M}_{(-)} = \mathfrak{M}_{+\infty}$. In this event, the argument in the proof of Proposition 3.3 shows that

$$\{\alpha_t|_{M(p_{-\infty}-p_{+\infty})}\}_{t\in\mathbf{R}}$$

is trivial on $M(p_{-\infty} - p_{+\infty})$ which, in turn, means that

$$p_{-\infty} - p_{+\infty} \in M^\alpha \cap \mathscr{Z}(M).$$

Since M^α is a factor, we have $p_{-\infty} - p_{+\infty} = 1$, a contradiction. Therefore, either $\mathfrak{M}_{(+)}$ or $\mathfrak{M}_{(-)}$ is not left-reducing. Set $p = p_{-\infty} - p_{+\infty}$. If $\mathfrak{M}_{(+)}$ is not left-reducing, then by Theorem 3.1, there is a unitary group $\{V_t\}_{t\in\mathbf{R}}$ on $R_pL^2(M)$ such that

$$L_{\alpha_t(x)}R_p = V_tL_xV_t^*, \quad x \in M.$$

Since $L_p = R_p$, we have

$$L_{\alpha_t(p)}L_p = V_tL_pV_t^* = V_tV_t^*V_tV_t^* = L_p,$$

and so $\alpha_t(p) = p$ for all $t \in \mathbf{R}$. Thus

$$p \in \mathscr{Z}(M) \cap M^\alpha.$$

Since M^α is a factor, $p = 1$, which implies that $p_{-\infty} = 1$ and $p_{+\infty} = 0$. Hence $\mathfrak{M}$ is left-pure and left full, and the proof is complete.

COROLLARY 3.5. *Suppose that $\{\alpha_t\}_{t\in\mathbf{R}}$ is not trivial and M^α is a factor. If $\mathfrak{M}$ is a two-sided invariant subspace of $L^2(M)$ which is not left-reducing, then $\mathfrak{M}_{(+)}$ is not left-reducing.*

Proof. Let $\mathfrak{M}$ be a two-sided invariant subspace of $L^2(M)$ which is not left-reducing. Applying Corollary 3.4, we know that $\mathfrak{M}$ is left-pure and left-full. If $\mathfrak{M}_{(+)}$ were left-reducing, then $\mathfrak{M}_{(+)} = L^2(M)$ because $\mathfrak{M}$ is left-full. By Theorem 3.1, then, there is a strongly continuous unitary group $\{V_t\}_{t\in\mathbf{R}}$ on $L^2(M)$, whose spectrum is nonnegative, which implements $\{\alpha_t\}_{t\in\mathbf{R}}$. By the corollary to Theorem 3.1 of [**2**], there is a unitary group $\{w_t\}_{t\in\mathbf{R}}$ in M such that

$$L_{\alpha_t(x)} = L_{w_t}L_xL_{w_t^*}.$$

This unitary group must lie in the center of M^α, and since M^α is a factor, we conclude that $\{\alpha_t\}_{t\in\mathbf{R}}$ is trivial on M. This contradiction completes the proof.

4. Maximality of $H^\infty(\alpha)$. Finite algebras. In this section, we suppose that M is a σ-finite, finite von Neumann algebra. Recall that this is tantamount to assuming that there is a faithful, normal, finite trace τ on M. We fix one such trace for the remainder of this section and we assume that there is a τ-preserving, σ-weakly continuous flow $\{\alpha_t\}_{t\in\mathbf{R}}$ on M. We note that $L^2(M)$ coincides with Segal's L^2-space, [**20**], constructed from τ, $L^2(M, \tau)$, and we note that since M is finite and $\{\alpha_t\}_{t\in\mathbf{R}}$ preserves τ, there is a faithful, normal, $\{\alpha_t\}_{t\in\mathbf{R}}$-invariant expectation ϵ_0 from M onto M^α. It results from this, as is proved in [**7**], [**8**], and [**27**], that $H^\infty(\alpha)$ is a finite, maximal, subdiagonal algebra in M with respect to ϵ_0 in the sense of Arveson [**1**]. Finally, we note that since $\{\alpha_t\}_{t\in\mathbf{R}}$ is τ-preserving, the canonical unitary group $\{U_t\}_{t\in\mathbf{R}}$ on $L^2(M, \tau)$ determined by $\{\alpha_t\}_{t\in\mathbf{R}}$ as in Section 2 is given by the formula

$$U_t(x) = \alpha_t(x), \quad x \in M.$$

This observation yields

LEMMA 4.1. *If* $\{E[t, \infty)\}_{t\in\mathbf{R}}$ *is the spectral measure for* $\{\alpha_t\}_{t\in\mathbf{R}}$, *then*

$$E[t, \infty)L^2(M, \tau) = [M^\alpha([t, \infty))]_2.$$

Proof. From elementary spectral theory we know that

$$E[t, \infty)L^2(M, \tau) = \{x \in L^2(M, \tau) | \mathrm{Sp}_U(x) \subseteq [t, \infty)\}.$$

Since $\{U_t\}_{t\in\mathbf{R}}$ is an extension of $\{\alpha_t\}_{t\in\mathbf{R}}$,

$$E[t, \infty)L^2(M, \tau) \cap M = M^\alpha([t, \infty)).$$

Since $E[t, \infty)L^2(M, \tau)$ is a left-invariant subspace of $L^2(M, \tau)$, Theorem 1 of [**16**] implies that

$$E[t, \infty)L^2(M, \tau) = [M^\alpha([t, \infty))]_2,$$

and completes the proof.

THEOREM 4.2. *Let M be a von Neumann algebra with a faithful, normal, finite trace τ on M, and let $\{\alpha_t\}_{t\in\mathbf{R}}$ be a σ-weakly continuous flow on M such that $\tau \circ \alpha_t = \tau$, for all $t \in \mathbf{R}$. If M^α is a factor, then $H^\infty(\alpha)$ is a maximal σ-weakly closed subalgebra of M.*

Proof. Suppose that there is a σ-weakly closed subalgebra B of M such that $H^\infty(\alpha) \subseteq B \subseteq M$. Form $\mathfrak{M} = [B]_2$ and apply Theorem 1 of [**16**] to conclude that

$$\mathfrak{M} \neq L^2(M, \tau).$$

Moreover, $\mathfrak{M}$ is a two-sided invariant subspace of $L^2(M, \tau)$ which is not left-reducing since $1 \in B$. By Corollary 3.4, $\mathfrak{M}$ is left-pure and left-full, and so, by Corollary 3.5, $\mathfrak{M}_{(+)}$ is not left-reducing. Therefore, by Theorem 3.1, there is a unitary group $\{V_t\}_{t\in\mathbf{R}}$ and a cocycle $\{a_t\}_{t\in\mathbf{R}}$ of unitary operators in M such that

$$V_t = R_{a_t}U_t, \quad \mathfrak{M}_{(+)} = F[0, \infty)L^2(M, \tau), \quad \text{and}$$
$$\mathfrak{M}_{(-)} = F(0, \infty)L^2(M, \tau),$$

where F is the spectral measure of $\{V_t\}_{t\in\mathbf{R}}$. Since

$$\mathfrak{M}_{(-)} \subseteq \mathfrak{M} \subseteq \mathfrak{M}_{(+)},$$

the projection of $L^2(M, \tau)$ onto $\mathfrak{M}$, $P_{\mathfrak{M}}$, commutes with the spectral measure $\{F[t, \infty)\}_{t\in\mathbf{R}}$, and so $P_{\mathfrak{M}}$ commutes with $\{V_t\}_{t\in\mathbf{R}}$. Hence $V_t\mathfrak{M} = \mathfrak{M}$ for all $t \in \mathbf{R}$. Put

$$\widetilde{B} = \{x \in M | L_x\mathfrak{M} \subseteq \mathfrak{M}\}.$$

Then $\widetilde{B}$ is a σ-weakly closed subalgebra of M that contains B and satisfies $[\widetilde{B}]_2 = [B]_2$, as may be seen from the proof of Theorem 4.1 of [**10**]. By Theorem 1 of [**16**], then, we find that $\widetilde{B} = B$. Next observe that for all $x \in B$,

$$L_{\alpha_t(x)}\mathfrak{M} = V_tL_xV_t^*\mathfrak{M} = V_tL_x\mathfrak{M} \subseteq V_t\mathfrak{M} = \mathfrak{M},$$

for all $t \in \mathbf{R}$, and so B is $\{\alpha_t\}_{t\in\mathbf{R}}$-invariant. Since

$$\begin{aligned} U_sF[t, \infty)U_s^*L^2(M, \tau) &= U_sF[t, \infty)L^2(M, \tau) \\ &= U_s \bigwedge_{r<t}[\mathfrak{L}[r, \infty)\mathfrak{M}]_2 \subseteq \bigwedge_{r<t}[\mathfrak{L}[r, \infty)\mathfrak{M}]_2 \\ &= F[t, \infty)L^2(M, \tau), \end{aligned}$$

we see that

$$U_sF[t, \infty)U_s^* \leqq F[t, \infty) \quad \text{for all } t \in \mathbf{R}.$$

This implies that equality holds and, therefore, that $\{V_t\}_{t\in\mathbf{R}}$ and $\{U_t\}_{t\in\mathbf{R}}$ commute. As a result, we find that $\{a_t\}_{t\in\mathbf{R}}$ is a unitary group in M^α. Since each of the spaces $F[t, \infty)L^2(M, \tau)$ is right-invariant, each is $R(M^\alpha)$-invariant. Thus, for all t, $F[t, \infty)$ lies in $R(M^\alpha)'$ and so, therefore, does $\{V_t\}_{t\in\mathbf{R}}$. From this we conclude that

$$R_{a_t} \in R(M^\alpha) \cap R(M^\alpha)',$$

and since M^α is a factor, there must be a real number λ such that $a_t = e^{i\lambda t}$. Thus

$$V_t = e^{i\lambda t}U_t \quad \text{for all } t \in \mathbf{R}.$$

If E denotes the spectral measure of $\{U_t\}_{t\in\mathbf{R}}$, then we conclude from this and Lemma 4.1 that

$$\begin{aligned} [M^\alpha((-\lambda, \infty))]_2 &= E(-\lambda, \infty)L^2(M, \tau) \\ &= F(0, \infty)L^2(M, \tau) \subseteq \mathfrak{M} \subseteq F[0, \infty)L^2(M, \tau) \\ &= [M^\alpha([-\lambda, \infty))]_2. \end{aligned}$$

Since $\mathfrak{M} \supseteq H^2(\alpha)$, we have $\lambda > 0$. Since $[M^\alpha([-\lambda, \infty))]_2$ is a left-invariant subspace and $M^\alpha([-\lambda, \infty))$ is σ-weakly closed, Theorem 1 of [**16**] implies that

$$M^\alpha((-\lambda, \infty)) \subseteq B \subseteq M^\alpha([-\lambda, \infty)).$$

We must now consider two cases:

(1) $M^{\alpha}((-\lambda, \infty)) \neq H^{\infty}(\alpha)$; and

(2) $M^{\alpha}((-\lambda, \infty)) = H^{\infty}(\alpha)$.

Case (1). In this case, we have that for $-\lambda < t < 0$,

$$\begin{aligned}[M^{\alpha}([t-\lambda, \infty))]_2 &= E[t-\lambda, \infty)L^2(M, \tau) = F[t, \infty)L^2(M, \tau)\\ &= \bigwedge_{-\lambda<s<t} [\mathfrak{L}[s, \infty)F[0, \infty)L^2(M, \tau)]_2\\ &\subseteq \bigwedge_{-\lambda<s<t} [\mathfrak{L}[s, \infty)[B]_2]_2\\ &\subseteq [B]_2 \subseteq F[0, \infty)L^2(M, \tau).\end{aligned}$$

From this and Theorem 1 of [**16**], we conclude that $M^{\alpha}((-2\lambda, \infty))$, which is the σ-weak closure of

$$\bigcup_{-\lambda<t} M^{\alpha}([t-\lambda, \infty)),$$

is contained in B. Repeating this argument, we conclude that $M^{\alpha}((-n\lambda, \infty))$ is contained in B for all $n > 0$ and, therefore, that $B = M$, a contradiction.

Case (2). Since M^{α} is a factor, $\mathrm{Sp}(\alpha)$ is a subgroup of $\mathbf{R}$ by Proposition 16.1 of [**23**]. If

$$M^{\alpha}((-\lambda, \infty)) = H^{\infty}(\alpha),$$

then we conclude that

$$\mathrm{Sp}(\alpha) = \{n\lambda\}_{n\in\mathbf{Z}}.$$

Set

$$B(-\lambda) = B \cap M^{\alpha}(\{-\lambda\}).$$

Since $B \neq H^{\infty}(\alpha)$, $B(-\lambda) \neq \{0\}$. As in the proof of Theorem 2.3 of [**18**], there is a unitary operator u in $M^{\alpha}(\{-\lambda\})$ such that

$$M^{\alpha}(\{-\lambda\}) = M^{\alpha}u = uM^{\alpha}.$$

Consequently,

$$H^{\infty}(\alpha) = u^*M^{\alpha}([-\lambda, \infty)) \supseteq u^*B \supseteq u^*M^{\alpha}((-\lambda, \infty)) = H_0^{\infty}(\alpha).$$

Now u^*B is a σ-weakly closed two-sided ideal in $H^{\infty}(\alpha)$ containing $H_0^{\infty}(\alpha)$ properly. Therefore, $\epsilon_0(u^*B)$ is a nonzero ideal in M^{α}. Since M^{α} is a finite factor, and therefore algebraically simple, we see that

$$\epsilon_0(u^*B) = M^{\alpha}.$$

Therefore,

$$u^*B = H^{\infty}(\alpha) \quad \text{and} \quad B = uH^{\infty}(\alpha) = M^{\alpha}([-\lambda, \infty)).$$

In particular, $u \in B$ and we have

$$B \supseteq uB = M^{\alpha}([-2\lambda, \infty)) = u^2 H^{\infty}(\alpha).$$

Repeating this argument yields the inclusion

$$B \supseteq M^{\alpha}([-n\lambda, \infty)) \quad \text{for all } n > 0,$$

which implies that $B = M$, again a contradiction. This completes the proof.

5. Analytic crossed products. The continuous case. Let N be a von Neumann algebra on a Hilbert space H, and let $\{\beta_t\}_{t \in \mathbf{R}}$ be a σ-weakly continuous flow on N. Recall that the crossed product, $N \times_{\beta} \mathbf{R}$, determined by N and $\{\beta_t\}_{t \in \mathbf{R}}$ is the von Neumann algebra on the Hilbert space $L^2(\mathbf{R}, H)$ generated by the operators $\pi(x)$, $x \in N$, and $\lambda(s)$, $s \in \mathbf{R}$, defined by the equations

$$(\pi(x)f)(t) = \beta_{-t}(x)f(t), \quad f \in L^2(\mathbf{R}, H), \ t \in \mathbf{R},$$

and

$$(\lambda(s)f)(t) = f(t - s), \quad f \in L^2(\mathbf{R}, H), \ t \in \mathbf{R}.$$

The automorphism group $\{\alpha_t\}_{t \in \mathbf{R}}$ of $N \times_{\beta} \mathbf{R}$ which is dual to $\{\beta_t\}_{t \in \mathbf{R}}$ in the sense of Takesaki [**25**] is implemented by the unitary representation of $\mathbf{R}$, $\{S_t\}_{t \in \mathbf{R}}$, defined by the formula

$$(S_t f)(s) = e^{ist} f(s), \quad f \in L^2(\mathbf{R}, H);$$

that is,

$$\alpha_t(y) = S_t y S_t^*, \quad \text{for all } y \in N \times_{\beta} \mathbf{R}.$$

The group $\{\alpha_t\}_{t \in \mathbf{R}}$ is characterized by the equations

$$\alpha_t(\pi(x)) = \pi(x), \quad x \in N, \ t \in \mathbf{R},$$

$$\alpha_s(\lambda(t)) = e^{-ist}\lambda(t), \ s, t \in \mathbf{R}.$$

In particular,

$$\pi(N) = \{y \in N \times_{\beta} \mathbf{R} | \alpha_t(y) = y, \text{ for all } t \in \mathbf{R}\}.$$

For simplicity, we write M for $N \times_{\beta} \mathbf{R}$. The following proposition was proved by M. McAsey and the first author in the context of C^*-crossed products in [**9**]. In the von Neumann algebra setting, there is a simpler proof.

PROPOSITION 5.1. *Let $N \times_{\beta} \mathbf{R}_+$ denote the σ-weakly closed subalgebra generated by $\pi(N)$ and $\{\lambda(s)\}_{s \in \mathbf{R}_+}$. Then the three spaces, $H^{\infty}(\alpha)$, $N \times_{\beta} \mathbf{R}_+$ and $H_0^{\infty}(\alpha)$ coincide.*

Proof. Since $\pi(N) = M^{\alpha}$ and since $\mathrm{Sp}_{\alpha}(\lambda(s)) = s$, it is clear that

$$N \times_\beta \mathbf{R}_+ \subseteq H^\infty(\alpha).$$

To see that $H_0^\infty(\alpha) \subseteq N \times_\beta \mathbf{R}_+$, choose $x \in M$ with compact spectrum contained in $(0, \infty)$. Then choose an f in $L^1(\mathbf{R})$ such that $\hat{f}(t) = 1$, for all $t \in \mathrm{Sp}_\alpha(x)$ and such that the support of $\hat{f}$, supp $\hat{f}$, is compact and contained in $(0, \infty)$. By Proposition 14.2 (9) of [**25**], $\alpha(f)x = x$. Since M is the σ-weak closure of the linear span, L, of $\pi(N)$ and $\{\lambda(s)\}_{s \in \mathbf{R}}$, there is a net $\{x_\gamma\}_{\gamma \in \Gamma}$ in L that converges σ-weakly to x. Then $\{\alpha(f)x_\gamma\}_{\gamma \in \Gamma}$ converges σ-weakly to $\alpha(f)x = x$. Since

$$\mathrm{Sp}_\alpha(\alpha(f))x_\gamma) \subseteq \mathrm{supp}\ \hat{f} \cap \mathrm{Sp}_\alpha(x_\gamma) \subseteq (0, \infty),$$

by Proposition 14.2(3) of [**23**], and since $\alpha(f)x_\gamma$ belongs to L, we see that $x \in N \times_\beta \mathbf{R}_+$. Since $H_0^\infty(\alpha)$ is the σ-weak closure of the set of all $x \in M$ with compact spectra in $(0, \infty)$, we conclude that

$$H_0^\infty(\alpha) \subseteq N \times_\beta \mathbf{R}_+.$$

To complete the proof, we need only show that

$$H^\infty(\alpha) \subseteq H_0^\infty(\alpha).$$

But if $x \in H^\infty(\alpha)$, then for all $t > 0$,

$$\lambda(t)x \in H_0^\infty(\alpha).$$

Since $\{\lambda(t)\}_{t \in \mathbf{R}}$ is a strongly continuous unitary group, $\lambda(t)x$ converges σ-weakly to x as $t \to 0$. Thus $x \in H_0^\infty(\alpha)$, and the proof is complete.

By Proposition 5.1, $H^\infty(\alpha)$ is the σ-weakly closed subalgebra of M generated by $\pi(N)$ and $\{\lambda(t)\}_{t \in \mathbf{R}_+}$, and so, as in [**10-12**], we call $H^\infty(\alpha)$ the *analytic crossed product* (formerly, non-self-adjoint crossed product) determined by N and $\{\beta_t\}_{t \in \mathbf{R}}$. Since there are no $\{\alpha_t\}_{t \in \mathbf{R}}$-invariant, faithful, normal conditional expectations of M onto $M^\alpha = \pi(N)$, $H^\infty(\alpha)$ is not a subdiagonal algebra in the sense of [**1**]. However, we do have a necessary and sufficient condition for $H^\infty(\alpha)$ to be maximal among the σ-weakly closed subalgebras of M.

THEOREM 5.2. *With the notation as above, $H^\infty(\alpha)$ is a maximal σ-weakly closed subalgebra of M if and only if N is a factor.*

The proof rests on several lemmas. First recall that the formula

$$\epsilon(x) = \int_{-\infty}^{\infty} \alpha_t(x)dt, \quad x \in M_+$$

defines a faithful, normal, semi-finite, operator-valued weight ϵ from M onto $\pi(N)$ by Lemma 5.2 of [**5**]. We denote $\{x \in M | \epsilon(x^*x) \in M\}$ by $\mathfrak{F}$. As is shown in Section 1 of [**4**], $\{\alpha_t\}_{t \in \mathbf{R}}$ has a natural extension to the extended positive part of M_+, $\hat{M}_+$. We keep the same notation for

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this extension. Since $\epsilon \circ \alpha_t = \epsilon$, $t \in \mathbf{R}$, and since there is a faithful normal semi-finite weight ψ_0 on $\pi(N)$, we obtain an $\{\alpha_t\}_{t\in\mathbf{R}}$- invariant, faithful, normal, semi-finite weight on M through the formula

$$\psi(x) = \psi_0(\epsilon(x)), \quad x \in M_+$$

(see [**23**, Proposition 11.6]). In this section, we shall have to investigate the subspaces of space $L^2(M)$ associated with M and ψ that are invariant under $H^\infty(\alpha)$.

LEMMA 5.3. *Put* $\mathfrak{F}_+ = H^\infty(\alpha) \cap \mathfrak{F}$. *Then* $\mathfrak{F}_+$ *is* σ*-weakly dense in* $H^\infty(\alpha)$.

Proof. Set

$$A = \{x \in H^\infty(\alpha) \,|\mathrm{sp}_\alpha(x) \text{ is compact in } (0, \infty)\}.$$

Then, by Proposition 5.1, A is σ-weakly dense in $H^\infty(\alpha)$. Let $x \in A$ and choose $f \in L^1(\mathbf{R})$ with compactly supported Fourier transform such that supp $\hat{f} \subseteq (0, \infty)$ and such that $\alpha(f)x = x$. Since $\mathfrak{F}$ is σ-weakly dense in M, there is a net $\{x_\gamma\}_{\gamma\in\Gamma}$ in $\mathfrak{F}$ converging σ-weakly to x. Then $\alpha(f)x_\gamma$ converges σ-weakly to $\alpha(f)x = x$. By Lemma 21.3 of [**23**],

$$\alpha(f)x_\gamma \in A \cap \mathfrak{F} \subseteq \mathfrak{F}_+, \quad \text{for all } \gamma \in \Gamma.$$

Hence $\mathfrak{F}_+$ is σ-weakly dense in $H^\infty(\alpha)$ and the proof is complete.

LEMMA 5.4. *If* B *is a* σ*-weakly closed subalgebra of* M *containing* $H^\infty(\alpha)$, *then* $B \cap \mathfrak{F}$ *is* σ*-weakly dense in* B.

Proof. Since $\mathfrak{F}_+$ is σ-weakly dense in $H^\infty(\alpha)$, by Lemma 5.3 there is a net $\{e_\gamma\}_{\gamma\in\Gamma}$ in $\mathfrak{F}_+$ converging σ-weakly to 1. Since $\mathfrak{F}$ is a left ideal in M, $xe_\gamma \in B \cap \mathfrak{F}$ and xe_γ converges σ-weakly to x. This completes the proof.

LEMMA 5.5. *If* $\mathfrak{M}$ *is a left-invariant subspace of* $L^2(M)$, *then* $\mathfrak{M}$ *is completely normalized.*

Proof. If $\mathfrak{M}_{(+)} \neq \mathfrak{M}_{(-)}$, then by Proposition 5.5 of [**8**], the vector state determined by any unit vector in $\mathfrak{M}_{(+)} \ominus \mathfrak{M}_{(-)}$ is $\{\alpha_t\}_{t\in\mathbf{R}}$-invariant. But there are no $\{\alpha_t\}_{t\in\mathbf{R}}$-invariant normal states of M. For, if φ is one, then since $\mathfrak{F}$ is σ-weakly dense in M, there is an x in $\mathfrak{F}$ such that $\varphi(x^*x) \neq 0$. We then have

$$\varphi(\epsilon(x^*x)) = \varphi\left(\int_{-\infty}^{\infty} \alpha_t(x^*x)\right)dt = \int_{-\infty}^{\infty} \varphi(\alpha_t(x^*x))dt$$
$$= \int_{-\infty}^{\infty} \varphi(x^*x)dt = \infty.$$

This contradiction completes the proof.

LEMMA 5.6. *If N is not a factor and if $\{\beta_t\}_{t\in\mathbf{R}}$ acts ergodically on the center, $\mathscr{Z}(N)$, of N, then there is a family $\{e_t\}_{t<0}$ of projections in $\mathscr{Z}(N)$ such that*

$$e_{t+s} = e_t\beta_t(e_s), \quad s, t < 0,$$

and $0 < e_t < 1$ *for some* $t < 0$.

Proof. First we note that $\mathscr{Z}(N)$ is nonatomic. Indeed, if p is a minimal projection in $\mathscr{Z}(N)$, then so is $\beta_t(p)$ for all t. Consequently, for each $t \in \mathbf{R}$, $\beta_t(p)p = 0$ or p. Since $\{\beta_t\}_{t\in\mathbf{R}}$ is σ-weakly continuous, $\beta_t(p)$ converges to p σ-weakly as $t \to 0$. It follows that $\beta_t(p) = p$ for all t in a neighborhood of 0 and, therefore, for all $t \in \mathbf{R}$. Since $\{\beta_t\}_{t\in\mathbf{R}}$ is ergodic on $\mathscr{Z}(N)$, $p = 1$, and so N is a factor. This contradiction shows that there are no atoms in $\mathscr{Z}(N)$.

Next we observe that there is a faithful normal state on $\mathscr{Z}(N)$. (Note: We do not assume a priori that N is σ-finite and therefore, that $\mathscr{Z}(N)$ is σ-finite.) Let φ_0 be any normal state of $\mathscr{Z}(N)$ and let $s(\varphi_0)$ be the support projection of φ_0. Then

$$\beta_t(s(\varphi_0)) = s(\varphi_0 \circ \beta_{-t}) \quad \text{for all } t \in \mathbf{R}.$$

By ergodicity,

$$\bigvee_{t\in\mathbf{R}} s(\varphi_0 \circ \beta_t) = 1,$$

but also by the σ-weak continuity of $\{\beta_t\}_{t\in\mathbf{R}}$,

$$\bigvee_{t\in Q} s(\varphi_0 \circ \beta_t) = 1.$$

If $\{t_n\}_{n\in 1}^{\infty}$ is a counting of the rationals, then

$$\sum_{n=1}^{\infty} \left(\frac{1}{2^n}\right)\varphi_0 \circ \beta_{t_n}$$

is a faithful normal state on $\mathscr{Z}(N)$.

By Cohen's famous factorization theorem,

$$\{\alpha(f)x | f \in L^1(\mathbf{R}),\ x \in \mathscr{Z}(N)\}$$

is a $\{\alpha_t\}_{t\in\mathbf{R}}$-invariant, σ-weakly dense, C^*-subalgebra of $\mathscr{Z}(N)$ on which $\{\beta_t\}_{t\in\mathbf{R}}$ is strongly continuous. If Ω is the maximal ideal space of this subalgebra, then there is a continuous, one-parameter group of homeomorphisms, $\{T_t\}_{t\in\mathbf{R}}$, of Ω, and, from what was just noted above, there is a nonatomic, quasi-invariant, ergodic, probability measure μ on Ω, with $\mathrm{supp}(\mu) = \Omega$, such that

$$\Gamma(\beta_t(x))(\omega) = \Gamma(x)(T_t\omega) \text{ a.e. } (\mu),$$

where Γ is the canonical extension of the Gelfand transform to all of $\mathscr{Z}(N)$, mapping isomorphically onto $L^\infty(\mu)$. Let E be an open subset of Ω such that $0 < \mu(E) < 1$. Since μ is nonatomic and the function, $(t, \omega) \to T_t\omega$, from $\mathbf{R} \times \Omega$ to Ω is continuous, there is an open subset W of E (so $\mu(W) \neq 0$), and a positive δ such that if $|t| < \delta$, and if $\omega \in W$, then $T_t\omega \in E$. Thus

$$W \subseteq \bigcap_{|t|<\delta} T_tE \subseteq E.$$

For each $t < 0$, we set

$$E_t = \bigcap_{t \leqq s \leqq 0} T_sE.$$

Then

$$E_{t+s} = E_t \cap T_t(E_s) \quad \text{for } s, t < 0$$

and from the last inclusion, we conclude that there is a t such that $0 < \mu(E_t) < 1$. So, if we define

$$e_t = \Gamma^{-1}(1_{E_t}), \quad t < 0,$$

then we obtain the desired family of projections, $\{e_t\}_{t<0}$, in $\mathscr{Z}(N)$.

Proof of Theorem 5.2. Suppose that N is a factor and that B is a σ-weakly closed subalgebra of M such that $H^\infty(\alpha) \subseteq B \subsetneq M$. Then there are nonzero vectors ξ and η in $L^2(M)$ such that

$$\omega_{\xi,\eta}(y) = (L_y\xi, \eta) = 0, \quad \text{for all } y \in B.$$

Form $\mathfrak{M} = [L(B)\xi]_2$. Then $\mathfrak{M}$ is a left-invariant subspace of $L^2(M)$ which is not left reducing since $\omega_{\xi,\eta} \neq 0$. By Lemma 5.5, $\mathfrak{M}$ is completely normalized. Set

$$\mathfrak{M}_1 = \bigwedge_{t \in \mathbf{R}} [\mathfrak{L}[t, \infty)\mathfrak{M}]_2$$

to obtain a left-pure, left-invariant subspace of $L^2(M)$ that is completely normalized. By Theorem 3.1, there is a projection $p \in M$ and there is a strongly continuous representation $\{V_t\}_{t \in \mathbf{R}}$ of $\mathbf{R}$ on $R_pL^2(M)$ such that

$$\mathfrak{M} = F[0, \infty)R_pL^2(M),$$

where F is the spectral measure for $\{V_t\}_{t \in \mathbf{R}}$. Set

$$\widetilde{B} = \{x \in M | L_x\mathfrak{M} \subseteq \mathfrak{M}\}.$$

Since $R_p > F[0, \infty)$, we have, for all $x \in \widetilde{B}$,

$$\begin{aligned} L_{\alpha_t(x)}\mathfrak{M} &= L_{\alpha_t(x)}F[0, \infty)L^2(M) \\ &= L_{\alpha_t(x)}R_pL^2(M) = V_tL_xV_t^*F[0, \infty)R_pL^2(M) \end{aligned}$$

$$\subseteq V_t L_x F[0, \infty) R_p L^2(M) \subseteq V_t F[0, \infty) R_p L^2(M)$$
$$= F[0, \infty) L^2(M) = \mathfrak{M}.$$

Thus $\widetilde{B}$ is an $\{\alpha_t\}_{t\in\mathbf{R}}$-invariant, σ-weakly closed subalgebra of M such that $M \supseteq \widetilde{B} \subseteq B$ ($\widetilde{B} \neq M$ because $\mathfrak{M}$ is not left-reducing). (We will see later, in Lemma 5.7, that the introduction of $\widetilde{B}$ really is superfluous; B already is invariant.) By Lemma 5.4, $\widetilde{B} \cap \mathfrak{F}$ is σ-weakly dense in $\widetilde{B}$. Since $\widetilde{B}$ ($\supseteq H^\infty(\alpha)$) is $\{\alpha_t\}_{t\in\mathbf{R}}$-invariant, and $\{\alpha_t\}_{t\in\mathbf{R}}$ is an integrable action, there is an element $x \in \widetilde{B}$ such that

$$\mathrm{Sp}_\alpha(x) = \{t\} \quad \text{for some } t < 0$$

(cf. [**23**, Section 21.3]). But then there is an $x_0 \in N$ such that $x = \pi(x_0)\lambda(t)$. So,

$$\pi(N)\pi(x_0)\pi(N)\lambda(t) = \pi(N)\pi(x_0)\lambda(t)\pi(N) = \pi(N) \times \pi(N) \subseteq \widetilde{B}.$$

Since N is a factor, and since $\pi(N)\pi(x_0)\pi(N)$ is a two-sided ideal of $\pi(N)$, we conclude from this inclusion that

$$\pi(N)\lambda(t) \subseteq \widetilde{B}$$

and, therefore, that $\lambda(t) \in \widetilde{B}$. This implies that $\lambda(s) \in \widetilde{B}$, for all $s \in \mathbf{R}$. Indeed, if $s > t$, then

$$\lambda(s) = \lambda(t)\lambda(s - t) \in \widetilde{B},$$

while if $s < t$, then, choosing a positive integer n such that $s \geqq nt$, we conclude that

$$\lambda(s) = \lambda(t)^n\lambda(s - nt) \in \widetilde{B}.$$

Since $\widetilde{B}$ contains $\pi(N)$, as well, we reach, finally, the contradiction that $\widetilde{B} = M$. Thus $H^\infty(\alpha)$ is a maximal σ-weakly closed subalgebra of M.

For the converse, suppose that N is not a factor. If $\{\beta_t\}_{t\in\mathbf{R}}$ is not ergodic on $\mathfrak{F}(N)$, then for any proper, $\{\beta_t\}_{t\in\mathbf{R}}$-invariant projection p in $\mathfrak{F}(N)$, it is evident that

$$\pi(p)H^\infty(\alpha) \oplus \pi(1 - p)M$$

is a proper σ-weakly closed subalgebra of M containing $H^\infty(\alpha)$ properly. Consequently, without loss of generality, we may suppose that $\{\beta_t\}_{t\in\mathbf{R}}$ acts ergodically on $\mathscr{Z}(N)$. By Lemma 5.6, then, there is a family, $\{e_t\}_{t<0}$, of central projections in N such that

$$e_{t+s} = e_t\beta_t(e_s),$$

for all $s, t < 0$, and $0 < e_t < 1$ for some $t < 0$. We set $e_t = 1$ for $t \geqq 0$, and let B denote the σ-weak closure of the linear span of

$$H^\infty(\alpha) \quad \text{and} \quad \{\pi(e_t)\pi(N)\lambda(t)\}_{t<0}.$$

Evidently, $H^{\infty}(\alpha) \subsetneqq B \subsetneqq M$, and so, to complete the proof, it suffices to prove that B is an algebra. If t, $s < 0$, then

$$(\pi(e_t)\pi(N)\lambda(t))(\pi(e_s)\pi(N)\lambda(s))$$
$$= \pi(e_t)\pi(\beta_t(e_s))\pi(N)\lambda(t + s)$$
$$= \pi(e_{t+s})\pi(N)\lambda(t + s),$$

while if $t < 0$ and $s \geqq 0$, then

$$(\pi(e_t)\pi(N)\lambda(t))\pi(N)\lambda(s)$$
$$= \pi(e_t)\pi(N)\lambda(t + s)$$
$$\subseteq \pi(e_{t+s})\pi(N)\lambda(t + s).$$

These two computations show that B is closed under multiplication and complete the proof.

This proof of Theorem 5.2 suggests the form of all the σ-weakly closed super-algebras of $H^{\infty}(\alpha)$ when $H^{\infty}(\alpha)$ is not a maximal σ-weakly closed subalgebra of M. Our objective in the remainder of the paper is to show that the suggestion is correct. We need a series of lemmas; in them, B will denote a fixed σ-weakly closed subalgebra of M such that $H^{\infty}(\alpha) \subsetneqq B \subsetneqq M$.

LEMMA 5.7. *B is $\{\alpha_t\}_{t\in\mathbf{R}}$-invariant.*

Proof. As in the proof of Theorem 5.2, we can find proper, $\{\alpha_t\}_{t\in\mathbf{R}}$-invariant, σ-weakly closed subalgebras of M that contain B. We let $\widetilde{B}$ be the smallest such algebra. If $\widetilde{B} \neq B$, then there are nonzero vectors, ξ and η, in $L^2(M)$ such that

$$\omega_{\xi,\eta}(x) = 0 \quad \text{for all } x \in B$$

while

$$\omega_{\xi,\eta}(x_0) \neq 0 \quad \text{for some } x_0 \in \widetilde{B}.$$

Let

$$C = \{x \in M | L_x[B\xi]_2 \subseteq [B\xi]_2\}.$$

As in the proof of Theorem 5.2, C is a proper σ-weakly closed subalgebra of M containing B that is $\{\alpha_t\}_{t\in\mathbf{R}}$-invariant. Therefore $\widetilde{B} \subseteq C$. Since $[C\xi]_2 = [B\xi]_2$, we have $[\widetilde{B}\xi]_2 = [B\xi]_2$. But then,

$$\omega_{\xi,\eta}(x) = 0 \quad \text{for all } x \in \widetilde{B}.$$

This contradiction completes the proof.

LEMMA 5.8. *For each $t < 0$, there is a central projection e_t in N such that*

$$B \cap M^{\alpha}(\{t\}) = \pi(e_t)\pi(N)\lambda(t).$$

Proof. Let

$$B_t = B \cap M^{\alpha}(\{t\}) \quad \text{for all } t < 0.$$

Since $\pi(N) = M^{\alpha}(\{0\}) \subseteq B$,

$$\pi(N)B_t\pi(N) = B_t.$$

Thus $B_t\lambda(-t)$ is a σ-weakly closed two-sided ideal in $\pi(N)$ and we conclude that there is a central projection e_t in N such that

$$B_t\lambda(-t) = \pi(e_t)\pi(N).$$

This completes the proof.

LEMMA 5.9. *For all $s, t < 0$, $e_{t+s} = e_t\beta_t(e_s)$.*

Proof. Since

$$\pi(\beta_t(e_s)) = \lambda(t)\pi(e_s)\lambda(t)^*,$$

we have

$$\begin{aligned}
&\pi(e_t)\pi(\beta_t(e_s))\pi(N)\lambda(t + s)\\
&= \pi(e_t)\pi(N)\pi(\beta_t(e_s))\pi(N)\lambda(t)\lambda(s)\\
&= (\pi(e_t)\pi(N)\lambda(t))(\pi(e_s)\pi(N)\lambda(s))\\
&= B_tB_s \subseteq B_{t+s} = \pi(e_{t+s})\pi(N)\lambda(t + s).
\end{aligned}$$

Thus $e_t\beta_t(e_s) \leqq e_{t+s}$. To prove equality, observe that since

$$B_{s+t}\lambda(-s) \subseteq B_t,$$

we have $e_{t+s} \leqq e_t$. On the other hand, since $t < 0$, we have

$$\begin{aligned}
&\pi(\beta_{-t}(e_{t+s}))\pi(N)\lambda(s)\\
&= \lambda(-t)\pi(e_{t+s})\pi(N)\lambda(t + s)\\
&= \lambda(-t)B_{t+s} \subseteq B_s = \pi(e_s)\pi(N)\lambda(s).
\end{aligned}$$

This implies that $\beta_{-t}(e_{t+s}) \leqq e_s$, and so $e_{t+s} \leqq \beta_t(e_s)$. Thus

$$e_{t+s} = e_t\beta_t(e_s) \quad \text{for all } s, t < 0,$$

and the proof is complete.

LEMMA 5.10. *B is the σ-weakly closed linear span of $\{B_t\}_{t<0}$ and $H^{\infty}(\alpha)$.*

Proof. This is an immediate consequence of the Fourier inversion theorem (cf. [**23**, Corollary 21.3]).

https://doi.org/10.4153/CJM-1987-005-4 Published online by Cambridge University Press

Combining Lemmas 5.7-10 with a calculation in the proof of Theorem 5.2 gives a proof of our last result.

THEOREM 5.11. *If B is a σ-weakly closed subalgebra of M containing $H^{\infty}(\alpha)$, then there is a family, $\{e_t\}_{t \in \mathbf{R}}$, of central projections in N such that*

$$e_{t+s} = e_t\beta_t(e_s),\ s,\ t < 0,\ e_t = 1,\ t \geqq 0,$$

and such that

$$B \cap M^{\alpha}(\{t\}) = \pi(e_t)\pi(N)\lambda(t).$$

Conversely, given such a family $\{e_t\}_{t \in \mathbf{R}}$, the σ-weakly closed linear span of the spaces

$$\{\pi(e_t)\pi(N)\lambda(t)\}_{t \in \mathbf{R}}$$

is a σ-weakly closed subalgebra of M containing $H^{\infty}(\alpha)$. Moreover, the correspondence between subalgebras B and families $\{e_t\}_{t \in \mathbf{R}}$ is bijective.

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