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W-GROUPS AND NEAR-RING MODULES

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ABSTRACT. W-groups are treated in the more general context of near-ring modules. It is shown that they share properties of d.g. modules with respect to central series. In particular, results of Magomaev are obtained and used to show that the group-theoretic and module-theoretic central series coincide for W-groups.

1. **Introduction.** For basic definitions and results on near-rings and their modules see [1]. In this paper all near-rings R (and their modules) will be unitary and will satisfy the axiom $x \cdot 0=0$ for all $x \in R$.

A ring W is called a binomial ring if it is an integral domain containing Z for which $r \in W$ implies

$$\binom{r}{n} = \frac{r(r-1)\cdots(r-n+1)}{n!} \in W \quad \text{for all} \quad n \in Z^+.$$

A W-group is then a locally nilpotent group G which is a W near-ring module subject to the following axioms:

$$W_1 \quad r(m+x-m) = m+rx-m \quad \text{for all} \quad r \in R,\, m, x \in G$$

$$W_2 \quad rx_1+rx_2+\cdots+rx_n = \sum_{i=1}^{c} \binom{r}{i} t_i(x) \quad \text{where} \quad r \in W,$$

c is the class of the nilpotent group generated by $x_1, \ldots, x_n$, and the t_i are the Petresco words defined by the recurrence relations

$$\sum_{i=1}^{k} \binom{k}{i} t_i(x) = kx_1+kx_2+\cdots+kx_n \qquad k = 1, 2, \ldots$$

Examples and elementary properties of W-groups are found in [2]. It is the purpose of this paper to derive properties of the upper central series of W-groups which agree with those in [3] by viewing them as near-ring modules; and further to show that the group-theoretic and module-theoretic central series coincide. This was shown for the d.g. case in [4] and in this connection we note that, as in the d.g. case, normal W-subgroups of a W-group are W-submodules [2].

2. **Upper Central Series.** We begin by considering near-ring modules ${}_RM$ which have the property

$$W_1 \quad r(m+x-m) = m+rx-m \quad \text{for all} \quad r \in R,\, m, x \in M.$$

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These will be called R-powered modules. Note that every inner automorphism of $(M, +)$ is then an R-module automorphism. Let $\{Z_Z^i\}$ be the upper central series of $(M, +)$ and write $Z_Z^1 = Z_Z$.

LEMMA 1. *If ${}_RM$ is R-powered, the centralizer $C_Z(A)$ of any subset A of M is an R-subgroup.*

Proof. $C_Z(A)$ is certainly a subgroup and if $x = a + x - a$ for any $a \in A$ then $rx = r(a + x - a) = a + rx - a$ by W_1 so $rx \in C_Z(A)$.

In [4] an upper central series of R-submodules of M was defined inductively by: $Z_R^0 = (0)$ and for $i \geq 1$ $Z_R^i = \{x \mid r(x + sy) - rx - rsy \in Z_R^{i-1}$ for all $y \in M$, $r, s \in R\}$. Equivalently $Z_R^i / Z_R^{i-1} = Z_R^1(M/Z_R^{i-1})$. Now define a central series of R-subgroups of an R-powered module M inductively as follows: Put $Z^1 = Z_Z^1$. Since the Z_R^{i-1} are R-submodules, each factor M/Z_R^{i-1} is an R-module which is R-powered since M is. Thus its group-theoretic center is a normal R-subgroup by lemma 1 and so by the isomorphism theorems [1] has the form Z^i/Z_R^{i-1} for some normal R-subgroup Z^i of M. This defines Z^i for $i \geq 2$. Thus we have $Z_R^i \subseteq Z^i \subseteq Z_Z^i$ and in the d.g. case equality holds [4]. Similarly for W-groups we have

THEOREM 1. *If G is a W-group, $Z_W^i = Z_Z^i$.*

Proof. Since $Z_W^1 \subseteq Z_Z^1$, we suppose $m \in Z_Z^1$, and show $r(b + sm) = rsm + rb$. Now $rb + rsm = r(b + sm) + \Sigma_2^c \binom{r}{i} t_i$ where c is the nilpotency class of $\langle b, sm \rangle$. By lemma 1 $sm \in Z_Z^1$ so $c = 1$ and $rb + rsm = rsm + rb = r(b + sm)$.

Inductively suppose $Z_Z^{i-1} = Z_W^{i-1}$. Then

$$Z_W^i / Z_W^{i-1} = Z_W^1(G/Z_W^{i-1}) = Z_Z^1(G/Z_W^{i-1})$$

by the first part of the proof

$$= Z_Z(G/Z_Z^{i-1}) = Z_Z^i/Z_Z^{i-1} = Z_Z^i/Z_W^{i-1}$$

Hence $Z_W^i = Z_Z^i$.

COROLLARY ([3]). *If G is a W-group, the terms and factors of the group-theoretic upper central series are W-groups.*

3. **Lower Central Series.** In [4] the commutator of two R-subgroups A, B of ${}_RM$ was defined by $[A, B]_R =$ the R-subgroup generated by $\{r(a + b) - ra - rb \mid r \in R, a \in A, b \in B\}$. It was shown that in the d.g. case $[A, B]_R = [A, B]_Z$ and the following result was used:

PROPOSITION 1. ([4]) *If A, B are normal subgroups of the (multiplicative) group G then for all $a \in A$, $b \in B$, $n \in Z^+$*

$$\left(\prod_1^n b_i a_i\right)\left(\prod_1^n b_i\right)^{-1}\left(\prod_1^n a_i\right)^{-1} \in [A, B]_Z.$$

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This result depended on the obvious fact that

$$x \in [A, B]_Z \Rightarrow gxg^{-1} \in [A, B]_Z \quad \text{for all} \quad g \in G. \tag{$*$}$$

PROPOSITION 2. *If A, B are normal subgroups of G then for all $a \in A$, $b \in B$, $n \in Z^+$ $(\prod_1^n a_ib_i)(\prod_1^n b_i)^{-1}(\prod_1^n a_i)^{-1} \in [A, B]_Z$.*

Proof. For $n=1$ $abb^{-1}a^{-1}=e \in [A, B]$. For $n \geq 2$,

$$\left(\prod_1^n a_ib_i\right)\left(\prod_1^n b_i\right)^{-1}\left(\prod_1^n a_i\right)^{-1} = a_1\left(\prod_2^n b_{i-1}a_i\right)\left(\prod_2^n b_{i-1}\right)^{-1}\left(\prod_2^n a_i\right)^{-1} a_1^{-1} \in [A, B]_Z$$

by Proposition 1 and $(*)$.

THEOREM 2. *If A, B are normal subgroups of the W-group G and $[A, B]_Z$ is a W-subgroup, then $[A, B]_Z = [A, B]_W$.*

Proof. Since $[A, B]_Z \subseteq [A, B]_W$, it remains to show each generator of $[A, B]_W$ is in $[A, B]_Z$. Now $rb+ra=r(b+a)+\sum_2^c \binom{r}{i} t_i$ where c is the nilpotency class of the subgroup generated by a and b. Therefore

$$\begin{aligned} r(a+b)-ra-rb = r(a+b)-(rb+ra) &= r(a+b)-\left[r(b+a)+\sum_2^c \binom{r}{i} t_i\right] \\ &= r(a+b)-\sum_2^c \binom{r}{i} t_i - r(b+a). \end{aligned}$$

Since $[A, B]_Z$ is a W-submodule of G,

$$r(m-a-b+a+b)-rm \in [A, B]_Z \quad \text{for all} \quad a \in A,\ b \in B,\ m \in G,\ r \in W.$$

Taking $m = b+a$, $r(a+b)-r(b+a) \in [A, B]_Z$ (†)

Since $n(-b-a)+na+nb \in [A, B]_Z$ for all $n \in Z^+$ by (the additive form of) Proposition 2 therefore $t_2=2(-b-a)+2a+2b \in [A, B]_Z$. Inductively, if $t_i \in [A, B]_Z$ for all $i<n$ then $\binom{n}{i} t_i \in [A, B]_Z$ (since $[A, B]_Z$ is assumed to be a W-subgroup) so $t_n = -\binom{n}{n-1} t_{n-1} - \cdots - \binom{n}{2} t_2 + n(-b-a)+na+nb \in [A, B]_Z$. Hence

$$x = \sum_{i=2}^c \binom{r}{i} t_i \in [A, B]_Z.$$

Finally $r(a+b)-ra-rb=r(a+b)-r(b+a)+r(b+a)-x-r(b+a)$ is in $[A, B]_Z$ by $(*)$ and (†) and the proof is complete.

Let $\{G_R^i\}$, $\{G_Z^i\}$ denote the lower central series of R-submodules ([4]) and subgroups respectively. Magomaev has shown [3] that if G is nilpotent, every G_Z^i is a W-subgroup of G. Using this we have

THEOREM 3. *If G is a nilpotent W-group, $G_W^i = G_Z^i$ for all i.*

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Proof. Since $G_Z^1=[G, G]_Z$ is an R-submodule, $G_Z^1=G_W^1$ by Theorem 2. Inductively if $G_Z^{i-1}=G_W^{i-1}$ then

$$G_Z^i = [G, G_Z^{i-1}]_Z = [G, G_W^{i-1}]_Z = [G, G_W^{i-1}]_W$$

by Theorem 2

$$= G_W^i.$$

REMARK. As with the upper central series, one can define a series $\{G^i\}$ intermediate to $\{G_Z^i\}$ and $\{G_R^i\}$ by putting $G^i=[G, G_R^{i-1}]_Z$. Theorem 3 shows that in a nilpotent W-group the three series coincide.

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