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ON A CONSTRUCTION OF COMPLETE SIMPLY-CONNECTED RIEMANNIAN MANIFOLDS WITH NEGATIVE CURVATURE

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Let M be a complete simply-connected riemannian manifold of even dimension m. J. Dodziuk and I.M. Singer ([D1]) have conjectured that $H_2^p(M) = 0$ if $p \neq m/2$ and dim $H_2^{m/2}(M) = \infty$, where $H_2^*(M)$ is the space of L_2-harmonic forms on M.

Recently, M. T. Anderson ([An]) constructed manifolds which are counterexamples to the J. Dodziuk-I. M. Singer conjecture. In this paper, we will discuss how to construct complete simply-connected riemannian manifolds with negative sectional curvature, by the idea of M. T. Anderson and a private advice of J. Dodziuk ([D2]).

THEOREM. *Let B be a complele riemannian C^∞-manifold with C^∞-connected boundary ∂B and f a C^∞-function on B. Suppose that B and f satisfy the following conditions;*

(B.1) *B has the riemannian simple double $2B$, that is the canonically endowed continuous metric of $2B$ is smooth.*

(B.2) *The sectional curvature K_B of B is negative, or $B := [0, \infty)$,*

(B.3) *B is simply-connected,*

(F.1) *f is a function of the geodesic distance r from ∂B,*

(F.2) *f is an odd function of r on a neighborhood of $r = 0$ and satisfies that $f'(0) = 1$, $f''(r) > 0$ for $r > 0$, and $f'''(0) > 0$.*

Let $M := (B \setminus \partial B) \times_{f|_{B \setminus \partial B}} S^n(1)$. Then there is the unique complete simply-connected riemannian manifold $\mathscr{M}$ with negative curvature which is the completion of M.

Remark. Any function on $[0, \infty)$ can be considered as a function satisfying (F.1) under the assumptions (B.1)-(B.3).

Manifolds are supposed to be connected paracompact Hausdorff spaces.

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1.

J. Kazdan-F. Warner ([K-W]) proved that, for a C^∞-metric g on $R^2\setminus\{0\}$, there is a C^∞-metric $\tilde{g}$ on R^2 such that $\tilde{g}$ restricted to $R^2\setminus\{0\}$ is g. First, we will generalize their result.

LEMMA 1.1 (cf. [K-W], [O-N, p. 31]). *If $f(t)$ is a real valued C^∞-even function on R, then $f(r)$ is a C^∞-function on R^n, where $r:=((x^1)^2+\cdots+(x^n)^2)^{1/2}$.*

LEMMA 1.2. *Let $f\colon R^m\times R^{1+n}\to R$ be a continuous function. If f satisfies the following conditions;*

(1.2.1) *f is of class C^∞ on $(R^m\times R^{1+n})\setminus(R^m\times\{0\})$,*

(1.2.2) *f is invariant under $\{I_m\}\times O(n+1)$, where $\{I_m\}$ is the unit group on R^m and $O(n+1)$ is the rotation group on R^{1+n},*

(1.2.3) *f is of class C^∞ on $R^m\times l$ for any straight line $l\subset R^{1+n}$ through the origin, then f is of class C^∞ on $R^m\times R^{1+n}$.*

Proof. We introduce two coordinates on $R^m\times R^{1+n}$, one is the usual Cartesian coordinates $(x^1,\cdots,x^m,z^0,z^1,\cdots,z^n)$ and one is $(x^1,\cdots,x^m,r,y^1,\cdots,y^n)$ where $(r,y^1,\cdots,y^n)$, $(r>0)$, the polar coordinates on R^{1+n}. By (1.2.2), we can consider that f is a function with only $(x^1,\cdots,x^m,r)$ variables.

Step 1. We take a point $x_o:=(x_o^1,\cdots,x_o^m)$ and fix it. (1.2.3), (1.2.2) and Lemma 1.1 imply that $f_o(r):=f(x_o,r)$, $r:=((z^1)^2+\cdots+(z^n)^2)^{1/2}$, can be considered to be of class C^∞ on R^{1+n}. Since $(\partial/\partial x^i)f$ are invariant under $\{I_m\}\times O(n+1)$ and are of class C^∞ on $R^m\times l$ for a fixed l, if we choose any sequence $\{(x_n,z_n)\}$ in $R^m\times R^{1+n}$ converging to $(x_o,0)$, then we have

$$\begin{aligned}&\left|\left(\frac{\partial}{\partial x^j}\right)f(x_n,z_n)-\left(\frac{\partial}{\partial x^j}\right)f(x_o,0)\right|\\&\qquad=\left|\left(\frac{\partial}{\partial x^j}\right)f(x_n,\pi(z_n))-\left(\frac{\partial}{\partial x^j}\right)f(x_o,0)\right|\longrightarrow 0\\&\qquad\qquad((x_n,z_n)\longrightarrow(x_o,0))\,,\end{aligned}$$

where $\pi\colon R^{1+n}\to l_+:=\{r\in l\,|\,r\geq 0\}$ is the canonical projection. Thus, together with by (1.2.1), we have that $(\partial/\partial x^j)f$ are continuous on $R^m\times R^{1+n}$, and, inductively, $(\partial^{\alpha_1+\cdots+\alpha_k}/(\partial x^{i_1})^{\alpha_1}\cdots(\partial x^{i_k})^{\alpha_k})f$ are continuuos on $R^m\times R^{1+n}$.

Step 2. We set

$$F(x, r) := \left(\frac{\partial^{\alpha_1+\cdots+\alpha_k}}{(\partial x^{i_1})^{\alpha_1}\cdots(\partial x^{i_k})^{\alpha_k}}\right) f(x, r),\ \alpha_1 + \cdots + \alpha_k \geq 0\,.$$

Note that $F_0(r)$ is of class C^∞ on R^{1+n}. For example, since

$$\frac{\partial^2}{\partial z^\alpha \partial z^\beta} F(x, r) = \begin{cases} \left(\dfrac{1}{r}\dfrac{\partial}{\partial r}\right)^2 F(x, r) z^\alpha z^\beta, & \alpha \neq \beta\,, \\ \left(\dfrac{1}{r}\dfrac{\partial}{\partial r}\right)^2 F(x, r)(z^\alpha)^2 + \dfrac{1}{r}\dfrac{\partial}{\partial r} F(x, r), & \alpha = \beta\,, \end{cases}$$

and $(1/r\cdot\partial/\partial r)^p F(x, r)$ $(p = 0, 1, 2, \cdots)$ are even functions in r, by the same way as Step 1, $(\partial^2/\partial z^\alpha \partial z^\beta) F(x, r)$ are continuous on $R^m \times R^{1+n}$. More generally, we have

$$\left(\frac{\partial^{\beta_1+\cdots+\beta_s+\alpha_1+\cdots+\alpha_k}}{(\partial z^{j_1})^{\beta_1}\cdots(\partial z^{j_s})^{\beta_s}(\partial x^{i_1})^{\alpha_1}\cdots(\partial x^{i_k})^{\alpha_k}}\right) f \qquad (\beta_1 + \cdots + \beta_s > 0,\ \alpha_1 + \cdots + \alpha_k \geq 0)$$

are continuous on $R^m \times R^{1+n}$. Therefore, f is of class C^∞ on $R^m \times R^{1+n}$. □

PROPOSITION 1.3. *Let B be a complete riemannian manifold with C^∞-boundary ∂B and f a C^∞-function on B. Suppose that B and f satisfy the following conditions;*

(1.B.1) *B has the riemannian simple double $2B$,*

(1.F.1) *$f(x) > 0$ if $x \in B \setminus \partial B$, and f is an odd function on a neighbourhood of ∂B of the arc-length r in the inner normal direction to ∂B.*

(1.F.2) *$\|\operatorname{grad} f\|(x) = 1$ if $x \in \partial B$.*

Let M be $(B\setminus\partial B) \times_{f|B\setminus\partial B} S^n(1)$. Then there is the unique complete riemannian manifold $\mathscr{M}$ without boundary such that $\mathscr{M}$ is the completion of M.

Proof. Let (U, φ) be a local path of ∂B and N the ε-collar neighborhood of U in B, We define a manifold $\mathscr{N}$ by

$$\mathscr{N} := (N\setminus U) \times_{f|N/U} S^n(1)\,.$$

Imbedding of $S^n(1)$ into R^{1+n}, we define a diffeomorphism Ψ of $\mathscr{N}$ into $R^m \times R^{1+n}$ by

$$\Psi : ((x, \exp rX), y) \longrightarrow (\varphi(x), r(y))\,,$$

where $X \in T_x B$ is the unit inner normal vector to ∂B and $0 < r < \varepsilon$.

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We take the riemannian metric g of $\Psi(\mathcal{N})$ so that Ψ may become an isometry. Note that g can be extended to the continuous metric $\bar{g}$ of $\overline{\Psi(\mathcal{N})}$ by the natural way. We have only to show that $\bar{g}$ is of class C^∞ at the origin. Let $(x^1, \cdots, x^m, x^{m+1}, \cdots, x^{m+1+n})$ be the Cartesian coordinates of $R^m \times R^{1+n}$. And we adopt the ranges of indices;

$$1 \leq i, j \leq m \quad \text{and} \quad m+1 \leq \alpha, \beta \leq m+1+n\,.$$

It is clear from Lemma 1.2 that $\bar{g}_{ij} := \bar{g}(\partial/\partial x^i, \partial/\partial x^j)$ is of class C^∞. It follows from Lemma 1.2 again that $(1/r)\bar{g}(\partial/\partial x^i, \partial/\partial r)$ is of class C^∞. Therefore $\bar{g}_{i\alpha} := \bar{g}(\partial/\partial x^i, \partial/\partial x^\alpha) = x^\alpha(1/r)\bar{g}(\partial/\partial x^i, \partial/\partial r)$ is of class C^∞. Finally, we have that

$$\begin{aligned}
\bar{g}_{\alpha\beta} &:= \bar{g}(\partial/\partial x^\alpha, \partial/\partial x^\beta) \\
&= \tilde{g}_{\alpha\beta} + \frac{f^2(x, r) - r^2}{r^4} r^4 g_{S^n}(\partial/\partial x^\alpha, \partial/\partial x^\beta) \\
&= \tilde{g}_{\alpha\beta} + \frac{f^2(x, r) - r^2}{r^4} (r^2 \tilde{g}_{\alpha\beta} - x^\alpha x^\beta)
\end{aligned}$$

where $\tilde{g}$ is the standard metric on $R^m \times R^{1+n}$. It follows from Lemma 1.2 that $(f^2 - r^2)/r^4$ is of class C^∞. Therefore, $\bar{g}_{\alpha\beta}$ is of class C^∞. □

Remark 1.4 ([B] p. 269). If $m = 0$ in Proposition 1.3, we can get a theorem of J. Kazdan-F. Warner; If we identify $\{x \in R^{1+n} \mid 0 < |x| < \varepsilon\}$ with $(0, \varepsilon) \times S^n$ in polar coordinates, the C^∞-riemannian metric $dt^2 + \varphi(t)^2 \hat{g}_0$ (where t is the parameter on $(0, \varepsilon)$ and $\hat{g}_o$ a metric on S^n) extends to a C^∞-riemannian metric on $\{x \in R^n \mid |x| < \varepsilon\}$ if and only if $\hat{g}_o$ is λg_{can} where g_{can} is the canonical metric on S^n and λ some positive constant, and $(1/\lambda)\varphi$ is the restriction on $(0, \varepsilon)$ of *a* C^∞ odd function on $(0, \varepsilon)$ with $(1/\lambda)\varphi'(0) = 1$.

OBSERVATION 1.5. Since $\mathscr{M}$ is a completion of M as a metric space, by means of theory of metric spaces, we can see that the condition (1.B.1) is necessary for the existence of $\mathscr{M}$. The condition (1.B.1) is strictly stronger than the condition that ∂B is totally geodesic. For example, consider the surface of revolution of the graph

$$x \in [0, \infty) \longrightarrow x^3 - 3x^2 + 6 \in R\,.$$

2.

LEMMA 2.1 ([B-O]). *Let* $M := B \times_f F$ *be a warped product with a warping function* f *where* B *and* F *are any riemannian manifolds. Let* π_1

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and π_2 be the natural projections of M onto B and F respectively. Let Π be a 2-plane tangent to M at x and $\{X+V, Y+W\}$ an orthonormal basis for Π, where $X, Y \in T_{\pi_1(x)}B$ and $V, W \in T_{\pi_2(x)}F$. The sectional curvature $K(\Pi)$ of Π in M is given by

$$K(\Pi) = K^1_{X,Y} + K^2_{X,Y,V,W} + K^3_{V,W}\,,$$

where

$$\begin{aligned} K^1_{X,Y} &:= K_B(X, Y)\|X \wedge Y\|^2_B\,, \\ K^2_{X,Y,V,W} &:= -f(\pi_1(x))\{\|W\|^2_F((\nabla_B)^2 f)(X, X) - 2\langle V, W\rangle_F((\nabla_B)^2 f)(X, Y) \\ &\qquad + \|V\|^2_F((\nabla_B)^2 f)(Y, Y)\}\,, \\ K^3_{V,W} &:= f^2(\pi_1(x))\{K_F(V, W) - \|\operatorname{grad} f\|^2_B\}\|V \wedge W\|^2_F\,, \end{aligned}$$

and $\nabla_{(\cdot)}$ and $K_{(\cdot)}$ are the covariant derivative and the sectional curvature of $(\,.\,)$ respectively and $(\nabla_B)^2 f$ is the Hessian of f.

We shall prove Theorem. By the conditions of B, there is a diffeomorphism $\Psi\colon \partial B \times [0, \infty) \to B$ such that, for any $x \in \partial B$, $\tau_x(r) := \Psi(x, r)$ is the geodesic parametrized by the arc-length r, starting at x and normal to ∂B. (Thus, Remark after Theorem holds.) Moreover, we have that $\pi_1(\mathscr{M}) = \pi_1(\partial B \times R^{1+n}) = \pi_1(\partial B) = 0$, because ∂B is simply-connected by the conditions. Since Lemma 2.1, (B.2) and (F.2) imply that K^1, K^2 and K^3 are non-positive on M and at least one of them is strictly negative on M, it is enough to show that at least one of K^1, K^2 and K^3 is strictly negative if $r \to 0$. Let x_0 be any point of ∂B and X_r, Y_r, V_r, W_r any vector fields along $\tau_{x_0}(r)$, where X_r, Y_r are horizontal and V_r, W_r are vertical if $r \neq 0$.

Case 1. The case that X_0 and Y_0 are linearly independent. We have

$$K^1_{X_0, Y_0} < 0\,.$$

Case 2. The case that V_0 and W_0 are linearly independent. (F.1) and (F.2) imply that

$$f^2(r) = r^2 + 2ar^4 + \cdots\,, \qquad a > 0$$

and

$$\begin{aligned} \|\operatorname{grad} f(r)\|^2_B &\geq \langle \operatorname{grad} f(r), \partial/\partial r\rangle^2_B \\ &= \left(\frac{\partial f}{\partial r}\right)^2 \\ &= 1 + 6ar^2 + \cdots\,. \end{aligned}$$

Then we have

$$\frac{1-\|\operatorname{grad} f(r)\|_B^2}{f^2(r)} \leq \frac{1-(1+6ar^2+\cdots)}{r^2+2ar^4+\cdots}$$
$$= \frac{-6a+O(r)}{1+O(r)}.$$

Therefore we have

$$\lim_{r\to 0} K^3_{V_r,W_r} \leq -6a < 0.$$

Case 3. The case except Case 1 and Case 2. We can choose X_r, Y_r, V_r and W_r such that $Y_r = c_1 X_r$ and $W_r = c_2 V_r$, where c_1 and c_2 are constants with $c_1 \neq c_2$. Let Π_r be the 2-plane spanned by the orthonormal basis $\{X_r + V_r,\ Y_r + W_r\}$. Then we have

$$K(\Pi_r) = -\frac{((\nabla_B)^2 f)_{X_r,X_r}}{f(r)\langle X_r, X_r\rangle_B}.$$

To get $\lim_{r\to 0} K(\Pi_r) < 0$, it is enough to show that

$$\lim_{r\to 0} \frac{((\nabla_B)^2 f)_{X_r,X_r}}{f(r)} > 0$$

under the assumption $\|X_r\|_B = 1$.

$$\frac{((\nabla_B)^2 f)_{X_r,X_r}}{f(r)} = \frac{f''(r)(\nabla_{X_r} r)^2 + f'(r)(\nabla^2 r)_{X_r,X_r}}{f(r)},$$

and (F.2) imply the claim. Therefore we have Theorem.

EXAMPLE 2.2 (cf. [M]). Let R^m be given a negatively curved metric, and $B := [0, \infty) \times_\varphi R^m$ the warped product with the warping function φ such that (1) φ is a C^∞-even function in a neighbourhood of 0, (2) $\varphi > 0$, and (3) $\varphi'' > 0$. Then B satisfies the conditions of Theorem.

Comment of counter example of M. T. Anderson. If, in Theorem, we set the following, we can get his example; $2B := H^{2p}(-a^2)$, $\partial B :=$ the totally geodesic hyperplane H^{2p-1} of $H^{2p}(-a^2)$ and $f(r) := \sinh r$.

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References

[A] M. F. Atiyah, Elliptic operators, discrete groups and von Neumann algebras, Astérisque, **32–33** (1976), 43–72.

[An] M. T. Anderson, L^2-harmonic forms and a conjecture of Dodziuk-Singer, Bull. Amer. Math. Soc., **13** (1985), 163–165.

[B] A. L. Besse, Einstein manifolds, Ergeb. Math. Grenzgeb. 3. folge. Band **10**, 1987, Springer-Verlag.

[B-O] R. L. Bishop-B. O'Neill, Manifolds of negative curvature, Trans. Amer. Math. Soc., **145** (1969), 1–49.

[D1] J. Dodziuk, L^2-harmonic forms on rotationally symmetric Riemannian manifolds, Proc. of Amer. Math. Soc., **77** (1979), 395–400.

[D2] J. Dodzuik, A private letter.

[K-W] J. L. Kazdan-F. W. Warner, Curvature functions for open 2-manifolds, Ann. of Math., **99** (1974), 203–219.

[M] H. Mori, Remarks on the Riemannian metric on the double of a manifold with boundary, preprint.

[O-N] T. Ochiai-J. Noguchi, Geometric Function Theory (in Japanese), Iwanami, 1986.

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