THE NUMBER OF k-COLOURED GRAPHS ON LABELLED NODES

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Introduction. By a labelled graph we shall mean a set of "nodes," distinguishable from one another and denoted by $A_1, A_2, \ldots,$ and a collection of "edges" viz., pairs of nodes. We say that an edge "joins" the pair of nodes which specifies it. We further stipulate that at most one edge joins any two nodes, and that no edge joins a node to itself.

By a "colouring" of a graph in k colours we shall mean a mapping of the nodes of the graph *onto* a set of k colours $C_1, C_2, \ldots, C_k$ such that no two nodes which are joined by an edge are mapped onto the same colour. A graph so coloured in exactly k colours will be called a k-coloured graph. Since it is usually possible to colour a graph in more than one way, there will, in general, be many k-coloured graphs corresponding to a given graph.

The object of this paper is to derive an expression for the number of labelled k-coloured graphs on a given number of nodes. This is a generalization of a result given by Gilbert (**2**, § 1). Suppose we are given a set of n nodes $A_1, A_2, \ldots, A_n$, a set of positive non-zero integers $n_1, n_2, \ldots, n_k$ such that $n_1 + n_2 + \ldots + n_k = n$ and a set of integers $e_{\alpha\beta}(\alpha, \beta = 1, 2, \ldots, k)$. We shall count the number of k-coloured graphs on these n nodes which are such that n_α nodes are allocated the colour C_α and $e_{\alpha\beta}$ edges join nodes allocated the colour C_α to nodes allocated the colour C_β, $(\alpha, \beta = 1, 2, \ldots, k)$. We let $E = \sum_{\alpha<\beta} e_{\alpha\beta}$ be the total number of edges.

First allocate the colours to the various nodes. This is possible in

$$\frac{n!}{n_1!n_2!\ldots n_k!}$$

different ways. Next consider the number of ways of choosing $e_{\alpha\beta}$ edges joining nodes coloured in C_α and C_β. There are $n_\alpha n_\beta$ possible edges, so the choice can be made in

$$\binom{n_\alpha n_\beta}{e_{\alpha\beta}}$$

different ways. Thus the total number of graphs is

$$(1) \qquad \frac{n!}{n_1!n_2!\ldots n_k!} \prod_{\alpha<\beta} \binom{n_\alpha n_\beta}{e_{\alpha\beta}}.$$

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To find the number of graphs having E edges we must sum expression (1) over all sets $\{e_{\alpha\beta}\}$ such that $\sum e_{\alpha\beta} = E$. Since

$$\binom{n_\alpha n_\beta}{e_{\alpha\beta}}$$

is the coefficient of

$$t^{e_{\alpha\beta}} \quad \text{in} \quad (1+t)^{n_\alpha n_\beta},$$

this sum is the coefficient of t^E in

$$\frac{n!}{n_1!n_2! \ldots n_k!} \prod_{\alpha<\beta} (1+t)^{n_\alpha n_\beta} = \frac{n!}{n_1!n_2! \ldots n_k!} (1+t)^{\Sigma\, n_\alpha n_\beta}$$

and is therefore

$$(2) \qquad \frac{n!}{n_1!n_2! \ldots n_k!} \binom{\frac{1}{2}n^2 - \frac{1}{2}\sum n_\alpha{}^2}{E}$$

since

$$\sum n_\alpha n_\beta = \tfrac{1}{2}(\sum n_\alpha)^2 - \tfrac{1}{2}\sum n_\alpha{}^2.$$

In the special case when there are n colours and each node receives a different colour we have $n_1 = n_2 = \ldots = n_n = 1$ and (2) reduces to

$$n!\binom{\frac{1}{2}n(n-1)}{E}.$$

Since, under these conditions, every graph on n nodes can be coloured in $n!$ different ways, the number of graphs on n labelled nodes having E edges is seen to be

$$\binom{\frac{1}{2}n(n-1)}{E}.$$

This result is easily obtained directly (**2**, p. 405).

To find the total number of k-coloured graphs on n nodes and E edges we need to sum (2) over all sets $\{n_\alpha\}$ such that $\sum n_\alpha = n$. Thus we obtain

$$\sum_{(n)} \frac{n!}{n_1!n_2! \ldots n_k!} \binom{\frac{1}{2}n^2 - \frac{1}{2}\sum n_\alpha{}^2}{E}$$

but it does not appear that this formula is very amenable to manipulation.

Let us now remove the restriction on the number of edges in the graph, and consider the number of k-coloured graphs (whatever the number of edges) which are associated in the above way with the set $\{n_\alpha\}$. This number is obtained by summing (2) for all possible values of E, and is thus

$$(3) \qquad \frac{n!}{n_1!n_2! \ldots n_k!} 2^{\frac{1}{2}n^2 - \frac{1}{2}\Sigma n_\alpha{}^2}.$$

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The total number of k-coloured graphs on n labelled nodes can now be found. We denote it by $F_n(k)$, and we see that

$$F_n(k) = \sum_{(n)} \frac{n!}{n_1!n_2! \ldots n_n!} 2^{\frac{1}{2}n^2 - \frac{1}{2}\Sigma n_\alpha^2}$$

$$= n!2^{\frac{1}{2}n^2} \text{ times the coefficient of } x^n \text{ in}$$

$$\left[\sum_{s=1}^{\infty} \frac{2^{-\frac{1}{2}s^2}}{s!} x^s\right]^k.$$

Hence

$$\sum_{n=1}^{\infty} 2^{-\frac{1}{2}n^2} F_n(k) \frac{x^n}{n!} = \left[\sum_{s=1}^{\infty} \frac{2^{-\frac{1}{2}s^2}}{s!} x^s\right]^k \tag{4}$$

from which $F_n(k)$ may be calculated.

2. For convenient calculation of $F_n(k)$ we may write (4) as

$$\sum_{n=1}^{\infty} 2^{-\frac{1}{2}n^2} F_n(k) \frac{x^n}{n!} = \left(\sum_{r=1}^{\infty} 2^{-\frac{1}{2}r^2} F_r(k-1) \frac{x^r}{r!}\right)\left(\sum_{s=1}^{\infty} 2^{-\frac{1}{2}s^2} \frac{x^s}{s!}\right)$$

whence, equating coefficients of x^n, we obtain

$$F_n(k) = \sum_{r=1}^{n-1} \binom{n}{r} 2^{r(n-r)} F_r(k-1) \tag{5}$$

which gives the numbers of k-coloured graphs in terms of the numbers of $(k-1)$-coloured graphs. Some values of $F_n(k)$ are given in Table I.

TABLE I

n \ k	1	2	3	4	5	6	7
1	1	0	0	0	0	0	0
2	1	4	0	0	0	0	0
3	1	24	48	0	0	0	0
4	1	160	1152	1536	0	0	0
5	1	1440	30720	122880	122880	0	0
6	1	18304	1152000	10813440	29491200	23592960	0
7	1	330624	65630208	1348730880	7707033600	15854469120	10569646080

3. If we wish to count the total number of graphs coloured in k or fewer colours, we proceed as before but remove the restriction that the n_α's are non-zero, and allow them to be any non-negative integers. Denoting the required number of graphs by $M_n(k)$ we obtain

$$\sum_{n=0}^{\infty} 2^{-\frac{1}{2}n^2} M_n(k) \frac{x^n}{n!} = \left[\sum_{s=0}^{\infty} 2^{-\frac{1}{2}s^2} \frac{x^s}{s!}\right]^k. \tag{6}$$

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By the method of § 2 we obtain from (6) the relation

$$M_n(k) = \sum_{r=0}^{n} \binom{n}{r} 2^{r(n-r)} M_r(k-1) \tag{7}$$

with $M_0(k) = 1$.

$M_n(k)$, unlike $F_n(k)$, is a polynomial in k of degree n. This follows either from (7) by mathematical induction, or from the fact that $M_n(k)$ is the sum of the chromatic polynomials* of all graphs on n nodes, each polynomial being counted as many times as there are ways of labelling the corresponding graph. Some values of $M_n(k)$ are given in Table II.

TABLE II

n \ k	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9
2	1	6	15	28	45	66	91	120	153
3	1	26	123	340	725	1326	2191	3368	4905
4	1	162	1635	7108	20805	48486	97447	176520	296073
5	1	1442	35043	254404	1058885	3216486	7986727		
6	1	18306	1206915	15531268					
7	1	330626	66622083						

The first four polynomials are

$$M_1(k) = k,$$
$$M_2(k) = 2k^2 - k,$$
$$M_3(k) = 8k^3 - 12k^2 + 5k,$$

and

$$M_4(k) = 64k^4 - 192k^3 + 208k^2 - 79k.$$

4. If $f_n(k)$ denotes the number of *connected* k-coloured graphs, it can be shown by the methods used in **(2)** that

$$\sum_{n=0}^{\infty} F_n(k) \frac{x^n}{n!} = \exp\left\{ \sum_{n=1}^{\infty} f_n(k) \frac{x^n}{n!} \right\} \tag{8}$$

with $F_0(k) = 1$.

Differentiating both sides of (8) and equating coefficients of x^{n-1} we obtain

$$F_n(k) = \sum_{r=1}^{n} \binom{n-1}{r-1} F_{n-r}(k) f_r(k)$$

or

*For the definition of the chromatic polynomial of a graph see **(1)**.

$$f_n(k) = F_n(k) - \sum_{r=1}^{n-1} \binom{n-1}{r-1} F_{n-r}(k) f_r(k) \tag{9}$$

giving $f_n(k)$ in terms of $f_{n-1}(k), f_{n-2}(k)$, when $F_n(k), F_{n-1}(k), \ldots$, are known.

References

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2. E. N. Gilbert, *Enumeration of labelled graphs*, Can. J. Math., *8* (1956), 405–411.

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