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A finitely generated, infinitely related group with trivial multiplicator

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We exhibit a 3-generator metabelian group which is not finitely related but has a trivial multiplicator.

1.

The purpose of this note is to establish the existence of a finitely generated group which is not finitely related, but whose multiplicator is finitely generated. This settles negatively a question which has been open for a few years (it was first brought to my attention by Michel Kervaire and Joan Landman Dyer in 1964, but I believe it is somewhat older). The group is given in the following theorem.

THEOREM. *The* 3-*generator, metabelian group*

$$G = \left\langle a, b, t;\ t^{-1}at = a^4,\ tbt^{-1} = b^2,\ [a, t^{-i}bt^i] = 1 \quad (i = 0, \pm 1, \ldots)\right\rangle \tag{1}$$

*is not finitely related but its multiplicator** $M(G)$ *is zero.*

The proof of this theorem will be carried out in two parts. First we prove, in §2, that $M(G) = 0$. Then, in §3, we prove G is not finitely related.

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* That is the second homology group of G with integral coefficients.

2.

Let F be the free group on x, y, z and let R be the normal subgroup of F generated by the elements

$$z^{-1}xzx^{-4},\ zyz^{-1}y^{-2},\ [x,\ z^{-i}yz^{i}]\quad (i = 0,\ \pm 1,\ \ldots)\ .$$

Then

$$M(G) \cong ([F,\ F]\cap R)/[F,\ R]\ .$$

Put

$$\bar{F} = F/[F,\ R],\ \bar{R} = R/[F,\ R],\ \bar{x} = x[F,\ R],\ \bar{y} = y[F,\ R],\ \bar{z} = z[F,\ R]\ .$$

Furthermore let N be the normal subgroup of F generated by x and y. Then N contains R (and hence $[F,\ R]$). We put

$$\bar{N} = N/[F,\ R]\ .$$

Our objective is to prove

$$[\bar{F},\ \bar{F}]\cap\bar{R} = 1\ .$$

In order to do so we prove first that $\bar{N}$ is abelian. Clearly $\bar{N}$ is a central extension of $\bar{R}$ by the direct product of two copies of the additive group of dyadic fractions, that is, the subgroup of the additive group of rational numbers consisting of rationals of the form $\frac{r}{2^s}$ where r and s are integers. Putting

$$\bar{x}_i = \bar{z}^{-i}\bar{x}\bar{z}^{i}\ ,\quad \bar{y}_i = \bar{z}^{-i}\bar{y}\bar{z}^{i}\quad (i = 0,\ \pm 1,\ \ldots)\ ,$$

we see that $\bar{N}$ is generated by the elements $\bar{x}_i$, $\bar{y}_i$, where here i is allowed to range over all the integers. Notice that

$$\bar{z}^{-1}\bar{x}_i\bar{z} = \bar{x}_i^{4}\bar{r}_i\ ,\quad \left(\bar{z}^{-1}\bar{y}_j\bar{z}\right)^2 = \bar{y}_j\bar{s}_j\quad (\bar{r}_i,\ \bar{s}_j\in\bar{R})$$

for any choice of integers i and j. Hence, remembering $\bar{R}$ is central in $\bar{F}$, we find

$$\begin{aligned}[\bar{x}_i,\ \bar{y}_j] &= \bar{z}^{-1}[\bar{x}_i,\ \bar{y}_j]\bar{z} = \left[\bar{z}^{-1}\bar{x}_i\bar{z},\ \bar{z}^{-1}\bar{y}_j\bar{z}\right] = \left[\bar{x}_i^{4}\bar{r}_i,\ \bar{z}^{-1}\bar{y}_j\bar{z}\right] = \left[\bar{x}_i^{4},\ \bar{z}^{-1}\bar{y}_j\bar{z}\right]\\ &= \left(\left[\bar{x}_i,\ \bar{z}^{-1}\bar{y}_j\bar{z}\right]^2\right)^2 = \left[\bar{x}_i,\ \left(\bar{z}^{-1}\bar{y}_j\bar{z}\right)^2\right]^2 = [\bar{x}_i,\ \bar{y}_j\bar{s}_j]^2 = [\bar{x}_i,\ \bar{y}_j]^2\ .\end{aligned}$$

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So

$$[\bar{x}_i, \bar{y}_j] = 1$$

for all i and j. Hence

$$\bar{R} = gp\left(\bar{z}^{-1}\bar{x}\bar{z}\bar{x}^{-4}, \bar{z}\bar{y}\bar{z}^{-1}\bar{y}^{-2}\right) .$$

Since

$$\left(\bar{z}^{-1}\bar{x}\bar{z}\bar{x}^{-4}\right)^{l}\left(\bar{z}\bar{y}\bar{z}^{-1}\bar{y}^{-2}\right)^{m} \in \bar{F}' \text{ only if } l = m = 0 ,$$

it follows that

$$[\bar{F}, \bar{F}] \cap \bar{R} = 1 .$$

Consequently $M(G)$ is trivial, as required.

3.

We complete the proof of the theorem by showing that G is not finitely related. Let us suppose the contrary. Then by a theorem of Neumann [2] finitely many of the given defining relations of G suffice to define G. Thus we may present G in the form

$$G = \left\langle a, b, t; t^{-1}at = a^4, tbt^{-1} = b^2, [a, t^{-i}bt^{i}] = 1 \quad (-n \le i \le n, n > 0 \text{ a fixed integer})\right\rangle .$$

However since $tbt^{-1} = b^2$ the relations

$$[a, t^{-i}bt^{i}] = 1 \quad (-n \le i \le n)$$

all follow from the single relation

$$[a, t^{n}bt^{-n}] = 1$$

(because the elements $t^{i}bt^{-i}$ $(-n < i \le n)$ are powers of $t^{n}bt^{-n}$). Thus, replacing b by $t^{n}bt^{-n}$ if necessary, it follows that G can be presented in the form

$$(2) \qquad G = \left\langle a, b, t; t^{-1}at = a^4, tbt^{-1} = b^2, [a, b] = 1\right\rangle .$$

Observe now, as we in fact observed earlier, that if H is the normal

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subgroup of G generated by a and b then H is the direct product of two copies of the dyadic fractions and is therefore abelian (*cf.* the presentation (1)). But by the Reidemeister-Schreier method for finding generators and defining relations for a subgroup of a group given by generators and defining relations it is easy enough to obtain a presentation for H from the presentation (2) (see Magnus, Karrass and Solitar [1], p. 91 *sqq*). Indeed let us put

$$a_i = t^{-i}at^i \ , \quad b_i = t^{-i}bt^i \quad (i = 0, \pm 1, \ldots) \ .$$

Then

$$(3) \quad H = \left\langle \ldots, a_i, \ldots, \ldots, b_i, \ldots; \ldots, a_{i+1} = a_i^4, \ldots, \ldots, b_i = b_{i+1}^2, \ldots, \ldots, [a_i, b_i] = 1, \ldots \right\rangle .$$

Notice in the presentation above the subscript i ranges over all the integers. Our aim is to show that the group presented by (3) is non-abelian, thereby contradicting the fact that H is indeed an abelian subgroup of G. This, in turn, contradicts the assumption that G is finitely related and so completes the proof of our theorem.

The easiest way to prove H (as presented by (3)) is non-abelian is to represent it as an ascending union of generalised free products. To this end let

$$A_i = \langle \tilde{a}_i \rangle \ \text{and} \ B_i = \langle \tilde{b}_i \rangle \quad (i = 0, \pm 1, \ldots)$$

be infinite cyclic groups. Let

$$H_0 = A_0 \times B_0$$

be the direct product of A_0 and B_0. We define now H_1 to be the generalised free product of H_0 and $A_1 \times B_1$ identifying $\tilde{a}_0^4$ with $\tilde{a}_1$ and $\tilde{b}_0$ with $\tilde{b}_1^2$:

$$H_1 = \{H_0 * (A_1 \times B_1); \ \tilde{a}_0^4 = \tilde{a}_1, \ \tilde{b}_0 = \tilde{b}_1^2\} \ .$$

Observe that H_1 is non-abelian since

$$\tilde{a}_0\tilde{b}_1 \neq \tilde{b}_1\tilde{a}_0 \ .$$

We define similarly (and inductively)

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$$H_{i+1} = \left\{H_i * (A_{i+1} \times B_{i+1});\ \tilde{a}_i^4 = \tilde{a}_{i+1},\ \tilde{b}_i = \tilde{b}_{i+1}^2\right\}$$

and thence

$$H_\infty = \bigcup_{i=0}^{\infty} H_i .$$

It follows that H_∞ may be presented in the form

$$H_\infty = \langle \tilde{a}_0, \tilde{a}_1, \ldots, \tilde{b}_0, \tilde{b}_1, \ldots;\ \tilde{a}_0^4 = \tilde{a}_1,\ \tilde{a}_1^4 = \tilde{a}_2, \ldots,$$
$$\tilde{b}_0 = \tilde{b}_1^2,\ \tilde{b}_1 = \tilde{b}_2^2, \ldots, [\tilde{a}_0, \tilde{b}_0] = 1,\ [\tilde{a}_1, \tilde{b}_1] = 1, \ldots\rangle .$$

Now we put

$$K_0 = H_\infty$$

and define

$$K_1 = \left\{K_0 * (A_{-1} \times B_{-1});\ \tilde{a}_0 = \tilde{a}_{-1}^4,\ \tilde{b}_0^2 = \tilde{b}_{-1}\right\} .$$

We define similarly (and inductively)

$$K_{i+1} = \left\{K_i * (A_{-(i+1)} \times B_{-(i+1)});\ \tilde{a}_{-i} = \tilde{a}_{-(i+1)}^4,\ \tilde{b}_{-i}^2 = \tilde{b}_{-(i+1)}\right\} .$$

Finally we put

$$\tilde{H}\ (= K_\infty) = \bigcup_{i=0}^{\infty} K_i .$$

It follows that $\tilde{H}$ is a non-abelian group (since it contains H_1). Moreover $\tilde{H}$ can clearly be presented in the form

$$(4)\quad \tilde{H} = \left\langle \ldots, \tilde{a}_{-1}, \tilde{a}_0, \tilde{a}_1, \ldots, \ldots, \tilde{b}_{-1}, \tilde{b}_0, \tilde{b}_1, \ldots;\right.$$
$$\left. \ldots, \tilde{a}_{i+1} = \tilde{a}_i^4, \ldots, \ldots, \tilde{b}_i = \tilde{b}_{i+1}^2, \ldots, \ldots, [\tilde{a}_i, \tilde{b}_i] = 1, \ldots\right\rangle$$

where here the subscript i is allowed to range over all possible integers. A comparison of (3) with (4) immediately shows

$$H \cong \tilde{H} .$$

So H is non-abelian. This completes the proof of our theorem.

References

[1] Wilhelm Magnus; Abraham Karrass; Donald Solitar, *Combintorial group theory: Presentations of groups in terms of generators and relations* (Interscience [John Wiley & Sons], New York, London, Sydney, 1966).

[2] B.H. Neumann, "Some remarks on infinite groups", *J. London Math. Soc.* 12 (1937), 120-127.

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