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Smooth Maps and Real Algebraic Morphisms

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Abstract. Let X be a compact nonsingular real algebraic variety and let Y be either the blowup of $\mathbb{P}^n(\mathbb{R})$ along a linear subspace or a nonsingular hypersurface of $\mathbb{P}^m(\mathbb{R}) \times \mathbb{P}^n(\mathbb{R})$ of bidegree $(1,1)$. It is proved that a $\mathcal{C}^\infty$ map $f\colon X \to Y$ can be approximated by regular maps if and only if $f^*\big(H^1(Y,\mathbb{Z}/2)\big) \subseteq H^1_{\mathrm{alg}}(X,\mathbb{Z}/2)$, where $H^1_{\mathrm{alg}}(X,\mathbb{Z}/2)$ is the subgroup of $H^1(X,\mathbb{Z}/2)$ generated by the cohomology classes of algebraic hypersurfaces in X. This follows from another result on maps into generalized flag varieties.

Throughout this note the term *real algebraic variety* designates a locally ringed space isomorphic to a Zariski locally closed subset of $\mathbb{P}^n(\mathbb{R})$, for some n, endowed with the Zariski topology and the sheaf of $\mathbb{R}$-valued regular functions. It is well known that every real algebraic variety is isomorphic to a Zariski closed subvariety of $\mathbb{R}^n$ for some n [2, Proposition 3.2.10, Theorem 3.4.4] (note that real algebraic varieties defined above are called affine real algebraic varieties in [2]). Morphisms between real algebraic varieties will be called *regular maps*. Every real algebraic variety carries also the Euclidean topology, that is, the topology induced by the usual metric topology on $\mathbb{R}$. Unless explicitly stated otherwise, all topological notions related to real algebraic varieties will refer to the Euclidean topology.

Given two nonsingular real algebraic varieties X and Y, we regard the set $\mathcal{R}(X,Y)$ of all regular maps from X into Y as a subset of the space $\mathcal{C}^\infty(X,Y)$ of all $\mathcal{C}^\infty$ maps from X into Y, endowed with the $\mathcal{C}^\infty$ topology (the weak $\mathcal{C}^\infty$ topology in the terminology used in [4]). It follows from the classical Weierstrass approximation theorem that $\mathcal{R}(X,Y)$ is dense in $\mathcal{C}^\infty(X,Y)$, provided that $Y = \mathbb{R}^p$. A useful description of the closure of $\mathcal{R}(X,Y)$ in $\mathcal{C}^\infty(X,Y)$, where Y is a Grassmann variety or a flag variety, is given in [2], [3], [6], [7]. In the present note we generalize these results.

We say that a $\mathcal{C}^\infty$ map $f\colon X \to Y$ can be *approximated in the $\mathcal{C}^\infty$ topology by regular maps* if f belongs to the closure of $\mathcal{R}(X,Y)$ in $\mathcal{C}^\infty(X,Y)$. For every continuous map $g\colon X \to Y$, let $g^*\colon H^1(Y,\mathbb{Z}/2) \to H^1(X,\mathbb{Z}/2)$ denote the induced homomorphism. Assuming that X is compact, we define $H^1_{\mathrm{alg}}(X,\mathbb{Z}/2)$ to be the subgroup of $H^1(X,\mathbb{Z}/2)$ generated by the cohomology classes Poincaré dual to the homology classes represented by algebraic hypersurfaces of X (that is, Zariski closed subvarieties of X of pure codimension 1), *cf.* [2].

Recall that if Y is an algebraic hypersurface of $\mathbb{P}^m(\mathbb{R}) \times \mathbb{P}^n(\mathbb{R})$, then the ideal of the polynomial ring $\mathbb{R}[S,T]$, with $S = (S_0,\dots,S_m)$ and $T = (T_0,\dots,T_n)$, generated by all the bihomogeneous polynomials vanishing on Y is generated by one bihomogeneous polynomial, whose bidegree (p,q) is called the bidegree of Y (an element F of $\mathbb{R}[S,T]$ is said to be bihomogeneous of bidegree (p,q) if F is a homogeneous polynomial of degree p in S over the ring $\mathbb{R}[T]$ and a homogeneous polynomial of degree q in T over the ring $\mathbb{R}[S]$).

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Theorem 1 *Let X be a compact nonsingular real algebraic variety. Let Y be either the blowup of $\mathbb{P}^n(\mathbb{R})$ along a linear subspace or a nonsingular algebraic hypersurface of $\mathbb{P}^m(\mathbb{R}) \times \mathbb{P}^n(\mathbb{R})$ of bidegree $(1,1)$. Given a $\mathcal{C}^\infty$ map $f\colon X \to Y$, the following conditions are equivalent:*

(a) f can be approximated in the $\mathcal{C}^\infty$ topology by regular maps;
(b) f is homotopic to a regular map from X into Y;
(c) $f^*\big(H^1(Y,\mathbb{Z}/2)\big) \subseteq H^1_{\mathrm{alg}}(X,\mathbb{Z}/2)$.

We shall derive Theorem 1 from a more general result, whose statement requires more preparation.

Let $\mathbb{K}$ denote $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ (the quaternions). An *algebraic $\mathbb{K}$-vector bundle* on a real algebraic variety X is a triple $\xi = (E, \pi, X)$, where the total space E is a real algebraic variety, the projection $\pi\colon E \to X$ is a regular map, each fiber $E_x = \pi^{-1}(x)$ is a $\mathbb{K}$-vector space for x in X, and the usual local triviality condition is satisfied. We shall sometimes denote the total space of ξ by $E(\xi)$. It is known that every algebraic $\mathbb{K}$-vector bundle on X is generated by global sections (*cf.* [5] for a simple proof; the reader should keep in mind that algebraic $\mathbb{K}$-vector bundles considered here are often called strongly algebraic $\mathbb{K}$-vector bundles in the literature [1], [2], [3], [5], [6]). A topological $\mathbb{K}$-vector bundle on X is said to *admit an algebraic structure* if it is topologically isomorphic to an algebraic $\mathbb{K}$-vector bundle on X. Below we assume that all $\mathbb{K}$-vector bundles are of constant rank.

Given a finite dimensional $\mathbb{K}$-vector space V and a positive integer q, we denote by $\mathbb{G}_q(V)$ the Grassmann space of all q-dimensional $\mathbb{K}$-vector subspaces of V. As in [2, Sections 3.4, 13.3], we shall always regard $\mathbb{G}_q(V)$ as a real algebraic variety. The universal $\mathbb{K}$-vector bundle $\gamma_q(V)$ on $\mathbb{G}_q(V)$ is algebraic.

More generally, fix a real algebraic variety Z. Given an algebraic $\mathbb{K}$-vector bundle ξ on Z, we regard

$$\mathbb{G}_q(\xi) = \bigcup_{z\in Z} \mathbb{G}_q\big(E(\xi)_z\big)$$

as a real algebraic variety. We shall make use of the universal algebraic $\mathbb{K}$-vector bundle $\gamma_{q,\xi}$ on $\mathbb{G}_q(\xi)$, whose total space is

$$E(\gamma_{q,\xi}) = \{(L, v) \in \mathbb{G}_q(\xi) \times E(\xi) \mid v \in L\}$$

and the projection $\pi_{q,\xi}\colon E(\gamma_{q,\xi}) \to \mathbb{G}_q(\xi)$ is defined by $\pi_{q,\xi}(L, v) = L$.

An s-tuple $\underline{\xi} = (\xi_1, \dots, \xi_s)$ is said to be a *system of algebraic $\mathbb{K}$-vector bundles on Z* if each ξ_i is an algebraic $\mathbb{K}$-vector bundle on Z, and ξ_j is an algebraic $\mathbb{K}$-vector subbundle of ξ_{j+1} for $1 \le j \le s-1$. Given an s-tuple of integers $\underline{k} = (k_1, \dots, k_s)$ satisfying $1 \le k_1 \le \cdots \le k_s$, we set

$$\mathbb{F}(\underline{k}, \underline{\xi}) = \{(L_1, \dots, L_s) \in \mathbb{G}_{k_1}(\xi_1) \times \cdots \times \mathbb{G}_{k_s}(\xi_s) \mid L_1 \subseteq \cdots \subseteq L_s\}.$$

We also define

$$\rho\colon \mathbb{F}(\underline{k}, \underline{\xi}) \to Z,$$
$$\rho(L_1, \dots, L_s) = z, \quad \text{where} \quad L_i \subset E(\xi_i)_z \quad \text{for } 1 \le i \le s,$$
$$\rho_i\colon \mathbb{F}(\underline{k}, \underline{\xi}) \to \mathbb{G}_{k_i}(\xi_i), \quad \rho_i(L_1, \dots, L_s) = L_i, \quad 1 \le i \le s.$$

Theorem 2 *Let X be a compact nonsingular real algebraic variety. If Z is nonsingular, then given a $\mathcal{C}^\infty$ map $f\colon X \to \mathbb{F}(\underline{k}, \underline{\xi})$, the following conditions are equivalent:*

(a) *f can be approximated in the $\mathcal{C}^\infty$ topology by regular maps;*
(b) *$\rho \circ f$ can be approximated in the $\mathcal{C}^\infty$ topology by regular maps and the pullback topological $\mathbb{K}$-vector bundle $(\rho_i \circ f)^*\gamma_{k_i,\xi_i}$ on X admits an algebraic structure for all $1 \leq i \leq s$.*

Proof It is obvious that (a) implies (b).

Suppose now that (b) holds. We may assume that ξ_s is an algebraic $\mathbb{K}$-vector subbundle of the trivial algebraic $\mathbb{K}$-vector bundle ε^n on Z with total space $Z \times \mathbb{K}^n$ (*cf.* [2, Theorem 12.1.7]). By [2, Proposition 12.1.11], there exist algebraic $\mathbb{K}$-vector subbundles $\eta_1, \ldots, \eta_s$ of ε^n such that $\xi_i \oplus \eta_i = \varepsilon^n$ for $1 \leq i \leq s$ and η_{j+1} is an algebraic $\mathbb{K}$-vector subbundle of η_j for $1 \leq j \leq s-1$. Note that

$$\mathbb{F}(\underline{k}, \underline{\xi}) \subseteq Z \times \mathbb{F}(\underline{k}, \mathbb{K}^n),$$

where

$$\mathbb{F}(\underline{k}, \mathbb{K}^n) = \{(L_1, \ldots, L_s) \in \mathbb{G}_{k_1}(\mathbb{K}^n) \times \cdots \times \mathbb{G}_{k_s}(\mathbb{K}^n) \mid L_1 \subseteq \cdots \subseteq L_s\}.$$

The subset U of $Z \times \mathbb{G}_{k_1}(\mathbb{K}^n) \times \cdots \times \mathbb{G}_{k_s}(\mathbb{K}^n)$ that consists of all the elements $(z, L_1, \ldots, L_s)$ such that

$$(\{z\} \times L_i) \cap E(\eta_i)_z = \{0\} \quad \text{for } 1 \leq i \leq s$$

is Zariski open and contains $\mathbb{F}(\underline{k}, \underline{\xi})$. Furthermore, if $\varphi_i\colon \varepsilon^n = \xi_i \oplus \eta_i \to \xi_i$ is the standard projection morphism of algebraic $\mathbb{K}$-vector bundles, then the map

$$r\colon U \cap \big(Z \times \mathbb{F}(\underline{k}, \mathbb{K}^n)\big) \to \mathbb{F}(\underline{k}, \underline{\xi}),$$
$$r(z, L_1, \ldots, L_s) = \big(\varphi_1(\{z\} \times L_1), \ldots, \varphi_s(\{x\} \times L_s)\big)$$

is regular and the identity on $\mathbb{F}(\underline{k}, \underline{\xi})$. It follows that f can be approximated in the $\mathcal{C}^\infty$ topology by regular maps if and only if $g = e \circ f$ can be approximated in the $\mathcal{C}^\infty$ topology by regular maps, where $e\colon \mathbb{F}(\underline{k}, \underline{\xi}) \hookrightarrow Z \times \mathbb{F}(\underline{k}, \mathbb{K}^n)$ is the inclusion map.

Let

$$\sigma\colon Z \times \mathbb{F}(\underline{k}, \mathbb{K}^n) \to Z,$$
$$\sigma_i\colon Z \times \mathbb{F}(\underline{k}, \mathbb{K}^n) \to \mathbb{G}_{k_i}(\mathbb{K}^n), \quad 1 \leq i \leq s$$

be the canonical projections. By [6, Theorem 1.2], g can be approximated in the $\mathcal{C}^\infty$ topology by regular maps if and only if $\sigma \circ g$ can be approximated in the $\mathcal{C}^\infty$ topology by regular maps and the pullback $\mathbb{K}$-vector bundle $(\sigma_i \circ g)^*\gamma_{k_i}(\mathbb{K})$ on X admits an algebraic structure for all $1 \leq i \leq s$. Condition (a) follows since $\rho \circ f = \sigma \circ g$ and $(\rho_i \circ f)^*\gamma_{k_i,\xi_i} = (\sigma_i \circ g)^*\gamma_{k_i}(\mathbb{K}^n)$. ∎

Corollary 3 *With the notation as in Theorem 2, assume that $Z = \mathbb{G}_q(\mathbb{K}^n)$. Given a $\mathcal{C}^\infty$ map $f\colon X \to \mathbb{F}(\underline{k}, \underline{\xi})$, the following conditions are equivalent:*

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(a) f *can be approximated in the* $\mathcal{C}^\infty$ *topology by regular maps;*
(b) f *is homotopic to a regular map from* X *into* $\mathbb{F}(\underline{k}, \underline{\xi})$*;*
(c) *The pullpack* $\mathbb{K}$*-vector bundles* $(\rho \circ f)^*\gamma_q(\mathbb{K}^n)$ *and* $(\rho_i \circ f)^*\gamma_{k_i,\xi_i}$ *on* X *admit algebraic structures,* $1 \le i \le s$.

Proof Obviously, (a) implies (b) and (b) implies (c).

If $(\rho \circ f)^*\gamma_q(\mathbb{K}^n)$ admits an algebraic structure, then $\rho \circ f$ can be approximated in the $\mathcal{C}^\infty$ topology by regular maps (this is proved in [3, Theorem 2.5] and also follows from Theorem 2 with Z consisting of one point and $s = 1$). Therefore, in view of Theorem 2, (c) implies (a). ∎

We shall also isolate another special case of Theorem 2. Given integers m and n satisfying $0 \le m \le n$, we identify $\mathbb{K}^m$ with the subset $\mathbb{K}^m \times \{0\}$ of $\mathbb{K}^n$; thus $\mathbb{K}^m \subseteq \mathbb{K}^n$. Let $\underline{k} = (k_1, \dots, k_s)$ and $\underline{n} = (n_1, \dots, n_s)$ be s-tuples of integers $1 \le k_1 \le \dots \le k_s$, $1 \le n_1 \le \dots \le n_s$. Set

$$\mathbb{F}(\underline{k}, \underline{n}; \mathbb{K}) = \{(L_1, \dots, L_s) \in \mathbb{G}_{k_1}(\mathbb{K}^{n_1}) \times \dots \times \mathbb{G}_{k_s}(\mathbb{K}^{n_s}) \mid L_1 \subseteq \dots \subseteq L_s\}$$

and let $\pi_i \colon \mathbb{F}(\underline{k}, \underline{n}; \mathbb{K}) \to \mathbb{G}_{k_i}(\mathbb{K}^{n_i})$ be defined by $\pi_i(L_1, \dots, L_s) = L_i$ for $1 \le i \le s$.

Corollary 4 *Let* X *be a compact nonsingular real algebraic variety. Given a* $\mathcal{C}^\infty$ *map* $f \colon X \to \mathbb{F}(\underline{k}, \underline{n}; \mathbb{K})$*, the following conditions are equivalent:*

(a) f *can be approximated in the* $\mathcal{C}^\infty$ *topology by regular maps;*
(b) f *is homotopic to a regular map from* X *into* $\mathbb{F}(\underline{k}, \underline{n}; \mathbb{K})$*;*
(c) *The pullback* $\mathbb{K}$*-vector bundle* $(\pi_i \circ f)^*\gamma_{k_i}(\mathbb{K}^n)$ *on* X *admits an algebraic structure for all* $1 \le i \le s$.

Proof It suffices to apply Theorem 2 with Z consisting of one point. ∎

Recall that a topological $\mathbb{R}$-line bundle λ on a compact nonsingular real algebraic variety X admits an algebraic structure if and only if its first Stiefel-Whitney class $w_1(\lambda)$ belongs to $H^1_{\mathrm{alg}}(X, \mathbb{Z}/2)$ (*cf.* [2, Théorème 12.4.8]).

Similarly, let η be a topological $\mathbb{R}$-vector bundle on X. Assume that rank $\eta = n$ and η is a subbundle of a trivial $\mathbb{R}$-vector bundle ε on X of rank $n + 1$. We assert that η admits an algebraic structure if and only if $w_1(\eta)$ belongs to $H^1_{\mathrm{alg}}(X, \mathbb{Z}/2)$. Indeed, let μ be a topological $\mathbb{R}$-line bundle on X such that $\eta \oplus \mu = \varepsilon$. By [2, Proposition 12.3.5], η admits an algebraic structure if and only if μ does. The assertion follows from the remark above since $w_1(\eta) = w_1(\mu)$.

These two facts will be repeatedly used without explicit reference.

Proof of Theorem 1 Case 1. Suppose that Y is the blowup of $\mathbb{P}^n(\mathbb{R})$ along a linear subspace H. Without loss of generality we may assume that

$$H = \{(x_0 : \dots : x_n) \in \mathbb{P}^n(\mathbb{R}) \mid x_0 = \dots = x_r = 0\}$$

for some r, $1 \le r \le n-1$. Then Y consists of all the points $\big((x_0 : \cdots : x_n), (y_0 : \cdots : y_r)\big)$ in $\mathbb{P}^n(\mathbb{R}) \times \mathbb{P}^r(\mathbb{R})$ such that $(x_0, \ldots, x_r)$ belongs to the vector subspace of $\mathbb{R}^{r+1}$ generated by $(y_0, \ldots, y_r)$.

Let $\xi = \gamma_1(\mathbb{R}^{r+1}) \oplus \varepsilon^{n-r}$, where ε^{n-r} is the trivial $\mathbb{R}$-vector bundle on $\mathbb{P}^r(\mathbb{R}) = \mathbb{G}_1(\mathbb{R}^{r+1})$ with total space $\mathbb{G}_1(\mathbb{R}^{r+1}) \times \mathbb{R}^{n-r}$. Then the real algebraic varieties Y and $\mathbb{G}_1(\xi)$ are isomorphic, and therefore we assume below that $Y = \mathbb{G}_1(\xi)$. Let $\rho\colon \mathbb{G}_1(\xi) \to \mathbb{G}_1(\mathbb{R}^{r+1})$ be the canonical projection. Let $f\colon X \to Y = \mathbb{G}_1(\xi)$ be a $\mathcal{C}^\infty$ map. The pullback $\mathbb{R}$-line bundles $(\rho \circ f)^*\gamma_1(\mathbb{R}^{r+1})$ and $f^*\gamma_{1,\xi}$ on X admit algebraic structures if and only if the cohomology classes

$$w_1\big((\rho \circ f)^*\gamma_1(\mathbb{R}^{r+1})\big) = f^*\Big(\rho^*\Big(w_1\big(\gamma_1(\mathbb{R}^{r+1})\big)\Big)\Big) \quad \text{and} \quad w_1(f^*\gamma_{1,\xi}) = f^*\big(w_1(\gamma_{1,\xi})\big)$$

belong to $H^1_{\mathrm{alg}}(X, \mathbb{Z}/2)$. Since $\rho^*\Big(w_1\big(\gamma_1(\mathbb{R}^{r+1})\big)\Big)$ and $w_1(\gamma_{1,\xi})$ generate $H^1(Y, \mathbb{Z}/2)$, the latter condition is equivalent to $f^*\big(H^1(Y, \mathbb{Z}/2)\big) \subseteq H^1_{\mathrm{alg}}(X, \mathbb{Z}/2)$. In view of Corollary 3, the proof of Case 1 is complete.

Case 2. Suppose that Y is a nonsingular hypersurface of $\mathbb{P}^m(\mathbb{R}) \times \mathbb{P}^n(\mathbb{R})$ of bidegree $(1,1)$. We may assume that $m \le n$. Since Y is nonsingular, one easily sees that it is isomorphic to

$$\{\big((x_0 : \cdots : x_m), (y_0 : \cdots : y_n)\big) \in \mathbb{P}^m(\mathbb{R}) \times \mathbb{P}^n(\mathbb{R}) \mid x_0 y_0 + \cdots + x_m y_m = 0\}$$

and hence also isomorphic to $\mathbb{F}\big((1,n), (m+1, n+1); \mathbb{R}\big)$ (we use the same notation as in Corollary 4). Below we assume $Y = \mathbb{F}\big((1,n), (m+1, n+1); \mathbb{R}\big)$. Let $\pi_1\colon Y \to \mathbb{G}_1(\mathbb{R}^{m+1})$ and $\pi_2\colon Y \to \mathbb{G}_n(\mathbb{R}^{n+1})$ be the canonical projections. The pullback $\mathbb{R}$-line bundle $(\pi_1 \circ f)^*\gamma_1(\mathbb{R}^{m+1})$ on X admits an algebraic structure if and only if the cohomology class

$$w_1\big((\pi_1 \circ f)^*\gamma_1(\mathbb{R}^{m+1})\big) = f^*\Big(\pi_1^*\Big(w_1\big(\gamma_1(\mathbb{R}^{m+1})\big)\Big)\Big)$$

belongs to $H^1_{\mathrm{alg}}(X, \mathbb{Z}/2)$.

Similarly, the pullback $\mathbb{R}$-vector bundle $(\pi_2 \circ f)^*\gamma_n(\mathbb{R}^{n+1})$ on X admits an algebraic structure if and only if the cohomology class

$$w_1\big((\pi_2 \circ f)^*\gamma_n(\mathbb{R}^{n+1})\big) = f^*\Big(\pi_2^*\Big(w_1\big(\gamma_n(\mathbb{R}^{n+1})\big)\Big)\Big)$$

belongs to $H^1_{\mathrm{alg}}(X, \mathbb{Z}/2)$ (note that rank $\gamma_n(\mathbb{R}^{n+1}) = n$ and $\gamma_n(\mathbb{R}^{n+1})$ is a subbundle of a trivial bundle of rank $n+1$).

Since $\pi_1^*\Big(w_1\big(\gamma_1(\mathbb{R}^{m+1})\big)\Big)$ and $\pi_2^*\Big(w_1\big(\gamma_n(\mathbb{R}^{n+1})\big)\Big)$ generate $H^1(Y, \mathbb{Z}/2)$, it follows that $(\pi_1 \circ f)^*\gamma_1(\mathbb{R}^{m+1})$ and $(\pi_2 \circ f)^*\gamma_n(\mathbb{R}^{n+1})$ admit algebraic structures if and only if $f^*\big(H^1(Y, \mathbb{Z}/2)\big) \subseteq H^1_{\mathrm{alg}}(X, \mathbb{Z}/2)$. In order to complete the proof it suffices to apply Corollary 4. ■

Some approximation results proved above can be extended to maps defined on varieties that are not necessarily compact.

We say that a nonsingular real algebraic variety Y has *property* (A) *relative to a nonsingular real algebraic variety* X if given a $\mathcal{C}^\infty$ map $f\colon X \to Y$, the following conditions are equivalent:

(a) f can be approximated in the $\mathcal{C}^\infty$ topology by regular maps;
(b) f is homotopic to a regular map from X into Y.

Theorem 5 *Let Y be a compact nonsingular real algebraic variety. If Y has property* (A) *relative to every compact nonsingular real algebraic variety, then Y has property* (A) *relative to every nonsingular real algebraic variety.*

Proof One can repeat, virtually word for word, the proof of [6, Theorem 1.4]. ■

Observe that Theorem 5 is applicable in the situations described in Theorem 1 and Corollaries 3 and 4.

Remark 6 Given two real algebraic varieties X and Y, we can regard $\mathcal{R}(X, Y)$ as a subset of the space $\mathcal{C}(X, Y)$ of all continuous maps from X and Y, endowed with the $\mathcal{C}^0$ topology (that is, the compact-open topology).

Assume that X and Z are as in Theorem 2, except that they are not necessarily nonsingular. Given a continuous map $f\colon X \to \mathbb{F}(\underline{k}, \underline{\xi})$, the following conditions are equivalent:

(a) f can be approximated in the $\mathcal{C}^0$ topology by regular maps;
(b) $\rho \circ f$ can be approximated in the $\mathcal{C}^0$ topology by regular maps and the pullback topological $\mathbb{K}$-vector bundle $(\rho_i \circ f)^*\gamma_{k_i,\xi_i}$ on X admits an algebraic structure for all $1 \le i \le s$.

One proves this statement by modifying in a straightforward manner the proof of Theorem 2.

There are also obvious versions of Corollaries 3 and 4, with X not necessarily nonsingular and f continuous.

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