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ON EVALUATION FORMULAS FOR DOUBLE L-VALUES

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In this paper, we give some evaluation formulas for the values of double L-series of Tornheim's type, in terms of the Dirichlet L-values and the Riemann zeta values at positive integers. As special cases, these give the formulas for double L-values given by Terhune.

1. INTRODUCTION

Let $\mathbb{N}$ be the set of natural numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{Z}$ the ring of rational integers, $\mathbb{Q}$ the field of rational numbers, $\mathbb{R}$ the field of real numbers and $\mathbb{C}$ the field of complex numbers.

Let χ, ψ be primitive Dirichlet characters. We consider the double L-series of Tornheim's type defined by

$$\mathcal{L}(k,l,d;\chi,\psi) = \sum_{m,n=1}^{\infty} \frac{\chi(m)\psi(m+n)}{m^k n^l (m+n)^d}, \tag{1.1}$$

where $k,l,d \in \mathbb{N}_0$ with $k+d>1$, $l+d>1$, $k+l+d>2$. This can be a character analogue of the Tornheim double series

$$T(k,l,d) = \sum_{m,n=1}^{\infty} \frac{1}{m^k n^l (m+n)^d} \tag{1.2}$$

defined in [**8**].

Tornheim showed that $T(k,l,N-k-l)$ can be expressed as a polynomial in $\{\zeta(j) \mid 2 \leqslant j \leqslant N\}$ with rational coefficients when N is odd and $N \geqslant 3$, where $\zeta(s)$ is the Riemann zeta function. This essentially includes Euler's consideration for $T(k,0,N-k)$ which is called the Euler sum (see, for example, [**2**]). Independently, Mordell also considered these series in [**5**]. Recently Huard, Williams and Zhang Nan-Yue gave an explicit formula for $T(k,l,N-k-l)$ as a rational linear combination of the products $\zeta(2j)\zeta(N-2j)$ $(0 \leqslant j \leqslant (N-3)/2)$ when N is odd, $N \geqslant 3$, and $k,l \in \mathbb{N}_0$ satisfying $1 \leqslant k+l \leqslant N-1$, $k \leqslant N-2$ and $l \leqslant N-2$ (see [**3**]). Recently Matsumoto considered

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$T(s_1, s_2, s_3)$ as a meromorphic function of complex variables $(s_1, s_2, s_3) \in \mathbb{C}^3$ in [4]. Note that he further considered multiple zeta functions of Tornheim's type.

$\mathcal{L}(k, 0, d; \chi, \psi)$ is what is called the double L-series (see, for example, [1, 6, 7]). Recently Terhune proved that if $\chi(-1)\psi(-1) = (-1)^{k+d+1}$ then $\mathcal{L}(k, 0, d; \chi, \psi)$ is a polynomial in the values of polylogarithms at pth roots of unity with $\mathbb{Q}(\zeta_m)$-coefficients, where $m, p \in \mathbb{N}$ are determined by χ, ψ (see [6, 7]). As concrete examples, he listed some evaluation formulas for $\mathcal{L}(k, 0, d; \chi_3, \chi_0)$, $\mathcal{L}(k, 0, d; \chi_0, \chi_3)$ and $\mathcal{L}(k, 0, d; \chi_5, \chi_0)$ when $k + d$ is even and odd, respectively, where χ_0 is the trivial character, χ_3 and χ_5 are the quadratic character of conductor 3 and 5, respectively.

The aim of this paper is to give explicit evaluation formulas for $\mathcal{L}(k, l, d; \chi_0, \chi)$ for an arbitrary primitive Dirichlet character $\chi \neq \chi_0$, when $\chi(-1) = (-1)^{k+l+d+1}$ (see Theorem 3.1). In particular when $l = 0$, we obtain evaluation formulas for double L-series $\mathcal{L}(k, 0, d; \chi_0, \chi)$ and $\mathcal{L}(k, 0, d; \chi, \chi_0)$, when $\chi(-1) = (-1)^{k+d+1}$. These include Terhune's formulas for double L-series given in [6, 7].

In order to prove our assertion, we make use of our previous result, namely the evaluation formulas for

$$\sum_{m,n=1}^{\infty} \frac{(-1)^{m+n} \sin((m+n)\theta)}{m^k n^l (m+n)^d}$$

for $\theta \in [-\pi, \pi]$ when $k + l + d$ is even (see [9, Proposition 3.1]).

As concrete examples, we give evaluation formulas for $\mathcal{L}(k, l, d; \chi_0, \chi_3)$ when $k+l+d$ is even, in terms of the values of $\zeta(s)$ and the Dirichlet L-series $L(s, \chi_3)$ at positive integers (see Proposition 4.1). For example, we have

$$\mathcal{L}(1, 1, 2; \chi_0, \chi_3) = 2L(4, \chi_3) - \frac{26\sqrt{3}}{81}\pi\zeta(3), \tag{1.3}$$

$$\mathcal{L}(2, 0, 2; \chi_3, \chi_0) = L(4, \chi_3) + \frac{2\pi^2}{9}L(2, \chi_3) - \frac{26\sqrt{3}}{81}\pi\zeta(3). \tag{1.4}$$

Note that $L(1, \chi_3) = \sqrt{3}\pi/9$. (1.4) was obtained by Terhune.

2. Preliminaries

In this section, we quote some results from [9] as follows. For $u \in \mathbb{R}$ with $1 \leqslant u \leqslant 1 + \delta$ and $s \in \mathbb{R}$, we define

$$\phi(s; u) := \sum_{m=1}^{\infty} \frac{(-u)^{-m}}{m^s}. \tag{2.1}$$

If $u > 1$, then $\phi(s; u)$ is convergent absolutely for any $s \in \mathbb{R}$. In the case when $u = 1$, let $\phi(s) := \phi(s; 1) = (2^{1-s} - 1)\zeta(s)$.

We denote the pth derivative of $\sin(X)$ by $\sin^{(p)}(X)$. Furthermore we denote $\sin^{(p)}(X)\big|_{X=m\theta}$ by $\sin^{(p)}(m\theta)$ for $m \in \mathbb{N}_0$. Then we define

$$\mathcal{I}_p(\theta;k;u) := \sum_{m=1}^{\infty} \frac{(-u)^{-m}\sin^{(p)}(m\theta)}{m^k}$$

for $p \in \mathbb{N}_0$, $k \in \mathbb{N}$, $\theta \in [-\pi,\pi]$ and $u \in [1, 1+\delta]$. Let $\lambda_j = \{1+(-1)^j\}/2$ for $j \in \mathbb{Z}$. Then we have the following lemma (see [**9**, Proposition 3.1]).

Lemma 2.1. *Let $k,l,d \in \mathbb{N}_0$ with $k+d>1$, $l+d>1$, $k+l+d>2$, $d \geqslant 2$ and $\theta \in [-\pi,\pi]$. Suppose $k+l+d \equiv 0 \pmod 2$. Then*

$$\begin{aligned}(2.2)\sum_{m,n=1}^{\infty} &\frac{(-1)^{m+n}\sin((m+n)\theta)}{m^k n^l (m+n)^d} \\ &= \sum_{j=0}^{k}\phi(k-j)(-1)^j\lambda_{k+j}\sum_{\nu=0}^{j}\binom{d-1+j-\nu}{j-\nu}\frac{(-\theta)^\nu}{\nu!}\mathcal{I}_\nu(\theta;d+j+l-\nu;1) \\ &\quad+\sum_{j=0}^{l}\phi(l-j)(-1)^j\lambda_{l+j}\sum_{\nu=0}^{j}\binom{d-1+j-\nu}{j-\nu}\frac{(-\theta)^\nu}{\nu!}\mathcal{I}_\nu(\theta;d+j+k-\nu;1) \\ &\quad+\sum_{n=0}^{[(d-2)/2]}\beta_{2n+1-d}(k,l;1)\frac{(-1)^n\theta^{2n+1}}{(2n+1)!},\end{aligned}$$

where

$$\begin{aligned}(2.3)\quad \beta_{-N-1}(k,l;1)\lambda_{k+l+N} &= -2\sum_{\nu=0}^{N}\phi(N-\nu)\lambda_{N+\nu}\lambda_{k+l+\nu} \\ &\times\Bigg\{(-1)^k\sum_{\rho=0}^{[k/2]}\phi(2\rho)\sum_{\mu=0}^{[(k-2\rho-1)/2]}\binom{\nu+k-2\rho-2\mu}{k-2\rho-2\mu-1} \\ &\times\zeta(k+l+\nu-2\rho-2\mu+1)\frac{(-1)^\mu\pi^{2\mu}}{(2\mu+1)!} \\ &\quad+(-1)^l\sum_{\rho=0}^{[l/2]}\phi(2\rho)\sum_{\mu=0}^{[(l-2\rho-1)/2]}\binom{\nu+l-2\rho-2\mu}{l-2\rho-2\mu-1} \\ &\times\zeta(k+l+\nu-2\rho-2\mu+1)\frac{(-1)^\mu\pi^{2\mu}}{(2\mu+1)!}\Bigg\}\end{aligned}$$

for $N \in \mathbb{N}_0$.

Proof: It follows from Lemma 2.2 in [**9**] that each side of Equation (3.1) in [**9**] is uniformly convergent with respect to $u \in [1, 1+\delta]$ because $d \geqslant 2$ and $\theta \in [-\pi,\pi]$. So (3.1) in [**9**] holds for $u=1$. Note that we assumed $k,l,d \in \mathbb{N}$ in Proposition 3.1 of [**9**]. However, we can see that [**9**, Proposition 3.1] holds for $k,l,d \in \mathbb{N}_0$ satisfying the conditions in the statement of Lemma 2.1. Hence we obtain (2.2). From (3.6) in [9], we obtain (2.3). ∎

When $d \geqslant 3$, we can differentiate (2.2) with respect to θ because of its uniform convergency. Using the known relation

$$-\binom{x-1}{y-1}+\binom{x}{y}=\binom{x-1}{y},$$

and replacing $d-1$ with d, we have the following.

Lemma 2.2. *Let $k, l, d \in \mathbb{N}_0$ with $k+d>1$, $l+d>1$, $k+l+d>2$, $d \geqslant 2$ and $\theta \in [-\pi,\pi]$. Suppose $k+l+d \equiv 1 \pmod 2$. Then*

$$(2.4)\quad \sum_{m,n=1}^{\infty}\frac{(-1)^{m+n}\cos((m+n)\theta)}{m^k n^l (m+n)^d}$$
$$=\sum_{j=0}^{k}\phi(k-j)(-1)^j\lambda_{k+j}\sum_{\nu=0}^{j}\binom{d-1+j-\nu}{j-\nu}\frac{(-\theta)^\nu}{\nu!}\mathcal{I}_{\nu+1}(\theta;d+j+l-\nu;1)$$
$$+\sum_{j=0}^{l}\phi(l-j)(-1)^j\lambda_{l+j}\sum_{\nu=0}^{j}\binom{d-1+j-\nu}{j-\nu}\frac{(-\theta)^\nu}{\nu!}\mathcal{I}_{\nu+1}(\theta;d+j+k-\nu;1)$$
$$+\sum_{n=0}^{[(d-1)/2]}\beta_{2n-d}(k,l;1)\frac{(-1)^n\theta^{2n}}{2n!}.$$

For simplicity, for $p \in \{0,1\}$, we let

$$(2.5)\qquad \mathcal{A}_p(\theta;k,l,d)=\sum_{j=0}^{k}\phi(k-j)(-1)^j\lambda_{k+j}$$
$$\times\sum_{\nu=0}^{j}\binom{d-1+j-\nu}{j-\nu}\frac{(-\theta)^\nu}{\nu!}\mathcal{I}_{\nu+p}(\theta;d+j+l-\nu;1)$$
$$+\sum_{j=0}^{l}\phi(l-j)(-1)^j\lambda_{l+j}$$
$$\times\sum_{\nu=0}^{j}\binom{d-1+j-\nu}{j-\nu}\frac{(-\theta)^\nu}{\nu!}\mathcal{I}_{\nu+p}(\theta;d+j+k-\nu;1).$$

Example 2.3. Putting $(k,l)=(1,1)$ in (2.3), we have $\beta_{-1}(1,1;1)=\zeta(3)$. Furthermore, putting $(k,l)=(2,0)$, we have $\beta_{-1}(2,0;1)=-\zeta(3)$.

3. Evaluation formulas

Let χ be the primitive Dirichlet character with conductor $f>1$. It is well-known that

$$(3.1)\qquad \chi(n)=\frac{1}{\tau(\overline{\chi})}\sum_{a=1}^{f}\overline{\chi}(a)e^{2\pi i a n/f}$$

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for $n \in \mathbb{Z}$, where $\overline{\chi} = \chi^{-1}$ and $\tau(\chi) = \sum_{a=1}^{f} \chi(a) e^{2\pi i a/f}$ (see, for example, [**10**, Lemma 4.7]). Hence we have

$$\chi(n) = \frac{2}{\tau(\overline{\chi})} \sum_{a=1}^{[f/2]} \overline{\chi}(a) \cos(2\pi a n/f) \quad (\text{if } \chi(-1) = 1), \tag{3.2}$$

$$\chi(n) = \frac{2i}{\tau(\overline{\chi})} \sum_{a=1}^{[f/2]} \overline{\chi}(a) \sin(2\pi a n/f) \quad (\text{if } \chi(-1) = -1). \tag{3.3}$$

Furthermore we can check that if f is even then

$$\sin\left(\frac{2\pi n}{f}\left(\frac{f}{2} - b\right)\right) = (-1)^n \sin\left(\frac{2\pi n b}{f}\right), \tag{3.4}$$

$$\cos\left(\frac{2\pi n}{f}\left(\frac{f}{2} - b\right)\right) = (-1)^n \cos\left(\frac{2\pi n b}{f}\right), \tag{3.5}$$

and if f is odd then

$$\sin\left(\frac{2\pi n}{f}\left(\frac{f+1}{2} - b\right)\right) = (-1)^n \sin\left(\frac{2\pi n}{f}\left(b - \frac{1}{2}\right)\right), \tag{3.6}$$

$$\cos\left(\frac{2\pi n}{f}\left(\frac{f+1}{2} - b\right)\right) = (-1)^n \cos\left(\frac{2\pi n}{f}\left(b - \frac{1}{2}\right)\right). \tag{3.7}$$

For $k \in \mathbb{N}$ and $z \in \mathbb{C}$ with $|z| \leqslant 1$, we consider the polylogarithms defined by

$$Li(k; z) = \sum_{m=1}^{\infty} \frac{z^m}{m^k}. \tag{3.8}$$

From the relation

$$\sin^{(p)} x = \frac{i^{p-1}}{2}\left(e^{ix} + (-1)^{p-1} e^{-ix}\right),$$

we have

$$\mathcal{I}_p(\theta; k; 1) = \frac{i^{p-1}}{2}\left\{Li(k; -e^{i\theta}) + (-1)^{p-1} Li(k; -e^{-i\theta})\right\} \tag{3.9}$$

for $p \in \mathbb{N}_0$.

THEOREM 3.1. *Let $k, l, d \in \mathbb{N}_0$ with $k+d > 1$, $l+d > 1$, $k+l+d > 2$ and $d \geqslant 2$, and χ be a Dirichlet character with conductor $f > 1$ and with $\chi(-1) = (-1)^{k+l+d+1}$. If $\chi(-1) = 1$ and f is even, then*

$$\mathcal{L}(k, l, d; \chi_0, \chi) = \frac{2}{\tau(\overline{\chi})} \sum_{b=1}^{f/2-1} \overline{\chi}\left(\frac{f}{2} - b\right)\left\{\mathcal{A}_1\left(\frac{2\pi b}{f}; k, l, d\right) + \sum_{n=0}^{[(d-1)/2]} \beta_{2n-d}(k, l; 1) \frac{(-1)^n (2\pi b/f)^{2n}}{2n!}\right\}. \tag{3.10}$$

If $\chi(-1)=-1$ and f is even, then

$$\mathcal{L}(k,l,d;\chi_0,\chi)=\frac{2i}{\tau(\overline{\chi})}\sum_{b=1}^{f/2-1}\overline{\chi}\Big(\frac{f}{2}-b\Big)\Big\{\mathcal{A}_0\Big(\frac{2\pi(b-1/2)}{f};k,l,d\Big)+\sum_{n=0}^{[(d-2)/2]}\beta_{2n+1-d}(k,l;1)\frac{(-1)^n(2\pi(b-1/2)/f)^{2n+1}}{(2n+1)!}\Big\}.$$

If $\chi(-1)=1$ and f is odd, then

$$(3.11)\ \mathcal{L}(k,l,d;\chi_0,\chi)=\frac{2}{\tau(\overline{\chi})}\sum_{b=1}^{(f-1)/2}\overline{\chi}\Big(\frac{f+1}{2}-b\Big)\times\Big\{\mathcal{A}_1\Big(\frac{2\pi b}{f};k,l,d\Big)+\sum_{n=0}^{[(d-1)/2]}\beta_{2n-d}(k,l;1)\frac{(-1)^n(2\pi b/f)^{2n}}{2n!}\Big\}.$$

If $\chi(-1)=-1$ and f is odd, then

$$(3.12)\ \mathcal{L}(k,l,d;\chi_0,\chi)=\frac{2i}{\tau(\overline{\chi})}\sum_{b=1}^{(f-1)/2}\overline{\chi}\Big(\frac{f+1}{2}-b\Big)\times\Big\{\mathcal{A}_0\Big(\frac{2\pi(b-1/2)}{f};k,l,d\Big)+\sum_{n=0}^{[(d-2)/2]}\beta_{2n+1-d}(k,l;1)\frac{(-1)^n(2\pi(b-1/2)/f)^{2n+1}}{(2n+1)!}\Big\}.$$

Note that

$$(3.13)\ \mathcal{A}_p(\theta;k,l,d)=\sum_{j=0}^{k}\phi(k-j)(-1)^j\lambda_{k+j}\sum_{\nu=0}^{j}\binom{d-1+j-\nu}{j-\nu}\frac{(-\theta)^\nu}{\nu!}\times\frac{i^{\nu+p-1}}{2}\{Li(d+j+l-\nu;-e^{i\theta})+(-1)^{\nu+p-1}Li(d+j+l-\nu;-e^{-i\theta})\}$$
$$+\sum_{j=0}^{l}\phi(l-j)(-1)^j\lambda_{l+j}\sum_{\nu=0}^{j}\binom{d-1+j-\nu}{j-\nu}\frac{(-\theta)^\nu}{\nu!}\times\frac{i^{\nu+p-1}}{2}\{Li(d+j+k-\nu;-e^{i\theta})+(-1)^{\nu+p-1}Li(d+j+k-\nu;-e^{-i\theta})\},$$

and $\beta_{-j}(k,l;1)$ is defined by (2.3).

PROOF: If f is even (respecitvely, odd) then we put $a=f/2-b$ (respectively, $a=(f+1)/2-b$) in (3.2) and (3.3). By combining Lemma 2.1, Lemma 2.2 and (3.2)-(3.9), we obtain (3.10)-(3.12). ∎

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REMARK 3.2. In particular when $l = 0$, we obtain the evaluation formulas for $\mathcal{L}(k,0,d;\chi_0,\chi)$ when $\chi(-1) = (-1)^{k+d+1}$. Furthermore, by using

$$\begin{aligned}(3.14)\qquad L(p,\chi)\zeta(q) &= \Bigg(\sum_{0<m_1<m_2} + \sum_{m_1>m_2>0} + \sum_{0<m_1=m_2}\Bigg)\frac{\chi(m_1)}{m_1{}^p m_2{}^q}\\ &= \mathcal{L}(p,0,q;\chi,\chi_0) + \mathcal{L}(q,0,p;\chi_0,\chi) + L(p+q,\chi),\end{aligned}$$

we obtain the evaluation formulas for $\mathcal{L}(k,0,d;\chi,\chi_0)$.

4. THE CASE OF CONDUCTOR 3

As concrete examples, we give the evaluation formulas for $\mathcal{L}(k,l,d;\chi_3)$ in terms of the Dirichlet L-values and the Riemann zeta values as follows.

PROPOSITION 4.1. *Let $k,l,d \in \mathbb{N}_0$ with $k+d>1$, $l+d>1$, $k+l+d>2$ and $d \geqslant 2$. Suppose $k+l+d$ is even. Then*

$$\begin{aligned}(4.1)\ &\mathcal{L}(k,l,d;\chi_0,\chi_3)\\ &= \frac{1}{\sqrt{3}}\Bigg[\sum_{j=0}^{k}\phi(k-j)(-1)^j\lambda_{k+j}\\ &\times\Bigg\{\sqrt{3}\sum_{\mu=0}^{[j/2]}\binom{d-1+j-2\mu}{j-2\mu}\frac{(-1)^\mu(\pi/3)^{2\mu}}{(2\mu)!}L(d+j+l-2\mu,\chi_3)\\ &\quad-\sum_{\mu=0}^{[(j-1)/2]}\binom{d-2+j-2\mu}{j-2\mu-1}\frac{(-1)^\mu(\pi/3)^{2\mu+1}}{(2\mu+1)!}\psi(d+j+l-2\mu-1)\Bigg\}\\ &\quad+\sum_{j=0}^{l}\phi(l-j)(-1)^j\lambda_{l+j}\\ &\times\Bigg\{\sqrt{3}\sum_{\mu=0}^{[j/2]}\binom{d-1+j-2\mu}{j-2\mu}\frac{(-1)^\mu(\pi/3)^{2\mu}}{(2\mu)!}L(d+j+k-2\mu,\chi_3)\\ &\quad-\sum_{\mu=0}^{[(j-1)/2]}\binom{d-2+j-2\mu}{j-2\mu-1}\frac{(-1)^\mu(\pi/3)^{2\mu+1}}{(2\mu+1)!}\psi(d+j+k-2\mu-1)\Bigg\}\Bigg]\\ &\quad-\frac{2}{\sqrt{3}}\sum_{n=0}^{[(d-2)/2]}\beta_{2n+1-d}(k,l;1)\frac{(-1)^n(\pi/3)^{2n+1}}{(2n+1)!},\end{aligned}$$

where $\phi(s) = (2^{1-s}-1)\zeta(s)$, $\psi(s) = (1-3^{1-s})\zeta(s)$.

PROOF: From (3.8) and (3.9), we can easily check that

$$(4.2)\qquad \mathcal{I}_{2\mu}\Big(\frac{\pi}{3};k;1\Big) = -\frac{\sqrt{3}}{2}(-1)^\mu L(k,\chi_3),$$

$$(4.3)\qquad \mathcal{I}_{2\mu+1}\Big(\frac{\pi}{3};k;1\Big) = -\frac{1}{2}(-1)^\mu\psi(k)$$

for $\mu \in \mathbb{N}_0$, where $\psi(s) = (1-3^{1-s})\zeta(s)$. By applying Theorem 3.1 with $\chi = \chi_3$ and $f = 3$, we obtain the assertion. ∎

EXAMPLE 4.2. Putting $(k,l) = (1,1)$ in (2.3), we have

$$\beta_{-2j-1}(1,1;1) = -2\sum_{\mu=0}^{j}\left(2^{1-2j+2\mu}-1\right)\zeta(2j-2\mu)\zeta(2\mu+3)$$

for $j \in \mathbb{N}_0$, because $\phi(0) = -1/2$. Putting $(k,l,d) = (1,1,2r)$ in (4.1), we have

$$\begin{aligned}\mathcal{L}(1,1,2r;\chi_0,\chi_3) = \frac{1}{\sqrt{3}}\Big\{(2\sqrt{3}\,r)L(2r+2,\chi_3) - \frac{\pi}{3}(1-3^{-2r})\zeta(2r+1)\Big\}\\ +\frac{4}{\sqrt{3}}\sum_{n=0}^{r-1}\sum_{\mu=0}^{r-1-n}(2^{3+2n+2\mu-2r}-1)\zeta(2r-2-2n-2\mu)\zeta(2\mu+3)\\ \times\frac{(-1)^n(\pi/3)^{2n+1}}{(2n+1)!}\end{aligned}$$

for $r \in \mathbb{N}$. Putting $r = 1$, we obtain (1.3). Furthermore, putting $r = 2$, we have

$$\mathcal{L}(1,1,4;\chi_0,\chi_3) = 4L(6,\chi_3) - \frac{242\sqrt{3}}{729}\pi\zeta(5) - \frac{8\sqrt{3}}{243}\pi^3\zeta(3).$$

In particular when $l = 0$ in (4.1), we can give some evaluation formulas for the double L-series attached to χ_3. For example, putting $(k,l,d) = (2,0,2)$ in (4.1) and using $\beta_{-1}(2,0;1) = -\zeta(3)$ (see Example 2.3), we have

$$\mathcal{L}(2,0,2;\chi_0,\chi_3) = -2L(4,\chi_3) - \frac{\pi^2}{18}L(2,\chi_3) + \frac{26\sqrt{3}}{81}\pi\zeta(3). \tag{4.4}$$

Using (3.14), we obtain (1.4).

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