Lakshmibai–Seshadri paths of level-zero shape and one-dimensional sums associated to level-zero fundamental representations

Satoshi Naito and Daisuke Sagaki

Abstract

We give an interpretation of the energy function and classically restricted one-dimensional sums associated to tensor products of level-zero fundamental representations of quantum affine algebras in terms of Lakshmibai–Seshadri paths of level-zero shape.

1. Introduction

A one-dimensional sum (1dsum for short) is a weighted sum over certain one-dimensional configurations, where the weights are given by the 'energy function', and arose from the study of solvable lattice models in statistical mechanics through Baxter's corner transfer matrix method. However, the crystal basis theory of Kashiwara provided an intrinsic definition of a 1dsum, and a conceptual proof of the fact that, in the infinite lattice size limit, a 1dsum tends to the character of a highest weight module over an affine Lie algebra (see [KKMMNN92]). The purpose of this paper is to give an interpretation of the energy function and hence of 'classically restricted' 1dsums associated to tensor products of certain finite-dimensional irreducible modules, called level-zero fundamental representations, over a quantum affine algebra via Lakshmibai–Seshadri paths (LS paths for short) of level-zero shape.

Let $\mathfrak{g}$ be an affine Lie algebra over $\mathbb{Q}$ with Cartan subalgebra $\mathfrak{h}$, and let $\mathfrak{g}_{I_0}$ be its finite-dimensional simple Lie subalgebra whose Dynkin diagram (with I_0 the set of vertices) is obtained from that of $\mathfrak{g}$ (with I the set of vertices) by removing a distinguished 0 vertex. Also, let $U_q(\mathfrak{g}) \supset U_q(\mathfrak{g}_{I_0})$ be the quantized universal enveloping algebras associated to $\mathfrak{g} \supset \mathfrak{g}_{I_0}$, and let $U_q'(\mathfrak{g})$ $(\subset U_q(\mathfrak{g}))$ be the quantum affine algebra (without derivation) associated to $\mathfrak{g}$.

In [KR90] (see also [Kir85, Kir87, Kir89] and [KKR88]), Kirillov and Reshetikhin conjectured the existence of a family of finite-dimensional representations of the Yangian $Y(\mathfrak{g}_{I_0})$ associated to $\mathfrak{g}_{I_0}$ for which tensor products of these representations admit the decomposition into irreducible $\mathfrak{g}_{I_0}$-modules, with each multiplicity given by a specific sum of products of binomial coefficients. Note that under the widely believed correspondence between the finite-dimensional representations of the Yangian $Y(\mathfrak{g}_0)$ and those of the nontwisted quantum affine algebra $U_q'(\mathfrak{g})$, this conjecture can be translated into the one for the untwisted quantum affine algebras; both of them are called the Kirillov–Reshetikhin (KR for short) conjecture. The KR conjecture has been the subject of a number of papers (see [KNT02, § 5.7] and references therein). Among them, we would like to mention [Nak03], [Her06], and [Her07], in which the KR conjecture (of 'type I' in the sense of [KNT02, § 5.7]) is first proved in nontwisted A, D, E cases, then in all the nontwisted cases, and finally in the general case.

Inspired by the seminal paper [KR90], it was conjectured in [HKOTY99] and [HKOTT02] that there exists a family of finite-dimensional irreducible $U_q'(\mathfrak{g})$-modules $W_s^{(i)}$ having crystal bases $\mathcal{B}^{i,s}$ indexed by $i \in I_0$ and $s \in \mathbb{Z}_{\geqslant 1}$ for which the decomposition into irreducible $U_q(\mathfrak{g}_{I_0})$-modules is prescribed by a special case of the KR conjecture (of 'type II' in the sense of [KNT02, §5.7]); the $U_q'(\mathfrak{g})$-module $W_s^{(i)}$ is called a KR module, and the crystal $\mathcal{B}^{i,s}$ is called a KR crystal. It should be mentioned that (the roots of) the Drinfeld polynomials of the KR module $W_s^{(i)}$ for each $i \in I_0$ and $s \in \mathbb{Z}_{\geqslant 1}$ are also specified up to a multiplicative constant (see [HKOTY99, Remark 2.2] and the comment preceding [HKOTY99, Conjecture 2.1]).

Furthermore, in [HKOTY99] and [HKOTT02], a 'fermionic formula' M was defined to be a suitable q-analogue of the tensor product multiplicity formula in the KR conjecture (of type II), which can be rewritten as a weighted sum over a set of combinatorial objects (called rigged configurations). Also, a classically restricted 1dsum X was introduced as the generating function in q of $U_q(\mathfrak{g}_{I_0})$-highest weight elements of a tensor product of the (conjectural) crystals $\mathcal{B}^{i,s}$, $i \in I_0$, $s \in \mathbb{Z}_{\geqslant 1}$, weighted by the energy function. The '$X = M$ conjecture' proposed in these papers asserts the equality between these two kinds of polynomials in q^{-1} with nonnegative integer coefficients.

The $X = M$ conjecture (including the construction of the crystals $\mathcal{B}^{i,s}$) has been studied intensively in recent years (see [Oka07, §§2.2 and 5.3] and references therein). Among them, we would like to mention [Kas02], in which level-zero fundamental representations $W(\varpi_i)$, $i \in I_0$, of $U_q'(\mathfrak{g})$ are introduced, where the ϖ_i, $i \in I_0$, are the level-zero fundamental weights for $\mathfrak{g}$. Furthermore, in [Kas02], it is proved that the level-zero fundamental representations $W(\varpi_i)$, $i \in I_0$, are finite-dimensional irreducible $U_q'(\mathfrak{g})$-modules having 'simple' crystal bases $\mathcal{B}(W(\varpi_i))$. Also, the Drinfeld polynomials of $W(\varpi_i)$ are computed in [Nak04, §3.1] (at least in the nontwisted case), and they coincide with those of $W_1^{(i)}$ specified in [HKOTY99] and [HKOTT02] for all $i \in I_0$. Therefore, by [Nak03], [Her06], and [Her07], the decomposition of $W(\varpi_i)$ into irreducible $U_q(\mathfrak{g}_{I_0})$-modules turns out to be the one for $W_1^{(i)}$ prescribed by a special case of the KR conjecture (of type I) for every $i \in I_0$. Thus, through enough evidence (see also [Cha01], [HN06], [Kas05], [FL06], and [FL07]), it is confirmed that the level-zero fundamental representation $W(\varpi_i)$ is indeed the (conjectural) KR module $W_s^{(i)}$ with $s = 1$, and hence the crystal basis $\mathcal{B}(W(\varpi_i))$ of $W(\varpi_i)$ is the KR crystal $\mathcal{B}^{i,1}$ for every $i \in I_0$.

Here we recall that most of all the known proofs of the $X = M$ conjecture are given by establishing a weight-preserving bijection from the set of $U_q(\mathfrak{g}_{I_0})$-highest weight elements of a tensor product of KR crystals $\mathcal{B}^{i,s}$ to the set of rigged configurations (see [Sch07] and references therein; [KSS02], [Sch05], [SSh06], [SSt06], [BFKL06], and also [FSS07] to name a few of the recent ones). In these proofs, the KR crystal $\mathcal{B}^{i,s}$ is constructed as a certain direct sum (prescribed by the KR conjecture) of highest weight crystals for $U_q(\mathfrak{g}_{I_0})$; the action of the Kashiwara operators corresponding to the 0 vertex are defined on it, by an argument that depends on the type of $\mathfrak{g}_{I_0}$ and requires a lot of work based on an explicit combinatorial realization of highest weight crystals for $U_q(\mathfrak{g}_{I_0})$. Hence it is highly desirable to have a type-independent combinatorial realization of the KR crystals $\mathcal{B}^{i,s}$, $i \in I_0$, $s \in \mathbb{Z}_{\geqslant 1}$, and the corresponding realization of the energy function on tensor products of them, which would provide a general approach to the $X = M$ conjecture.

Contrastingly, in Littelmann's path model, the root operators are universally defined for all paths, and hence we obtain a path model for KR modules if we choose the right set of paths; the advantage of this approach is that, when it can be used, it enables us to study KR crystals in a way completely independent of the type of $\mathfrak{g}_{I_0}$. In fact, in a series of papers [NS03, NS05, NS06], we studied the crystal $\mathbb{B}(\lambda)$ of all LS paths of shape λ for a level-zero integral weight λ of the form $\lambda = \sum_{i \in I_0} \lambda^{(i)} \varpi_i$ with $\lambda^{(i)} \in \mathbb{Z}_{\geqslant 0}$; we showed that the associated crystal $\mathbb{B}(\lambda)_{\mathrm{cl}}$ is isomorphic to the tensor product crystal $\bigotimes_{i \in I_0} \mathbb{B}(\varpi_i)_{\mathrm{cl}}^{\otimes \lambda^{(i)}}$, and that each $\mathbb{B}(\varpi_i)_{\mathrm{cl}}$ is isomorphic as a crystal to $\mathcal{B}(W(\varpi_i)) = \mathcal{B}^{i,1}$ for $i \in I_0$. Hence the crystal $\mathbb{B}(\lambda)_{\mathrm{cl}}$ is isomorphic to the tensor product

1526

crystal $\bigotimes_{i \in I_0} (\mathcal{B}^{i,1})^{\otimes \lambda^{(i)}}$. Here the crystal $\mathbb{B}(\lambda)_{\mathrm{cl}}$ is defined to be the set of all paths of the form $\mathrm{cl}(\pi) = \mathrm{cl} \circ \pi$, $\pi \in \mathbb{B}(\lambda)$, where the map $\mathrm{cl} : \mathfrak{h}^*_{\mathbb{R}} \twoheadrightarrow \mathfrak{h}^*_{\mathbb{R}}/\mathbb{R}\delta$ is the canonical projection, with $\delta \in \mathfrak{h}^*$ the null root of $\mathfrak{g}$ and $\mathfrak{h}^*_{\mathbb{R}} := \mathbb{R} \otimes_{\mathbb{Q}} \mathfrak{h}^*$. Thus, we have an explicit combinatorial realization of tensor products of KR crystals $\mathcal{B}^{i,s}$, $i \in I_0$, with $s = 1$ in the framework of Littelmann's path model.

Now, the remaining task is to give an interpretation of the energy function on tensor products of KR crystals $\mathcal{B}^{i,1}$, $i \in I_0$, in terms of Littelmann's path model. The purpose of this paper is to provide such an interpretation. Let λ be a level-zero integral weight as above. We define a function (called the degree function) $\mathrm{Deg} : \mathbb{B}(\lambda)_{\mathrm{cl}} \to \mathbb{Z}_{\leqslant 0}$ as follows (see § 3.1 for details). For $\eta \in \mathbb{B}(\lambda)_{\mathrm{cl}}$, the degree $\mathrm{Deg}(\eta) \in \mathbb{Z}$ is defined to be (the negative of) the coefficient of the null root δ in a specific expression of the weight $\pi_\eta(1) \in \mathfrak{h}^*$ of a certain distinguished element $\pi_\eta \in \mathbb{B}_0(\lambda)$ such that $\mathrm{cl}(\pi_\eta) = \eta$, where $\mathbb{B}_0(\lambda)$ denotes the connected component of $\mathbb{B}(\lambda)$ containing the straight line path: $t \mapsto t\lambda$, $t \in [0,1]$. Take an arbitrary sequence $\mathbf{i} = (i_1, i_2, \ldots, i_n)$ of elements of I_0 (with repetitions allowed). Then, as above, the tensor product crystal $\mathcal{B}^{i_1,1} \otimes \mathcal{B}^{i_2,1} \otimes \cdots \otimes \mathcal{B}^{i_n,1} \simeq \mathbb{B}(\varpi_{i_1}) \otimes \mathbb{B}(\varpi_{i_2}) \otimes \cdots \otimes \mathbb{B}(\varpi_{i_n}) =: \mathbb{B}_{\mathbf{i}}$ is isomorphic to the crystal $\mathbb{B}(\lambda)_{\mathrm{cl}}$ associated to the crystal $\mathbb{B}(\lambda)$ of all LS paths of shape $\lambda := \sum_{k=1}^{n} \varpi_{i_k}$. Our main result (Theorem 4.1.1) of this paper states that the energy function on the tensor product $\mathbb{B}_{\mathbf{i}}$ of KR crystals with $s = 1$ can be identified (through the isomorphism above between $\mathbb{B}_{\mathbf{i}}$ and $\mathbb{B}(\lambda)_{\mathrm{cl}}$) with the degree function Deg on $\mathbb{B}(\lambda)_{\mathrm{cl}}$, up to a specific additive constant (see § 4.1 for its explicit definition). Furthermore, we obtain an expression (Corollary 5.1.1) for classically restricted 1dsums in terms of LS paths. In particular, by restricting ourselves to the case of $A_{\ell-1}^{(1)}$, we obtain an expression (Corollary 5.2.4) for Kostka–Foulkes polynomials in terms of LS paths.

This paper is organized as follows. In § 2, we first fix our notation for quantum affine algebras. Then we briefly review some standard facts on LS path crystals with weight lattice P or P_{cl}, and fundamental results on simple P_{cl}-crystals for quantum affine algebras. In § 3, we define our 'degree function' on P_{cl}-crystals of LS paths of level-zero shape, and show some of its basic properties. In § 4, we first state our main result (Theorem 4.1.1) describing the energy function associated to tensor products of level-zero fundamental representations. Then, we give a proof of it after showing a key proposition to our proof. In § 5, we mention the relation to classically restricted 1dsums and Kostka–Foulkes polynomials.

2. Preliminaries

2.1 Affine Lie algebras and quantum affine algebras

Let $A = (a_{ij})_{i,j \in I}$ be a generalized Cartan matrix of affine type. Throughout this paper, we assume that the elements of the index set I are numbered as in [Kac90, § 4.8, Tables Aff 1–Aff 3]. Take a special vertex $0 \in I$ as in these tables, and set $I_0 := I \setminus \{0\}$. Let $\mathfrak{g} = \mathfrak{g}(A)$ be the affine Lie algebra associated to the Cartan matrix $A = (a_{ij})_{i,j \in I}$ of affine type over the field $\mathbb{Q}$ of rational numbers, and let $\mathfrak{h}$ be its Cartan subalgebra. Note that $\mathfrak{h} = (\bigoplus_{j \in I} \mathbb{Q} h_j) \oplus \mathbb{Q} d$, where $\Pi^{\vee} := \{h_j\}_{j \in I} \subset \mathfrak{h}$ is the set of simple coroots, and $d \in \mathfrak{h}$ is the scaling element. Also, we denote by $\Pi := \{\alpha_j\}_{j \in I} \subset \mathfrak{h}^* := \mathrm{Hom}_{\mathbb{Q}}(\mathfrak{h}, \mathbb{Q})$ the set of simple roots, and by $\Lambda_j \in \mathfrak{h}^*$, $j \in I$, the fundamental weights; note that $\alpha_j(d) = \delta_{j,0}$ and $\Lambda_j(d) = 0$ for $j \in I$. Let $\delta = \sum_{j \in I} a_j \alpha_j \in \mathfrak{h}^*$ and $c = \sum_{j \in I} a_j^{\vee} h_j \in \mathfrak{h}$ be the null root and the canonical central element of $\mathfrak{g}$, respectively. Here we should note that $a_0 = 2$ if $\mathfrak{g}$ is of type $A_{2\ell}^{(2)}$, and $a_0 = 1$ otherwise. We define the Weyl group W of $\mathfrak{g}$ by $W = \langle r_j \mid j \in I \rangle \subset \mathrm{GL}(\mathfrak{h}^*)$, where $r_j \in \mathrm{GL}(\mathfrak{h}^*)$ is the simple reflection associated to α_j for $j \in I$, and then define the set Δ^{re} of real roots by $\Delta^{\mathrm{re}} = W\Pi$. The set of positive real roots is denoted by $\Delta_+^{\mathrm{re}} \subset \Delta^{\mathrm{re}}$. Also, let us denote by $(\cdot, \cdot)$ a nondegenerate, symmetric, (W-) invariant bilinear form on $\mathfrak{h}^*$ normalized in such a way that $(\alpha_i, \alpha_j) \in \mathbb{Z}$ for all $i, j \in I$.

We take a dual weight lattice $P^\vee$ and a weight lattice P as follows:

$$P^\vee = \left(\bigoplus_{j\in I} \mathbb{Z}h_j\right) \oplus \mathbb{Z}d \subset \mathfrak{h} \quad \text{and} \quad P = \left(\bigoplus_{j\in I} \mathbb{Z}\Lambda_j\right) \oplus \mathbb{Z}a_0^{-1}\delta \subset \mathfrak{h}^*. \qquad (2.1.1)$$

It is clear that P contains all the simple roots α_j, $j \in I$, and that $P \cong \operatorname{Hom}_{\mathbb{Z}}(P^\vee, \mathbb{Z})$. The quintuplet $(A, \Pi, \Pi^\vee, P, P^\vee)$ is called a Cartan datum for the generalized Cartan matrix $A = (a_{ij})_{i,j\in I}$ of affine type (see [HK02, Definition 2.1.1]).

Let $\mathfrak{g}_{I_0}$ be the canonical finite-dimensional Lie subalgebra of $\mathfrak{g}$ generated by x_j, y_j, $j \in I_0$, and h_j, $j \in I$, where x_j (respectively, y_j) is a nonzero root vector of $\mathfrak{g}$ corresponding to the simple root α_j (respectively, $-\alpha_j$); note that $\mathfrak{h}_{I_0} := \bigoplus_{j\in I} \mathbb{Q}h_j$ is the Cartan subalgebra of $\mathfrak{g}_{I_0}$. We denote by $\mathring{W}$ the subgroup of W generated by r_j, $j \in I_0$, which can be thought of as the Weyl group of the Lie subalgebra $\mathfrak{g}_{I_0} \subset \mathfrak{g}$. Let $w_0 \in \mathring{W}$ denote the longest element of $\mathring{W}$.

DEFINITION 2.1.1. An integral weight $\lambda \in P$ is said to be *level-zero* if $\lambda(c) = 0$. In addition, a level-zero integral weight $\lambda \in P$ is said to be I_0-*dominant* (respectively, *strictly I_0-dominant*) if $\lambda(h_j) \geqslant 0$ (respectively, $\lambda(h_j) > 0$) for all $j \in I_0$.

For each $i \in I_0 = I \setminus \{0\}$, we define a level-zero fundamental weight $\varpi_i \in P$ by

$$\varpi_i = \Lambda_i - a_i^\vee \Lambda_0. \qquad (2.1.2)$$

Note that the ϖ_i is actually a level-zero, I_0-dominant integral weight; in fact, $\varpi(c) = 0$ and $\varpi_i(h_j) = \delta_{i,j}$ for $j \in I_0$. In particular, the restriction of the ϖ_i, $i \in I_0$, to the Cartan subalgebra $\mathfrak{h}_{I_0}$ of $\mathfrak{g}_{I_0}$ can be thought of as the fundamental weights for $\mathfrak{g}_{I_0} \subset \mathfrak{g}$. We set

$$P_+^0 := \sum_{i\in I_0} \mathbb{Z}_{\geqslant 0}\varpi_i. \qquad (2.1.3)$$

Recall from [Kas02, (5.6)] (see also [NS08, Lemmas 2.5 and 2.6]) that, for each $\lambda \in P_+^0$, the equality $W\lambda \cap (\lambda + \mathbb{Z}\delta) = \lambda + \mathbb{Z}d_\lambda\delta$ holds for some positive integer $d_\lambda \in \mathbb{Z}_{>0}$, and set $\mathring{Q}_+ := \sum_{j\in I_0} \mathbb{Z}_{\geqslant 0}\alpha_j$. The next lemma follows immediately from [NS08, Lemma 2.3.2 and Remark 4.1.1] and the proof of [NS08, Lemma 2.3.3] by noting the linear independence of α_j, $j \in I_0$, and δ.

LEMMA 2.1.2. Let $\lambda \in P_+^0$. Then, for each $w \in W$, there exist unique $\beta \in \mathring{Q}_+$ and $k \in \mathbb{Z}$ such that $w\lambda = \lambda - \beta + kd_\lambda\delta$. Moreover, $\lambda - \beta = w'\lambda$ for some $w' \in \mathring{W}$.

Let $\lambda \in P$ be an integral weight, and $\mu, \nu \in W\lambda$. Following [Lit95, §4], we write $\mu > \nu$ if there exist a sequence $\mu = \nu_0, \nu_1, \ldots, \nu_n = \nu$ of elements of $W\lambda$ and a sequence $\xi_1, \xi_2, \ldots, \xi_n$ of positive real roots such that $\nu_k = r_{\xi_k}(\nu_{k-1})$ and such that $\nu_{k-1}(\xi_k^\vee) \in \mathbb{Z}_{<0}$ for all $1 \leqslant k \leqslant n$, where $\xi_k^\vee \in \mathfrak{h}$ denotes the dual root of $\xi_k \in \Delta_+^{\mathrm{re}}$, and r_{ξ_k} denotes the associated reflection; we write $\mu \geqslant \nu$ if $\mu > \nu$ or $\mu = \nu$.

Remark 2.1.3. Let $\lambda \in P_+^0$, and let $\nu, \nu' \in W\lambda$ be such that $\nu > \nu'$. Write ν and ν' as $\nu = \lambda - \beta + kd_\lambda\delta$ and $\nu' = \lambda - \beta' + k'd_\lambda\delta$ for $\beta, \beta' \in \mathring{Q}_+$ and $k, k' \in \mathbb{Z}$ (see Lemma 2.1.2), respectively. Then we deduce from [NS08, Remark 2.4.3(1)] that either $k < k'$ holds, or $k = k'$ and $\beta - \beta' \in \mathring{Q}_+ \setminus \{0\}$ holds.

Now, let $\mathrm{cl} : \mathfrak{h}^* \twoheadrightarrow \mathfrak{h}^*/\mathbb{Q}\delta$ be the canonical projection from $\mathfrak{h}^*$ onto $\mathfrak{h}^*/\mathbb{Q}\delta$. We define a classical weight lattice P_{cl} and a classical dual weight lattice $P_{\mathrm{cl}}^\vee$ by

$$P_{\mathrm{cl}} = \mathrm{cl}(P) = \bigoplus_{j\in I} \mathbb{Z}\,\mathrm{cl}(\Lambda_j) \quad \text{and} \quad P_{\mathrm{cl}}^\vee = \bigoplus_{j\in I} \mathbb{Z}h_j \subset P^\vee. \qquad (2.1.4)$$

We see that $P_{\mathrm{cl}} \simeq P/(\mathbb{Q}\delta \cap P)$, and that P_{cl} can be identified with $\operatorname{Hom}_{\mathbb{Z}}(P_{\mathrm{cl}}^\vee, \mathbb{Z})$ as a $\mathbb{Z}$-module by $(\mathrm{cl}(\lambda))(h) = \lambda(h)$ for $\lambda \in P$ and $h \in P_{\mathrm{cl}}^\vee$. The quintuple $(A, \mathrm{cl}(\Pi), \Pi^\vee, P_{\mathrm{cl}}, P_{\mathrm{cl}}^\vee)$ is called a classical

Cartan datum (see [HK02, § 10.1]). Note that there exists a natural action of the Weyl group W on $\mathfrak{h}^*/\mathbb{Q}\delta$ induced by the one on $\mathfrak{h}^*$, since $W\delta = \delta$. It is obvious that $w \circ \mathrm{cl} = \mathrm{cl} \circ w$ for all $w \in W$. If we set $\mathfrak{h}^{*0} := \{\lambda \in \mathfrak{h}^* \mid \lambda(c) = 0\}$, then there exists a (positive definite) symmetric bilinear form $(\cdot, \cdot)_{\mathrm{cl}}$ on $\mathrm{cl}(\mathfrak{h}^{*0}) = \mathfrak{h}^{*0}/\mathbb{Q}\delta$ induced by the restriction to $\mathfrak{h}^{*0}$ of the nondegenerate, symmetric bilinear form $(\cdot, \cdot)$ on $\mathfrak{h}^*$, since $(\delta, \mathfrak{h}^{*0}) = \{0\}$.

DEFINITION 2.1.4. An integral weight $\mu \in P_{\mathrm{cl}}$ is said to be *level-zero* if $\mu(c) = 0$. In addition, a level-zero integral weight $\mu \in P_{\mathrm{cl}}$ is said to be I_0-*dominant* (respectively, *strictly I_0-dominant*) if $\lambda(h_j) \geqslant 0$ (respectively, $\lambda(h_j) > 0$) for all $j \in I_0$.

Remark 2.1.5. Let $\lambda \in P_+^0$. It is easy to verify that $\mathrm{cl}(W\lambda) = \mathring{W}\,\mathrm{cl}(\lambda)$ (see the proof of [NS08, Lemma 2.3.3]). Also, we see that $\mathrm{cl}(\lambda)$ is the unique level-zero, I_0-dominant integral weight in $\mathrm{cl}(W\lambda) = \mathring{W}\,\mathrm{cl}(\lambda)$, and that $\mathrm{cl}(w_0\lambda) = w_0\,\mathrm{cl}(\lambda)$ is the unique element of $\mathrm{cl}(W\lambda) = \mathring{W}\,\mathrm{cl}(\lambda)$ such that $(\mathrm{cl}(w_0\lambda))(h_j) = (w_0\,\mathrm{cl}(\lambda))(h_j) \leqslant 0$ for all $j \in I_0$.

Let $U_q'(\mathfrak{g})$ be the quantized universal enveloping algebra of the affine Lie algebra $\mathfrak{g}$ with weight lattice P_{cl} over the field $\mathbb{Q}(q)$ of rational functions in q. We denote by x_j, y_j, $j \in I$, and q^h, $h \in P_{\mathrm{cl}}^\vee$, the canonical generators of $U_q'(\mathfrak{g})$, where x_j (respectively, y_j) corresponds to the simple root α_j (respectively, $-\alpha_j$) for $j \in I$.

2.2 Crystals of LS paths with weight lattice P or P_{cl}

A path (with weight in P) is, by definition, a piecewise linear, continuous map $\pi : [0,1] \to \mathfrak{h}_{\mathbb{R}}^* := \mathbb{R} \otimes_{\mathbb{Q}} \mathfrak{h}^*$ such that $\pi(0) = 0$ and $\pi(1) \in P \subset \mathbb{R} \otimes_{\mathbb{Z}} P = \mathfrak{h}_{\mathbb{R}}^*$. We denote by $\mathbb{P}$ the set of all paths $\pi : [0,1] \to \mathfrak{h}_{\mathbb{R}}^*$. For each $\pi_1, \pi_2 \in \mathbb{P}$, we define a path $\pi_1 \pm \pi_2 \in \mathbb{P}$ by $(\pi_1 \pm \pi_2)(t) = \pi_1(t) \pm \pi_2(t)$ for $t \in [0,1]$. For an integral weight $\nu \in P$, let π_ν denote the straight line path connecting $0 \in P$ with $\nu \in P$, i.e. $\pi_\nu(t) := t\nu$ for $t \in [0,1]$.

Remark 2.2.1. In [Lit94] and [Lit95], paths in $\mathbb{P}$ are considered modulo reparametrization. However, in this paper, we do not consider paths in $\mathbb{P}$ modulo reparametrization since there is no need to do so. Note that all results of [Lit94] and [Lit95] that we need in this paper still hold in this setting.

Let $\pi \in \mathbb{P}$. A pair $(\underline{\nu}; \underline{\sigma})$ of a sequence $\underline{\nu} : \nu_1, \nu_2, \ldots, \nu_s$ of elements of $\mathfrak{h}_{\mathbb{R}}^*$ and a sequence $\underline{\sigma} : 0 = \sigma_0 < \sigma_1 < \cdots < \sigma_s = 1$ of rational numbers is called an *expression* of $\pi \in \mathbb{P}$ if the following equation holds:

$$\pi(t) = \sum_{u'=1}^{u-1} (\sigma_{u'} - \sigma_{u'-1})\nu_{u'} + (t - \sigma_{u-1})\nu_u \quad \text{for } \sigma_{u-1} \leqslant t \leqslant \sigma_u,\ 1 \leqslant u \leqslant s. \tag{2.2.1}$$

In this case, we write $\pi = (\underline{\nu}; \underline{\sigma})$. An expression $(\nu_1, \nu_2, \ldots, \nu_s; \underline{\sigma})$ of π is said to be *reduced* if $\nu_u \neq \nu_{u+1}$ for any $u = 1, 2, \ldots, s-1$.

Remark 2.2.2 (see [NS08, Remark 2.5.2]). Let $\pi \in \mathbb{P}$. We easily see that there exists a unique reduced expression of π. Also, if $(\nu_1, \nu_2, \ldots, \nu_s; \sigma_0, \sigma_1, \ldots, \sigma_s)$ is an expression of π, then the reduced expression of π is obtained from this expression by 'omitting' the element ν_{u+1} such that $\nu_u = \nu_{u+1}$ and the corresponding rational number σ_u.

DEFINITION 2.2.3. Let $\pi = (\nu_1, \nu_2, \ldots, \nu_s; \underline{\sigma})$ be an expression of $\pi \in \mathbb{P}$. We call $\nu_1 \in \mathfrak{h}_{\mathbb{R}}^*$ (respectively, $\nu_s \in \mathfrak{h}_{\mathbb{R}}^*$) the *initial* (respectively, *final*) *direction* of π; it is easy to check that these elements $\nu_1, \nu_s \in \mathfrak{h}_{\mathbb{R}}^*$ do not depend on the choice of an expression of π. The initial (respectively, final) direction of π is denoted by $\iota(\pi)$ (respectively, $\kappa(\pi)$).

Remark 2.2.4. Let $\pi_1, \pi_2 \in \mathbb{P}$. We easily see that $\iota(\pi_1 \pm \pi_2) = \iota(\pi_1) \pm \iota(\pi_2)$ and $\kappa(\pi_1 \pm \pi_2) = \kappa(\pi_1) \pm \kappa(\pi_2)$.

1529

Let $\lambda \in P$ be an integral weight. Recall from [NS08, Definition 2.6.1] (see also [Lit95, §4]) that a Lakshmibai–Seshadri (LS) path of shape λ is a path $\pi \in \mathbb{P}$ having an expression of the form $\pi = (\nu_1, \nu_2, \ldots, \nu_s; \sigma_0, \sigma_1, \ldots, \sigma_s)$, where $\nu_1, \nu_2, \ldots, \nu_s \in W\lambda$, and where, for each $1 \leqslant u \leqslant s - 1$, there exists a '$\sigma_u$-chain' for (ν_u, ν_{u+1}) (see [Lit95, §4] and [NS08, Definition 2.4.5] for the definition of 'σ_u-chain'). We denote by $\mathbb{B}(\lambda)$ the set of all LS paths of shape λ.

Remark 2.2.5 (see [NS08, Remark 2.6.2]). Let $\lambda \in P$ be an integral weight. The straight line path $\pi_\nu(t) := t\nu$, $t \in [0, 1]$, is an element of $\mathbb{B}(\lambda)$ for all $\nu \in W\lambda$.

LEMMA 2.2.6. *Let* $\lambda \in P_+^0$. *If* $\pi \in \mathbb{B}(\lambda)$, *then* $\pi + \pi_{kd_\lambda\delta} \in \mathbb{B}(\lambda)$ *for all* $k \in \mathbb{Z}$.

Proof. Let $k \in \mathbb{Z}$. We know from [NS08, Lemma 2.7.4] that $\pi + \pi_{kd_\lambda\delta} \in \mathbb{B}(\lambda + kd_\lambda\delta)$. Also, because there exists $w \in W$ such that $w(\lambda) = \lambda + kd_\lambda\delta$ by the definition of d_λ, we deduce from the definition of LS paths that $\mathbb{B}(\lambda + kd_\lambda\delta) = \mathbb{B}(w(\lambda)) = \mathbb{B}(\lambda)$ (see [NS08, Remark 2.6.3(3)]). This proves the lemma. □

Throughout this paper, we use standard notation and terminology of the theory of (abstract) crystals without further mention; we refer the reader to [Kas95, §7] and [HK02, §4.5] for details.

Now, we recall from [Lit94, §§1.2 and 1.3] and [Lit95, §1] the definition of the root operators e_j and f_j, $j \in I$, on $\mathbb{B}(\lambda)$; see also [GL04, §5.1] for the presentation of the definition in this form. Let $\pi \in \mathbb{B}(\lambda)$, and $j \in I$. We set

$$H_j^\pi(t) := (\pi(t))(h_j) \quad \text{for } t \in [0, 1], \quad m_j^\pi := \min\{H_j^\pi(t) \mid t \in [0, 1]\}. \tag{2.2.2}$$

Then we define $e_j\pi$ as follows (note that $m_j^\pi \in \mathbb{Z}_{\leqslant 0}$ by [Lit95, Lemma 4.5d)]). If $m_j^\pi = 0$, then $e_j\pi := \mathbf{0}$, where the $\mathbf{0}$ is an additional element corresponding to '0' in the theory of crystals. If $m_j^\pi \leqslant -1$, then we define $e_j\pi \in \mathbb{P}$ by

$$(e_j\pi)(t) = \begin{cases} \pi(t) & \text{if } 0 \leqslant t \leqslant t_0, \\ \pi(t_0) + r_j(\pi(t) - \pi(t_0)) & \text{if } t_0 \leqslant t \leqslant t_1, \\ \pi(t) + \alpha_j & \text{if } t_1 \leqslant t \leqslant 1, \end{cases} \tag{2.2.3}$$

where we set

$$\begin{aligned} t_1 &:= \min\{t \in [0, 1] \mid H_j^\pi(t) = m_j^\pi\}, \\ t_0 &:= \max\{t \in [0, t_1] \mid H_j^\pi(t) = m_j^\pi + 1\}. \end{aligned} \tag{2.2.4}$$

Similarly, $f_j\pi \in \mathbb{P} \cup \{\mathbf{0}\}$ is defined as follows (note that $H_j^\pi(1) - m_j^\pi \in \mathbb{Z}_{\geqslant 0}$ by [Lit95, Lemma 4.5d)] and $\pi(1) \in P$). If $H_j^\pi(1) - m_j^\pi = 0$, then $f_j\pi := \mathbf{0}$. If $H_j^\pi(1) - m_j^\pi \geqslant 1$, then we define $f_j\pi \in \mathbb{P}$ by

$$(f_j\pi)(t) = \begin{cases} \pi(t) & \text{if } 0 \leqslant t \leqslant t_0, \\ \pi(t_0) + r_j(\pi(t) - \pi(t_0)) & \text{if } t_0 \leqslant t \leqslant t_1, \\ \pi(t) - \alpha_j & \text{if } t_1 \leqslant t \leqslant 1, \end{cases} \tag{2.2.5}$$

where we set

$$\begin{aligned} t_0 &:= \max\{t \in [0, 1] \mid H_j^\pi(t) = m_j^\pi\}, \\ t_1 &:= \min\{t \in [t_0, 1] \mid H_j^\pi(t) = m_j^\pi + 1\}. \end{aligned} \tag{2.2.6}$$

1530

Using the root operators e_j and f_j, $j \in I$, we can endow the set $\mathbb{B}(\lambda)$ of all LS paths of shape $\lambda \in P$ with the structure of a (P-weighted) crystal, i.e. the structure of a crystal associated to the Cartan datum $(A, \Pi, \Pi^\vee, P, P^\vee)$ (see [Lit95, §§ 2 and 4] and also [NS06, Theorems 1.2.3 and 1.4.5]). In fact, it follows from [Lit95, Lemma 2.1c] that

$$-m_j^\pi = \max\{l \in \mathbb{Z}_{\geqslant 0} \mid e_j^l \pi \neq \mathbf{0}\}, \tag{2.2.7}$$

$$H_j^\pi(1) - m_j^\pi = \max\{l \in \mathbb{Z}_{\geqslant 0} \mid f_j^l \pi \neq \mathbf{0}\}, \tag{2.2.8}$$

and hence that $\varepsilon_j(\pi) = -m_j^\pi$ and $\varphi_j(\pi) = H_j^\pi(1) - m_j^\pi$.

Let $\lambda \in P$ be an integral weight. For $\pi \in \mathbb{B}(\lambda)$, define a piecewise linear, continuous map $\mathrm{cl}(\pi) : [0, 1] \to \mathfrak{h}_\mathbb{R}^* / \mathbb{R}\delta$ by $(\mathrm{cl}(\pi))(t) = \mathrm{cl}(\pi(t))$ for $t \in [0, 1]$, where $\mathrm{cl} : \mathfrak{h}_\mathbb{R}^* \twoheadrightarrow \mathfrak{h}_\mathbb{R}^* / \mathbb{R}\delta$ is the canonical projection. We set $\mathbb{B}(\lambda)_{\mathrm{cl}} := \{\mathrm{cl}(\pi) \mid \pi \in \mathbb{B}(\lambda)\}$.

Remark 2.2.7. We see from Remark 2.2.5 that the straight line path $\eta_\mu(t) = t\mu$, $t \in [0, 1]$, is contained in $\mathbb{B}(\lambda)_{\mathrm{cl}}$ for all $\mu \in \mathrm{cl}(W\lambda) = \mathring{W}\,\mathrm{cl}(\lambda)$.

An expression and a reduced expression of $\eta \in \mathbb{B}(\lambda)_{\mathrm{cl}}$ are defined similarly to those for the case of $\mathbb{B}(\lambda)$. In addition, for $\eta \in \mathbb{B}(\lambda)_{\mathrm{cl}}$, we define the initial and final directions of η (which do not depend on the choice of an expression of η) as in Definition 2.2.3, and also denote the initial (respectively, final) direction of $\eta \in \mathbb{B}(\lambda)_{\mathrm{cl}}$ by $\iota(\eta)$ (respectively, $\kappa(\eta)$).

Remark 2.2.8. Let $\pi = (\nu_1, \nu_2, \ldots, \nu_s; \underline{\sigma})$ be an expression of $\pi \in \mathbb{B}(\lambda)$. Then, $\mathrm{cl}(\pi) \in \mathbb{B}(\lambda)_{\mathrm{cl}}$ has an expression $\mathrm{cl}(\pi) = (\mathrm{cl}(\nu_1), \mathrm{cl}(\nu_2), \ldots, \mathrm{cl}(\nu_s); \underline{\sigma})$. Hence it follows that

$$\begin{cases} \iota(\mathrm{cl}(\pi)) = \mathrm{cl}(\iota(\pi)) \in \mathrm{cl}(W\lambda) = \mathring{W}\,\mathrm{cl}(\lambda) \\ \kappa(\mathrm{cl}(\pi)) = \mathrm{cl}(\kappa(\pi)) \in \mathrm{cl}(W\lambda) = \mathring{W}\,\mathrm{cl}(\lambda) \end{cases} \quad \text{for every } \pi \in \mathbb{B}(\lambda).$$

Let $\eta \in \mathbb{B}(\lambda)_{\mathrm{cl}}$, and $j \in I$. We set $H_j^\eta(t) := (\eta(t))(h_j)$, $t \in [0, 1]$, and define m_j^η to be the minimum value of the function $H_j^\eta(t)$ in the interval $[0, 1]$. It is obvious that, for every $\pi \in \mathbb{B}(\lambda)$ and $j \in I$,

$$H_j^{\mathrm{cl}(\pi)}(t) = H_j^\pi(t) \text{ for all } t \in [0, 1], \text{ and hence } m_j^{\mathrm{cl}(\pi)} = m_j^\pi. \tag{2.2.9}$$

Remark 2.2.9. Let $\eta \in \mathbb{B}(\lambda)_{\mathrm{cl}}$, and $j \in I$. We see from [Lit95, Lemma 4.5d] along with (2.2.9) that $\eta(1) \in P_{\mathrm{cl}}$, and that all local minima of the function $H_j^\eta(t)$, $t \in [0, 1]$, are integers. In particular, the minimum value m_j^η of the function $H_j^\eta(t)$ in the interval $[0, 1]$ is a nonpositive integer, and $H_j^\eta(1) - m_j^\eta$ is a nonnegative integer.

We define $e_j\eta, f_j\eta \in \mathbb{B}(\lambda)_{\mathrm{cl}} \cup \{\mathbf{0}\}$ for $\eta \in \mathbb{B}(\lambda)_{\mathrm{cl}}$ and $j \in I$ in the same way as in the case of $\mathbb{B}(\lambda)$. We easily see from (2.2.9) that

$$\mathrm{cl}(e_j\pi) = e_j\,\mathrm{cl}(\pi), \quad \mathrm{cl}(f_j\pi) = f_j\,\mathrm{cl}(\pi) \quad \text{for } \pi \in \mathbb{B}(\lambda) \text{ and } j \in I, \tag{2.2.10}$$

where $\mathrm{cl}(\mathbf{0})$ is understood to be $\mathbf{0}$.

Remark 2.2.10. Let $\eta \in \mathbb{B}(\lambda)_{\mathrm{cl}}$, and $j \in I$. It follows from the definition of the root operator e_j that, if $e_j\eta \neq \mathbf{0}$, then the initial direction $\iota(e_j\eta)$ is equal either to $\iota(\eta)$ or to $r_j(\iota(\eta))$.

We know from [NS05, Theorem 2.4 and § 3.1] that the set $\mathbb{B}(\lambda)_{\mathrm{cl}}$ equipped with the root operators e_j and f_j, $j \in I$, is a P_{cl}-weighted crystal (P_{cl}-crystal for short), i.e. a crystal associated to the classical Cartan datum $(A, \mathrm{cl}(\Pi), \Pi^\vee, P_{\mathrm{cl}}, P_{\mathrm{cl}}^\vee)$, and that the following equations hold:

$$-m_j^\eta = \max\{l \in \mathbb{Z}_{\geqslant 0} \mid e_j^l \eta \neq \mathbf{0}\} = \varepsilon_j(\eta), \tag{2.2.11}$$

$$H_j^\eta(1) - m_j^\eta = \max\{l \in \mathbb{Z}_{\geqslant 0} \mid f_j^l \eta \neq \mathbf{0}\} = \varphi_j(\eta). \tag{2.2.12}$$

For each $\eta \in \mathbb{B}(\lambda)_{\mathrm{cl}}$ and $j \in I$, we set $e_j^{\max}\eta := e_j^{\varepsilon_j(\eta)}\eta \in \mathbb{B}(\lambda)_{\mathrm{cl}}$. The proof of the next lemma is similar to that of [Lit94, 5.3 Lemma] (see Remark 2.2.9).

1531

LEMMA 2.2.11. *Let $\lambda \in P$ be an integral weight, and let $\eta \in \mathbb{B}(\lambda)_{\mathrm{cl}}$, $j \in I$.*

(1) If the initial direction $\iota(\eta) \in P_{\mathrm{cl}}$ of η satisfies $(\iota(\eta))(h_j) < 0$, then $e_j \eta \neq \mathbf{0}$.

(2) For all $0 \leqslant l \leqslant \varepsilon_j(\eta) - 1$, we have $\iota(e_j^l \eta) = \iota(\eta)$.

(3) If $(\iota(\eta))(h_j) \leqslant 0$, then $\iota(e_j^{\max} \eta) = r_j(\iota(\eta))$.

Using Lemma 2.2.11(3), we can show the following lemma by induction on p.

LEMMA 2.2.12. *Let $\lambda \in P$ be an integral weight. Let $\eta \in \mathbb{B}(\lambda)_{\mathrm{cl}}$, and set $\mu := \iota(\eta) \in P_{\mathrm{cl}}$. If $j_1, j_2, \ldots, j_p \in I$ satisfy the condition that $(r_{j_{p'}} r_{j_{p'-1}} \cdots r_{j_1}(\mu))(h_{j_{p'+1}}) \leqslant 0$ for all $p' = 0, 1, \ldots, p-1$, then the initial direction of $e_{j_p}^{\max} e_{j_{p-1}}^{\max} \cdots e_{j_1}^{\max}(\eta) \in \mathbb{B}(\lambda)_{\mathrm{cl}}$ is equal to $r_{j_p} r_{j_{p-1}} \cdots r_{j_1}(\mu)$.*

LEMMA 2.2.13. *Let $\lambda \in P$ be an integral weight. Let $\eta \in \mathbb{B}(\lambda)_{\mathrm{cl}}$, and set $\mu := \kappa(\eta) \in P_{\mathrm{cl}}$. If $j \in I$ is such that $\mu(h_j) > 0$, then $f_j \eta \neq \mathbf{0}$ holds.*

Proof. From the assumption of the lemma, we deduce that $H_j^\pi(1) - m_j^\pi > 0$. Therefore, it follows from Remark 2.2.9 that $H_j^\pi(1) - m_j^\pi \geqslant 1$. Hence, by (2.2.12), we have $\varphi_j(\eta) \geqslant 1$, which implies that $f_j \eta \neq \mathbf{0}$. This proves the lemma. □

2.3 Regular crystals and simple crystals

For a proper subset J of I, we set $A_J := (a_{ij})_{i,j \in J}$, $\Pi_J := \{\alpha_j\}_{j \in J} \subset \Pi$, and $\Pi_J^\vee := \{h_j\}_{j \in J} \subset \Pi^\vee$. When we regard a P_{cl}-crystal $\mathcal{B}$ as a crystal associated to the Cartan datum $(A_J, \mathrm{cl}(\Pi_J), \Pi_J^\vee, P_{\mathrm{cl}}, P_{\mathrm{cl}}^\vee)$, we denote it by $\mathrm{res}_J \mathcal{B}$. Also, we denote by $U_q'(\mathfrak{g})_J$ the $\mathbb{Q}(q)$-subalgebra of $U_q'(\mathfrak{g})$ generated by x_j, y_j, $j \in J$, and q^h, $h \in P_{\mathrm{cl}}^\vee$. Recall that a P_{cl}-crystal $\mathcal{B}$ is said to be regular if, for every proper subset $J \subsetneqq I$, $\mathrm{res}_J \mathcal{B}$ is isomorphic to the crystal base of an integrable $U_q'(\mathfrak{g})_J$-module.

Remark 2.3.1. Let $\lambda \in P$ be a level-zero integral weight. We know from [NS05, Proposition 3.13] that $\mathbb{B}(\lambda)_{\mathrm{cl}}$ is a regular P_{cl}-crystal with finitely many elements.

If $\mathcal{B}$ is a regular P_{cl}-crystal equipped with the Kashiwara operators e_j and f_j, $j \in I$, then we set $e_j^{\max} b := e_j^{\varepsilon_j(b)} b \in \mathcal{B}$ for $b \in \mathcal{B}$ and $j \in I$, where $\varepsilon_j(b) := \max\{l \in \mathbb{Z}_{\geqslant 0} \mid e_j^l b \neq \mathbf{0}\}$. For regular P_{cl}-crystals $\mathcal{B}_1$ and $\mathcal{B}_2$, we define the tensor product P_{cl}-crystal $\mathcal{B}_1 \otimes \mathcal{B}_2$ of $\mathcal{B}_1$ and $\mathcal{B}_2$ as in [Kas95, §7.3] and [HK02, Definition 4.5.3]; note that $\mathcal{B}_1 \otimes \mathcal{B}_2$ is also a regular P_{cl}-crystal. The next lemma follows immediately from the tensor product rule for crystals.

LEMMA 2.3.2. *Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be regular P_{cl}-crystals, and $b_1 \in \mathcal{B}_1$, $b_2 \in \mathcal{B}_2$. Let $j \in I$.*

(1) We have $\varepsilon_j(b_1 \otimes b_2) \geqslant \varepsilon_j(b_1)$. Therefore, if $e_j(b_1 \otimes b_2) = \mathbf{0}$, then $e_j b_1 = \mathbf{0}$.

(2) Set $L := \varepsilon_j(b_1 \otimes b_2)$. Then, for $0 \leqslant l \leqslant L$, we have

$$e_j^l(b_1 \otimes b_2) = \begin{cases} b_1 \otimes e_j^l b_2 & \text{if } 0 \leqslant l \leqslant L - \varepsilon_j(b_1), \\ e_j^{l-L+\varepsilon_j(b_1)} b_1 \otimes e_j^{L-\varepsilon_j(b_1)} b_2 & \text{if } L - \varepsilon_j(b_1) \leqslant l \leqslant L. \end{cases}$$

In particular, we have $e_j^{\max}(b_1 \otimes b_2) = e_j^{\max} b_1 \otimes b_2'$ for some $b_2' \in \mathcal{B}_2$.

Let $\mathcal{B}$ be a regular P_{cl}-crystal. We define

$$\|b\| := \sqrt{(\mathrm{wt}(b), \mathrm{wt}(b))_{\mathrm{cl}}} \quad \text{for } b \in \mathcal{B}. \tag{2.3.1}$$

LEMMA 2.3.3. *Let $\mathcal{B}$ be a regular P_{cl}-crystal. For each $b \in \mathcal{B}$ and $j \in I$, we have $\|e_j^{\max} b\| \geqslant \|b\|$, with equality if and only if either $e_j b = \mathbf{0}$ or $f_j b = \mathbf{0}$ holds.*

Proof. Using the equation $\varphi_j(b) = (\mathrm{wt}(b))(h_j) + \varepsilon_j(b)$, we easily see that

$$\|e_j^{\max} b\|^2 = \|b\|^2 + \varepsilon_j(b) \varphi_j(b)(\alpha_j, \alpha_j).$$

1532

Therefore, the inequality $\|e_j^{\max}b\| \geqslant \|b\|$ follows immediately from the fact that $\varepsilon_j(b) \geqslant 0$, $\varphi_j(b) \geqslant 0$, and $(\alpha_j, \alpha_j) > 0$. Also, the equality holds if and only if $\varepsilon_j(b) = 0$ or $\varphi_j(b) = 0$, which is equivalent to saying that $e_j b = \mathbf{0}$ or $f_j b = \mathbf{0}$. This proves the lemma. $\qquad\square$

Let $\mathcal{B}$ be a regular P_{cl}-crystal. For each $j \in I$, we define $S_j : \mathcal{B} \to \mathcal{B}$ by

$$S_j b = \begin{cases} f_j^l b & \text{if } l := (\mathrm{wt}\, b)(h_j) \geqslant 0, \\ e_j^{-l} b & \text{if } l := (\mathrm{wt}\, b)(h_j) < 0. \end{cases} \tag{2.3.2}$$

We know from [Kas94, §7] that there exists a unique action $S : W \to \mathrm{Bij}(\mathcal{B})$, $w \mapsto S_w$, of the Weyl group W on the set $\mathcal{B}$ such that $S_{r_j} = S_j$ for all $j \in I$, where $\mathrm{Bij}(\mathcal{B})$ denotes the group of all bijections from the set $\mathcal{B}$ to itself; in fact, if $w = r_{j_1} r_{j_2} \cdots r_{j_p} \in W$ for $j_1, j_2, \ldots, j_p \in I$, then $S_w = S_{j_1} S_{j_2} \cdots S_{j_p}$. Note that $\mathrm{wt}(S_w b) = w(\mathrm{wt}(b))$ for all $w \in W$ and $b \in \mathcal{B}$.

DEFINITION 2.3.4 [AK97, §1.4]. Let $\mathcal{B}$ be a regular P_{cl}-crystal. An element $b \in \mathcal{B}$ is said to be *extremal* if, for every $w \in W$, either $e_j S_w b = \mathbf{0}$ or $f_j S_w b = \mathbf{0}$ holds for each $j \in I$.

Remark 2.3.5. It follows immediately from the definition above that, if $b \in \mathcal{B}$ is an extremal element, then $S_w b$ is an extremal element of weight $w(\mathrm{wt}(b))$ for each $w \in W$.

LEMMA 2.3.6. *Let $\mathcal{B}$ be a regular P_{cl}-crystal with finitely many elements. If $b \in \mathcal{B}$ satisfies the condition that $\|b\| = \max\{\|b'\| \mid b' \in \mathbb{B}(\lambda)_{\mathrm{cl}}\}$, then b is an extremal element.*

Proof. Let $w \in W$, and $j \in I$. Since $(\cdot, \cdot)_{\mathrm{cl}}$ is W-invariant, it follows that $\|S_w b\| = \|b\|$. Using this, we obtain

$$\begin{aligned} \|b\| &\geqslant \|e_j^{\max} S_w b\| \quad \text{by the maximality of } \|b\| \\ &\geqslant \|S_w b\| \quad \text{by Lemma 2.3.3} \\ &= \|b\|, \end{aligned}$$

and hence $\|e_j^{\max} S_w b\| = \|S_w b\|$. Therefore, by Lemma 2.3.3, either $e_j S_w b = \mathbf{0}$ or $f_j S_w b = \mathbf{0}$ holds. This proves the lemma. $\qquad\square$

DEFINITION 2.3.7. Let $\mathcal{B}$ be a regular P_{cl}-crystal with finitely many elements. The P_{cl}-crystal $\mathcal{B}$ is said to be *simple* if it satisfies the following conditions.

(1) The weights of elements of $\mathcal{B}$ are all level-zero.

(2) The set of all extremal elements of $\mathcal{B}$ is exactly the W-orbit of some extremal element of $\mathcal{B}$. Also, for each extremal element $b \in \mathcal{B}$, the subset $\mathcal{B}_{\mathrm{wt}(b)} \subset \mathcal{B}$ of all elements of weight $\mathrm{wt}(b)$ consists of a single element, i.e. $\mathcal{B}_{\mathrm{wt}(b)} = \{b\}$.

Remark 2.3.8. Let $\mathcal{B}$ be a simple P_{cl}-crystal. Then it follows from Remark 2.1.5 and Definition 2.3.7(2) that there exists a unique extremal element b of $\mathcal{B}$ such that $\mathrm{wt}(b) \in P_{\mathrm{cl}}$ is level-zero and I_0-dominant.

LEMMA 2.3.9. *The following hold.*

(1) *A simple P_{cl}-crystal is connected.*

(2) *A tensor product of simple P_{cl}-crystals is also a simple P_{cl}-crystal.*

(3) *Let $\mathcal{B}_1$, $\mathcal{B}_2$ be simple P_{cl}-crystals. Then there exists at most one isomorphism of P_{cl}-crystals from $\mathcal{B}_1$ to $\mathcal{B}_2$. In particular, any automorphism of a simple P_{cl}-crystal is necessarily the identity map.*

Proof. Parts (1) and (2) are precisely Lemmas 1.9 and 1.10 in [AK97], respectively. Let us show part (3). Let $\mathcal{B}_1$, $\mathcal{B}_2$ be simple P_{cl}-crystals, and let $\Phi : \mathcal{B}_1 \xrightarrow{\sim} \mathcal{B}_2$ be an isomorphism of P_{cl}-crystals

from $\mathcal{B}_1$ to $\mathcal{B}_2$. We see from Remark 2.3.8 that there exists a unique element $b_1 \in \mathcal{B}_1$ (respectively, $b_2 \in \mathcal{B}_2$) such that b_1 (respectively, b_2) is extremal, and such that $\mathrm{wt}(b_1)$ (respectively, $\mathrm{wt}(b_2)$) is level-zero and I_0-dominant. Since $\Phi : \mathcal{B}_1 \xrightarrow{\sim} \mathcal{B}_2$ is an isomorphism of P_{cl}-crystals, it follows that $\Phi(b_1)$ is extremal, and that $\mathrm{wt}(\Phi(b_1))$ is level-zero and I_0-dominant. Hence, from the uniqueness of such an element, we obtain $\Phi(b_1) = b_2$. Also, because a simple P_{cl}-crystal is connected by part (1), an isomorphism of P_{cl}-crystals from $\mathcal{B}_1$ to $\mathcal{B}_2$ is determined uniquely by the requirement that $\Phi(b_1) = b_2$. Thus the proof of the lemma is complete. $\qquad\square$

2.4 Tensor product decomposition and combinatorial R-matrices

We know from [NS06, Propositions 3.4.1 and 3.4.2] that, for each $i \in I_0$, the P_{cl}-crystal $\mathbb{B}(\varpi_i)_{\mathrm{cl}}$ is a simple P_{cl}-crystal isomorphic to the crystal basis $\mathcal{B}(W(\varpi_i))$ of the level-zero fundamental representation $W(\varpi_i)$, which is a finite-dimensional irreducible $U_q'(\mathfrak{g})$-module introduced in [Kas02, § 5.2]. Because the $W(\varpi_i)$, $i \in I_0$, are 'good' $U_q'(\mathfrak{g})$-modules in the sense of [Kas02, § 8], we deduce from [Kas02, Proposition 10.6] that, for each $i_1, i_2 \in I_0$, there exists a unique isomorphism (called a combinatorial R-matrix) $R_{\varpi_{i_1}, \varpi_{i_2}} : \mathbb{B}(\varpi_{i_1})_{\mathrm{cl}} \otimes \mathbb{B}(\varpi_{i_2})_{\mathrm{cl}} \xrightarrow{\sim} \mathbb{B}(\varpi_{i_2})_{\mathrm{cl}} \otimes \mathbb{B}(\varpi_{i_1})_{\mathrm{cl}}$ of P_{cl}-crystals (see also [Oka07, § 2.3]); the uniqueness follows from parts (2) and (3) of Lemma 2.3.9. Combining this fact and the tensor product decomposition theorem [NS05, Theorem 3.2], we obtain the following theorem.

THEOREM 2.4.1. *Let* $\mathbf{i} = (i_1, i_2, \ldots, i_n)$ *be an arbitrary sequence of elements of* I_0 *(with repetitions allowed), and set* $\lambda := \sum_{k=1}^{n} \varpi_{i_k} \in P_+^0$. *Then there exists a unique isomorphism of* P_{cl}-*crystals*,

$$\Psi_{\mathbf{i}} : \mathbb{B}(\lambda)_{\mathrm{cl}} \xrightarrow{\sim} \mathbb{B}_{\mathbf{i}} := \mathbb{B}(\varpi_{i_1})_{\mathrm{cl}} \otimes \mathbb{B}(\varpi_{i_2})_{\mathrm{cl}} \otimes \cdots \otimes \mathbb{B}(\varpi_{i_n})_{\mathrm{cl}}. \tag{2.4.1}$$

Remark 2.4.2. Let $\mathbf{i} = (i_1, i_2, \ldots, i_n)$ and $\lambda = \sum_{k=1}^{n} \varpi_{i_k} \in P_+^0$ be as in Theorem 2.4.1. It follows from Theorem 2.4.1 along with parts (1) and (2) of Lemma 2.3.9 that $\mathbb{B}(\lambda)_{\mathrm{cl}}$ is a simple P_{cl}-crystal isomorphic to the crystal basis of the tensor product $U_q'(\mathfrak{g})$-module $W_{\mathbf{i}} := W(\varpi_{i_1}) \otimes W(\varpi_{i_2}) \otimes \cdots \otimes W(\varpi_{i_n})$ of the level-zero fundamental representations $W(\varpi_{i_k})$, $1 \leqslant k \leqslant n$.

Remark 2.4.3. Let $\lambda \in P_+^0$. We know from [NS05, Lemma 3.19(1)] that the straight line path $\eta_{\mathrm{cl}(\lambda)}$ is an extremal element of $\mathbb{B}(\lambda)_{\mathrm{cl}}$, and that $S_w \eta_{\mathrm{cl}(\lambda)} = \eta_{w\,\mathrm{cl}(\lambda)}$ for each $w \in W$. Therefore, from Remark 2.4.2, we deduce (recalling the definition of simple P_{cl}-crystals) that each extremal element of $\mathbb{B}(\lambda)_{\mathrm{cl}}$ is a straight line path η_μ for some $\mu \in \mathrm{cl}(W\lambda) = \mathring{W}\,\mathrm{cl}(\lambda)$, and that the number of elements of weight μ in $\mathbb{B}(\lambda)_{\mathrm{cl}}$ is equal to one for all $\mu \in \mathrm{cl}(W\lambda) = \mathring{W}\,\mathrm{cl}(\lambda)$. In particular, the straight line path $\eta_{\mathrm{cl}(\lambda)}$ is the unique extremal element of $\mathbb{B}(\lambda)_{\mathrm{cl}}$ whose weight is level-zero and I_0-dominant, of which we made mention in Remark 2.3.8.

Remark 2.4.4. Let $\lambda \in P_+^0$. We see from [NS08, Lemma 2.6.4] that the weights of $\mathbb{B}(\lambda)_{\mathrm{cl}}$ are all contained in the set $\mathrm{cl}(\lambda) - a_0^{-1}\,\mathrm{cl}(\mathring{Q}_+)$.

Let $\mathbf{i} = (i_1, i_2, \ldots, i_n)$ and $\lambda \in P_+^0$ be as in Theorem 2.4.1. It is easily seen from Remark 2.4.3 and [AK97, Lemma 1.6(1)] that $\eta_{\mathrm{cl}(\varpi_{i_1})} \otimes \eta_{\mathrm{cl}(\varpi_{i_2})} \otimes \cdots \otimes \eta_{\mathrm{cl}(\varpi_{i_n})}$ is the unique extremal element of the simple P_{cl}-crystal $\mathbb{B}_{\mathbf{i}} = \mathbb{B}(\varpi_{i_1})_{\mathrm{cl}} \otimes \mathbb{B}(\varpi_{i_2})_{\mathrm{cl}} \otimes \cdots \otimes \mathbb{B}(\varpi_{i_n})_{\mathrm{cl}}$ whose weight is level-zero and I_0-dominant. Therefore, we deduce from Remark 2.4.3 and the proof of Lemma 2.3.9(3) that

$$\Psi_{\mathbf{i}}(\eta_{\mathrm{cl}(\lambda)}) = \eta_{\mathrm{cl}(\varpi_{i_1})} \otimes \eta_{\mathrm{cl}(\varpi_{i_2})} \otimes \cdots \otimes \eta_{\mathrm{cl}(\varpi_{i_n})}. \tag{2.4.2}$$

COROLLARY 2.4.5. *Let* $\lambda, \lambda' \in P_+^0$.

(1) *There exists a unique isomorphism* $\Psi_{\lambda, \lambda'} : \mathbb{B}(\lambda + \lambda')_{\mathrm{cl}} \xrightarrow{\sim} \mathbb{B}(\lambda)_{\mathrm{cl}} \otimes \mathbb{B}(\lambda')_{\mathrm{cl}}$ *of* P_{cl}-*crystals.*

(2) *There exists a unique isomorphism* $R_{\lambda, \lambda'} : \mathbb{B}(\lambda)_{\mathrm{cl}} \otimes \mathbb{B}(\lambda')_{\mathrm{cl}} \xrightarrow{\sim} \mathbb{B}(\lambda')_{\mathrm{cl}} \otimes \mathbb{B}(\lambda)_{\mathrm{cl}}$ *of* P_{cl}-*crystals.*

Proof. Part (1) follows from Theorem 2.4.1. Part (2) follows immediately from part (1). $\qquad\square$

1534

Let $\lambda, \lambda' \in P_+^0$. By the same reasoning as that yielding (2.4.2), we obtain

$$\Psi_{\lambda,\lambda'}(\eta_{\mathrm{cl}(\lambda+\lambda')}) = \eta_{\mathrm{cl}(\lambda)} \otimes \eta_{\mathrm{cl}(\lambda')}, \tag{2.4.3}$$

$$R_{\lambda,\lambda'}(\eta_{\mathrm{cl}(\lambda)} \otimes \eta_{\mathrm{cl}(\lambda')}) = \eta_{\mathrm{cl}(\lambda')} \otimes \eta_{\mathrm{cl}(\lambda)}. \tag{2.4.4}$$

2.5 Local energy functions

Let $\lambda, \lambda' \in P_+^0$, and let $\mathbf{i} = (i_1, i_2, \ldots, i_n)$, $\mathbf{i}' = (i_1', i_2', \ldots, i_{n'}')$ be sequences of elements of I_0 such that $\lambda = \sum_{k=1}^n \varpi_{i_k}$ and $\lambda' = \sum_{k'=1}^{n'} \varpi_{i_{k'}'}$, respectively. We define the tensor product $U_q'(\mathfrak{g})$-modules $W_{\mathbf{i}}$ and $W_{\mathbf{i}'}$ corresponding to $\mathbf{i}$ and $\mathbf{i}'$, respectively, as in Remark 2.4.2; note that both $W_{\mathbf{i}}$ and $W_{\mathbf{i}'}$ are 'good' $U_q'(\mathfrak{g})$-modules (in the sense of [Kas02, § 8]) by [Kas02, Proposition 8.7], and that $\mathbb{B}(\lambda)_{\mathrm{cl}}$ and $\mathbb{B}(\lambda')_{\mathrm{cl}}$ are isomorphic as a P_{cl}-crystal to the crystal bases of $W_{\mathbf{i}}$ and $W_{\mathbf{i}'}$, respectively. Therefore, by an argument similar to that in [Kas02, § 11], we obtain the following theorem (see also [Oka07, § 2.3]).

THEOREM 2.5.1. *Let $\lambda, \lambda' \in P_+^0$. Then, there exists a unique $\mathbb{Z}$-valued function (called a local energy function) $H_{\lambda,\lambda'} : \mathbb{B}(\lambda)_{\mathrm{cl}} \otimes \mathbb{B}(\lambda')_{\mathrm{cl}} \to \mathbb{Z}$ satisfying the following two conditions.*

(H1) *For each $\eta_1 \otimes \eta_2 \in \mathbb{B}(\lambda)_{\mathrm{cl}} \otimes \mathbb{B}(\lambda')_{\mathrm{cl}}$ and $j \in I$ such that $e_j(\eta_1 \otimes \eta_2) \neq \mathbf{0}$, the equation*

$$H_{\lambda,\lambda'}(e_j(\eta_1 \otimes \eta_2))$$
$$= \begin{cases} H_{\lambda,\lambda'}(\eta_1 \otimes \eta_2) + 1 & \text{if } j = 0, \text{ and if } e_0(\eta_1 \otimes \eta_2) = e_0\eta_1 \otimes \eta_2, \ e_0(\widetilde{\eta}_2 \otimes \widetilde{\eta}_1) = e_0\widetilde{\eta}_2 \otimes \widetilde{\eta}_1, \\ H_{\lambda,\lambda'}(\eta_1 \otimes \eta_2) - 1 & \text{if } j = 0, \text{ and if } e_0(\eta_1 \otimes \eta_2) = \eta_1 \otimes e_0\eta_2, \ e_0(\widetilde{\eta}_2 \otimes \widetilde{\eta}_1) = \widetilde{\eta}_2 \otimes e_0\widetilde{\eta}_1, \\ H_{\lambda,\lambda'}(\eta_1 \otimes \eta_2) & \text{otherwise,} \end{cases}$$
$$\tag{2.5.1}$$

holds, where we set $\widetilde{\eta}_2 \otimes \widetilde{\eta}_1 := R_{\lambda,\lambda'}(\eta_1 \otimes \eta_2) \in \mathbb{B}(\lambda')_{\mathrm{cl}} \otimes \mathbb{B}(\lambda)_{\mathrm{cl}}$.

(H2) *The equation $H_{\lambda,\lambda'}(\eta_{\mathrm{cl}(\lambda)} \otimes \eta_{\mathrm{cl}(\lambda')}) = 0$ holds.*

3. Degree function on LS path crystals

3.1 Definition of the degree function

Let $\lambda \in P_+^0$. Recall from Lemma 2.1.2 that every element ν of $W\lambda$ can be written uniquely in the form $\nu = \lambda - \beta + kd_\lambda\delta$ with $\beta \in \mathring{Q}_+$ and $k \in \mathbb{Z}$.

LEMMA 3.1.1. *Let $\pi \in \mathbb{B}(\lambda)$. If the initial direction $\iota(\pi) \in W\lambda$ of π is contained in $\lambda - \mathring{Q}_+$, then $\pi(1) \in P$ can be written uniquely in the form $\pi(1) = \lambda - a_0^{-1}\beta + a_0^{-1}K\delta$ with $\beta \in \mathring{Q}_+$ and $K \in \mathbb{Z}_{\geqslant 0}$.*

Proof. We know from [NS08, Lemma 2.6.4], along with the linear independence of α_j, $j \in I$, and δ, that $\pi(1) \in P$ can be written uniquely in the form $\pi(1) = \lambda - a_0^{-1}\beta + a_0^{-1}K\delta$ with $\beta \in \mathring{Q}_+$ and $K \in \mathbb{Z}$. Let us show that the coefficient $a_0^{-1}K$ of δ in this expression of $\pi(1) \in P$ is nonnegative. Let $\pi = (\nu_1, \nu_2, \ldots, \nu_s; \underline{\sigma})$ be an expression of $\pi \in \mathbb{B}(\lambda)$, and write each $\nu_u \in W\lambda$ for $1 \leqslant u \leqslant s$ as $\nu_u = \lambda - \beta_u + k_u d_\lambda\delta$, where $\beta_u \in \mathring{Q}_+$ and $k_u \in \mathbb{Z}$ (see Lemma 2.1.2); note that $k_1 = 0$ by the assumption of the lemma. It follows from the definition of LS paths that $\nu_1 \geqslant \nu_2 \geqslant \cdots \geqslant \nu_s$, and hence from Remark 2.1.3 that $0 = k_1 \leqslant k_2 \leqslant \cdots \leqslant k_s$. Observe by (2.2.1) that the coefficient $a_0^{-1}K$ of δ in the expression above of $\pi(1) \in P$ is equal to $\sum_{u=1}^s (\sigma_u - \sigma_{u-1})k_u d_\lambda$. Therefore, we conclude that $a_0^{-1}K$ is nonnegative, and hence so is K. This proves the lemma. $\qquad\square$

Let us denote by $\mathbb{B}_0(\lambda) \subset \mathbb{B}(\lambda)$ the connected component of the P-crystal $\mathbb{B}(\lambda)$ containing the straight line path π_λ. We know the following lemma from [NS08, Lemma 4.2.3].

Lemma 3.1.2. *Let $\eta \in \mathbb{B}(\lambda)_{\mathrm{cl}}$. Then, the set $\mathrm{cl}^{-1}(\eta) \cap \mathbb{B}_0(\lambda)$ is nonempty, where we set $\mathrm{cl}^{-1}(\eta) :=$ $\{\pi \in \mathbb{B}(\lambda) \mid \mathrm{cl}(\pi) = \eta\}$. Moreover, if we take an arbitrary $\pi \in \mathrm{cl}^{-1}(\eta) \cap \mathbb{B}_0(\lambda)$, then $\mathrm{cl}^{-1}(\eta) \cap \mathbb{B}_0(\lambda) =$ $\{\pi + \pi_{kd_\lambda\delta} \mid k \in \mathbb{Z}\}$.*

Proposition 3.1.3. *Let $\eta \in \mathbb{B}(\lambda)_{\mathrm{cl}}$. Then, the set $\mathrm{cl}^{-1}(\eta) \cap \mathbb{B}_0(\lambda)$ contains a unique element π_η such that $\iota(\pi_\eta) \in \lambda - \mathring{Q}_+$.*

Proof. Let us take $\pi \in \mathrm{cl}^{-1}(\eta) \cap \mathbb{B}_0(\lambda)$, and write its initial direction $\iota(\pi) \in W\lambda$ as $\iota(\pi) = \lambda - \beta + kd_\lambda\delta$ for $\beta \in \mathring{Q}_+$ and $k \in \mathbb{Z}$. Then we see from Lemma 3.1.2 that $\pi_\eta := \pi - \pi_{kd_\lambda\delta}$ is also contained in $\mathrm{cl}^{-1}(\eta) \cap \mathbb{B}_0(\lambda)$. Also, it is easy to show (see Remark 2.2.4) that $\iota(\pi_\eta)$ is equal to $\iota(\pi) - kd_\lambda\delta = \lambda - \beta$, and hence that $\iota(\pi_\eta) \in \lambda - \mathring{Q}_+$. This proves the existence of π_η. The uniqueness of π_η follows immediately from Lemma 3.1.2. This completes the proof of the proposition. □

Let $\eta \in \mathbb{B}(\lambda)_{\mathrm{cl}}$, and take $\pi_\eta \in \mathrm{cl}^{-1}(\eta) \cap \mathbb{B}_0(\lambda)$ of Proposition 3.1.3. Then, by Lemma 3.1.1, we can write $\pi_\eta(1) \in P$ in the form $\pi_\eta(1) = \lambda - a_0^{-1}\beta + a_0^{-1}K\delta$ with $\beta \in \mathring{Q}_+$ and $K \in \mathbb{Z}_{\geqslant 0}$. Now, we define the degree $\mathrm{Deg}(\eta) \in \mathbb{Z}_{\leqslant 0}$ of the $\eta \in \mathbb{B}(\lambda)_{\mathrm{cl}}$ by

$$\mathrm{Deg}(\eta) = -K. \tag{3.1.1}$$

Proposition 3.1.4. *Let $\eta \in \mathbb{B}(\lambda)_{\mathrm{cl}}$. Take $\pi \in \mathrm{cl}^{-1}(\eta) \cap \mathbb{B}(\lambda)$ such that $\iota(\pi) \in \lambda - \mathring{Q}_+$ and $\pi \neq \pi_\eta$.*
(1) If we write $\pi(1) \in P$ in the form $\pi(1) = \lambda - a_0^{-1}\beta' + a_0^{-1}K'\delta$ with $\beta' \in \mathring{Q}_+$ and $K' \in \mathbb{Z}_{\geqslant 0}$, then we have $-K' < \mathrm{Deg}(\eta)$.
(2) If we write the final directions $\kappa(\pi_\eta)$ and $\kappa(\pi)$ of π_η and π in the form $\kappa(\pi_\eta) = \lambda - \beta + kd_\lambda\delta$ and $\kappa(\pi) = \lambda - \beta' + k'd_\lambda\delta$ with $\beta, \beta' \in \mathring{Q}_+$ and $k, k' \in \mathbb{Z}$, respectively, then we have $k < k'$.

Remark 3.1.5. Part (1) of Proposition 3.1.4 characterizes the degree $\mathrm{Deg}(\eta) \in \mathbb{Z}_{\leqslant 0}$ of $\eta \in \mathbb{B}(\lambda)_{\mathrm{cl}}$ as the maximum of the nonpositive integers $-K$ for which $\pi(1) \in P$ is of the form $\pi(1) = \lambda - a_0^{-1}\beta + a_0^{-1}K\delta$ with $\beta \in \mathring{Q}_+$ and $K \in \mathbb{Z}_{\geqslant 0}$, where $\pi \in \mathrm{cl}^{-1}(\eta) \cap \mathbb{B}(\lambda)$ is such that $\iota(\pi) \in \lambda - \mathring{Q}_+$. Moreover, the maximum $\mathrm{Deg}(\eta)$ is attained only by $\pi_\eta \in \mathrm{cl}^{-1}(\eta) \cap \mathbb{B}_0(\lambda)$ of Proposition 3.1.3.

To prove Proposition 3.1.4, we need the following lemma, which can be shown by an argument in the proof of [NS08, Theorem 3.1.1].

Lemma 3.1.6. *Each connected component of $\mathbb{B}(\lambda)$ contains a unique element whose reduced expression is of the form*

$$(\lambda, \lambda + k_2 d_\lambda\delta, \ldots, \lambda + k_s d_\lambda\delta; \sigma_0, \sigma_1, \ldots, \sigma_s), \tag{3.1.2}$$

with $k_2, \ldots, k_s \in \mathbb{Z}$ and $0 = \sigma_0 < \sigma_1 < \cdots < \sigma_s = 1$.

Remark 3.1.7. It follows from the definition of LS paths that $\lambda > \lambda + k_2 d_\lambda\delta > \cdots > \lambda + k_s d_\lambda\delta$. Hence we see from Remark 2.1.3 that $0 < k_2 < \cdots < k_s$.

Proof of Proposition 3.1.4. Assume that $\pi \in \mathbb{B}(\lambda)$ lies in a connected component of $\mathbb{B}(\lambda)$ containing an LS path π' whose reduced expression is of the form (3.1.2). We see from Proposition 3.1.3 and the assumption of the lemma that π does not lie in $\mathbb{B}_0(\lambda)$, and hence that $s \geqslant 2$.

We set $\psi := (0, k_2 d_\lambda\delta, \ldots, k_s d_\lambda\delta; \sigma_0, \sigma_1, \ldots, \sigma_s)$; note that $\pi' = \pi_\lambda + \psi$. Let X be a monomial of X in the root operators e_j, f_j for $j \in I$ such that $\pi = X\pi'$. Then we have

$$\begin{aligned} X\pi_\lambda &= X(\pi' - \psi) = X\pi' - \psi \quad \text{by [NS08, Lemma 2.7.1]} \\ &= \pi - \psi. \end{aligned}$$

Since $\mathrm{cl}(\pi) = \eta$, it follows that $\mathrm{cl}(X\pi_\lambda) = \mathrm{cl}(\pi - \psi) = \mathrm{cl}(\pi) = \eta$. Hence we get $X\pi_\lambda \in \mathrm{cl}^{-1}(\eta) \cap \mathbb{B}_0(\lambda)$. Also, because $\iota(\pi) \in \lambda - \mathring{Q}_+$ and $\iota(\psi) = 0$, we see that $\iota(X\pi_\lambda) = \iota(\pi - \psi) = \iota(\pi) - \iota(\psi) \in \lambda - \mathring{Q}_+$. Hence it follows from Proposition 3.1.3 that $X\pi_\lambda = \pi_\eta$. Thus we obtain $\pi_\eta = \pi - \psi$, and hence

$$\mathrm{Deg}(\eta) = -K' + a_0 \times (\text{coefficient of } \delta \text{ in } \psi(1)) \quad \text{and} \quad k = k' - k_s.$$

Since $s \geqslant 2$ as seen above, we deduce from Remark 3.1.7 (using (2.2.1)) that the coefficient of δ in $\psi(1)$ is greater than 0. Therefore, we conclude that $-K' < \mathrm{Deg}(\eta)$. Furthermore, since $k_s > 0$ with $s \geqslant 2$, it follows that $k < k'$. This completes the proof of the proposition. $\qquad\square$

3.2 Behavior of the degree function under root operators

As in § 3.1, let $\lambda \in P_+^0$.

LEMMA 3.2.1. *The following hold.*

(1) *We have* $\mathrm{Deg}(\eta_{\mathrm{cl}(\lambda)}) = 0$.

(2) *Let* $\eta \in \mathbb{B}(\lambda)_{\mathrm{cl}}$, *and* $j \in I$. *If* $e_j\eta \neq \mathbf{0}$, *then (see Remark 2.2.10)*

$$\mathrm{Deg}(e_j\eta) = \begin{cases} \mathrm{Deg}(\eta) - 1 & \text{if } j = 0 \text{ and } \iota(e_0\eta) = \iota(\eta), \\ \mathrm{Deg}(\eta) - (\iota(\eta))(h_0) - 1 & \text{if } j = 0 \text{ and } \iota(e_0\eta) = r_0(\iota(\eta)), \\ \mathrm{Deg}(\eta) & \text{if } j \neq 0. \end{cases} \qquad (3.2.1)$$

Proof. Part (1) is obvious from the definition of Deg, since $\pi_{\eta_{\mathrm{cl}(\lambda)}} = \pi_\lambda$. Let us prove part (2). It is obvious that $e_j\pi_\eta \in \mathbb{B}_0(\lambda)$ since $\pi_\eta \in \mathbb{B}_0(\lambda)$ by definition. Also, we know from (2.2.10) that $\mathrm{cl}(e_j\pi_\eta) = e_j\,\mathrm{cl}(\pi_\eta) = e_j\eta$. Let us write $\pi_\eta(1) \in P$ in the form $\pi_\eta(1) = \lambda - a_0^{-1}\beta - a_0^{-1}\,\mathrm{Deg}(\eta)\delta$ with $\beta \in \mathring{Q}_+$.

First, assume that $j \neq 0$. We deduce from Remark 2.2.10 along with Lemma 2.1.2 that $\iota(e_j\pi_\eta) \in \lambda - \mathring{Q}_+$. Because $e_j\pi_\eta \in \mathbb{B}_0(\lambda)$ and $\mathrm{cl}(e_j\pi_\eta) = e_j\eta$, it follows from Proposition 3.1.3 that $\pi_{e_j\eta} = e_j\pi_\eta$. Since $j \neq 0$, we have

$$\pi_{e_j\eta}(1) = (e_j\pi_\eta)(1) = \pi_\eta(1) + \alpha_j = \lambda - \underbrace{(a_0^{-1}\beta - \alpha_j)}_{\in a_0^{-1}\mathring{Q}_+} - a_0^{-1}\,\mathrm{Deg}(\eta)\delta,$$

and hence $\mathrm{Deg}(e_j\eta) = \mathrm{Deg}(\eta)$.

Next, assume that $j = 0$ and $\iota(e_0\eta) = \iota(\eta)$. Then we deduce (using (2.2.9)) from the definitions of the root operator e_0 on $\mathbb{B}(\lambda)$ and the root operator e_0 on $\mathbb{B}(\lambda)_{\mathrm{cl}}$ that $\iota(e_0\pi_\eta) = \iota(\pi_\eta)$, and hence $\iota(e_0\pi_\eta) \in \lambda - \mathring{Q}_+$. Because $e_0\pi_\eta \in \mathbb{B}_0(\lambda)$ and $\mathrm{cl}(e_0\pi_\eta) = e_0\eta$, it follows from Proposition 3.1.3 that $\pi_{e_0\eta} = e_0\pi_\eta$. Now, define $\theta \in \mathring{Q}_+$ by $\theta = \delta - a_0\alpha_0$. Since $\alpha_0 = a_0^{-1}(\delta - \theta)$, we have

$$\begin{aligned} \pi_{e_0\eta}(1) = (e_0\pi_\eta)(1) &= \pi_\eta(1) + \alpha_0 \\ &= \lambda - a_0^{-1}\beta - a_0^{-1}\,\mathrm{Deg}(\eta)\delta + a_0^{-1}(\delta - \theta) \\ &= \lambda - a_0^{-1}(\beta + \theta) - a_0^{-1}(\mathrm{Deg}(\eta) - 1)\delta, \end{aligned}$$

and hence $\mathrm{Deg}(e_0\eta) = \mathrm{Deg}(\eta) - 1$.

Finally, assume that $j = 0$ and $\iota(e_0\eta) = r_0(\iota(\eta))$. Then we deduce (using (2.2.9)) from the definitions of the root operator e_0 on $\mathbb{B}(\lambda)$ and the root operator on $e_0\,\mathbb{B}(\lambda)_{\mathrm{cl}}$ that

$$\begin{aligned} \iota(e_0\pi_\eta) = r_0\iota(\pi_\eta) &= \iota(\pi_\eta) - (\iota(\pi_\eta))(h_0)\alpha_0 \\ &= \iota(\pi_\eta) + a_0^{-1}(\iota(\pi_\eta))(h_0)\theta - a_0^{-1}(\iota(\pi_\eta))(h_0)\delta, \end{aligned} \qquad (3.2.2)$$

where $\theta = \delta - a_0\alpha_0$ as above. Note that $\iota(\pi_\eta) + a_0^{-1}(\iota(\pi_\eta))(h_0)\theta \in \lambda - \sum_{j \in I_0}\mathbb{Q}\alpha_j$, since $\iota(\pi_\eta) \in \lambda - \mathring{Q}_+$. Because $\iota(e_0\pi_\eta) \in W\lambda$, it follows from Lemma 2.1.2 and the linear independence of α_j, $j \in I$, and δ that $\iota(\pi_\eta) + a_0^{-1}(\iota(\pi_\eta))(h_0)\theta \in \lambda - \mathring{Q}_+$ and $a_0^{-1}(\iota(\pi_\eta))(h_0) \in \mathbb{Z}d_\lambda$. Hence, $a_0^{-1}(\iota(\pi_\eta))(h_0) = kd_\lambda$ for some $k \in \mathbb{Z}$. Also, because $e_0\pi_\eta \in \mathbb{B}_0(\lambda)$ and $\mathrm{cl}(e_0\pi_\eta) = e_0\eta \in \mathbb{B}_0(\lambda)$, we deduce from Lemma 3.1.2

that $e_0\pi_\eta + \pi_{kd_\lambda\delta} \in \mathrm{cl}^{-1}(e_0\eta) \cap \mathbb{B}_0(\lambda)$. Furthermore, we have

$$
\begin{aligned}
\iota(e_0\pi_\eta + \pi_{kd_\lambda\delta}) &= \iota(e_0\pi_\eta) + \iota(\pi_{kd_\lambda\delta}) \quad \text{by Remark } 2.2.4 \\
&= \iota(\pi_\eta) + a_0^{-1}(\iota(\pi_\eta))(h_0)\theta - kd_\lambda\delta + kd_\lambda\delta \quad \text{by (3.2.2)} \\
&= \iota(\pi_\eta) + a_0^{-1}(\iota(\pi_\eta))(h_0)\theta,
\end{aligned}
$$

where the last element lies in $\lambda - \mathring{Q}_+$ as seen above. Therefore, by Proposition 3.1.3, we get $\pi_{e_0\eta} = e_0\pi_\eta + \pi_{kd_\lambda\delta}$. From this, we obtain

$$
\begin{aligned}
\pi_{e_0\eta}(1) &= (e_0\pi_\eta)(1) + \pi_{kd_\lambda\delta}(1) = \pi_\eta(1) + \alpha_0 + kd_\lambda\delta \\
&= \lambda - a_0^{-1}\beta - a_0^{-1}\operatorname{Deg}(\eta)\delta + a_0^{-1}(\delta - \theta) + kd_\lambda\delta \\
&= \lambda - a_0^{-1}(\beta + \theta) - a_0^{-1}(\operatorname{Deg}(\eta) - a_0 kd_\lambda - 1)\delta,
\end{aligned}
$$

and hence

$$
\operatorname{Deg}(e_0\eta) = \operatorname{Deg}(\eta) - a_0 kd_\lambda - 1 = \operatorname{Deg}(\eta) - (\iota(\pi_\eta))(h_0) - 1.
$$

Note that $(\iota(\pi_\eta))(h_0) = (\iota(\eta))(h_0)$ since $\mathrm{cl}(\iota(\pi_\eta)) = \iota(\mathrm{cl}(\pi_\eta)) = \iota(\eta)$ by Remark 2.2.8. Thus we conclude that $\operatorname{Deg}(e_0\eta) = \operatorname{Deg}(\eta) - (\iota(\eta))(h_0) - 1$, as desired. This completes the proof of the lemma. $\square$

LEMMA 3.2.2. *Let* $\eta \in \mathbb{B}(\lambda)_{\mathrm{cl}}$, *and* $j \in I$. *Assume that* $e_j\eta \neq \mathbf{0}$, *and that* $(\iota(\eta))(h_j) \leqslant 0$. *Then*,

$$
\operatorname{Deg}(e_j^{\max}\eta) = \begin{cases} \operatorname{Deg}(\eta) - \varepsilon_0(\eta) - (\iota(\eta))(h_0) & \text{if } j = 0, \\ \operatorname{Deg}(\eta) & \text{if } j \neq 0. \end{cases} \tag{3.2.3}
$$

Proof. If $j \neq 0$, then it follows immediately from Lemma 3.2.1(2) that $\operatorname{Deg}(e_j^{\max}\eta) = \operatorname{Deg}(\eta)$. Now assume that $j = 0$. If $\varepsilon_0(\eta) = 0$, i.e. $e_0\eta = \mathbf{0}$, then we see from Lemma 2.2.11(1) that $(\iota(\eta))(h_0) \geqslant 0$, which, when combined with the assumption of the lemma, implies that $(\iota(\eta))(h_0) = 0$. Hence we have

$$
\operatorname{Deg}(e_0^{\max}\eta) = \operatorname{Deg}(e_0^0\eta) = \operatorname{Deg}(\eta) = \operatorname{Deg}(\eta) - \underbrace{\varepsilon_0(\eta)}_{=0} - \underbrace{(\iota(\eta))(h_0)}_{=0}.
$$

Thus it remains to consider the case $\varepsilon_0(\eta) \geqslant 1$. From Lemmas 2.2.11(2) and 3.2.1(2), it follows that

$$
\operatorname{Deg}(e_0^{\varepsilon_0(\eta)-1}\eta) = \operatorname{Deg}(\eta) - \varepsilon_0(\eta) + 1. \tag{3.2.4}
$$

Therefore, by using parts (2) and (3) of Lemma 2.2.11, we see from Lemma 3.2.1(2) that

$$
\begin{aligned}
\operatorname{Deg}(e_0^{\max}\eta) &= \operatorname{Deg}(e_0^{\varepsilon_0(\eta)}\eta) = \operatorname{Deg}(e_0 e_0^{\varepsilon_0(\eta)-1}\eta) \\
&= \operatorname{Deg}(e_0^{\varepsilon_0(\eta)-1}\eta) - (\iota(\eta))(h_0) - 1 \\
&= \operatorname{Deg}(\eta) - \varepsilon_0(\eta) + 1 - (\iota(\eta))(h_0) - 1 \quad \text{by (3.2.4)} \\
&= \operatorname{Deg}(\eta) - \varepsilon_0(\eta) - (\iota(\eta))(h_0).
\end{aligned}
$$

This proves the lemma. $\square$

4. Relation between the energy function and the degree function

4.1 Main results

Let $\mathbf{i} = (i_1, i_2, \ldots, i_n)$ be an arbitrary sequence of elements of I_0, and define the tensor product P_{cl}-crystal $\mathbb{B}_{\mathbf{i}} := \mathbb{B}(\varpi_{i_1})_{\mathrm{cl}} \otimes \mathbb{B}(\varpi_{i_2})_{\mathrm{cl}} \otimes \cdots \otimes \mathbb{B}(\varpi_{i_n})_{\mathrm{cl}}$. For an element $\eta_1 \otimes \eta_2 \otimes \cdots \otimes \eta_n \in \mathbb{B}_{\mathbf{i}}$, we define $\eta_l^{(k)} \in \mathbb{B}(\varpi_{i_l})_{\mathrm{cl}}$, $1 \leqslant k < l \leqslant n$, as follows (see [HKOTY99, §3] and [HKOTT02, §3.3]). There

1538

exists a unique isomorphism

$$\mathbb{B}(\varpi_{i_k})_{\mathrm{cl}} \otimes \mathbb{B}(\varpi_{i_{k+1}})_{\mathrm{cl}} \otimes \cdots \otimes \mathbb{B}(\varpi_{i_{l-1}})_{\mathrm{cl}} \otimes \mathbb{B}(\varpi_{i_l})_{\mathrm{cl}}$$
$$\xrightarrow{\sim} \mathbb{B}(\varpi_{i_l})_{\mathrm{cl}} \otimes \mathbb{B}(\varpi_{i_k})_{\mathrm{cl}} \otimes \cdots \otimes \mathbb{B}(\varpi_{i_{l-2}})_{\mathrm{cl}} \otimes \mathbb{B}(\varpi_{i_{l-1}})_{\mathrm{cl}}$$

of P_{cl}-crystals, which is given as the composition $R_{\varpi_{i_k}, \varpi_{i_l}} \circ R_{\varpi_{i_{k+1}}, \varpi_{i_l}} \circ \cdots \circ R_{\varpi_{i_{l-1}}, \varpi_{i_l}}$ of combinatorial R-matrices (see § 2.4); for uniqueness, see Lemma 2.3.9(3). We define $\eta_l^{(k)}$ to be the first factor (which lies in $\mathbb{B}(\varpi_{i_l})_{\mathrm{cl}}$) of the image of $\eta_k \otimes \eta_{k+1} \otimes \cdots \otimes \eta_l \in \mathbb{B}(\varpi_{i_k})_{\mathrm{cl}} \otimes \mathbb{B}(\varpi_{i_{k+1}})_{\mathrm{cl}} \otimes \cdots \otimes \mathbb{B}(\varpi_{i_l})_{\mathrm{cl}}$ under the above isomorphism of P_{cl}-crystals. For convenience, we set $\eta_l^{(l)} := \eta_l$ for $1 \leqslant l \leqslant n$.

For each $1 \leqslant k \leqslant n$, take (and fix) an arbitrary element $\eta_k^{\flat} \in \mathbb{B}(\varpi_{i_k})_{\mathrm{cl}}$ such that $f_j \eta_k^{\flat} = \mathbf{0}$ for all $j \in I_0$. Note that such an element $\eta_k^{\flat} \in \mathbb{B}(\varpi_{i_k})_{\mathrm{cl}}$ actually exists. Indeed, for each $i \in I_0$, we know from Remark 2.2.7 that $\eta_{\widetilde{\varpi}_i} \in \mathbb{B}(\varpi_i)_{\mathrm{cl}}$, where $\widetilde{\varpi}_i := w_0 \operatorname{cl}(\varpi_i) \in P_{\mathrm{cl}}$ (see also Remark 2.1.5); it follows immediately from the definition of the root operators f_j, $j \in I_0$, that $f_j \eta_{\widetilde{\varpi}_i} = \mathbf{0}$ for all $j \in I_0$.

Now, following [HKOTY99, § 3] and [HKOTT02, § 3.3], we define the energy function $D_{\mathbf{i}} : \mathbb{B}_{\mathbf{i}} = \mathbb{B}(\varpi_{i_1})_{\mathrm{cl}} \otimes \mathbb{B}(\varpi_{i_2})_{\mathrm{cl}} \otimes \cdots \otimes \mathbb{B}(\varpi_{i_n})_{\mathrm{cl}} \to \mathbb{Z}$ by

$$D_{\mathbf{i}}(\eta_1 \otimes \eta_2 \otimes \cdots \otimes \eta_n)$$
$$= \sum_{1 \leqslant k < l \leqslant n} H_{\varpi_{i_k}, \varpi_{i_l}}(\eta_k \otimes \eta_l^{(k+1)}) + \sum_{k=1}^{n} H_{\varpi_{i_k}, \varpi_{i_k}}(\eta_k^{\flat} \otimes \eta_k^{(1)}). \tag{4.1.1}$$

Also, we define a constant $D_{\mathbf{i}}^{\mathrm{ext}} \in \mathbb{Z}$ by

$$D_{\mathbf{i}}^{\mathrm{ext}} = \sum_{k=1}^{n} H_{\varpi_{i_k}, \varpi_{i_k}}(\eta_k^{\flat} \otimes \eta_{\mathrm{cl}(\varpi_{i_k})}). \tag{4.1.2}$$

The main result of this paper is the following theorem.

THEOREM 4.1.1. *Let* $\mathbf{i} = (i_1, i_2, \ldots, i_n)$ *be an arbitrary sequence of elements of* I_0, *and set* $\lambda := \sum_{k=1}^{n} \varpi_{i_k} \in P_+^0$. *Then, for every* $\eta \in \mathbb{B}(\lambda)_{\mathrm{cl}}$, *the equation*

$$\operatorname{Deg}(\eta) = D_{\mathbf{i}}(\Psi_{\mathbf{i}}(\eta)) - D_{\mathbf{i}}^{\mathrm{ext}} \tag{4.1.3}$$

holds, where $\Psi_{\mathbf{i}} : \mathbb{B}(\lambda)_{\mathrm{cl}} \xrightarrow{\sim} \mathbb{B}_{\mathbf{i}}$ *is the isomorphism of* P_{cl}-*crystals in Theorem 2.4.1.*

We will establish Theorem 4.1.1 under the following plan. First, in § 4.2, we show some technical lemmas needed in subsections that follow. Next, in § 4.3, using these lemmas, we prove Proposition 4.3.1, which is the key to our proof (in § 4.4) of Theorem 4.1.2 below. Finally, in § 4.5, we prove Theorem 4.1.3 below, which, when combined with Theorem 4.1.2, establishes Theorem 4.1.1.

THEOREM 4.1.2. *Let* $\mathbf{i} = (i_1, i_2, \ldots, i_n)$ *be an arbitrary sequence of elements of* I_0, *and set* $\lambda := \sum_{k=1}^{n} \varpi_{i_k} \in P_+^0$. *Let* $\eta \in \mathbb{B}(\lambda)_{\mathrm{cl}}$, *and set* $\Psi_{\mathbf{i}}(\eta) := \eta_1 \otimes \eta_2 \otimes \cdots \otimes \eta_n \in \mathbb{B}_{\mathbf{i}} = \mathbb{B}(\varpi_{i_1})_{\mathrm{cl}} \otimes \mathbb{B}(\varpi_{i_2})_{\mathrm{cl}} \otimes \cdots \otimes \mathbb{B}(\varpi_{i_n})_{\mathrm{cl}}$. *Then, the following equation holds:*

$$\operatorname{Deg}(\eta) = \sum_{1 \leqslant k < l \leqslant n} H_{\varpi_{i_k}, \varpi_{i_l}}(\eta_k \otimes \eta_l^{(k+1)}) + \sum_{k=1}^{n} \operatorname{Deg}(\eta_k^{(1)}). \tag{4.1.4}$$

THEOREM 4.1.3. *Let* $i \in I_0$, *and let* $\eta^{\flat} \in \mathbb{B}(\varpi_i)_{\mathrm{cl}}$ *be an element of* $\mathbb{B}(\varpi_i)_{\mathrm{cl}}$ *such that* $f_j \eta^{\flat} = \mathbf{0}$ *for all* $j \in I_0$. *Then, for every* $\eta \in \mathbb{B}(\varpi_i)_{\mathrm{cl}}$, *the following equation holds:*

$$\operatorname{Deg}(\eta) = H_{\varpi_i, \varpi_i}(\eta^{\flat} \otimes \eta) - H_{\varpi_i, \varpi_i}(\eta^{\flat} \otimes \eta_{\mathrm{cl}(\varpi_i)}). \tag{4.1.5}$$

Remark 4.1.4. Assume that $\mathfrak{g}$ is a nontwisted affine Lie algebra (note that $a_0 = 1$), and let $i \in I_0$, $m \in \mathbb{Z}_{\geqslant 1}$. In [GL04, § 5.7 and Proposition 5.9], they constructed an isomorphism ψ of P-weighted

crystals from the affinization of the P_{cl}-crystal $(\mathbb{B}(\varpi_i)_{\mathrm{cl}})^{\otimes m}$ onto $\bigsqcup_{0 \leqslant M < m} \mathbb{B}_0(m\varpi_i + M\delta)$; see [GL04, § 2.8] and [NS08, § 4.1] for the affinization of a P_{cl}-crystal. Also, in [NS08, § 4.2], we constructed an isomorphism Θ of P-weighted crystals from the affinization of the P_{cl}-crystal $\mathbb{B}(m\varpi_i)_{\mathrm{cl}}$ onto $\bigsqcup_{0 \leqslant M < m} \mathbb{B}_0(m\varpi_i + M\delta)$. In fact, we can show, without making use of Theorem 4.1.1 (cf. the proof of [NS08, Proposition 4.14]), that the two isomorphisms ψ and Θ can be considered as identical under the identification $\mathbb{B}(m\varpi_i)_{\mathrm{cl}} \xrightarrow{\sim} (\mathbb{B}(\varpi_i)_{\mathrm{cl}})^{\otimes m}$ of P_{cl}-crystals given by Theorem 2.4.1. From this, using the explicit constructions of ψ and Θ (see the proof of [NS08, Proposition 4.14]), we obtain a description of $\mathrm{Deg}(\eta)$ for $\eta \in \mathbb{B}(m\varpi_i)_{\mathrm{cl}}$ in terms of local energy functions. This description of the degree function $\mathrm{Deg} : \mathbb{B}(\lambda)_{\mathrm{cl}} \to \mathbb{Z}$ is similar and closely related to the one in Theorem 4.1.1 for the special case that $\mathfrak{g}$ is a nontwisted affine Lie algebra and $\lambda = m\varpi_i$.

4.2 Some technical lemmas

Recall from [Kac90, Proposition 6.3] that a real root of $\mathfrak{g}$ lies either in $a_0^{-1}\mathring{Q}_+ + a_0^{-1}\mathbb{Z}\delta$ or in $-a_0^{-1}\mathring{Q}_+ + a_0^{-1}\mathbb{Z}\delta$.

LEMMA 4.2.1. Let $\lambda \in P_+^0$, and let $w \in W$, $j \in I$.

(1) If $(w(\mathrm{cl}(\lambda)))(h_j) < 0$, then the real root $w^{-1}(\alpha_j)$ lies in $-a_0^{-1}\mathring{Q}_+ + a_0^{-1}\mathbb{Z}\delta$.

(2) If the real root $w^{-1}(\alpha_j)$ lies in $-a_0^{-1}\mathring{Q}_+ + a_0^{-1}\mathbb{Z}\delta$, then $(w(\mathrm{cl}(\lambda)))(h_j) \leqslant 0$.

(3) Assume that λ is strictly I_0-dominant. Then, $(w(\mathrm{cl}(\lambda)))(h_j) < 0$ if and only if the real root $w^{-1}(\alpha_j)$ lies in $-a_0^{-1}\mathring{Q}_+ + a_0^{-1}\mathbb{Z}\delta$.

Proof. Since $(\alpha_j, \alpha_j) \in \mathbb{Z}_{>0}$ for all $j \in I$, and since

$$(w(\mathrm{cl}(\lambda)))(h_j) = (w(\lambda))(h_j) = \frac{2(w(\lambda), \alpha_j)}{(\alpha_j, \alpha_j)} = \frac{2(\lambda, w^{-1}(\alpha_j))}{(\alpha_j, \alpha_j)},$$

it follows that $(w(\mathrm{cl}(\lambda)))(h_j) < 0$ if and only if $(\lambda, w^{-1}(\alpha_j)) < 0$, and that $(w(\mathrm{cl}(\lambda)))(h_j) = 0$ if and only if $(\lambda, w^{-1}(\alpha_j)) = 0$. Also, since λ is level-zero and I_0-dominant, and since $(\alpha_j, \alpha_j) \in \mathbb{Z}_{>0}$ for all $j \in I_0$, we have

$$(\lambda, \pm a_0^{-1}\mathring{Q}_+ + a_0^{-1}\mathbb{Z}\delta) = (\lambda, \pm a_0^{-1}\mathring{Q}_+) \subset \pm\mathbb{Q}_{\geqslant 0}.$$

Now, all the assertions of the lemma follow immediately from the discussion above. $\square$

LEMMA 4.2.2. Let $\lambda \in P_+^0$, and let $\eta \in \mathbb{B}(\lambda)_{\mathrm{cl}}$, $j \in I$. Assume that η has an expression of the form $\eta = (\mu_1, \mu_2; 0, \sigma, 1)$, with $\mu_1, \mu_2 \in \mathrm{cl}(W\lambda) = \mathring{W}\,\mathrm{cl}(\lambda)$ and $0 < \sigma < 1$. If $\mu_1(h_j) < 0$, then $e_j^{\max}\eta = (r_j(\mu_1), \mu_2'; 0, \sigma, 1)$, where

$$\mu_2' := \begin{cases} \mu_2 & \text{if } \mu_2(h_j) \geqslant 0, \\ r_j\mu_2 & \text{if } \mu_2(h_j) < 0. \end{cases} \tag{4.2.1}$$

Proof. First, assume that $\mu_2(h_j) \geqslant 0$. Then, since $\mu_1(h_j) < 0$ by the assumption of the lemma, it follows that the function $H_j^\eta(t)$ is strictly decreasing in the interval $[0, \sigma]$, and $m_j^\eta = H_j^\eta(\sigma) < 0$; note that $\varepsilon_j(\eta) = -m_j^\eta$ by (2.2.11). For $0 \leqslant l \leqslant \varepsilon_j(\eta) = -m_j^\eta$, let $\sigma^{(l)}$ be the unique point in $[0, \sigma]$ such that $H_j^\eta(\sigma^{(l)}) = m_j^\eta + l$; observe that $0 = \sigma^{(\varepsilon_j(\eta))} < \sigma^{(\varepsilon_j(\eta)-1)} < \cdots < \sigma^{(0)} = \sigma$. Now it is easily shown by induction on l that, for $0 \leqslant l \leqslant \varepsilon_j(\eta)$,

$$(e_j^l\eta)(t) = \begin{cases} \eta(t) & \text{if } 0 \leqslant t \leqslant \sigma^{(l)}, \\ \eta(\sigma^{(l)}) + r_j(\eta(t) - \eta(\sigma^{(l)})) & \text{if } \sigma^{(l)} \leqslant t \leqslant \sigma, \\ \eta(t) + l\alpha_j & \text{if } \sigma \leqslant t \leqslant 1. \end{cases}$$

1540

In particular, by taking $l = \varepsilon_j(\eta)$, we have

$$(e_j^{\max}\eta)(t) = \begin{cases} r_j(\eta(t)) & \text{if } 0 \leqslant t \leqslant \sigma, \\ \eta(t) + \varepsilon_j(\eta)\alpha_j & \text{if } \sigma \leqslant t \leqslant 1, \end{cases}$$

which implies that $e_j^{\max}\eta = (r_j(\mu_1), \mu_2; 0, \sigma, 1)$, since $\eta = (\mu_1, \mu_2; 0, \sigma, 1)$.

Next, assume that $\mu_2(h_j) < 0$. Then, since $\mu_1(h_j) < 0$ by the assumption of the lemma, it follows that the function $H_j^\eta(t)$ is strictly decreasing in the interval $[0, 1]$, and $m_j^\eta = H_j^\eta(1) < 0$; note that $\varepsilon_j(\eta) = -m_j^\eta$ by (2.2.11). For $0 \leqslant l \leqslant \varepsilon_j(\eta) = -m_j^\eta$, let $\sigma^{(l)}$ be the unique point in $[0, 1]$ such that $H_j^\eta(\sigma^{(l)}) = m_j^\eta + l$; observe that $0 = \sigma^{(\varepsilon_j(\eta))} < \sigma^{(\varepsilon_j(\eta)-1)} < \cdots < \sigma^{(0)} = 1$. Now it is easily shown by induction on l that, for $0 \leqslant l \leqslant \varepsilon_j(\eta)$,

$$(e_j^l\eta)(t) = \begin{cases} \eta(t) & \text{if } 0 \leqslant t \leqslant \sigma^{(l)}, \\ \eta(\sigma^{(l)}) + r_j(\eta(t) - \eta(\sigma^{(l)})) & \text{if } \sigma^{(l)} \leqslant t \leqslant 1. \end{cases}$$

In particular, by taking $l = \varepsilon_j(\eta)$, we have

$$(e_j^{\max}\eta)(t) = r_j(\eta(t)) \quad \text{for } t \in [0, 1],$$

which implies that $e_j^{\max}\eta = (r_j(\mu_1), r_j(\mu_2); 0, \sigma, 1)$, since $\eta = (\mu_1, \mu_2; 0, \sigma, 1)$. This completes the proof of the lemma. $\qquad\square$

LEMMA 4.2.3. *Let* $\lambda \in P_+^0$. *Let* $\eta = (\mu_1, \mu_2, \ldots, \mu_s; \sigma_0, \sigma_1, \sigma_2, \ldots, \sigma_s)$ *be an expression of* $\eta \in \mathbb{B}(\lambda)_{\mathrm{cl}}$, *and assume that* $s \geqslant 2$.

(1) *The element* $\eta' := (\mu_1, \mu_2; \sigma_0, \sigma_1, \sigma_s)$ *is contained in* $\mathbb{B}(\lambda)_{\mathrm{cl}}$.

(2) *If* $j \in I$ *is such that* $\mu_1(h_j) < 0$, *then* $(e_j^{\max}\eta)(t) = (e_j^{\max}\eta')(t)$ *for all* $t \in [0, \sigma_2]$. *Hence, by Lemma 4.2.2,* $e_j^{\max}\eta \in \mathbb{B}(\lambda)_{\mathrm{cl}}$ *has an expression of the form*

$$e_j^{\max}\eta = (r_j\mu_1, \mu_2', \mu_3', \ldots, \mu_{s'}'; \sigma_0, \sigma_1, \sigma_2, \sigma_3', \ldots, \sigma_{s'}'), \tag{4.2.2}$$

where

$$\mu_2' = \begin{cases} \mu_2 & \text{if } \mu_2(h_j) \geqslant 0, \\ r_j\mu_2 & \text{if } \mu_2(h_j) < 0. \end{cases} \tag{4.2.3}$$

Proof. (1) Let $\pi \in \mathbb{B}(\lambda)$ be such that $\mathrm{cl}(\pi) = \eta$, and let

$$\pi = (\nu_1, \nu_2, \ldots, \nu_{s''}; \sigma_0'', \sigma_1'', \sigma_2'', \ldots, \sigma_{s''}'')$$

be an expression of π. By 'inserting' (see [NS08, Remark 2.5.2(2)]) σ_1 (respectively, σ_2) between σ_k'' and σ_{k+1}'' such that $\sigma_k'' < \sigma_1$ (respectively, $\sigma_2) < \sigma_{k+1}''$ if necessary, we may assume that there exist $1 \leqslant u_1 < u_2 \leqslant s''$ such that $\sigma_{u_1}'' = \sigma_1$ and $\sigma_{u_2}'' = \sigma_2$. By Remark 2.2.8 and the condition that $\mathrm{cl}(\pi) = \eta$, we have

$$\begin{aligned} \mathrm{cl}(\nu_u) &= \mu_1 \quad \text{for all } 1 \leqslant u \leqslant u_1, \\ \mathrm{cl}(\nu_u) &= \mu_2 \quad \text{for all } u_1 + 1 \leqslant u \leqslant u_2. \end{aligned} \tag{4.2.4}$$

Set $\pi' := (\nu_1, \nu_2, \ldots, \nu_{u_2}; \sigma_0'', \sigma_1'', \ldots, \sigma_{u_2-1}'', \sigma_{s''}'')$. Then we can easily deduce from the definition of LS paths (see also [Lit95, Lemma 4.5b)]) that $\pi' \in \mathbb{B}(\lambda)$. Furthermore, it is clear from Remark 2.2.8 and (4.2.4) that $\mathrm{cl}(\pi') = \eta'$. Thus we have proved that $\eta' \in \mathbb{B}(\lambda)_{\mathrm{cl}}$.

(2) For $0 \leqslant l \leqslant \varepsilon_j(\eta) - 1$, we set $\eta_l := e_j^l\eta \in \mathbb{B}(\lambda)_{\mathrm{cl}}$, and

$$t_1^{(l)} := \min\{t \in [0, 1] \mid H_j^{\eta_l}(t) = m_j^{\eta_l}\}, \quad t_0^{(l)} := \max\{t \in [0, t_1^l] \mid H_j^{\eta_l}(t) = m_j^{\eta_l} + 1\}.$$

By (2.2.11), we have

$$m_j^{\eta_l} = -\varepsilon_j(\eta_l) = -\varepsilon_j(e_j^l\eta) = -\varepsilon_j(\eta) + l = m_j^\eta + l \quad \text{for } 0 \leqslant l \leqslant \varepsilon_j(\eta) - 1. \tag{4.2.5}$$

1541

Now, let us assume that $\mu_2(h_j) \geqslant 0$. Since $\mu_1(h_j) < 0$ by the assumption of the lemma, it follows that $k_1 := H_j^\eta(\sigma_1) < 0$. Also, since $\mu_2(h_j) \geqslant 0$, we see that the function $H_j^\eta(t)$, $t \in [0,1]$, attains a local minimum at $t = \sigma_1$. Therefore, we obtain $k_1 \in \mathbb{Z}_{<0}$ by Remark 2.2.9. Observe that

$$H_j^\eta(t) \geqslant k_1 \quad \text{for all } t \in [0, \sigma_2]. \tag{4.2.6}$$

We set $l_1 := k_1 - m_j^\eta \in \mathbb{Z}_{\geqslant 0}$; note that $l_1 < -m_j^\eta = \varepsilon_j(\eta)$ since $k_1 \in \mathbb{Z}_{<0}$.

CLAIM. *For all $0 \leqslant l \leqslant l_1$, we have $\eta_l(t) = \eta(t)$ for all $t \in [0, \sigma_2]$.*

Proof of Claim. We show the assertion by induction on l. When $l = 0$, the assertion obviously holds. Assume that $0 < l \leqslant l_1$ and that $\eta_{l-1}(t) = \eta(t)$ for all $t \in [0, \sigma_2]$. Note that $\eta_l = e_j \eta_{l-1}$. Therefore, by the definition of the root operator e_j, it suffices to show that $\sigma_2 \leqslant t_0^{(l-1)}$. We see from (4.2.5) that

$$H_j^{\eta_{l-1}}(t_1^{(l-1)}) = m_j^{\eta_{l-1}} = m_j^\eta + l - 1 \leqslant m_j^\eta + l_1 - 1 = k_1 - 1. \tag{4.2.7}$$

Also, since $\eta_{l-1}(t) = \eta(t)$ for all $t \in [0, \sigma_2]$ by the induction hypothesis, it follows from (4.2.6) that $H_j^{\eta_{l-1}}(t) = H_j^\eta(t) \geqslant k_1$ for all $t \in [0, \sigma_2]$. Hence we deduce from (4.2.7) that $t_1^{(l-1)} \notin [0, \sigma_2]$, i.e. that $\sigma_2 < t_1^{(l-1)}$.

We have $H_j^{\eta_{l-1}}(\sigma_2) = H_j^\eta(\sigma_2) \geqslant k_1 = m_j^\eta + l_1 \geqslant m_j^\eta + l = m_j^{\eta_{l-1}} + 1$ as seen above, and $H_j^{\eta_{l-1}}(t_1^{(l-1)}) = m_j^{\eta_{l-1}} < m_j^{\eta_{l-1}} + 1$. Therefore, from the continuity of the function $H_j^{\eta_{l-1}}(t)$ in the interval $\sigma_2 \leqslant t \leqslant t_1^{(l-1)}$, we conclude that there exists $\sigma_2 \leqslant t' < t_1^{(l-1)}$ such that $H_j^{\eta_{l-1}}(t') = m_j^{\eta_{l-1}} + 1$. Hence it follows from the definition of $t_0^{(l-1)}$ that $t_0^{(l-1)} \geqslant t' \geqslant \sigma_2$. This proves the claim. $\square$

From the claim above, by taking $l = l_1$, we obtain $\eta_{l_1}(t) = \eta(t)$ for all $t \in [0, \sigma_2]$. Consequently, we see from the definition of $t_1^{(l_1)}$ that $t_1^{(l_1)} = \sigma_1$, since $m_j^{\eta_{l_1}} = m_j^\eta + l_1 = k_1$ and $H_j^\eta(\sigma_1) = k_1$. Therefore, as in the proof of Lemma 4.2.2, we can show (using $\eta(t) = \eta_{l_1}(t)$ for $t \in [0, \sigma_2]$) that

$$(e_j^{\max}\eta)(t) = (e_j^{-m_j^\eta}\eta)(t) = (e_j^{l_1 - k_1}\eta)(t) = (e_j^{-k_1}\eta_{l_1})(t)$$
$$= \begin{cases} r_j(\eta(t)) & \text{if } 0 \leqslant t \leqslant \sigma_1, \\ \eta(t) - k_1\alpha_j & \text{if } \sigma_1 \leqslant t \leqslant \sigma_2, \\ \eta_{l_1}(t) - k_1\alpha_j & \text{if } \sigma_2 \leqslant t \leqslant 1. \end{cases}$$

From this and Lemma 4.2.2, we conclude that $(e_j^{\max}\eta)(t) = (e_j^{\max}\eta')(t)$ for $t \in [0, \sigma_2]$.

The proof for the case $\mu_2(h_j) < 0$ is similar; we give only a sketch of the proof. Take the largest $u \in \{2, 3, \ldots, s\}$ such that $\mu_{u'}(h_j) < 0$ for all $1 \leqslant u' \leqslant u$. Then we see that the function $H_j^\eta(t)$, $t \in [0,1]$, attains a local minimum at $t = \sigma_u$, and hence that $k_u := H_j^\eta(\sigma_u) \in \mathbb{Z}_{<0}$ by Remark 2.2.9. We set $l_u := k_u - m_j^\eta \in \mathbb{Z}_{\geqslant 0}$. Exactly in the same way as above, we can show that $\eta(t) = \eta_{l_u}(t)$ for all $t \in [0, \sigma_u]$. Consequently, we have $t_1^{(l_u)} = \sigma_u$, since $\mu_{u'}(h_j) < 0$ for all $1 \leqslant u' \leqslant u$. Therefore, as in the proof of Lemma 4.2.2, we can show (using $\eta(t) = \eta_{l_u}(t)$ for $t \in [0, \sigma_u]$) that

$$(e_j^{\max}\eta)(t) = (e_j^{-k_u}\eta_{l_u})(t) = \begin{cases} r_j(\eta(t)) & \text{if } 0 \leqslant t \leqslant \sigma_u, \\ \eta_{l_u}(t) - k_u\alpha_j & \text{if } \sigma_u \leqslant t \leqslant 1. \end{cases}$$

From this and Lemma 4.2.2, we conclude that $(e_j^{\max}\eta)(t) = (e_j^{\max}\eta')(t)$ for $t \in [0, \sigma_2]$. This completes the proof of the lemma. $\square$

4.3 Key proposition to the proof of Theorem 4.1.2

The following proposition plays a key role in the proof of Theorem 4.1.2.

1542

PROPOSITION 4.3.1. *Let $\lambda \in P_+^0$, and let $\eta \in \mathbb{B}(\lambda)_{\mathrm{cl}}$ be an element of $\mathbb{B}(\lambda)_{\mathrm{cl}}$ such that $e_j\eta = \mathbf{0}$ for all $j \in I_0$. Then, there exists a sequence $j_1, j_2, \ldots, j_N \in I$ such that:*

(A) $e_{j_N}^{\max} e_{j_{N-1}}^{\max} \cdots e_{j_1}^{\max}\eta = \eta_{\mathrm{cl}(\lambda)}$;

(B) $r_{j_1} r_{j_2} \cdots r_{j_p}(\alpha_{j_{p+1}}) \in -a_0^{-1}\mathring{Q}_+ + a_0^{-1}\mathbb{Z}\delta$ *for $p = 0, 1, \ldots, N-1$.*

In order to prove Proposition 4.3.1, we need Lemmas 4.3.2 and 4.3.3 below.

LEMMA 4.3.2. *Let $\lambda \in \sum_{i \in I_0} \mathbb{Z}_{>0}\varpi_i$ be a strictly I_0-dominant integral weight. Let η be an element of $\mathbb{B}(\lambda)_{\mathrm{cl}}$ such that $\iota(\eta) = \mathrm{cl}(\lambda)$. Then, there exists a sequence $j_1, j_2, \ldots, j_N \in I$ such that:*

(A) $e_{j_N}^{\max} e_{j_{N-1}}^{\max} \cdots e_{j_1}^{\max}\eta = \eta_{\mathrm{cl}(\lambda)}$;

(B') $(r_{j_p} r_{j_{p-1}} \cdots r_{j_1}(\mathrm{cl}(\lambda)))(h_{j_{p+1}}) < 0$ *for $p = 0, 1, \ldots, N-1$.*

Proof. The crucial point in the proof is to consider the special case of $\eta \in \mathbb{B}(\lambda)_{\mathrm{cl}}$ having an expression of the form

$$\eta = (\mathrm{cl}(\lambda), \mu; 0, \sigma, 1), \quad \text{with } \mu \in \mathrm{cl}(W\lambda) = \mathring{W}\mathrm{cl}(\lambda) \text{ and } 0 < \sigma < 1, \quad (4.3.1)$$

and prove the assertion of the lemma in this case. Before doing this, we show how to deduce the general case from the special case (4.3.1). By Remark 2.3.1, the set of all elements of $[0,1]$ appearing as the σ_u in the reduced expression $(\underline{\nu}; \sigma_0, \sigma_1, \ldots, \sigma_s)$ of some element of $\mathbb{B}(\lambda)_{\mathrm{cl}}$ is a finite set. Let $\eta = (\mathrm{cl}(\lambda), \mu_2, \ldots, \mu_s; \sigma_0, \sigma_1, \sigma_2, \ldots, \sigma_s)$ be the reduced expression of η. Using the finiteness above, we show the assertion of Lemma 4.3.2 by descending induction on σ_1 in the reduced expression $(\mathrm{cl}(\lambda), \mu_2, \ldots, \mu_s; \sigma_0, \sigma_1, \sigma_2, \ldots, \sigma_s)$ of $\eta \in \mathbb{B}(\lambda)_{\mathrm{cl}}$. When $\sigma_1 = 1$, we have $\eta = \eta_{\mathrm{cl}(\lambda)}$, since $\iota(\eta) = \mathrm{cl}(\lambda)$ by the assumption of the lemma. Thus the assertion obviously holds. Assume that $\sigma_1 < 1$, or equivalently $s \geqslant 2$. Set $\eta' := (\mathrm{cl}(\lambda), \mu_2; \sigma_0, \sigma_1, \sigma_s)$; note that $\eta' \in \mathbb{B}(\lambda)_{\mathrm{cl}}$ by Lemma 4.2.3(1). It follows from the assertion in the special case (4.3.1) that there exists a sequence $j_1, j_2, \ldots, j_{N'} \in I$ satisfying conditions (A) and (B') for η'. We set $\eta'' := e_{j_{N'}}^{\max} e_{j_{N'-1}}^{\max} \cdots e_{j_1}^{\max}\eta$. Then, repeated use of Lemma 4.2.3(2) shows that

$$\eta''(t) = (e_{j_{N'}}^{\max} e_{j_{N'-1}}^{\max} \cdots e_{j_1}^{\max}\eta')(t) = \eta_{\mathrm{cl}(\lambda)}(t) = t\,\mathrm{cl}(\lambda) \quad \text{for } t \in [0, \sigma_2]. \quad (4.3.2)$$

Hence the initial direction $\iota(\eta'')$, which equals $r_{j_{N'}} r_{j_{N'-1}} \cdots r_{j_1}(\mathrm{cl}(\lambda))$ by Lemma 4.2.3(2), must be equal to $\mathrm{cl}(\lambda)$. Therefore, if $\eta'' = (\mathrm{cl}(\lambda), \mu'_2, \ldots, \mu'_{s'}; \sigma'_0, \sigma'_1, \sigma'_2, \ldots, \sigma'_{s'})$ is the reduced expression of η'', then we see from (4.3.2) that $\sigma'_1 \geqslant \sigma_2 > \sigma_1$. Hence, by the induction hypothesis, there exists a sequence $j_{N'+1}, j_{N'+2}, \ldots, j_N \in I$ satisfying conditions (A) and (B') for η''. Thus we obtain a sequence

$$j_1, j_2, \ldots, j_{N'}, j_{N'+1}, j_{N'+2}, \ldots, j_N \in I$$

satisfying conditions (A) and (B') for η (note that $r_{j_{N'}} r_{j_{N'-1}} \cdots r_{j_1}(\mathrm{cl}(\lambda)) = \iota(\eta'') = \mathrm{cl}(\lambda)$). This proves the assertion of the lemma in the general case.

Now, it remains to prove the assertion of the lemma in the special case (4.3.1). Since $\mathbb{B}(\lambda)_{\mathrm{cl}}$ is a finite set (see Remark 2.3.1), it follows that $\max\{\|\eta'\| \mid \eta' \in \mathbb{B}(\lambda)_{\mathrm{cl}}\} < \infty$, where $\|\eta\|$ is as defined in (2.3.1); recall that $\mathrm{wt}(\eta)$ is given by $\mathrm{wt}(\eta) = \eta(1) \in P_{\mathrm{cl}}$. We will show the assertion by descending induction on $\|\eta\|$. When $\|\eta\| = \max\{\|\eta'\| \mid \eta' \in \mathbb{B}(\lambda)_{\mathrm{cl}}\}$, it follows from Lemma 2.3.6 that η is an extremal element. Since the initial direction $\iota(\eta)$ of η is equal to $\mathrm{cl}(\lambda)$ by assumption, we deduce from Remark 2.4.3 that $\eta = \eta_{\mathrm{cl}(\lambda)}$ (hence there is nothing to prove).

Assume that $\|\eta\| < \max\{\|\eta'\| \mid \eta' \in \mathbb{B}(\lambda)_{\mathrm{cl}}\}$. Set $\Lambda := w_0\lambda$, where $w_0 \in \mathring{W}$ is the longest element of $\mathring{W}$. Then it is easy to check that $\Lambda(h_j) \in \mathbb{Z}_{<0}$ for all $j \in I_0$. Note that $\mu \in \mathrm{cl}(W\lambda) = \mathring{W}\mathrm{cl}(\lambda)$ satisfies the condition that $\mu(h_j) \leqslant 0$ for all $j \in I_0$ if and only if $\mu = \mathrm{cl}(\Lambda)$ (see Remark 2.1.5).

We see from [AK97, Lemma 1.4] that there exists a sequence $j_1, j_2, \ldots, j_{N'} \in I$ such that:

(a) $r_{j_{N'}} r_{j_{N'-1}} \cdots r_{j_1}(\mathrm{cl}(\lambda)) = \mathrm{cl}(\Lambda)$;

(b) $(r_{j_p} r_{j_{p-1}} \cdots r_{j_1}(\mathrm{cl}(\lambda)))(h_{j_{p+1}}) < 0$ for $p = 0, 1, \ldots, N'-1$.

Set $\eta' := e_{j_{N'}}^{\max} e_{j_{N'-1}}^{\max} \cdots e_{j_1}^{\max} \eta \in \mathbb{B}(\lambda)_{\mathrm{cl}}$. Then it follows from Lemma 2.3.3 that

$$\|\eta'\| \geqslant \|\eta\|. \tag{4.3.3}$$

Furthermore, from (b), we deduce by repeated application of Lemma 4.2.2 that

$$e_{j_{N'}}^{\max} e_{j_{N'-1}}^{\max} \cdots e_{j_1}^{\max} \eta = (r_{j_{N'}} r_{j_{N'-1}} \cdots r_{j_1}(\mathrm{cl}(\lambda)), \mu'; 0, \sigma, 1) \quad \text{for some } \mu' \in \mathrm{cl}(W\lambda),$$

and hence from (a) that

$$\begin{aligned}
\eta' = e_{j_{N'}}^{\max} e_{j_{N'-1}}^{\max} \cdots e_{j_1}^{\max} \eta &= (r_{j_{N'}} r_{j_{N'-1}} \cdots r_{j_1}(\mathrm{cl}(\lambda)), \mu'; 0, \sigma, 1) \\
&= (\mathrm{cl}(\Lambda), \mu'; 0, \sigma, 1).
\end{aligned}$$

Case 1: $\mu' = \mathrm{cl}(\Lambda)$. In this case, we have $\eta' = (\mathrm{cl}(\Lambda), \mathrm{cl}(\Lambda); 0, \sigma, 1) = \eta_{\mathrm{cl}(\Lambda)}$. Let $w_0 = r_{j_{N'+1}} r_{j_{N'+2}} \cdots r_{j_{N-1}} r_{j_N}$ be a reduced expression of $w_0 \in \mathring{W}$; note that $j_{N'+1}, j_{N'+2}, \ldots, j_{N-1}, j_N \in I_0$. Then, since $\Lambda = w_0 \lambda$, it follows that

(c) $r_{j_N} r_{j_{N-1}} \cdots r_{j_{N'+1}}(\mathrm{cl}(\Lambda)) = \mathrm{cl}(\lambda)$.

In addition, using [Kac90, Lemma 3.11b)], we obtain

(d) $(r_{j_p} r_{j_{p-1}} \cdots r_{j_{N'+1}}(\mathrm{cl}(\Lambda)))(h_{j_{p+1}}) < 0$ for all $p = N', N' + 1, \ldots, N - 1$.

From (c) and (d), we see by repeated use of Lemma 4.2.2 that

$$\begin{aligned}
e_{j_N}^{\max} e_{j_{N-1}}^{\max} \cdots e_{j_{N'+1}}^{\max} e_{j_{N'}}^{\max} e_{j_{N'-1}}^{\max} \cdots e_{j_1}^{\max} \eta &= e_{j_N}^{\max} e_{j_{N-1}}^{\max} \cdots e_{j_{N'+1}}^{\max} \eta' \\
&= e_{j_N}^{\max} e_{j_{N-1}}^{\max} \cdots e_{j_{N'+1}}^{\max} (\mathrm{cl}(\Lambda), \mathrm{cl}(\Lambda); 0, \sigma, 1) = (\mathrm{cl}(\lambda), \mathrm{cl}(\lambda); 0, \sigma, 1) = \eta_{\mathrm{cl}(\lambda)}.
\end{aligned}$$

Therefore, the sequence $j_1, j_2, \ldots, j_N \in I$ satisfies condition (A). Also, it follows from (a), (b), and (d) that $(r_{j_p} r_{j_{p-1}} \cdots r_{j_1}(\mathrm{cl}(\lambda)))(h_{j_{p+1}}) < 0$ for all $p = 0, 1, \ldots, N - 1$, and hence that the sequence $j_1, j_2, \ldots, j_N \in I$ satisfies condition (B').

Case 2: $\mu' \neq \mathrm{cl}(\Lambda)$. In this case, we can take (and fix) $j_{N'+1} \in I_0$ such that $\mu'(h_{j_{N'+1}}) > 0$. Then, there exists a reduced expression of $w_0 \in \mathring{W}$ of the form $w_0 = r_{j_{N'+1}} r_{j_{N'+2}} \cdots r_{j_{N''}}$; note that $j_{N'+1}, j_{N'+2}, \ldots, j_{N''} \in I_0$. Since $\Lambda = w_0 \lambda$, it follows that

(e) $r_{j_{N''}} r_{j_{N''-1}} \cdots r_{j_{N'+1}}(\mathrm{cl}(\Lambda)) = \mathrm{cl}(\lambda)$.

In addition, using [Kac90, Lemma 3.11b)], we obtain

(f) $(r_{j_p} r_{j_{p-1}} \cdots r_{j_{N'+1}}(\mathrm{cl}(\Lambda)))(h_{j_{p+1}}) < 0$ all for $p = N', N' + 1, \ldots, N'' - 1$.

If we set $\eta'' := e_{j_{N''}}^{\max} e_{j_{N''-1}}^{\max} \cdots e_{j_{N'+1}}^{\max} \eta'$, then $\|\eta''\| > \|\eta\|$. Indeed, since $(\mathrm{cl}(\Lambda))(h_{j_{N'+1}}) < 0$ by (f) with $p = N'$, it follows from Lemma 2.2.11(1) that $e_{j_{N'+1}} \eta' \neq \mathbf{0}$. Also, since $\mu'(h_{j_{N'+1}}) > 0$, it follows from Lemma 2.2.13 that $f_{j_{N'+1}} \eta' \neq \mathbf{0}$. Therefore, we obtain from Lemma 2.3.3

$$\|e_{j_{N'+1}}^{\max} \eta'\| > \|\eta'\|. \tag{4.3.4}$$

Hence we have

$$\begin{aligned}
\|\eta''\| &\geqslant \|e_{j_{N'+1}}^{\max} \eta'\| \quad \text{by Lemma 2.3.3} \\
&> \|\eta'\| \quad \text{by (4.3.4)} \\
&\geqslant \|\eta\| \quad \text{by (4.3.3)}.
\end{aligned}$$

Furthermore, from (e) and (f), we see, by applying Lemma 4.2.2 successively, that

$$\begin{aligned}
\eta'' = e_{j_{N''}}^{\max} e_{j_{N''-1}}^{\max} \cdots e_{j_{N'+1}}^{\max} \eta' &= (r_{j_{N''}} r_{j_{N''-1}} \cdots r_{j_{N'+1}}(\mathrm{cl}(\Lambda)), \mu''; 0, \sigma, 1) \\
&= (\mathrm{cl}(\lambda), \mu''; 0, \sigma, 1)
\end{aligned}$$

for some $\mu'' \in \mathrm{cl}(W\lambda)$. Therefore, by the induction hypothesis, there exists a sequence $j_{N''+1}$, $j_{N''+2}, \ldots, j_N \in I$ satisfying conditions (A) and (B$'$) for η''. It is easily checked by (a), (b), (e), (f), and the induction hypothesis that the sequence

$$j_1, j_2, \ldots, j_{N'}, j_{N'+1}, j_{N'+2}, \ldots, j_{N''}, j_{N''+1}, j_{N''+2}, \ldots, j_N \in I$$

satisfies conditions (A) and (B$'$) for η. This proves the assertion in the special case (4.3.1), and hence completes the proof of the lemma. $\qquad\square$

If $\lambda \in \sum_{i \in I_0} \mathbb{Z}_{>0} \varpi_i$ is strictly I_0-dominant, then we see from Lemma 4.2.1(3) that condition (B$'$) of Lemma 4.3.2 can be replaced by condition (B) of Proposition 4.3.1. Namely, we have the following result.

LEMMA 4.3.3. *Let* $\lambda \in \sum_{i \in I_0} \mathbb{Z}_{>0} \varpi_i$ *be a strictly I_0-dominant integral weight. Let η be an element of $\mathbb{B}(\lambda)_{\mathrm{cl}}$ such that $\iota(\eta) = \mathrm{cl}(\lambda)$. Then, there exists a sequence $j_1, j_2, \ldots, j_N \in I$ satisfying conditions (A) and (B) of Proposition 4.3.1.*

Finally, let us give a proof of Proposition 4.3.1.

Proof of Proposition 4.3.1. We set $\rho := \sum_{i \in I_0} \varpi_i \in P_+^0$; note that $\rho \in P_+^0$ is strictly I_0-dominant, and hence so is $\lambda + \rho \in \sum_{i \in I} \mathbb{Z}_{>0} \varpi_i$. Since $e_j \eta = \mathbf{0}$ for all $j \in I_0$ by assumption and $e_j \eta_{\mathrm{cl}(\rho)} = \mathbf{0}$ for all $j \in I_0$ by the definition of the root operators e_j, we see from the tensor product rule for crystals that $\eta \otimes \eta_{\mathrm{cl}(\rho)} \in \mathbb{B}(\lambda)_{\mathrm{cl}} \otimes \mathbb{B}(\rho)_{\mathrm{cl}}$ also satisfies the condition that $e_j(\eta \otimes \eta_{\mathrm{cl}(\rho)}) = \mathbf{0}$ for all $j \in I_0$. Recall from Corollary 2.4.5(1) that there exists an isomorphism $\Psi_{\lambda,\rho} : \mathbb{B}(\lambda + \rho)_{\mathrm{cl}} \xrightarrow{\sim} \mathbb{B}(\lambda)_{\mathrm{cl}} \otimes \mathbb{B}(\rho)_{\mathrm{cl}}$ of P_{cl}-crystals. Set $\eta' := \Psi_{\lambda,\rho}^{-1}(\eta \otimes \eta_{\mathrm{cl}(\rho)}) \in \mathbb{B}(\lambda + \rho)_{\mathrm{cl}}$. Then, clearly $e_j \eta' = \mathbf{0}$ for all $j \in I_0$. Hence it follows from Lemma 2.2.11(1) (see also Remark 2.2.8) that the initial direction $\iota(\eta') \in \mathrm{cl}(W(\lambda + \rho)) = \mathring{W} \mathrm{cl}(\lambda + \rho)$ of η' is level-zero and I_0-dominant, and hence from Remark 2.1.5 that $\iota(\eta') = \mathrm{cl}(\lambda + \rho)$. Since $\lambda + \rho \in \sum_{i \in I} \mathbb{Z}_{>0} \varpi_i$ is strictly I_0-dominant, we know from Lemma 4.3.3 that there exists a sequence $j_1, j_2, \ldots, j_N \in I$ satisfying conditions (A) and (B) for η'. Now, it remains to show that $e_{j_N}^{\max} e_{j_{N-1}}^{\max} \cdots e_{j_1}^{\max} \eta = \eta_{\mathrm{cl}(\lambda)}$, i.e. that the sequence $j_1, j_2, \ldots, j_N \in I$ also satisfies condition (A) for η. We have

$$
\begin{aligned}
e_{j_N}^{\max} e_{j_{N-1}}^{\max} \cdots e_{j_1}^{\max}(\eta \otimes \eta_{\mathrm{cl}(\rho)}) &= e_{j_N}^{\max} e_{j_{N-1}}^{\max} \cdots e_{j_1}^{\max}(\Psi_{\lambda,\rho}(\eta')) \\
&= \Psi_{\lambda,\rho}(e_{j_N}^{\max} e_{j_{N-1}}^{\max} \cdots e_{j_1}^{\max} \eta') = \Psi_{\lambda,\rho}(\eta_{\mathrm{cl}(\lambda+\rho)}) \quad \text{by condition (A) for } \eta' \\
&= \eta_{\mathrm{cl}(\lambda)} \otimes \eta_{\mathrm{cl}(\rho)} \quad \text{by (2.4.3)}.
\end{aligned}
$$

Also, we see from Lemma 2.3.2(2) that

$$e_{j_N}^{\max} e_{j_{N-1}}^{\max} \cdots e_{j_1}^{\max}(\eta \otimes \eta_{\mathrm{cl}(\rho)}) = (e_{j_N}^{\max} e_{j_{N-1}}^{\max} \cdots e_{j_1}^{\max} \eta) \otimes \eta'' \quad \text{for some } \eta'' \in \mathbb{B}(\rho)_{\mathrm{cl}}.$$

Thus, we obtain $e_{j_N}^{\max} e_{j_{N-1}}^{\max} \cdots e_{j_1}^{\max} \eta = \eta_{\mathrm{cl}(\lambda)}$. This establishes Proposition 4.3.1. $\qquad\square$

4.4 Proof of Theorem 4.1.2

In this (and the next) subsection, the degree function Deg is defined on several different crystals $\mathbb{B}(\mu)_{\mathrm{cl}}$, $\mu \in P_+^0$. Accordingly, we write $\mathrm{Deg}_\mu(\eta)$ instead of $\mathrm{Deg}(\eta)$ for $\eta \in \mathbb{B}(\mu)_{\mathrm{cl}}$ with $\mu \in P_+^0$ to emphasize that η is an element of $\mathbb{B}(\mu)_{\mathrm{cl}}$.

Let $\lambda, \lambda' \in P_+^0$. For $\eta_1 \otimes \eta_2 \in \mathbb{B}(\lambda)_{\mathrm{cl}} \otimes \mathbb{B}(\lambda')_{\mathrm{cl}}$, we define (see Theorem 2.5.1)

$$D_{\lambda,\lambda'}(\eta_1 \otimes \eta_2) := H_{\lambda,\lambda'}(\eta_1 \otimes \eta_2) + \mathrm{Deg}_\lambda(\eta_1) + \mathrm{Deg}_{\lambda'}(\widetilde{\eta}_2), \tag{4.4.1}$$

where $\widetilde{\eta}_2 \in \mathbb{B}(\lambda')_{\mathrm{cl}}$ is defined by $\widetilde{\eta}_2 \otimes \widetilde{\eta}_1 = R_{\lambda,\lambda'}(\eta_1 \otimes \eta_2)$ (see Corollary 2.4.5(2)). From Theorem 2.5.1(H2) and Lemma 3.2.1(1), we deduce by use of (2.4.4) that

$$D_{\lambda,\lambda'}(\eta_{\mathrm{cl}(\lambda)} \otimes \eta_{\mathrm{cl}(\lambda')}) = H_{\lambda,\lambda'}(\eta_{\mathrm{cl}(\lambda)} \otimes \eta_{\mathrm{cl}(\lambda')}) + \mathrm{Deg}_\lambda(\eta_{\mathrm{cl}(\lambda)}) + \mathrm{Deg}_{\lambda'}(\eta_{\mathrm{cl}(\lambda')}) = 0. \tag{4.4.2}$$

LEMMA 4.4.1. *Let* $\lambda, \lambda' \in P_+^0$, *and let* $\eta_1 \otimes \eta_2 \in \mathbb{B}(\lambda)_{\mathrm{cl}} \otimes \mathbb{B}(\lambda')_{\mathrm{cl}}$. *The following hold:*
(1) $D_{\lambda,\lambda'}(e_j^{\max}(\eta_1 \otimes \eta_2)) = D_{\lambda,\lambda'}(\eta_1 \otimes \eta_2)$ *for all* $j \in I_0$;
(2) *if* $(\iota(\eta_1))(h_0) \leqslant 0$ *and* $(\iota(\widetilde{\eta}_2))(h_0) \leqslant 0$, *then*

$$D_{\lambda,\lambda'}(e_0^{\max}(\eta_1 \otimes \eta_2)) = D_{\lambda,\lambda'}(\eta_1 \otimes \eta_2) - (\iota(\eta_1) + \iota(\widetilde{\eta}_2))(h_0) - \varepsilon_0(\eta_1 \otimes \eta_2). \qquad (4.4.3)$$

Proof. Let $j \in I$. We see from Lemma 2.3.2(2) that

$$e_j^{\max}(\eta_1 \otimes \eta_2) = e_j^{\max}\eta_1 \otimes \eta_2' \quad \text{for some } \eta_2' \in \mathbb{B}(\lambda')_{\mathrm{cl}},$$

and hence that

$$\begin{aligned}
R_{\lambda,\lambda'}(e_j^{\max}(\eta_1 \otimes \eta_2)) &= e_j^{\max} R_{\lambda,\lambda'}(\eta_1 \otimes \eta_2) = e_j^{\max}(\widetilde{\eta}_2 \otimes \widetilde{\eta}_1) \\
&= e_j^{\max}\widetilde{\eta}_2 \otimes \eta_1' \quad \text{for some } \eta_1' \in \mathbb{B}(\lambda)_{\mathrm{cl}}.
\end{aligned}$$

Therefore, from the definition (4.4.1) of $D_{\lambda,\lambda'}$, we obtain

$$D_{\lambda,\lambda'}(e_j^{\max}(\eta_1 \otimes \eta_2)) = H_{\lambda,\lambda'}(e_j^{\max}(\eta_1 \otimes \eta_2)) + \mathrm{Deg}_\lambda(e_j^{\max}\eta_1) + \mathrm{Deg}_{\lambda'}(e_j^{\max}\widetilde{\eta}_2). \qquad (4.4.4)$$

From (4.4.4), part (1) follows immediately by Theorem 2.5.1(H1) and Lemma 3.2.1(2). Let us prove part (2). We give a proof only for the case $\varepsilon_0(\eta_1) \geqslant \varepsilon_0(\widetilde{\eta}_2)$; the proof for the case $\varepsilon_0(\eta_1) < \varepsilon_0(\widetilde{\eta}_2)$ is similar. By Lemma 3.2.2,

$$\begin{cases}
\mathrm{Deg}_\lambda(e_0^{\max}\eta_1) = \mathrm{Deg}_\lambda(\eta_1) - \varepsilon_0(\eta_1) - (\iota(\eta_1))(h_0), \\
\mathrm{Deg}_{\lambda'}(e_0^{\max}\widetilde{\eta}_2) = \mathrm{Deg}_{\lambda'}(\widetilde{\eta}_2) - \varepsilon_0(\widetilde{\eta}_2) - (\iota(\widetilde{\eta}_2))(h_0).
\end{cases} \qquad (4.4.5)$$

For simplicity of notation, we set $L := \varepsilon_0(\eta_1 \otimes \eta_2)$; note that $L = \varepsilon_0(\widetilde{\eta}_2 \otimes \widetilde{\eta}_1)$ since $\widetilde{\eta}_2 \otimes \widetilde{\eta}_1 = R_{\lambda,\lambda'}(\eta_1 \otimes \eta_2)$, and that $L \geqslant \varepsilon_0(\eta_1), \varepsilon_0(\widetilde{\eta}_2)$ by Lemma 2.3.2(1). It follows from Lemma 2.3.2(2) that, for $0 \leqslant l \leqslant L$,

$$e_0^l(\eta_1 \otimes \eta_2) = \begin{cases}
\eta_1 \otimes e_0^l\eta_2 & \text{if } 0 \leqslant l \leqslant L - \varepsilon_0(\eta_1), \\
e_0^{l-L+\varepsilon_0(\eta_1)}\eta_1 \otimes e_0^{L-\varepsilon_0(\eta_1)}\eta_2 & \text{if } L - \varepsilon_0(\eta_1) \leqslant l \leqslant L,
\end{cases}$$

and

$$e_0^l(\widetilde{\eta}_2 \otimes \widetilde{\eta}_1) = \begin{cases}
\widetilde{\eta}_2 \otimes e_0^l\widetilde{\eta}_1 & \text{if } 0 \leqslant l \leqslant L - \varepsilon_0(\widetilde{\eta}_2), \\
e_0^{l-L+\varepsilon_0(\widetilde{\eta}_2)}\widetilde{\eta}_2 \otimes e_0^{L-\varepsilon_0(\widetilde{\eta}_2)}\widetilde{\eta}_1 & \text{if } L - \varepsilon_0(\widetilde{\eta}_2) \leqslant l \leqslant L.
\end{cases}$$

Therefore, we deduce from Theorem 2.5.1(H1) that, for $0 \leqslant l' \leqslant L - \varepsilon_0(\eta_1)$,

$$H_{\lambda,\lambda'}(e_0^{l'}(\eta_1 \otimes \eta_2)) = H_{\lambda,\lambda'}(\eta_1 \otimes \eta_2) - l'. \qquad (4.4.6)$$

Similarly, we see from Theorem 2.5.1(H1) that, for $0 \leqslant l' \leqslant \varepsilon_0(\eta_1) - \varepsilon_0(\widetilde{\eta}_2)$,

$$\begin{aligned}
H_{\lambda,\lambda'}(e_0^{L-\varepsilon_0(\eta_1)+l'}(\eta_1 \otimes \eta_2)) &= H_{\lambda,\lambda'}(e_0^{L-\varepsilon_0(\eta_1)}(\eta_1 \otimes \eta_2)) \\
&= H_{\lambda,\lambda'}(\eta_1 \otimes \eta_2) - L + \varepsilon_0(\eta_1) \quad \text{by (4.4.6) with } l' = L - \varepsilon_0(\eta_1),
\end{aligned} \qquad (4.4.7)$$

and that, for $0 \leqslant l' \leqslant \varepsilon_0(\widetilde{\eta}_2)$,

$$\begin{aligned}
H_{\lambda,\lambda'}(e_0^{L-\varepsilon_0(\widetilde{\eta}_2)+l'}(\eta_1 \otimes \eta_2)) &= H_{\lambda,\lambda'}(e_0^{L-\varepsilon_0(\widetilde{\eta}_2)}(\eta_1 \otimes \eta_2)) + l' \\
&= H_{\lambda,\lambda'}(\eta_1 \otimes \eta_2) - L + \varepsilon_0(\eta_1) + l' \quad \text{by (4.4.7) with } l' = \varepsilon_0(\eta_1) - \varepsilon_0(\widetilde{\eta}_2).
\end{aligned} \qquad (4.4.8)$$

Finally, by taking $l' = \varepsilon_0(\widetilde{\eta}_2)$ in (4.4.8), we conclude that

$$\begin{aligned}
H_{\lambda,\lambda'}(e_0^{\max}(\eta_1 \otimes \eta_2)) &= H_{\lambda,\lambda'}(e_0^L(\eta_1 \otimes \eta_2)) \\
&= H_{\lambda,\lambda'}(\eta_1 \otimes \eta_2) - L + \varepsilon_0(\eta_1) + \varepsilon_0(\widetilde{\eta}_2) \\
&= H_{\lambda,\lambda'}(\eta_1 \otimes \eta_2) - \varepsilon_0(\eta_1 \otimes \eta_2) + \varepsilon_0(\eta_1) + \varepsilon_0(\widetilde{\eta}_2).
\end{aligned} \qquad (4.4.9)$$

By substituting (4.4.5) and (4.4.9) into (4.4.4), and then comparing the resulting equation with (4.4.1), we obtain the desired equation (4.4.3). This proves the lemma. $\qquad \square$

PROPOSITION 4.4.2. *Let* $\lambda, \lambda' \in P_+^0$, *and let* $\eta \in \mathbb{B}(\lambda + \lambda')_{\mathrm{cl}}$. *Then,*

$$\mathrm{Deg}_{\lambda+\lambda'}(\eta) = D_{\lambda,\lambda'}(\Psi_{\lambda,\lambda'}(\eta)), \tag{4.4.10}$$

where $\Psi_{\lambda,\lambda'} : \mathbb{B}(\lambda + \lambda')_{\mathrm{cl}} \xrightarrow{\sim} \mathbb{B}(\lambda)_{\mathrm{cl}} \otimes \mathbb{B}(\lambda')_{\mathrm{cl}}$ *is the isomorphism of* P_{cl}*-crystals in Corollary 2.4.5(1).*

Proof. If $j \in I_0$, then it follows from Lemmas 3.2.1(2) and 4.4.1(1) that $\mathrm{Deg}_{\lambda+\lambda'}(e_j^{\max}\eta) = \mathrm{Deg}_{\lambda+\lambda'}(\eta)$ and $D_{\lambda,\lambda'}(\Psi_{\lambda,\lambda'}(e_j^{\max}\eta)) = D_{\lambda,\lambda'}(e_j^{\max}(\Psi_{\lambda,\lambda'}(\eta))) = D_{\lambda,\lambda'}(\Psi_{\lambda,\lambda'}(\eta))$. Therefore, we may assume that $\eta \in \mathbb{B}(\lambda + \lambda')_{\mathrm{cl}}$ satisfies the condition that $e_j\eta = \mathbf{0}$ for all $j \in I_0$, since the P_{cl}-crystal $\mathbb{B}(\lambda + \lambda')_{\mathrm{cl}}$ is regular. Then it follows from Lemma 2.2.11(1) (see also Remark 2.2.8) that $\iota(\eta) \in \mathrm{cl}(W\lambda) = \mathring{W}\,\mathrm{cl}(\lambda)$ is level-zero and I_0-dominant, and hence from Remark 2.1.5 that $\iota(\eta) = \mathrm{cl}(\lambda + \lambda')$. Set $\eta_1 \otimes \eta_2 := \Psi_{\lambda,\lambda'}(\eta) \in \mathbb{B}(\lambda)_{\mathrm{cl}} \otimes \mathbb{B}(\lambda')_{\mathrm{cl}}$, and $\widetilde{\eta}_2 \otimes \widetilde{\eta}_1 := R_{\lambda,\lambda'}(\eta_1 \otimes \eta_2)$. Since $e_j\eta = \mathbf{0}$ for all $j \in I_0$, we have $e_j(\eta_1 \otimes \eta_2) = \mathbf{0}$ and $e_j(\widetilde{\eta}_2 \otimes \widetilde{\eta}_1) = \mathbf{0}$ for all $j \in I_0$, which implies that $e_j\eta_1 = \mathbf{0}$ and $e_j\widetilde{\eta}_2 = \mathbf{0}$ for all $j \in I_0$ by Lemma 2.3.2(1). Hence an argument similar to the above shows that $\iota(\eta_1) = \mathrm{cl}(\lambda)$ and $\iota(\widetilde{\eta}_2) = \mathrm{cl}(\lambda')$.

By Proposition 4.3.1, there exists a sequence $j_1, j_2, \ldots, j_N \in I$ satisfying conditions (A) and (B) for $\eta \in \mathbb{B}(\lambda + \lambda')_{\mathrm{cl}}$. Condition (B) implies that

$$(w^{(p)}(\mathrm{cl}(\lambda + \lambda')))(h_{j_{p+1}}) \leqslant 0 \quad \text{for } p = 0, 1, \ldots, N-1, \tag{4.4.11}$$

by Lemma 4.2.1(2), where we set $w^{(p)} := r_{j_p} r_{j_{p-1}} \cdots r_{j_1}$ for $0 \leqslant p \leqslant N$. Therefore, we see from Lemma 2.2.12 that

$$\iota(E^{(p)}\eta) = w^{(p)}(\mathrm{cl}(\lambda + \lambda')) \quad \text{for } p = 0, 1, \ldots, N, \tag{4.4.12}$$

where we set $E^{(p)} := e_{j_p}^{\max} e_{j_{p-1}}^{\max} \cdots e_{j_1}^{\max}$ for $0 \leqslant p \leqslant N$. Because $\iota(\eta_1) = \mathrm{cl}(\lambda)$ and $\iota(\widetilde{\eta}_2) = \mathrm{cl}(\lambda')$, an argument similar to the above shows that

$$\begin{cases} \iota(E^{(p)}\eta_1) = w^{(p)}(\mathrm{cl}(\lambda)) & \text{for } p = 0, 1, \ldots, N, \\ (w^{(p)}(\mathrm{cl}(\lambda)))(h_{j_{p+1}}) \leqslant 0 & \text{for } p = 0, 1, \ldots, N-1, \end{cases} \tag{4.4.13}$$

and

$$\begin{cases} \iota(E^{(p)}\widetilde{\eta}_2) = w^{(p)}(\mathrm{cl}(\lambda')) & \text{for } p = 0, 1, \ldots, N, \\ (w^{(p)}(\mathrm{cl}(\lambda')))(h_{j_{p+1}}) \leqslant 0 & \text{for } p = 0, 1, \ldots, N-1. \end{cases} \tag{4.4.14}$$

Now we have

$$\begin{aligned}
0 &= \mathrm{Deg}_{\lambda+\lambda'}(\eta_{\mathrm{cl}(\lambda+\lambda')}) \quad \text{by Lemma 3.2.1(1)} \\
&= \mathrm{Deg}_{\lambda+\lambda'}(e_{j_N}^{\max} e_{j_{N-1}}^{\max} \cdots e_{j_1}^{\max}\eta) \quad \text{by condition (A) for } \eta \\
&= \mathrm{Deg}_{\lambda+\lambda'}(\eta) - \sum_{1 \leqslant p \leqslant N; j_p = 0} \{\varepsilon_0(E^{(p-1)}\eta) + (w^{(p-1)}(\mathrm{cl}(\lambda + \lambda')))(h_{j_p})\}
\end{aligned}$$

by Lemma 3.2.2 along with (4.4.11) and (4.4.12),

and hence

$$\mathrm{Deg}_{\lambda+\lambda'}(\eta) = \sum_{1 \leqslant p \leqslant N; j_p = 0} \{\varepsilon_0(E^{(p-1)}\eta) + (w^{(p-1)}(\mathrm{cl}(\lambda + \lambda')))(h_{j_p})\}. \tag{4.4.15}$$

Here, by Lemma 2.3.2(2),

$$E^{(p)}(\eta_1 \otimes \eta_2) = E^{(p)}\eta_1 \otimes \eta_2' \quad \text{for some } \eta_2' \in \mathbb{B}(\lambda')_{\mathrm{cl}},$$

$$\begin{aligned}
R_{\lambda,\lambda'}(E^{(p)}(\eta_1 \otimes \eta_2)) &= E^{(p)}(R_{\lambda,\lambda'}(\eta_1 \otimes \eta_2)) = E^{(p)}(\widetilde{\eta}_2 \otimes \widetilde{\eta}_1) \\
&= E^{(p)}\widetilde{\eta}_2 \otimes \eta_1' \quad \text{for some } \eta_1' \in \mathbb{B}(\lambda)_{\mathrm{cl}}.
\end{aligned}$$

1547

S. Naito and D. Sagaki

Therefore, we see that

$$
\begin{aligned}
0 &= D_{\lambda,\lambda'}(\eta_{\mathrm{cl}(\lambda)} \otimes \eta_{\mathrm{cl}(\lambda')}) \quad \text{by (4.4.2)} \\
&= D_{\lambda,\lambda'}(e_{j_N}^{\max} e_{j_{N-1}}^{\max} \cdots e_{j_1}^{\max}(\eta_1 \otimes \eta_2)) \quad \text{by condition (A) for } \eta \text{ and (2.4.3)} \\
&= D_{\lambda,\lambda'}(\eta_1 \otimes \eta_2) - \sum_{1 \leqslant p \leqslant N; j_p=0} \{\varepsilon_0(E^{(p-1)}(\eta_1 \otimes \eta_2)) + (w^{(p-1)}(\mathrm{cl}(\lambda + \lambda')))(h_{j_p})\}
\end{aligned}
$$

by Lemma 4.4.1 along with (4.4.13) and (4.4.14).

Also, since $\Psi_{\lambda,\lambda'}(E^{(p)}\eta) = E^{(p)}\Psi_{\lambda,\lambda'}(\eta) = E^{(p)}(\eta_1 \otimes \eta_2)$, it follows that $\varepsilon_0(E^{(p)}\eta) = \varepsilon_0(E^{(p)}(\eta_1 \otimes \eta_2))$ for all $p = 0, 1, \ldots, N$. Thus we obtain

$$
D_{\lambda,\lambda'}(\eta_1 \otimes \eta_2) = \sum_{1 \leqslant p \leqslant N; j_p=0} \{\varepsilon_0(E^{(p-1)}\eta) + (w^{(p-1)}(\mathrm{cl}(\lambda + \lambda')))(h_{j_p})\}. \tag{4.4.16}
$$

Equation (4.4.10) follows immediately from (4.4.15) and (4.4.16). This completes the proof of the proposition. □

Now we are ready to prove Theorem 4.1.2.

Proof of Theorem 4.1.2. We proceed by induction on the length n of the sequence $\mathbf{i} = (i_1, i_2, \ldots, i_n)$. When $n = 1$, the assertion obviously holds. Assume that $n > 1$, and set $\mathbf{i}' := (i_1, i_2, \ldots, i_{n-1})$, $\lambda' := \lambda - \varpi_{i_n} \in P_+^0$. Recall from Theorem 2.4.1 and Corollary 2.4.5(1) that there exist isomorphisms

$$
\Psi_{\mathbf{i}'} : \mathbb{B}(\lambda')_{\mathrm{cl}} \xrightarrow{\sim} \mathbb{B}_{\mathbf{i}'} = \mathbb{B}(\varpi_{i_1})_{\mathrm{cl}} \otimes \mathbb{B}(\varpi_{i_2})_{\mathrm{cl}} \otimes \cdots \otimes \mathbb{B}(\varpi_{i_{n-1}})_{\mathrm{cl}}
$$

and

$$
\Psi_{\lambda',\varpi_{i_n}} : \mathbb{B}(\lambda)_{\mathrm{cl}} \xrightarrow{\sim} \mathbb{B}(\lambda')_{\mathrm{cl}} \otimes \mathbb{B}(\varpi_{i_n})_{\mathrm{cl}}
$$

of P_{cl}-crystals. Let $\eta \in \mathbb{B}(\lambda)_{\mathrm{cl}}$. We set $\eta' \otimes \eta'' := \Psi_{\lambda',\varpi_{i_n}}(\eta) \in \mathbb{B}(\lambda')_{\mathrm{cl}} \otimes \mathbb{B}(\varpi_{i_n})_{\mathrm{cl}}$, and

$$
\eta_1 \otimes \eta_2 \otimes \cdots \otimes \eta_n := \Psi_{\mathbf{i}}(\eta) \in \mathbb{B}_{\mathbf{i}} = \mathbb{B}(\varpi_{i_1})_{\mathrm{cl}} \otimes \mathbb{B}(\varpi_{i_2})_{\mathrm{cl}} \otimes \cdots \otimes \mathbb{B}(\varpi_{i_n})_{\mathrm{cl}}.
$$

Note that both of $\Psi_{\mathbf{i}}$ and $(\Psi_{\mathbf{i}'} \otimes \mathrm{id}) \circ \Psi_{\lambda',\varpi_{i_n}}$ are isomorphisms of P_{cl}-crystals from $\mathbb{B}(\lambda)_{\mathrm{cl}}$ to $\mathbb{B}_{\mathbf{i}}$. Since the P_{cl}-crystals $\mathbb{B}(\lambda)_{\mathrm{cl}}$ and $\mathbb{B}_{\mathbf{i}}$ are both simple, it follows from Lemma 2.3.9(3) that $\Psi_{\mathbf{i}} = (\Psi_{\mathbf{i}'} \otimes \mathrm{id}) \circ \Psi_{\lambda',\varpi_{i_n}}$, and hence that

$$
\eta'' = \eta_n \quad \text{and} \quad \Psi_{\mathbf{i}'}(\eta') = \eta_1 \otimes \eta_2 \otimes \cdots \otimes \eta_{n-1}. \tag{4.4.17}
$$

We see from Proposition 4.4.2 that

$$
\mathrm{Deg}_\lambda(\eta) = D_{\lambda',\varpi_{i_n}}(\eta' \otimes \eta'') = H_{\lambda',\varpi_{i_n}}(\eta' \otimes \eta'') + \mathrm{Deg}_{\lambda'}(\eta') + \mathrm{Deg}_{\varpi_{i_n}}(\widetilde{\eta}''), \tag{4.4.18}
$$

where we set $\widetilde{\eta}'' \otimes \widetilde{\eta}' := R_{\lambda',\varpi_{i_n}}(\eta' \otimes \eta'')$. Here we remark that the element $\eta_n^{(1)} \in \mathbb{B}(\varpi_{i_n})_{\mathrm{cl}}$ (for the definition of $\eta_n^{(1)}$, see §4.1) is the first factor of the image of $\eta_1 \otimes \eta_2 \otimes \cdots \otimes \eta_n \in \mathbb{B}_{\mathbf{i}}$ under the (unique) isomorphism $(\mathrm{id} \otimes \Psi_{\mathbf{i}'}) \circ R_{\lambda',\varpi_{i_n}} \circ \Psi_{\lambda',\varpi_{i_n}} \circ \Psi_{\mathbf{i}}^{-1}$ of P_{cl}-crystals, which is obtained as follows:

$$
\mathbb{B}(\varpi_{i_1})_{\mathrm{cl}} \otimes \mathbb{B}(\varpi_{i_2})_{\mathrm{cl}} \otimes \cdots \otimes \mathbb{B}(\varpi_{i_n})_{\mathrm{cl}} = \mathbb{B}_{\mathbf{i}} \xrightarrow[\sim]{\Psi_{\mathbf{i}}^{-1}} \mathbb{B}(\lambda)_{\mathrm{cl}} \xrightarrow[\sim]{\Psi_{\lambda',\varpi_{i_n}}} \mathbb{B}(\lambda')_{\mathrm{cl}} \otimes \mathbb{B}(\varpi_{i_n})_{\mathrm{cl}}
$$

$$
\xrightarrow[\sim]{R_{\lambda',\varpi_{i_n}}} \mathbb{B}(\varpi_{i_n})_{\mathrm{cl}} \otimes \mathbb{B}(\lambda')_{\mathrm{cl}} \xrightarrow[\sim]{\mathrm{id} \otimes \Psi_{\mathbf{i}'}} \mathbb{B}(\varpi_{i_n})_{\mathrm{cl}} \otimes \mathbb{B}(\varpi_{i_1})_{\mathrm{cl}} \otimes \cdots \otimes \mathbb{B}(\varpi_{i_{n-1}})_{\mathrm{cl}}.
$$

Also, it is easy to check that this first factor is identical to $\widetilde{\eta}'' \in \mathbb{B}(\varpi_{i_n})_{\mathrm{cl}}$. Thus we have $\eta_n^{(1)} = \widetilde{\eta}''$. Furthermore, using (4.4.17), we see from [Oka07, Lemma 5.2] that

$$
H_{\lambda',\varpi_{i_n}}(\eta' \otimes \eta'') = \sum_{1 \leqslant k \leqslant n-1} H_{\varpi_{i_k},\varpi_{i_n}}(\eta_k \otimes \eta_n^{(k+1)}). \tag{4.4.19}
$$

1548

Now, the induction hypothesis along with (4.4.17) implies that

$$\mathrm{Deg}_{\lambda'}(\eta') = \sum_{1 \leqslant k < l \leqslant n-1} H_{\varpi_{i_k}, \varpi_{i_l}}(\eta_k \otimes \eta_l^{(k+1)}) + \sum_{k=1}^{n-1} \mathrm{Deg}_{\varpi_{i_k}}(\eta_k^{(1)}). \qquad (4.4.20)$$

By substituting (4.4.19) and (4.4.20) into (4.4.18), and using the fact that $\eta_n^{(1)} = \widetilde{\eta}''$, we obtain

$$\begin{aligned}
\mathrm{Deg}_\lambda(\eta) &= H_{\lambda', \varpi_{i_n}}(\eta' \otimes \eta'') + \mathrm{Deg}_{\lambda'}(\eta') + \mathrm{Deg}_{\varpi_{i_n}}(\widetilde{\eta}'') \\
&= \sum_{1 \leqslant k \leqslant n-1} H_{\varpi_{i_k}, \varpi_{i_n}}(\eta_k \otimes \eta_n^{(k+1)}) \\
&\quad + \sum_{1 \leqslant k < l \leqslant n-1} H_{\varpi_{i_k}, \varpi_{i_l}}(\eta_k \otimes \eta_l^{(k+1)}) + \sum_{k=1}^{n-1} \mathrm{Deg}_{\varpi_{i_k}}(\eta_k^{(1)}) + \mathrm{Deg}_{\varpi_{i_n}}(\eta_n^{(1)}) \\
&= \sum_{1 \leqslant k < l \leqslant n} H_{\varpi_{i_k}, \varpi_{i_l}}(\eta_k \otimes \eta_l^{(k+1)}) + \sum_{k=1}^{n} \mathrm{Deg}_{\varpi_{i_k}}(\eta_k^{(1)}).
\end{aligned}$$

This completes the proof of (4.1.4), thereby establishing Theorem 4.1.2. $\qquad \square$

4.5 Proof of Theorem 4.1.3

In this subsection, we continue to write $\mathrm{Deg}_\mu(\eta)$ instead of $\mathrm{Deg}(\eta)$ in the case that η is an element of $\mathbb{B}(\mu)_{\mathrm{cl}}$ for $\mu \in P_+^0$.

Fix an arbitrary $i \in I_0$. Recall from Corollary 2.4.5(1) that there exists an isomorphism $\Psi_{\varpi_i, \varpi_i} : \mathbb{B}(2\varpi_i)_{\mathrm{cl}} \xrightarrow{\sim} \mathbb{B}(\varpi_i)_{\mathrm{cl}} \otimes \mathbb{B}(\varpi_i)_{\mathrm{cl}}$ of P_{cl}-crystals. The next lemma follows from the proof of [NS06, Proposition 3.4.4].

LEMMA 4.5.1. *Let $\eta_1 \otimes \eta_2 \in \mathbb{B}(\varpi_i)_{\mathrm{cl}} \otimes \mathbb{B}(\varpi_i)_{\mathrm{cl}}$. Then, the preimage $\Psi_{\varpi_i, \varpi_i}^{-1}(\eta_1 \otimes \eta_2) \in \mathbb{B}(2\varpi_i)_{\mathrm{cl}}$ of $\eta_1 \otimes \eta_2 \in \mathbb{B}(\varpi_i)_{\mathrm{cl}} \otimes \mathbb{B}(\varpi_i)_{\mathrm{cl}}$ under the isomorphism $\Psi_{\varpi_i, \varpi_i}$ is identical to the concatenation $\eta_1 * \eta_2$ of η_1 and η_2 defined by*

$$(\eta_1 * \eta_2)(t) = \begin{cases} \eta_1(2t) & \text{if } 0 \leqslant t \leqslant 1/2, \\ \eta_1(1) + \eta_2(2t-1) & \text{if } 1/2 \leqslant t \leqslant 1. \end{cases} \qquad (4.5.1)$$

In addition, from [NS06, Proposition 3.2.2], we have the next result.

LEMMA 4.5.2. *The set $\mathbb{B}(2\varpi_i)$ is identical to the set of all concatenations $\pi_1 * \pi_2$ of LS paths $\pi_1, \pi_2 \in \mathbb{B}(\varpi_i)$ such that $\kappa(\pi_1) \geqslant \iota(\pi_2)$. Here, the concatenation $\pi_1 * \pi_2$ is defined by the same formula as (4.5.1), with η_1 and η_2 replaced by π_1 and π_2, respectively.*

LEMMA 4.5.3. *Let $\eta \in \mathbb{B}(2\varpi_i)_{\mathrm{cl}}$, and set $\eta_1 \otimes \eta_2 := \Psi_{\varpi_i, \varpi_i}(\eta) \in \mathbb{B}(\varpi_i)_{\mathrm{cl}} \otimes \mathbb{B}(\varpi_i)_{\mathrm{cl}}$. Then,*

$$\mathrm{Deg}_{2\varpi_i}(\eta) = H_{\varpi_i, \varpi_i}(\eta_1 \otimes \eta_2) + 2\,\mathrm{Deg}_{\varpi_i}(\eta_1). \qquad (4.5.2)$$

Proof. Applying Theorem 4.1.2 to the case in which $\mathbf{i} = (i, i)$ and hence $\lambda = 2\varpi_i$, we obtain

$$\begin{aligned}
\mathrm{Deg}_{2\varpi_i}(\eta) &= H_{\varpi_i, \varpi_i}(\eta_1 \otimes \eta_2^{(2)}) + \mathrm{Deg}_{\varpi_i}(\eta_1^{(1)}) + \mathrm{Deg}_{\varpi_i}(\eta_2^{(1)}) \\
&= H_{\varpi_i, \varpi_i}(\eta_1 \otimes \eta_2) + \mathrm{Deg}_{\varpi_i}(\eta_1) + \mathrm{Deg}_{\varpi_i}(\eta_2^{(1)}),
\end{aligned}$$

since $\eta_1^{(1)} = \eta_1$ and $\eta_2^{(2)} = \eta_2$ by definition. Also, from Lemma 2.3.9(3) and the definition of $\eta_2^{(1)}$, we deduce that $\eta_2^{(1)} = \eta_1$, since the P_{cl}-crystal $\mathbb{B}(\varpi_i)_{\mathrm{cl}} \otimes \mathbb{B}(\varpi_i)_{\mathrm{cl}}$ is simple. Thus we obtain (4.5.2). This proves the lemma. $\qquad \square$

Let us fix an (arbitrary) element $\eta^\flat \in \mathbb{B}(\varpi_i)_{\mathrm{cl}}$ such that $f_j \eta^\flat = \mathbf{0}$ for all $j \in I_0$. Then it follows from (2.2.10) that $f_j \pi_{\eta^\flat} = \mathbf{0}$ for all $j \in I_0$. Therefore, by Lemma 2.2.13, the final direction

$\nu^{\flat} := \kappa(\pi_{\eta^{\flat}}) \in W\varpi_i$ of $\pi_{\eta^{\flat}}$ satisfies the condition that $\nu^{\flat}(h_j) \leqslant 0$ for all $j \in I_0$. Hence $\mathrm{cl}(\nu^{\flat}) = w_0(\mathrm{cl}(\varpi_i))$ by Remark 2.1.5. From this, we deduce (using Lemma 2.1.2) that $\nu^{\flat} \in W\varpi_i$ can be written as $\nu^{\flat} = w_0\varpi_i + k^{\flat}d_{\varpi_i}\delta$ for some $k^{\flat} \in \mathbb{Z}$.

LEMMA 4.5.4. *Let* $\eta \in \mathbb{B}(\varpi_i)_{\mathrm{cl}}$, *and set* $\eta' := \Psi_{\varpi_i,\varpi_i}^{-1}(\eta^{\flat} \otimes \eta) \in \mathbb{B}(2\varpi_i)_{\mathrm{cl}}$. *The following hold:*

(1) *the path* $\pi_{\eta^{\flat}} * (\pi_{\eta} + \pi_{k^{\flat}d_{\varpi_i}\delta})$ *lies in* $\mathrm{cl}^{-1}(\eta') \cap \mathbb{B}(2\varpi_i)$;

(2) $\pi_{\eta'} = \pi_{\eta^{\flat}} * (\pi_{\eta} + \pi_{k^{\flat}d_{\varpi_i}\delta})$.

Proof. (1) First, note that η' is identical to the concatenation $\eta^{\flat} * \eta$ by Lemma 4.5.1. Also, by the definition of concatenations (see (4.5.1)),

$$\mathrm{cl}(\pi_{\eta^{\flat}} * (\pi_{\eta} + \pi_{k^{\flat}d_{\varpi_i}\delta})) = \mathrm{cl}(\pi_{\eta^{\flat}}) * \mathrm{cl}(\pi_{\eta} + \pi_{k^{\flat}d_{\varpi_i}\delta}).$$

Since $\mathrm{cl}(\pi_{\eta^{\flat}}) = \eta^{\flat}$ and $\mathrm{cl}(\pi_{\eta} + \pi_{k^{\flat}d_{\varpi_i}\delta}) = \mathrm{cl}(\pi_{\eta}) = \eta$, we have

$$\mathrm{cl}(\pi_{\eta^{\flat}} * (\pi_{\eta} + \pi_{k^{\flat}d_{\varpi_i}\delta})) = \mathrm{cl}(\pi_{\eta^{\flat}}) * \mathrm{cl}(\pi_{\eta} + \pi_{k^{\flat}d_{\varpi_i}\delta}) = \eta^{\flat} * \eta = \eta',$$

and hence $\pi_{\eta^{\flat}} * (\pi_{\eta} + \pi_{k^{\flat}d_{\varpi_i}\delta}) \in \mathrm{cl}^{-1}(\eta')$.

Next, we show that $\pi_{\eta^{\flat}} * (\pi_{\eta} + \pi_{k^{\flat}d_{\varpi_i}\delta}) \in \mathbb{B}(2\varpi_i)$. Note that $\pi_{\eta} + \pi_{k^{\flat}d_{\varpi_i}\delta} \in \mathbb{B}(\varpi_i)$ by Lemma 2.2.6. Therefore, by Lemma 4.5.2, it suffices to show that $\iota(\pi_{\eta} + \pi_{k^{\flat}d_{\varpi_i}\delta}) \leqslant \nu^{\flat} = \kappa(\pi_{\eta^{\flat}})$. Let us write $\iota(\pi_{\eta}) \in \varpi_i - \mathring{Q}_+$ as $\iota(\pi_{\eta}) = w\varpi_i$ for some $w \in \mathring{W}$ (see Lemma 2.1.2). Since $w_0 \in \mathring{W}$ is greater than or equal to $w \in \mathring{W}$ with respect to the usual Bruhat ordering on the (finite) Weyl group $\mathring{W}$ of $\mathfrak{g}_{I_0}$, it follows (see [Lit95, Remark 4.2]) that $w_0\varpi_i \geqslant w\varpi_i$. From this, using the definition of the ordering $\geqslant$ on $W\varpi_i$, we see that

$$\nu^{\flat} = w_0\varpi_i + k^{\flat}d_{\varpi_i}\delta \geqslant w\varpi_i + k^{\flat}d_{\varpi_i}\delta = \iota(\pi_{\eta}) + k^{\flat}d_{\varpi_i}\delta. \tag{4.5.3}$$

Hence, by Remark 2.2.4, we obtain

$$\iota(\pi_{\eta} + \pi_{k^{\flat}d_{\varpi_i}\delta}) = \iota(\pi_{\eta}) + \iota(\pi_{k^{\flat}d_{\varpi_i}\delta}) = \iota(\pi_{\eta}) + k^{\flat}d_{\varpi_i}\delta \leqslant w_0\varpi_i + k^{\flat}d_{\varpi_i}\delta = \nu^{\flat}.$$

This proves part (1).

(2) First, by the definitions of $\mathrm{Deg}_{\varpi_i}(\eta^{\flat})$ and $\mathrm{Deg}_{\varpi_i}(\eta)$, we can write $\pi_{\eta^{\flat}}(1) \in P$ and $\pi_{\eta}(1) \in P$ as $\pi_{\eta^{\flat}}(1) = \varpi_i - a_0^{-1}\beta^{\flat} - a_0^{-1}\mathrm{Deg}_{\varpi_i}(\eta^{\flat})$ and $\pi_{\eta}(1) = \varpi_i - a_0^{-1}\beta - a_0^{-1}\mathrm{Deg}_{\varpi_i}(\eta)$ for some $\beta^{\flat}, \beta \in \mathring{Q}_+$, respectively. Hence we have

$$(\pi_{\eta^{\flat}} * (\pi_{\eta} + \pi_{k^{\flat}d_{\varpi_i}\delta}))(1) = \pi_{\eta^{\flat}}(1) + (\pi_{\eta} + \pi_{k^{\flat}d_{\varpi_i}\delta})(1) = \pi_{\eta^{\flat}}(1) + \pi_{\eta}(1) + \pi_{k^{\flat}d_{\varpi_i}\delta}(1)$$

$$= \{\varpi_i - a_0^{-1}\beta^{\flat} - a_0^{-1}\mathrm{Deg}_{\varpi_i}(\eta^{\flat})\} + \{\varpi_i - a_0^{-1}\beta - a_0^{-1}\mathrm{Deg}_{\varpi_i}(\eta)\} + k^{\flat}d_{\varpi_i}\delta$$

$$= 2\varpi_i - a_0^{-1}(\beta^{\flat} + \beta) - a_0^{-1}(\mathrm{Deg}_{\varpi_i}(\eta^{\flat}) + \mathrm{Deg}_{\varpi_i}(\eta) - a_0 k^{\flat}d_{\varpi_i})\delta. \tag{4.5.4}$$

Now, in view of Proposition 3.1.4(1), it suffices to show the following:

$$\iota(\pi_{\eta^{\flat}} * (\pi_{\eta} + \pi_{k^{\flat}d_{\varpi_i}\delta})) \in 2\varpi_i - \mathring{Q}_+, \tag{4.5.5}$$

$$\mathrm{Deg}_{2\varpi_i}(\eta') \leqslant \mathrm{Deg}_{\varpi_i}(\eta^{\flat}) + \mathrm{Deg}_{\varpi_i}(\eta) - a_0 k^{\flat}d_{\varpi_i}. \tag{4.5.6}$$

As for (4.5.5), we easily deduce from the definition of concatenations that $\iota(\pi_{\eta^{\flat}} * (\pi_{\eta} + \pi_{k^{\flat}d_{\varpi_i}\delta})) = 2\iota(\pi_{\eta^{\flat}})$, where the right-hand side lies in $2\varpi_i - \mathring{Q}_+$ since $\iota(\pi_{\eta^{\flat}}) \in \varpi_i - \mathring{Q}_+$ by definition.

Let us show (4.5.6). Set $\pi_1(t) := \pi_{\eta'}(t/2)$ and $\pi_2(t) := \pi_{\eta'}((t+1)/2) - \pi_{\eta'}(1/2)$ for $t \in [0,1]$. Then it is obvious that $\pi_{\eta'} = \pi_1 * \pi_2$. Also, we see from the proof of [NS06, Proposition 3.2.2] that $\pi_1, \pi_2 \in \mathbb{B}(\varpi_i)$. Furthermore, since $\eta^{\flat} * \eta = \eta' = \mathrm{cl}(\pi_{\eta'}) = \mathrm{cl}(\pi_1) * \mathrm{cl}(\pi_2)$, we deduce that $\mathrm{cl}(\pi_1) = \eta^{\flat}$ and $\mathrm{cl}(\pi_2) = \eta$.

Since $\iota(\pi_{\eta'}) \in 2\varpi_i - \mathring{Q}_+$, and since $\iota(\pi_1) = \frac{1}{2}\iota(\pi_{\eta'})$ by the definition of $\pi_1 \in \mathbb{B}(\varpi_i)$, it follows from Lemma 2.1.2 that $\iota(\pi_1) \in W\varpi_i$ lies in $\varpi_i - \mathring{Q}_+$. If we write $\pi_1(1) \in P$ in the form $\pi_1(1) = \varpi_i - a_0^{-1}\beta_1 + a_0^{-1}K_1\delta$ with $\beta_1 \in \mathring{Q}_+$ and $K_1 \in \mathbb{Z}_{\geqslant 0}$ (see Lemma 3.1.1), then by Proposition 3.1.4(1) applied to $\mathrm{cl}(\pi_1) = \eta^\flat \in \mathbb{B}(\varpi_i)_{\mathrm{cl}}$, we have

$$-K_1 \leqslant \mathrm{Deg}_{\varpi_i}(\eta^\flat). \tag{4.5.7}$$

Also, if we write $\kappa(\pi_1) \in W\varpi_i$ in the form $\varpi_i - \beta_1' + k_1'd_{\varpi_i}\delta$ with $\beta_1' \in \mathring{Q}_+$ and $k_1' \in \mathbb{Z}$, then by Proposition 3.1.4(2) applied to $\mathrm{cl}(\pi_1) = \eta^\flat \in \mathbb{B}(\varpi_i)_{\mathrm{cl}}$, we have

$$k_1' \geqslant k^\flat, \tag{4.5.8}$$

where we recall from the discussion preceding this lemma that $\nu^\flat = \kappa(\pi_{\eta^\flat}) \in W\varpi_i$ equals $w_0\varpi_i + k^\flat d_{\varpi_i}\delta$.

Since $\pi_{\eta'} = \pi_1 * \pi_2 \in \mathbb{B}(2\varpi_i)$, it follows from Lemma 4.5.2 that $\iota(\pi_2) \leqslant \kappa(\pi_1) = \varpi_i - \beta_1' + k_1'd_{\varpi_i}\delta$. Hence, by Lemma 2.1.2 and Remark 2.1.3, $\iota(\pi_2) \in W\varpi_i$ is of the form $\iota(\pi_2) = \varpi_i - \beta_2' + k_2'd_{\varpi_i}\delta$, with $\beta_2' \in \mathring{Q}_+$ and $k_2' \in \mathbb{Z}$ such that

$$k_2' \geqslant k_1'. \tag{4.5.9}$$

If we set $\pi_2' := \pi_2 - \pi_{k_2'd_{\varpi_i}\delta}$, then it follows from Lemma 2.2.6 that $\pi_2' \in \mathbb{B}(\varpi_i)$. Also, we have $\mathrm{cl}(\pi_2') = \mathrm{cl}(\pi_2) = \eta$, and by Remark 2.2.4, $\iota(\pi_2') = \iota(\pi_2) - k_2'd_{\varpi_i}\delta = \varpi_i - \beta_2' \in \varpi_i - \mathring{Q}_+$. Hence, if we write $\pi_2'(1) \in P$ in the form $\pi_2'(1) = \varpi_i - a_0^{-1}\beta_2 + a_0^{-1}K_2\delta$ with $\beta_2 \in \mathring{Q}_+$ and $K_2 \in \mathbb{Z}_{\geqslant 0}$ (see Lemma 3.1.1), then by Proposition 3.1.4(1) applied to $\mathrm{cl}(\pi_2') = \eta \in \mathbb{B}(\varpi_i)_{\mathrm{cl}}$, we have

$$-K_2 \leqslant \mathrm{Deg}_{\varpi_i}(\eta). \tag{4.5.10}$$

From the above, we see that

$$\begin{aligned}
\pi_{\eta'}(1) = (\pi_1 * \pi_2)(1) &= \pi_1(1) + \pi_2(1) = \pi_1(1) + \pi_2'(1) + k_2'd_{\varpi_i}\delta \\
&= (\varpi_i - a_0^{-1}\beta_1 + a_0^{-1}K_1\delta) + (\varpi_i - a_0^{-1}\beta_2 + a_0^{-1}K_2\delta) + k_2'd_{\varpi_i}\delta \\
&= 2\varpi_i - a_0^{-1}(\beta_1 + \beta_2) + a_0^{-1}(K_1 + K_2 + a_0k_2'd_{\varpi_i})\delta,
\end{aligned}$$

and hence that $\mathrm{Deg}_{2\varpi_i}(\eta') = -K_1 - K_2 - a_0k_2'd_{\varpi_i}$. By (4.5.7), (4.5.10), and the inequalities (4.5.8) and (4.5.9), we obtain

$$\mathrm{Deg}_{2\varpi_i}(\eta') = -K_1 - K_2 - a_0k_2'd_{\varpi_i} \leqslant \mathrm{Deg}_{\varpi_i}(\eta^\flat) + \mathrm{Deg}_{\varpi_i}(\eta) - a_0k^\flat d_{\varpi_i},$$

which is the desired inequality (4.5.6). This completes the proof of part (2). $\qquad\square$

Now we are ready to prove Theorem 4.1.3.

Proof of Theorem 4.1.3. Let $\eta \in \mathbb{B}(\varpi_i)_{\mathrm{cl}}$, and set $\eta' := \Psi_{\varpi_i,\varpi_i}^{-1}(\eta^\flat \otimes \eta) \in \mathbb{B}(2\varpi_i)_{\mathrm{cl}}$. It follows from Lemma 4.5.3 that

$$\mathrm{Deg}_{2\varpi_i}(\eta') = H_{\varpi_i,\varpi_i}(\eta^\flat \otimes \eta) + 2\mathrm{Deg}_{\varpi_i}(\eta^\flat). \tag{4.5.11}$$

Also, we deduce from Lemma 4.5.4(2) and the definition of $\mathrm{Deg}(\eta')$, by using (4.5.4), that

$$\mathrm{Deg}_{2\varpi_i}(\eta') = \mathrm{Deg}_{\varpi_i}(\eta^\flat) + \mathrm{Deg}_{\varpi_i}(\eta) - a_0k^\flat d_{\varpi_i}. \tag{4.5.12}$$

Hence, by combining (4.5.11) and (4.5.12), we obtain

$$H_{\varpi_i,\varpi_i}(\eta^\flat \otimes \eta) = \mathrm{Deg}_{\varpi_i}(\eta) - \mathrm{Deg}_{\varpi_i}(\eta^\flat) - a_0k^\flat d_{\varpi_i}. \tag{4.5.13}$$

In particular, by taking $\eta = \eta_{\mathrm{cl}(\varpi_i)} \in \mathbb{B}(\varpi_i)_{\mathrm{cl}}$ in (4.5.13), we obtain

$$H_{\varpi_i,\varpi_i}(\eta^\flat \otimes \eta_{\mathrm{cl}(\varpi_i)}) = \mathrm{Deg}_{\varpi_i}(\eta_{\mathrm{cl}(\varpi_i)}) - \mathrm{Deg}_{\varpi_i}(\eta^\flat) - a_0k^\flat d_{\varpi_i}.$$

1551

Since $\mathrm{Deg}_{\varpi_i}(\eta_{\mathrm{cl}(\varpi_i)}) = 0$ by Lemma 3.2.1(1), it follows that

$$H_{\varpi_i,\varpi_i}(\eta^\flat \otimes \eta_{\mathrm{cl}(\varpi_i)}) = -\mathrm{Deg}_{\varpi_i}(\eta^\flat) - a_0 k^\flat d_{\varpi_i}. \tag{4.5.14}$$

Equation (4.1.5) follows immediately from (4.5.13) and (4.5.14). This establishes Theorem 4.1.3. □

5. Relation to one-dimensional sums

In this section, as an application of Theorem 4.1.1, we provide an expression for classically restricted one-dimensional sums (1dsums for short) and, in the case of type $A^{(1)}_{\ell-1}$, an expression for Kostka–Foulkes polynomials in terms of LS paths.

5.1 Expression for one-dimensional sums in terms of LS paths

Here we recall from [HKOTY99, Conjecture 2.1] and [HKOTT02, Conjecture 2.1] a conjectural family of finite-dimensional irreducible $U'_q(\mathfrak{g})$-modules $W^{(i)}_s$ having crystal bases $\mathcal{B}^{i,s}$ indexed by $i \in I_0$ and $s \in \mathbb{Z}_{\geqslant 1}$ for which the decomposition into irreducible $U_q(\mathfrak{g}_{I_0})$-modules is prescribed by the KR conjecture, presented in [KR90]; the $U'_q(\mathfrak{g})$-module $W^{(i)}_s$ is called a KR module, and the crystal $\mathcal{B}^{i,s}$ is called a KR crystal. Through enough evidence (see Introduction and references therein), it is confirmed that the level-zero fundamental representation $W(\varpi_i)$ is indeed the (conjectural) KR module $W^{(i)}_s$ with $s = 1$, and hence the crystal basis $\mathcal{B}(W(\varpi_i))$ of $W(\varpi_i)$ is the KR crystal $\mathcal{B}^{i,1}$ for every $i \in I_0$. Thus, the (conjectural) KR crystal $\mathcal{B}^{i,s}$ with $s = 1$ is obtained as the crystal $\mathbb{B}(\varpi_i)_{\mathrm{cl}}$ for all $i \in I_0$.

In [HKOTY99] and [HKOTT02], a q-analogue of the KR conjecture (called the $X = M$ conjecture) was proposed by introducing a classically restricted one-dimensional sum (1dsum for short) X associated to a tensor product of KR crystals $\mathcal{B}^{i,s}$, $i \in I_0$, $s \in \mathbb{Z}_{\geqslant 1}$, and a fermionic formula M, which is a specific sum of products of q-binomial coefficients. (We will not pursue the fermionic formula M in this paper.) Following the definition in [HKOTY99, §3] and [HKOTT02, §3.3] of classically restricted 1dsums, we define a classically restricted 1dsum associated to a tensor product of simple crystals $\mathbb{B}(\varpi_i)_{\mathrm{cl}} \simeq \mathcal{B}(W(\varpi_i))$, $i \in I_0$, with $\mathbb{B}(\varpi_i)_{\mathrm{cl}}$ in place of $\mathcal{B}^{i,1}$ for $i \in I_0$, as follows. Let $\mathbf{i} = (i_1, i_2, \ldots, i_n)$ be a sequence of elements of I_0, and $\mathbb{B}_{\mathbf{i}} = \mathbb{B}(\varpi_{i_1})_{\mathrm{cl}} \otimes \mathbb{B}(\varpi_{i_2})_{\mathrm{cl}} \otimes \cdots \otimes \mathbb{B}(\varpi_{i_n})_{\mathrm{cl}}$. Then, for an element $\mu \in \mathrm{cl}(P^0_+) = \sum_{i \in I_0} \mathbb{Z}_{\geqslant 0}\, \mathrm{cl}(\varpi_i)$, the classically restricted 1dsum $X(\mathbb{B}_{\mathbf{i}}, \mu; q)$ is defined by

$$X(\mathbb{B}_{\mathbf{i}}, \mu; q) = \sum_{\substack{b \in \mathbb{B}_{\mathbf{i}} \\ e_j b = \mathbf{0} \ (j \in I_0) \\ \mathrm{wt}\, b = \mu}} q^{D_{\mathbf{i}}(b)},$$

where $D_{\mathbf{i}} : \mathbb{B}_{\mathbf{i}} \to \mathbb{Z}$ is as defined in (4.1.1). Because the isomorphism $\Psi_{\mathbf{i}} : \mathbb{B}(\lambda)_{\mathrm{cl}} \xrightarrow{\sim} \mathbb{B}_{\mathbf{i}}$ of P_{cl}-crystals is compatible with the root operators e_j, $j \in I_0$, on $\mathbb{B}(\lambda)_{\mathrm{cl}}$ and the root operators e_j, $j \in I_0$, on $\mathbb{B}_{\mathbf{i}}$, we obtain the following corollary of Theorem 4.1.1.

COROLLARY 5.1.1. *Let $\mathbf{i} = (i_1, i_2, \ldots, i_n)$ be a sequence of elements of I_0, and set $\lambda := \sum_{k=1}^n \varpi_{i_k} \in P^0_+$. For every $\mu \in \mathrm{cl}(P^0_+) = \sum_{i \in I_0} \mathbb{Z}_{\geqslant 0}\, \mathrm{cl}(\varpi_i)$, the following equation holds:*

$$\sum_{\substack{\eta \in \mathbb{B}(\lambda)_{\mathrm{cl}} \\ e_j \eta = \mathbf{0} \ (j \in I_0) \\ \eta(1) = \mu}} q^{\mathrm{Deg}(\eta)} = q^{-D^{\mathrm{ext}}_{\mathbf{i}}} X(\mathbb{B}_{\mathbf{i}}, \mu; q). \tag{5.1.1}$$

Remark 5.1.2 (see also [HKOTT02, Proposition 3.9]). Let $\lambda \in P^0_+$, and let $\mu \in \mathrm{cl}(P^0_+)$. We see from Corollary 5.1.1 that $q^{-D^{\mathrm{ext}}_{\mathbf{i}}} X(\mathbb{B}_{\mathbf{i}}, \mu; q)$ does not depend on the choice of the sequence $\mathbf{i} = (i_1, i_2, \ldots, i_n)$ of elements of I_0 such that $\lambda = \sum_{k=1}^n \varpi_{i_k}$.

5.2 Expression for Kostka–Foulkes polynomials in terms of LS paths

Let $i \in I_0$. Recall the element $\eta_{\widetilde{\varpi}_i} \in \mathbb{B}(\varpi_i)_{\mathrm{cl}}$ introduced in §4.1, where $\widetilde{\varpi}_i = w_0 \, \mathrm{cl}(\varpi_i) \in P_{\mathrm{cl}}$. We note that if the crystal $\mathrm{res}_{I_0} \mathbb{B}(\varpi_i)_{\mathrm{cl}}$ is connected (see §2.3 for the definition of $\mathrm{res}_{I_0} \mathbb{B}(\varpi_i)_{\mathrm{cl}}$), then the element $\eta_{\widetilde{\varpi}_i}$ is the unique element of $\mathbb{B}(\varpi_i)_{\mathrm{cl}}$ such that $f_j \eta_{\widetilde{\varpi}_i} = \mathbf{0}$ for all $j \in I_0$, since the P_{cl}-crystal $\mathbb{B}(\varpi_i)_{\mathrm{cl}}$ is regular.

LEMMA 5.2.1. *Let $i \in I_0$, and assume that the crystal $\mathrm{res}_{I_0} \mathbb{B}(\varpi_i)_{\mathrm{cl}}$ is connected. Then, the equation H_{ϖ_i,ϖ_i}*
$(\eta_{\widetilde{\varpi}_i} \otimes \eta) = 0$ *holds for all $\eta \in \mathbb{B}(\varpi_i)_{\mathrm{cl}}$.*

Remark 5.2.2. Since the P_{cl}-crystal $\mathbb{B}(\varpi_i)_{\mathrm{cl}}$ is regular, we see that $\mathrm{res}_{I_0} \mathbb{B}(\varpi_i)_{\mathrm{cl}}$ is connected if and only if $W(\varpi_i)$ is irreducible when regarded as a $U'_q(\mathfrak{g})_{I_0}$-module by restriction. Therefore, we deduce from [Kas05, Lemma 4.3(ii)] that, if $a_i^\vee = 1$, then $\mathrm{res}_{I_0} \mathbb{B}(\varpi_i)_{\mathrm{cl}}$ is connected.

Proof of Lemma 5.2.1. Since $\mathrm{res}_{I_0} \mathbb{B}(\varpi_i)_{\mathrm{cl}}$ is connected by the assumption of the lemma, there exists a monomial X in the root operators f_j for $j \in I_0$ such that $X\eta = \eta_{\widetilde{\varpi}_i}$. Since $f_j \eta_{\widetilde{\varpi}_i} = \mathbf{0}$ for all $j \in I_0$, it follows from the tensor product rule for crystals that

$$X(\eta_{\widetilde{\varpi}_i} \otimes \eta) = \eta_{\widetilde{\varpi}_i} \otimes X\eta = \eta_{\widetilde{\varpi}_i} \otimes \eta_{\widetilde{\varpi}_i}.$$

Furthermore, we see from the proof of [AK97, Lemma 1.6(1)] that

$$S_{w_0} X(\eta_{\widetilde{\varpi}_i} \otimes \eta) = S_{w_0}(\eta_{\widetilde{\varpi}_i} \otimes \eta_{\widetilde{\varpi}_i}) = (S_{w_0} \eta_{\widetilde{\varpi}_i}) \otimes (S_{w_0} \eta_{\widetilde{\varpi}_i}),$$

and from Remark 2.4.3 that $S_{w_0} \eta_{\widetilde{\varpi}_i} = \eta_{\mathrm{cl}(\varpi_i)}$. Since $w_0 \in \mathring{W}$, it follows from the definition of S_{w_0} (see (2.3.2)) that there exists a monomial X' in the root operators e_j, f_j for $j \in I_0$ such that $S_{w_0} X(\eta_{\widetilde{\varpi}_i} \otimes \eta) = X'X(\eta_{\widetilde{\varpi}_i} \otimes \eta)$. Therefore, we obtain

$$
\begin{aligned}
0 &= H_{\varpi_i,\varpi_i}(\eta_{\mathrm{cl}(\varpi_i)} \otimes \eta_{\mathrm{cl}(\varpi_i)}) \quad \text{by Theorem 2.5.1 (H2)} \\
&= H_{\varpi_i,\varpi_i}((S_{w_0}\eta_{\widetilde{\varpi}_i}) \otimes (S_{w_0}\eta_{\widetilde{\varpi}_i})) \\
&= H_{\varpi_i,\varpi_i}(S_{w_0} X(\eta_{\widetilde{\varpi}_i} \otimes \eta)) \\
&= H_{\varpi_i,\varpi_i}(X'X(\eta_{\widetilde{\varpi}_i} \otimes \eta)) \\
&= H_{\varpi_i,\varpi_i}(\eta_{\widetilde{\varpi}_i} \otimes \eta) \quad \text{by Theorem 2.5.1 (H1).}
\end{aligned}
$$

This proves the lemma. □

Remark 5.2.3. Let $\mathbf{i} = (i_1, i_2, \ldots, i_n)$ be a sequence of elements of I_0 such that $a_{i_k}^\vee = 1$ for all $1 \leqslant k \leqslant n$. Then we know from Remark 5.2.2 that $\mathrm{res}_{I_0} \mathbb{B}(\varpi_{i_k})_{\mathrm{cl}}$ is connected for all $1 \leqslant k \leqslant n$. Therefore, we see from Lemma 5.2.1 and the definitions of $D_{\mathbf{i}}$ and $D_{\mathbf{i}}^{\mathrm{ext}}$ that, for every

$$\eta_1 \otimes \eta_2 \otimes \cdots \otimes \eta_n \in \mathbb{B}_{\mathbf{i}} = \mathbb{B}(\varpi_{i_1})_{\mathrm{cl}} \otimes \mathbb{B}(\varpi_{i_2})_{\mathrm{cl}} \otimes \cdots \otimes \mathbb{B}(\varpi_{i_n})_{\mathrm{cl}},$$

we have

$$D_{\mathbf{i}}(\eta_1 \otimes \eta_2 \otimes \cdots \otimes \eta_n) = \sum_{1 \leqslant k < l \leqslant n} H_{\varpi_{i_k},\varpi_{i_l}}(\eta_k \otimes \eta_l^{(k+1)}) \quad \text{and} \quad D_{\mathbf{i}}^{\mathrm{ext}} = 0. \tag{5.2.1}$$

Now we restrict our attention to the case in which $\mathfrak{g}$ is of type $A_{\ell-1}^{(1)}$. Because $a_i^\vee = 1$ for all $i \in I_0$ in the case of type $A_{\ell-1}^{(1)}$, we know from Remark 5.2.2 that $\mathrm{res}_{I_0} \mathbb{B}(\varpi_i)_{\mathrm{cl}}$ is connected for all $i \in I_0$. In fact, we can check by direct calculation that every ϖ_i is 'minuscule', i.e. that $\varpi_i(\beta^\vee) \in \{0, \pm 1\}$ for all $\beta \in \mathring{\Delta} := \mathring{W}\Pi_{I_0} \subset \Delta^{\mathrm{re}}$, where $\Pi_{I_0} = \{\alpha_j\}_{j \in I_0}$, and hence that $\varpi_i(\xi^\vee) \in \{0, \pm 1\}$ for all $\xi \in \Delta_+^{\mathrm{re}}$. Using this fact, we deduce from the definition of LS paths that

$$\mathbb{B}(\varpi_i)_{\mathrm{cl}} = \{\eta_\mu \mid \mu \in \mathrm{cl}(W\varpi_i) = \mathring{W}\,\mathrm{cl}(\varpi_i)\} \tag{5.2.2}$$

1553

for every $i \in I_0$. Also, it follows from the definition of the root operators e_j (respectively, f_j), $j \in I$, that, for $\mu \in \mathring{W} \operatorname{cl}(\varpi_i)$ and $j \in I$, the following hold:

(1) $e_j \eta_\mu \neq \mathbf{0}$ (respectively, $f_j \eta_\mu \neq \mathbf{0}$) if and only if $\mu(h_j) = -1$ (respectively, $= 1$);

(2) if $e_j \eta_\mu \neq \mathbf{0}$ (respectively, $f_j \eta_\mu \neq \mathbf{0}$), then $e_j \eta_\mu$ (respectively, $f_j \eta_\mu$) $= \eta_{r_j \mu}$, with $r_j(\mu) = \mu + \operatorname{cl}(\alpha_j)$ (respectively, $= \mu - \operatorname{cl}(\alpha_j)$). $\qquad(5.2.3)$

Here we recall from [NY97, §§ 2.1 and 2.4] the P_{cl}-crystals $B_{\operatorname{cl}(\varpi_i)}$ (adapted to our notation) for $i \in I_0 = \{1, 2, \ldots, \ell - 1\}$. Let $i \in I_0 = \{1, 2, \ldots, \ell - 1\}$. Then, the crystal $B_{\operatorname{cl}(\varpi_i)}$ consists of all Young diagrams of shape (1^ℓ) having i dots (in all) with at most one dot in each box. The weight $\operatorname{wt}(b) \in P_{\mathrm{cl}}$ of the element $b \in B_{\operatorname{cl}(\varpi_i)}$ having dots exactly in the j_1th box, j_2th box, $\ldots$, j_ith box with $1 \leqslant j_1 < j_2 < \cdots < j_i \leqslant \ell$, is given by $\gamma_{j_1} + \gamma_{j_2} + \cdots + \gamma_{j_i} \in P_{\mathrm{cl}}$, where we set $\gamma_j := \operatorname{cl}(\varpi_j) - \operatorname{cl}(\varpi_{j-1})$ for $1 \leqslant j \leqslant \ell$, with $\varpi_0 = \varpi_\ell = 0 \in P$.

Now, for each $i \in I_0 = \{1, 2, \ldots, \ell - 1\}$, we define a map $\Phi_i : B_{\operatorname{cl}(\varpi_i)} \to \mathbb{B}(\varpi_i)_{\mathrm{cl}}$ by $\Phi_i(b) = \eta_{\operatorname{wt}(b)}$ for $b \in B_{\operatorname{cl}(\varpi_i)}$, which is easily seen to be a (well-defined) bijection that preserves weights, since the subgroup $\mathring{W} \subset W$ is isomorphic to the symmetric group S_ℓ permuting the γ_j, $1 \leqslant j \leqslant \ell$, and since $\operatorname{cl}(\varpi_i) = \gamma_1 + \gamma_2 + \cdots + \gamma_i$. Actually, the map $\Phi_i : B_{\operatorname{cl}(\varpi_i)} \to \mathbb{B}(\varpi_i)_{\mathrm{cl}}$ is an isomorphism of P_{cl}-crystals. Indeed, from the definition in [NY97, § 2.4] of the Kashiwara operators e_j (respectively, f_j), $j \in I$, on $B_{\operatorname{cl}(\varpi_i)}$, we see that, for $b \in B_{\operatorname{cl}(\varpi_i)}$ and $j \in I$, $e_j b \neq \mathbf{0}$ (respectively, $f_j b \neq \mathbf{0}$) holds if and only if $(\operatorname{wt}(b))(h_j) = -1$ (respectively, $= 1$), since $\gamma_j = \operatorname{cl}(\varpi_j) - \operatorname{cl}(\varpi_{j-1}) = \operatorname{cl}(\Lambda_j) - \operatorname{cl}(\Lambda_{j-1})$ for $2 \leqslant j \leqslant \ell - 1$, and $\gamma_1 = \operatorname{cl}(\varpi_1) = \operatorname{cl}(\Lambda_1) - \operatorname{cl}(\Lambda_0)$, $\gamma_\ell = -\operatorname{cl}(\varpi_{\ell-1}) = -\operatorname{cl}(\Lambda_{\ell-1}) + \operatorname{cl}(\Lambda_0)$. Also, if $e_j b \neq \mathbf{0}$ (respectively, $f_j b \neq \mathbf{0}$), then we have $\operatorname{wt}(e_j b) = \operatorname{wt}(b) + \operatorname{cl}(\alpha_j)$ (respectively, $\operatorname{wt}(f_j b) = \operatorname{wt}(b) - \operatorname{cl}(\alpha_j)$). Therefore, from (5.2.3), we conclude that $\Phi_i : B_{\operatorname{cl}(\varpi_i)} \to \mathbb{B}(\varpi_i)_{\mathrm{cl}}$ is an isomorphism of P_{cl}-crystals.

Let $\mathbf{i} = (i_1, i_2, \ldots, i_n)$ be an arbitrary sequence of elements of $I_0 = \{1, 2, \ldots, \ell-1\}$ such that $i_1 \geqslant i_2 \geqslant \cdots \geqslant i_n$, and set $\lambda := \sum_{k=1}^{n} \varpi_{i_k} \in P_+^0$; we denote this sequence $\mathbf{i} = (i_1, i_2, \ldots, i_n)$ by λ^+ when we regard it as a partition (or Young diagram). In the following, we identify an element $\mu \in \operatorname{cl}(P_+^0)$ that is the weight of an element of the P_{cl}-crystal $\mathbb{B}_{\mathbf{i}}$ with the partition $(^{(1)}\mu, {}^{(2)}\mu, \ldots, {}^{(\ell)}\mu) \in (\mathbb{Z}_{\geqslant 0})^\ell$ of $|\lambda^+| := i_1 + i_2 + \cdots + i_n$ such that $\mu = \sum_{i=1}^{\ell-1} (^{(i)}\mu - {}^{(i+1)}\mu) \operatorname{cl}(\varpi_i)$ and $\sum_{i=1}^{\ell} {}^{(i)}\mu = |\lambda^+|$; note that the elements $\operatorname{cl}(\varpi_i)$, $1 \leqslant i \leqslant \ell - 1$, of $\mathfrak{h}^*/\mathbb{Q}\delta$ are linearly independent over $\mathbb{Q}$. Because $\mathbb{B}(\varpi_i)_{\mathrm{cl}}$ is isomorphic as a P_{cl}-crystal to $B_{\operatorname{cl}(\varpi_i)}$ through the map $\Phi_i : B_{\operatorname{cl}(\varpi_i)} \xrightarrow{\sim} \mathbb{B}(\varpi_i)_{\mathrm{cl}}$ and, in addition, $\operatorname{res}_{I_0} \mathbb{B}(\varpi_i)_{\mathrm{cl}}$ is connected for every $i \in I_0 = \{1, 2, \ldots, \ell - 1\}$, we deduce from [NY97, Corollary 4.3] along with (5.2.1) that, for every $\mu \in \operatorname{cl}(P_+^0)$ that is the weight of an element of $\mathbb{B}_{\mathbf{i}}$, the Kostka–Foulkes polynomial $K_{\mu^t, \lambda^+}(q)$ (as defined in [Mac95, ch. III, § 6]) associated to the conjugate (or transpose) μ^{t} of the partition μ and the partition λ^+ is identical to the following 1dsum:

$$X(\mathbb{B}_{\mathbf{i}}, \mu; q^{-1}) = \sum_{\substack{b \in \mathbb{B}_{\mathbf{i}} \\ e_j b = \mathbf{0} \ (j \in I_0) \\ \operatorname{wt} b = \mu}} q^{-D_{\mathbf{i}}(b)}. \qquad (5.2.4)$$

By combining this fact and Corollary 5.1.1 (along with (5.2.1)), we obtain the following expression for Kostka–Foulkes polynomials in terms of LS paths.

Corollary 5.2.4. *Assume that $\mathfrak{g}$ is of type $A_{\ell-1}^{(1)}$, and keep the notation above. Let $\mu \in \operatorname{cl}(P_+^0) = \sum_{i \in I_0} \mathbb{Z}_{\geqslant 0} \operatorname{cl}(\varpi_i)$ be the weight of an element of the P_{cl}-crystal $\mathbb{B}_{\mathbf{i}}$. Then, the following equation holds:*

$$K_{\mu^t, \lambda^+}(q) = \sum_{\substack{\eta \in \mathbb{B}(\lambda)_{\mathrm{cl}} \\ e_j \eta = \mathbf{0} (j \in I_0) \\ \eta(1) = \mu}} q^{-\operatorname{Deg}(\eta)}.$$

REFERENCES

AK97　T. Akasaka and M. Kashiwara, *Finite-dimensional representations of quantum affine algebras*, Publ. Res. Inst. Math. Sci. **33** (1997), 839–867.

BFKL06　G. Benkart, I. Frenkel, S.-J. Kang and H. Lee, *Level 1 perfect crystals and path realizations of basic representations at $q = 0$*, Int. Math. Res. Not. **2006** (2006), art. ID 10312.

Cha01　V. Chari, *On the fermionic formula and the Kirillov–Reshetikhin conjecture*, Int. Math. Res. Not. **2001** (2001), 629–654.

FL06　G. Fourier and P. Littelmann, *Tensor product structure of affine Demazure modules and limit constructions*, Nagoya Math. J. **182** (2006), 171–198.

FL07　G. Fourier and P. Littelmann, *Weyl modules, Demazure modules, KR-modules, crystals, fusion products and limit constructions*, Adv. Math. **211** (2007), 566–593.

FSS07　G. Fourier, A. Schilling and M. Shimozono, *Demazure structure inside Kirillov–Reshetikhin crystals*, J. Algebra **309** (2007), 386–404.

GL04　J. Greenstein and P. Lamprou, *Path model for quantum loop modules of fundamental type*, Int. Math. Res. Not. **2004** (2004), 675–711.

HKOTT02　G. Hatayama, A. Kuniba, M. Okado, T. Takagi and Z. Tsuboi, *Paths, crystals and fermionic formulae*, in *MathPhys odyssey 2001: integrable models and beyond*, Progress in Mathematical Physics, vol. 23, eds M. Kashiwara and T. Miwa (Birkhäuser, Boston, 2002), 205–272.

HKOTY99　G. Hatayama, A. Kuniba, M. Okado, T. Takagi and Y. Yamada, *Remarks on fermionic formula*, in *Recent developments in quantum affine algebras and related topics*, Contemporary Mathematics, vol. 248, eds N. Jing and K. C. Misra (American Mathematical Society, Providence, RI, 1999), 243–291.

Her06　D. Hernandez, *The Kirillov–Reshetikhin conjecture and solutions of T-systems*, J. reine angew. Math. **596** (2006), 63–87.

Her07　D. Hernandez, *Kirillov–Reshetikhin conjecture: the general case*, Preprint (2007), arXiv:0704.2838v3.

HN06　D. Hernandez and H. Nakajima, *Level 0 monomial crystals*, Nagoya Math. J. **184** (2006), 85–153.

HK02　J. Hong and S.-J. Kang, *Introduction to quantum groups and crystal bases*, Graduate Studies in Mathematics, vol. 42 (American Mathematical Society, Providence, RI, 2002).

Kac90　V. G. Kac, *Infinite dimensional Lie algebras*, third edition (Cambridge University Press, Cambridge, 1990).

KKMMNN92　S.-J. Kang, M. Kashiwara, K. C. Misra, T. Miwa, T. Nakashima and A. Nakayashiki, *Affine crystals and vertex models*, in *Infinite analysis: Part A*, Advanced Series in Mathematical Physics, vol. 16, eds A. Tsuchiya, T. Eguchi and M. Jimbo (World Scientific, River Edge, NJ, 1992), 449–484.

Kas94　M. Kashiwara, *Crystal bases of modified quantized enveloping algebra*, Duke Math. J. **73** (1994), 383–413.

Kas95　M. Kashiwara, *On crystal bases*, in *Representations of groups*, CMS Conference Proceedings, vol. 16, eds B. N. Allison and G. H. Cliff (American Mathematical Society, Providence, RI, 1995), 155–197.

Kas02　M. Kashiwara, *On level-zero representations of quantized affine algebras*, Duke Math. J. **112** (2002), 117–175.

Kas05　M. Kashiwara, *Level zero fundamental representations over quantized affine algebras and Demazure modules*, Publ. Res. Inst. Math. Sci. **41** (2005), 223–250.

Kir85　A. N. Kirillov, *Combinatorial identities and completeness of states of the Heisenberg magnet*, J. Soviet Math. **30** (1985), 2298–2310.

Kir87　A. N. Kirillov, *Completeness of states of the generalized Heisenberg magnet*, J. Soviet Math. **36** (1987), 115–128.

Kir89   A. N. Kirillov, *Identities for the Rogers dilogarithmic function connected with simple Lie algebras*, J. Soviet Math. **47** (1989), 2450–2459.

KKR88   S. V. Kerov, A. N. Kirillov and N. Yu. Reshetikhin, *Combinatorics, the Bethe ansatz and representations of the symmetric group*, J. Soviet Math. **41** (1988), 916–924.

KNT02   A. Kuniba, T. Nakanishi and Z. Tsuboi, *The canonical solutions of the Q-systems and the Kirillov–Reshetikhin conjecture*, Comm. Math. Phys. **227** (2002), 155–190.

KR90   A. N. Kirillov and N. Yu. Reshetikhin, *Representations of Yangians and multiplicities of the inclusion of the irreducible components of the tensor product of representations of simple Lie algebras*, J. Soviet Math. **52** (1990), 3156–3164.

KSS02   A. N. Kirillov, A. Schilling and M. Shimozono, *A bijection between Littlewood–Richardson tableaux and rigged configurations*, Selecta Math. (N.S.) **8** (2002), 67–135.

Lit94   P. Littelmann, *A Littlewood–Richardson rule for symmetrizable Kac–Moody algebras*, Invent. Math. **116** (1994), 329–346.

Lit95   P. Littelmann, *Paths and root operators in representation theory*, Ann. of Math. (2) **142** (1995), 499–525.

Mac95   I. G. Macdonald, *Symmetric functions and Hall polynomials*, second edition, Oxford Mathematical Monographs, (Clarendon Press, Oxford, New York, 1995).

NS03   S. Naito and D. Sagaki, *Path model for a level-zero extremal weight module over a quantum affine algebra*, Int. Math. Res. Not. **2003** (2003), 1731–1754.

NS05   S. Naito and D. Sagaki, *Crystal of Lakshmibai–Seshadri paths associated to an integral weight of level zero for an affine Lie algebra*, Int. Math. Res. Not. **2005** (2005), 815–840.

NS06   S. Naito and D. Sagaki, *Path model for a level-zero extremal weight module over a quantum affine algebra. II*, Adv. Math. **200** (2006), 102–124.

NS08   S. Naito and D. Sagaki, *Crystal structure on the set of Lakshmibai–Seshadri paths of an arbitrary level-zero shape*, Proc. London Math. Soc. (3) **96** (2008), 582–622.

Nak03   H. Nakajima, *t-analogs of q-characters of Kirillov–Reshetikhin modules of quantum affine algebras*, Represent. Theory **7** (2003), 259–274.

Nak04   H. Nakajima, *Extremal weight modules of quantum affine algebras*, in *Representation theory of algebraic groups and quantum groups*, Advanced Studies in Pure Mathematics, vol. 40, eds T. Shoji *et al.* (Mathematical Society of Japan, Tokyo, 2004), 343–369.

NY97   A. Nakayashiki and Y. Yamada, *Kostka polynomials and energy functions in solvable lattice models*, Selecta Math. (N.S.) **3** (1997), 547–599.

Oka07   M. Okado, $X = M$ *conjecture*, in *Combinatorial aspect of integrable systems*, MSJ Memoirs, vol. 17, eds A. Kuniba and M. Okado (Mathematical Society of Japan, Tokyo, 2007), 43–73.

Sch05   A. Schilling, *A bijection between type $D_n^{(1)}$ crystals and rigged configurations*, J. Algebra **285** (2005), 292–334.

Sch07   A. Schilling, $X = M$ *theorem: fermionic formulas and rigged configurations under review*, in *Combinatorial aspect of integrable systems*, MSJ Memoirs, vol. 17, eds A. Kuniba and M. Okado (Mathematical Soceity of Japan, Tokyo, 2007), 75–104.

SSh06   A. Schilling and M. Shimozono, $X = M$ *for symmetric powers*, J. Algebra **295** (2006), 562–610.

SSt06   A. Schilling and P. Sternberg, *Finite-dimensional crystals $B^{2,s}$ for quantum affine algebras of type $D_n^{(1)}$*, J. Algebraic Combin. **23** (2006), 317–354.

Satoshi Naito   naito@math.tsukuba.ac.jp
Institute of Mathematics, University of Tsukuba, Tsukuba, Ibaraki 305-8571, Japan

Daisuke Sagaki   sagaki@math.tsukuba.ac.jp
Institute of Mathematics, University of Tsukuba, Tsukuba, Ibaraki 305-8571, Japan