Glasgow Math. J. **54** (2012) 261–271.
doi:10.1017/S0017089511000577.

ON AN EXAMPLE OF MUKAI

H. LANGE

Department Mathematik, Universität Erlangen-Nürnberg, Bismarckstraße $1\frac{1}{2}$, D-91054 Erlangen, Germany
e-mail: lange@mi.uni-erlangen.de

V. MERCAT

5 rue Delouvain, 75019 Paris, France
e-mail: vipani.mm@gmail.com

and P. E. NEWSTEAD

Department of Mathematical Sciences, University of Liverpool, Peach Street, Liverpool L69 7ZL, UK
e-mail: newstead@liv.ac.uk

(Received 9 February 2011; accepted 15 August 2011; first published online 12 December 2011)

Abstract. In this paper we use an example of Mukai to construct semistable bundles of rank 3 with six independent sections on a general curve of genus 9 or 11 with Clifford index strictly less than the Clifford index of the curve. The example also allows us to show the non-emptiness of some Brill–Noether loci with negative expected dimension.

2000 *Mathematics Subject Classification.* Primary 14H60

1. Introduction. Mukai stated in Proposition 2 in [**13**] that if C is a non-pentagonal curve of genus 9, then there exists a unique stable bundle E of rank 3 and determinant K_C with $h^0(E) = 6$ (actually Mukai said 'quasi-stable', which is what is now usually called 'polystable', but since $\deg E = 16$ is coprime to 3, this implies that E is stable). Computing the Clifford index $\gamma(E)$ as defined in [**6**] gives

$$\gamma(E) = \frac{1}{3}(16 - 6) = \frac{10}{3}.$$

Since C has Clifford index $\mathrm{Cliff}(C) = 4$, this contradicts the conjecture in [**12**] (see also Conjecture 9.3 in [**6**]). It further shows that the Brill–Noether locus $B(3, 16, 6)$, which has 'expected dimension' -11, is non-empty. It also sheds light on the main result of [**8**], which implies that any semistable bundle of rank 3 with $h^0 = 6$ on C has degree ≥ 15; in fact such a bundle of degree 15 cannot exist (see Proposition 5.2 and Comment 1), so Mukai's bundle has the minimum possible degree for a semistable bundle of rank 3 with $h^0 = 6$.

In fact, [**13**] contains no proofs. The above result is proved in [**14**], except that the stability of E is only indicated in Remark 5.7(2) in [**14**] (the full proof may be found in the addendum [**15**]). In this paper, we give a complete proof, show that a similar

All authors are the members of the research group, Vector Bundles on Algebraic Curves (VBAC).

result holds for genus 11 and consider possible generalisations and extensions. In the main theorem (Theorem 3.6) we give general conditions under which the Clifford index $\mathrm{Cliff}_n(C)$ defined in [**6**] is strictly smaller than $\mathrm{Cliff}(C)$ either for $n = 2$ or $n = 3$. The methods for the most part are those of Mukai.

In a postscript, we comment on developments since this paper was completed.

We would like to thank the referee for some helpful comments.

2. Background and preliminaries. Let C be a smooth projective curve of genus $g \geq 4$ defined over an algebraically closed field of characteristic zero. We denote by K_C the canonical line bundle on C. In [**6**], the classical Clifford index $\mathrm{Cliff}(C)$ of C was generalised to semistable bundles in two different ways, only one of which is needed in this paper. First we define for any vector bundle E of rank n and degree d,

$$\gamma(E) := \frac{1}{n}\left(d - 2(h^0(E) - n)\right) = \mu(E) - 2\frac{h^0(E)}{n} + 2,$$

where $\mu(E) = \frac{d}{n}$. Then the Clifford index $\mathrm{Cliff}_n(C)$ is defined by

$$\mathrm{Cliff}_n(C) := \min_E \left\{ \gamma(E) \;\middle|\; \begin{array}{c} E \text{ semistable of rank } n \\ h^0(E) \geq 2n, \;\; \mu(E) \leq g-1 \end{array} \right\}$$

(this invariant is denoted by γ_n' in [**6–10**]). We say that a bundle E *contributes* to $\mathrm{Cliff}_n(C)$ if it is semistable of rank n with $\mu(E) \leq g - 1$ and $h^0(E) \geq 2n$ and that E *computes* $\mathrm{Cliff}_n(C)$ if in addition $\gamma(E) = \mathrm{Cliff}_n(C)$. Note that $\mathrm{Cliff}_1(C) = \mathrm{Cliff}(C)$.

A conjecture was made in [**12**] concerning the maximum value of $h^0(E)$ for E, a semistable bundle of any given rank and degree; the most important part of this conjecture can be stated as follows.

CONJECTURE (Conjecture 9.3 in [**6**]). $\mathrm{Cliff}_n(C) = \mathrm{Cliff}(C)$.

The main purpose of this paper is to give examples to show that this conjecture can fail. For this purpose we use a bundle of rank 3 on a curve of genus 9 constructed by Mukai (see [**13–15**]). We show that the construction works also for genus 11.

In the remainder of this section, we introduce some notation that we shall need. First we recall the *gonality sequence* $d_1, d_2, \ldots, d_r, \ldots$ of C defined by

$$d_r := \min\{\deg L \mid L \text{ a line bundle on } C \text{ with } h^0(L) \geq r+1\}.$$

Note that a line bundle L of degree d_r with $h^0(L) \geq r + 1$ in fact has $h^0(L) = r + 1$ and is generated by its sections, so we have an exact evaluation sequence

$$0 \to E^* \to H^0(L) \otimes \mathcal{O}_C \to L \to 0. \tag{1}$$

The bundle E is often called the *dual span* of L. Note that $\mathrm{Cliff}(C)$ is the minimum value of $d_r - 2r$ taken over all r for which $d_r \leq g - 1$. We shall say that d_r *computes* $\mathrm{Cliff}(C)$ if $d_r \leq g - 1$ and $d_r - 2r = \mathrm{Cliff}(C)$. The numbers d_r satisfy the inequalities

$$d_r \leq g + r - \left[\frac{g}{r+1}\right] \tag{2}$$

with equality if C is a *Petri curve*, that is, a curve for which the multiplication map

$$H^0(L) \otimes H^0(L^* \otimes K_C) \longrightarrow H^0(K_C)$$

is injective for every line bundle L on C. It is important to note that the general curve of any given genus is a Petri curve (see, for example [**11**]).

We need also to recall the definitions of the higher rank Brill–Noether loci (see [**5**] for a survey of the theory and [**1**] for the notations we use here). Let $M(n, d)$ denote the moduli space of stable bundles of rank n and degree d on C. For any positive integer k, the Brill–Noether locus, $B(n, d, k)$, is defined by

$$B(n, d, k) = \{E \in M(n, d) \mid h^0(E) \geq k\}.$$

For every irreducible component Z of $B(n, d, k)$, we have

$$\dim Z \geq \beta(n, d, k) := n^2(g-1) - k(k - d + n(g-1)).$$

The number $\beta(n, d, k)$ is often called the *expected dimension* of $B(n, d, k)$.

Throughout the paper, C will denote a smooth projective curve of genus g and Clifford index $\mathrm{Cliff}(C) \geq 3$ (hence, $g \geq 7$ by (2)) defined over an algebraically closed field of characteristic zero. For a vector bundle G on C, the rank and degree of G will be denoted by r_G and d_G, respectively.

The following lemma of Paranjape–Ramanan will be used on several occasions.

LEMMA 2.1. *Let E be a vector bundle on C of rank $n \geq 2$ with $h^0(E) = n + s$ $(s \geq 1)$ such that E possesses no proper subbundle F with $h^0(F) > r_F$. Then $h^0(\det E) \geq ns + 1$ and so $d_E \geq d_{ns}$.*

Proof. This is a restatement of Lemma 3.9 in [**17**]. □

3. The main theorem.

Let C be a smooth curve with $\mathrm{Cliff}(C) \geq 3$ and n an integer, $n \geq 3$. We begin by taking two line bundles L_1, L_2 on C of degree d_{n-1} with $h^0(L_i) = n$. Let E_i denote the dual span of L_i (see (1)), so we have exact sequences

$$0 \to E_i^* \to H^0(L_i) \otimes \mathcal{O}_C \to L_i \to 0. \tag{3}$$

LEMMA 3.1. *Suppose $\frac{d_p}{p} \geq \frac{d_{p+1}}{p+1}$ for all $p < n-1$ and $d_{n-1} \neq (n-1)d_1$. Then, E_i is semistable and $h^0(E_i) = n$ for $i = 1, 2$.*

Proof. Dualising (3), we see at once that $h^0(E_i) \geq n$. The result now follows from Proposition 4.9(d) and Theorem 4.15(a) in [**6**]. □

Now consider non-trivial extensions

$$0 \to E_1 \to E \to L_2 \to 0. \tag{4}$$

Note that, if the hypotheses of Lemma 3.1 hold, then $h^0(E_1) = h^0(L_2) = n$, so $h^0(E) \leq 2n$.

LEMMA 3.2. *Suppose that $h^0(E_1) = n$. Then there exists a non-trivial extension* (4) *with $h^0(E) = 2n$ if and only if $h^0(E_2 \otimes E_1) > n^2$.*

Proof. Clearly $h^0(E) = 2n$ if and only if all sections of L_2 lift to E. We consider the dual of the sequence (3) for $i = 2$ tensored by E_1. If $\alpha : L_2 \to E_1$ is a non-zero homomorphism, then $H^0(L_2) \subset H^0(E_1)$. Since both spaces have dimension n, it follows that all sections of E_1 have the form $\alpha \circ s$ for some $s \in H^0(L_2)$, contradicting the fact that E_1 is generated. So $H^0(L_2^* \otimes E_1) = 0$ and, taking cohomology, we obtain an exact sequence,

$$0 \to H^0(L_2)^* \otimes H^0(E_1) \xrightarrow{\psi} H^0(E_2 \otimes E_1) \to H^1(L_2^* \otimes E_1) \xrightarrow{\varphi} H^0(L_2)^* \otimes H^1(E_1). \quad (5)$$

An extension (4) has the property that all sections of L_2 lift if and only if its class is in $\ker \varphi$. So there exists such an extension with this property if and only if ψ fails to be surjective. Since $H^0(L_2)^* \otimes H^0(E_1)$ has dimension n^2, the result follows. □

LEMMA 3.3. *Suppose E is as in Lemma 3.2. If $\gamma(E) < \mathrm{Cliff}(C)$, then $2d_{n-1} < nd_1$. Conversely, if d_1 computes $\mathrm{Cliff}(C)$ and $2d_{n-1} < nd_1$, then $\gamma(E) < \mathrm{Cliff}(C)$.*

Proof. We have $\gamma(E) = \frac{1}{n}(2d_{n-1} - 2n) = \frac{2d_{n-1}}{n} - 2$. Since $\mathrm{Cliff}(C) \leq d_1 - 2$ with equality if d_1 computes $\mathrm{Cliff}(C)$, the result follows. □

LEMMA 3.4. *Suppose C is a Petri curve of genus $g \geq 7$ and $n \geq 3$. Then $2d_{n-1} < nd_1$ except when $n = 3$, $g = 8, 10, 14$.*

Proof. This follows by direct computation from the formulae for d_r (see (2)). □

PROPOSITION 3.5. *Suppose that d_1 computes $\mathrm{Cliff}(C)$, $3d_1 \geq 2d_2$ and $\mathrm{Cliff}_2(C) = \mathrm{Cliff}(C)$. Suppose further that $n = 3$ and the extension* (4)

$$0 \to E_1 \to E \to L_2 \to 0$$

is non-trivial with $h^0(E) = 6$. Then E is semistable.

Proof. Suppose first that F is a subbundle of E of rank 2 contradicting semistability. Then $d_{E/F} < \frac{2d_2}{3}$. Since E is generated with $h^0(E^*) = 0$, the same holds for E/F, so $E/F \not\simeq \mathcal{O}$ and hence $h^0(E/F) \geq 2$ and $d_{E/F} \geq d_1$. This contradicts the hypothesis $3d_1 \geq 2d_2$.

Now suppose there is no subbundle of rank 2 contradicting semistability, but that L is a line subbundle with $d_L > \frac{2d_2}{3}$. If E/L is not stable, we can pull back a line subbundle of E/L to get a subbundle F of E of rank 2 with

$$d_F \geq \frac{1}{2}d_{E/L} + d_L = \frac{1}{2}(d_E + d_L) > \frac{4d_2}{3}.$$

This is a contradiction, so E/L is stable. Since $L \not\subset E_1$ and (4) is assumed non-trivial, we have $d_L < d_2$. So $h^0(L) \leq 2$ and $h^0(E/L) \geq 4$. It follows that E/L contributes to $\mathrm{Cliff}_2(C)$, so

$$d_{E/L} \geq 2\ \mathrm{Cliff}_2(C) + 4 = 2\ \mathrm{Cliff}(C) + 4 = 2d_1. \quad (6)$$

But $d_{E/L} < \frac{4d_2}{3}$, so this again contradicts the hypothesis $3d_1 \geq 2d_2$. □

THEOREM 3.6. *Suppose C is a curve for which d_1 computes $\mathrm{Cliff}(C)$, $3d_1 > 2d_2$ and there exist L_1, L_2 of degree d_2 with $h^0(L_i) = 3$ for which $h^0(E_2 \otimes E_1) > 9$. Then either $\mathrm{Cliff}_2(C) < \mathrm{Cliff}(C)$ or $\mathrm{Cliff}_3(C) < \mathrm{Cliff}(C)$.*

Proof. This follows from Lemmas 3.1, 3.2 and 3.3 and Proposition 3.5. □

REMARK 3.7. Suppose that all the hypotheses of Proposition 3.5 hold except that $\mathrm{Cliff}_2(C) < \mathrm{Cliff}(C)$. Then $\mathrm{Cliff}(C) \geq 5$ by Proposition 3.8 in **[6]** and hence $g \geq 11$; moreover, by Theorem 5.2 in **[6]**, we have $\mathrm{Cliff}_2(C) \geq \frac{d_4}{2} - 2$. It follows that the proof of the proposition is valid except that (6) must be replaced by

$$d_{E/L} \geq d_4. \tag{7}$$

If $3d_4 \geq 4d_2$, this contradicts the assumption that E is not semistable.

COROLLARY 3.8. *Suppose C is a curve for which d_1 computes* $\mathrm{Cliff}(C)$*, $3d_1 > 2d_2$, $3d_4 \geq 4d_2$ and there exist L_1, L_2 of degree d_2 with $h^0(L_i) = 3$ for which $h^0(E_2 \otimes E_1) > 9$. Then* $\mathrm{Cliff}_3(C) < \mathrm{Cliff}(C)$.

Proof. This follows from the proof of Theorem 3.6 and Remark 3.7. □

REMARK 3.9. The assumption $3d_4 \geq 4d_2$ holds for Petri curves of genus 12, 13, 14, 15, 18, 19 and 24 (also for genus ≤ 10, but in this case $\mathrm{Cliff}_2(C) = \mathrm{Cliff}(C)$, so the corollary does not lead to any improvement).

4. Curves of genus 9 and 11. To find examples of curves for which $\mathrm{Cliff}_3(C) < \mathrm{Cliff}(C)$, it remains only to choose C suitably and then show that there exist L_1, L_2 as in the statement of Theorem 3.6 such that $h^0(E_2 \otimes E_1) > 9$.

For $g = 9$, Mukai (Proposition 1.2 in **[14]**) proves this by a very special argument, which works also for $g = 11$. This is based on a result of Mumford **[16]**; for completeness and in view of possible generalisations, we give a proof using only Mumford's result.

PROPOSITION 4.1. *With the notations of the previous section, suppose $n = 3$ and let C be a Petri curve of genus 9 or 11. Then there exist E_1, E_2 as above such that $h^0(E_2 \otimes E_1) > 9$.*

Proof. Note first (see (2)) that $d_2 = g + 2 - \left[\frac{g}{3}\right] = g - 1$ in both cases. Take any line bundle L_1 of degree d_2 with $h^0(L_1) = 3$ and put $L_2 = L_1^* \otimes K_C$. Then $d_{L_2} = d_2$ and $h^0(L_2) = 3$. Moreover,

$$\det E_1 \otimes \det E_2 \simeq L_1 \otimes L_2 \simeq K_C.$$

The canonical homomorphism

$$(E_2 \otimes E_1) \otimes (E_2 \otimes E_1) \to \bigwedge^2 E_2 \otimes \bigwedge^2 E_1 \simeq K_C$$

defines a K_C-valued quadratic form Q on $E_2 \otimes E_1$. Now let L be a line subbundle of E_1, put $M = E_1/L$ and consider the family of extensions

$$0 \to L \to F \to M \to 0$$

with L and M fixed. Tensoring by E_2, we obtain a family

$$0 \to E_2 \otimes L \to E_2 \otimes F \to E_2 \otimes M \to 0. \tag{8}$$

https://doi.org/10.1017/S0017089511000577 Published online by Cambridge University Press

The quadratic form Q extends to a quadratic form on the family (8). It follows from the theorem in [**16**] (see Application (5) p. 186 in [**16**]) that

$$h^0(E_2 \otimes E_1) \equiv h^0(E_2 \otimes L) + h^0(E_2 \otimes M) \bmod 2.$$

On the other hand, by Serre duality and Riemann–Roch,

$$h^0(E_2 \otimes L) + h^0(E_2 \otimes M) = h^0(E_2 \otimes L) + h^1(E_2 \otimes L) \equiv \deg E_2 \bmod 2.$$

Now $\deg E_2 = g - 1$ is even, so $h^0(E_2 \otimes E_1)$ is also even. The result now follows from (5). □

REMARK 4.2. The only place where C being Petri is used here is in the requirement that $d_2 = g - 1$. For genus 9, this is true whenever Cliff(C) takes its maximum value 4. In fact, we have in this case $d_2 \geq \mathrm{Cliff}(C) + 4 = 8$ and (by (2)) $d_2 \leq g + 2 - [\frac{g}{3}] = 8$. We have also $d_1 = 6$ (in the language of Mukai's papers, C is 'non-pentagonal') and hence $3d_1 > 2d_2$.

THEOREM 4.3. *Let C be a curve of genus* 9 *with Clifford index* Cliff$(C) = 4$. *Then*

(i) $\mathrm{Cliff}_r(C) < \mathrm{Cliff}(C)$ *whenever r is divisible by* 3;

(ii) $3 \leq \mathrm{Cliff}_3(C) \leq \frac{10}{3}$;

(iii) *the expected dimension of the Brill–Noether locus $B(3, 16, 6)$ is -11, but this locus is non-empty.*

Proof. (i) If r is divisible by 3, $\mathrm{Cliff}_r(C) \leq \mathrm{Cliff}_3(C)$ by Lemma 2.2 in [**6**], so it is sufficient to prove that $\mathrm{Cliff}_3(C) < \mathrm{Cliff}(C)$. When $\mathrm{Cliff}(C) = 4$, we have $\mathrm{Cliff}_2(C) = \mathrm{Cliff}(C)$ by Proposition 3.8 in [**6**]. The result now follows from Theorem 3.6, Proposition 4.1 and Remark 4.2.

(ii) By Propositions 4.1 and 3.5, there exists a semistable bundle E of rank 3 and degree $2d_2 = 16$ with $h^0(E) = 6$. Hence, $\gamma(E) = \frac{10}{3}$, so $\mathrm{Cliff}_3(C) \leq \frac{10}{3}$. On the other hand, by Theorem 4.1 in [**8**], $\mathrm{Cliff}_3(C) \geq 3$.

(iii) We have $\beta(3, 16, 6) = 9g - 8 - 6(6 - 16 + 3g - 3) = -11$. The bundle E is semistable and therefore stable since $\gcd(3, 16) = 1$, so $B(3, 16, 6) \neq \emptyset$. □

For $g = 11$, there is more work to do. For a Petri curve of genus 11, we have $\mathrm{Cliff}(C) = 5$ and $d_4 = 13$. It follows from Theorem 5.2 in [**6**] that

$$\frac{9}{2} \leq \mathrm{Cliff}_2(C) \leq 5. \tag{9}$$

We wish to investigate the possibility that $\mathrm{Cliff}_2(C) = \frac{9}{2}$. We begin with a lemma which generalises part of Theorem 5.2 in [**6**].

LEMMA 4.4. *Let C be any smooth curve and F a semistable bundle of rank* 2 *and slope $\mu(F) \leq g - 1$ on C with $h^0(F) = n + s$, $s > 0$. Then*

$$\gamma(F) \geq \min\left\{\mathrm{Cliff}(C), \frac{d_{2s}}{2} - s\right\}.$$

Proof. If F has a line subbundle L with $h^0(L) \geq 2$, then, as in the first part of the proof of Theorem 5.2 in [**6**], we have $\gamma(F) \geq \mathrm{Cliff}(C)$. Otherwise, by Lemma 2.1, $d_F \geq d_{2s}$, giving $\gamma(F) \geq \frac{d_{2s}}{2} - s$. □

PROPOSITION 4.5. *Let C be a curve of genus* 11 *with* $\mathrm{Cliff}(C) = 5$ *and* $\mathrm{Cliff}_2(C) < \mathrm{Cliff}(C)$. *Then* $\mathit{Cliff}_2(C)$ *is computed by one or more generated stable bundles F of rank* 2 *and degree* 13 *with* $h^0(F) = 4$ *and by no other bundles.*

Proof. We have $\mathrm{Cliff}_2(C) = \frac{9}{2}$ by (9). Let F be a bundle computing $\mathrm{Cliff}_2(C)$ with $h^0(F) = 2 + s$. If $s \geq 3$, then by Lemma 4.4,

$$\gamma(F) \geq \min\left\{\mathrm{Cliff}(C), \frac{d_{2s}}{2} - s\right\} \geq \min\left\{5, \frac{d_6 + 2s - 6}{2} - s\right\} = \min\left\{5, \frac{d_6}{2} - 3\right\} = 5,$$

since $d_6 = 16$ by (2) and Lemma 4.6 in **[6]**, a contradiction. So $s = 2$, giving $h^0(F) = 4$ and $d_F = 13$. Since F is semistable and $\gcd(2, 13) = 1$, it is, in fact, stable. If F is not generated, then there exists a subsheaf F' of degree 12 with $h^0(F') = 4$; moreover, F' is a semistable bundle. Then $\gamma(F') = 4$, a contradiction. □

THEOREM 4.6. *Let C be a Petri curve of genus* 11. *Then,*

(i) $\mathrm{Cliff}_r(C) < \mathrm{Cliff}(C)$ *whenever r is divisible by* 3;

(ii) $\frac{11}{3} \leq \mathrm{Cliff}_3(C) \leq \frac{14}{3}$;

(iii) *the expected dimension of the Brill–Noether locus* $B(3, 20, 6)$ *is* -5, *but this locus is non-empty.*

Proof. (i) If $\mathrm{Cliff}_2(C) = \mathrm{Cliff}(C)$, this follows from Theorem 3.6, Lemma 3.4 and Proposition 4.1 together with Lemma 2.2 in **[6]**.

If $\mathrm{Cliff}_2(C) < \mathrm{Cliff}(C)$, Theorem 3.6 does not apply and, since $d_2 = 10$ and $d_4 = 13$, neither does Corollary 3.8. However, by Proposition 4.5, there exists a generated stable bundle F of rank 2 and degree 13 with $h^0(F) = 4$. Any line subbundle of F has degree ≤ 6 and hence $h^0 \leq 1$. So, by Lemma 2.1, we have $h^0(\det F) \geq 5$, and in fact this is an equality since $\deg \det F = 13 = d_4$.

Now let $L = \det F$ and consider extensions

$$0 \to F \to E \to K_C \otimes L^* \to 0. \tag{10}$$

It is clear that $\deg E = 20$ and that, if (10) does not split, then E is stable. Moreover, since $h^0(L) = 5$, it follows by Riemann–Roch that $h^0(K_C \otimes L^*) = 2$. Thus, $h^0(E) \leq 6$ with equality if and only if all sections of $K_C \otimes L^*$ lift to E. For this we require that the canonical homomorphism

$$H^1(K_C^* \otimes L \otimes F) \to \mathrm{Hom}(H^0(K_C \otimes L^*), H^1(F))$$

should fail to be injective. Equivalently, the dual homomorphism

$$H^0(K_C \otimes L^*) \otimes H^0(K_C \otimes F^*) \to H^0(K_C^2 \otimes L^* \otimes F^*) \tag{11}$$

should be non-surjective. We already know that $h^0(K_C \otimes L^*) = 2$ and, by Riemann–Roch,

$$h^0(K_C \otimes F^*) = h^1(F) = h^0(F) - 13 + 20 = 11.$$

So the dimension on the left-hand side of (11) is equal to 22. The bundle $K_C^2 \otimes L^* \otimes F^*$ is a stable bundle of rank 2 and degree $41 = 4g - 3$, so, by Riemann–Roch, the right-hand side of (11) has dimension $2g - 1 = 21$. Moreover, by the base point free pencil

https://doi.org/10.1017/S0017089511000577 Published online by Cambridge University Press

trick, the kernel of (11) is isomorphic to

$$H^0(K_C^* \otimes L \otimes K_C \otimes F^*) = H^0(\det F \otimes F^*) = H^0(F).$$

Since $h^0(F) = 4$, this completes the proof that (11) is not surjective.

(ii) The stable bundles E constructed in (i) have $\gamma(E) = \frac{1}{3}(20 - 6) = \frac{14}{3}$. On the other hand, since $\mathrm{Cliff}_2(C) \geq \frac{9}{2}$, we have $\mathrm{Cliff}_3(C) \geq \frac{10}{3}$ by Theorem 4.1 in [**8**]. Checking the proof of Theorem 4.1 in [**8**], one sees easily that one can replace $\frac{2\,\mathrm{Cliff}_2(C)+1}{3}$ in the statement of the theorem by $\min\left\{\frac{2\,\mathrm{Cliff}(C)+1}{3}, \frac{2\,\mathrm{Cliff}_2(C)+2}{3}\right\}$. So, in our case, $\mathrm{Cliff}_3(C) \geq \frac{11}{3}$.

(iii) We have $\beta(3, 20, 6) = 9g - 8 - 6(6 - 20 + 3g - 3) = -5$. The bundle E is semistable and therefore stable since $\gcd(3, 20) = 1$, so $B(3, 20, 6) \neq \emptyset$. □

5. Questions and comments.

In this section, we raise a number of interesting questions with some observations on possible answers.

Question 5.1. Can one find further examples of semistable bundles E of rank 3 with $h^0(E) = 6$ and $\gamma(E) < \mathrm{Cliff}(C)$?

In attempting to answer this, we first note the following.

Proposition 5.2. *Let E be a semistable bundle of rank 3 on C with $h^0(E) = 6$. Then,*

$$\gamma(E) \geq \min\left\{\frac{d_9}{3} - 2, d_1 - 2, \frac{2d_2}{3} - 2, \frac{d_4}{2} - 2\right\}. \tag{12}$$

Proof. If E has no proper subbundle F with $h^0(F) > r_F$, then $d_E \geq d_9$ by Lemma 2.1. So $\gamma_E \geq \frac{1}{3}(d_9 - 6) = \frac{d_9}{3} - 2$.

If E has a line subbundle L with $h^0(L) \geq 2$, then $d_L \geq d_1$, so by semistability $d_E \geq 3d_1$, giving $\gamma(E) \geq \frac{1}{3}(3d_1 - 6) = d_1 - 2$.

If E has a subbundle F of rank 2 with $h^0(F) = 3$ and no line subbundle with $h^0 \geq 2$, then $d_F \geq d_2$. Moreover, $h^0(E/F) \geq 3$, so $d_{E/F} \geq d_2$. Thus, $d_E \geq 2d_2$ and $\gamma(E) \geq \frac{2d_2}{3} - 2$.

Finally, if E has a subbundle F of rank 2 with $h^0(F) \geq 4$ but no line subbundle with $h^0 \geq 2$, then, by Lemma 2.1 again, we have $d_F \geq d_4$. So $d_E \geq \frac{3d_4}{2}$ and $\gamma(E) \geq \frac{d_4}{2} - 2$. □

Comment 1. The value $d_1 - 2$ in (12) can always be attained (by a direct sum of three line bundles of degree d_1 with $h^0 = 2$), but it is not clear whether it can be attained by a stable bundle; in any case, $d_1 - 2 \geq \mathrm{Cliff}(C)$, so this is not interesting from the point of view of Question 5.1. In general, the construction of (4) seems most likely to yield examples and the main obstacle is that we need to prove that $h^0(E_2 \otimes E_1) > 9$. For Petri curves of low genus, we have

- $g = 7$: (12) gives $\gamma(E) \geq \frac{8}{3}$. This could be attained only by a bundle of the form (4) and such bundles exist if and only if $h^0(E_2 \otimes E_1) > 9$. Apart from this, $\gamma(E) \geq \mathrm{Cliff}(C)$.
- $g = 8, 10$: (12) gives $\gamma(E) \geq d_1 - 2 = \mathrm{Cliff}(C)$, so there is nothing to prove.
- $g = 9$: (12) gives $\gamma(E) \geq \frac{10}{3}$ and this value can be attained by Theorem 4.3.

https://doi.org/10.1017/S0017089511000577 Published online by Cambridge University Press

- $g = 11$: (12) gives $\gamma(E) \geq \frac{13}{3}$. By Theorem 4.6, the value $\frac{14}{3}$ can be attained, but we do not know about $\frac{13}{3}$. Any bundle attaining this value would have to be of degree 19 and to have no proper subbundle F with $h^0(F) > r_F$.
- $g = 12$: (12) gives $\gamma(E) \geq \frac{14}{3}$. This value could be attained by a bundle of the form (4) or by a bundle possessing no subbundle F with $h^0(F) > r_F$; we do not know whether any such bundles exist.

COMMENT 2. If we restrict attention to stable bundles on Petri curves, Question 5.1 can be rephrased in terms of the Brill–Noether loci. In fact, on a Petri curve, we have (using (2)),

- g odd: $\beta(3, d, 6) < 0$ if and only if $d \leq 3d_1 - 1$;
- g even: $\beta(3, d, 6) < 0$ if and only if $d \leq 3d_1 + 1$. In this case the Brill–Noether locus $B(3, 3d_1 + 1, 6)$ is non-empty. (Take three line bundles L_1, L_2, L_3 of degree d_1 with $h^0(L_i) = 2$ and no two of the L_i isomorphic, and take E to be a general positive elementary transformation $0 \to L_1 \oplus L_2 \oplus L_3 \to E \to \tau \to 0$ with τ of length 1; then $E \in B(3, 3d_1 + 1, 6)$.)

We have also $\mathrm{Cliff}(C) = d_1 - 2$; so, for E of rank 3 with $h^0(E) = 6$, the condition $d_E \leq 3d_1 \pm 1$ is equivalent to $\gamma(E) \leq \mathrm{Cliff}(C) \pm \frac{1}{3}$. We, therefore, have the following version of Question 5.1.

QUESTION 5.3. Do there exist non-empty Brill–Noether loci $B(3, d, 6)$ on a Petri curve C with negative expected dimension in addition to those listed in Comment 2 above and those of Theorems 4.3 and 4.6?

COMMENT. The case $d_E = 3d_1$, where $\gamma(E) = \mathrm{Cliff}(C)$, is interesting here. There certainly exist semistable bundles in this case, but it is not clear whether stable bundles exist.

QUESTION 5.4. Can one calculate $\mathrm{Cliff}_3(C)$ precisely or at least obtain a better estimate than that of **[8]**?

COMMENT. For a Petri curve of genus 11, we have slightly improved the estimate of **[8]** in the proof of Theorem 4.6(ii) and this improvement applies to other curves. The arguments of **[8]** suggest that one example to be considered for a low value of $\gamma(E)$ would be a bundle of degree $2g + 3$ expressible in the form

$$0 \to F \to E \to L \to 0,$$

where F is a semistable bundle of rank 2 and degree d_2 with $h^0(F) = 3$, L is a line bundle with the maximal number of sections possible for its degree and $h^0(E) = 3 + h^0(L)$. Possible examples are

- C Petri of genus 7: $d_L = 10 = d_4$, so $h^0(E) = 8$ and $\gamma(E) = \frac{7}{3}$; if such a bundle exists, it computes $\mathrm{Cliff}_3(C)$;
- C Petri of genus 9: $d_L = 13 = d_5$, so $h^0(E) = 9$ and $\gamma(E) = 3$; again, if such a bundle exists, it computes $\mathrm{Cliff}_3(C)$;
- C Petri of genus 11: $d_L = 15$, $h^0(L) = 6$, so $h^0(E) = 9$ and $\gamma(E) = \frac{13}{3}$; even if such a bundle exists, it may not compute $\mathrm{Cliff}_3(C)$.

QUESTION 5.5. On a Petri curve of genus 11, does there exist a stable bundle of rank 2 and degree 13 with $h^0 = 4$?

COMMENT. As we have seen in Proposition 4.5, this is the only way in which one could have $\mathrm{Cliff}_2(C) < \mathrm{Cliff}(C)$ on such a curve. By Theorem 3.2 and Remark 5.4 in [**4**], such a bundle exists if and only if there is a non-degenerate morphism $C \to \mathbb{P}^4$ of degree 13 whose image is contained in a quadric.

QUESTION 5.6. What about $n = 4$?

COMMENT. There are now two obstacles to using the method of Section 3; both Propositions 3.5 and 4.1 use $n = 3$. It may be possible to generalise the first of these propositions; the second looks more problematic. Another possible method is to use extensions $0 \to E \to G \to M \to 0$ with E of rank 3 and degree $2d_2$ with $h^0(E) = 6$ and M a line bundle of degree d_1 with $h^0(M) = 2$; one still needs to prove semistability and show that the multiplication map

$$H^0(M) \otimes H^0(K_C \otimes E^*) \to H^0(M \otimes K_C \otimes E^*)$$

is not surjective.

QUESTION 5.7. Can one find examples with $\mathrm{Cliff}_3(C) > \mathrm{Cliff}_2(C)$ (or more generally $\mathrm{Cliff}_{n+1}(C) > \mathrm{Cliff}_n(C)$)?

COMMENT. Any example would show that the hypotheses of Theorem 2.4 in [**7**] can fail. Genus 11 is the first case where this might happen, but note that the bundles E constructed in the proof of Theorem 4.6 are all generated, so the conclusion of Theorem 2.4 in [**7**] could still hold.

6. Postscript. Since this paper was completed in September 2010, there have been remarkable developments in the construction of bundles providing counter-examples to Mercat's conjecture and relating them to Koszul cohomology, the maximal rank conjecture and the geometry of the moduli space of curves [**2, 3, 9, 10**]. It is interesting (and probably significant) to note that all currently known counter-examples involve curves lying on K3 surfaces. Most of this work concerns bundles of rank 2 and in particular (Theorem 1.3 in [**3**]) gives a negative answer to Question 5.5 for a general curve of genus 11. Paper [**3**] also contains a significant result for bundles of rank 3, showing that $\mathrm{Cliff}_3(C) < \mathrm{Cliff}(C)$ for curves of genus $g \geq 11$ of maximal Clifford index which lie on K3 surfaces, thus extending the results of this paper and providing an answer to Question 5.1 (Corollary 1.6 in [**3**]).

REFERENCES

1. S. B. Bradlow, O. García-Prada, V. Muñoz and P. E. Newstead, Coherent systems and Brill–Noether theory, *Int. J. Math.* **14** (2003), 683–733.

2. G. Farkas and A. Ortega, The maximal rank conjecture and rank two Brill–Noether theory, *Pure Appl. Math. Q.* **7**(4) (2011), 1265–1296.

3. G. Farkas and A. Ortega, Higher rank Brill–Noether theory on sections of K3 surfaces, arXiv:1102.0276.

4. I. Grzegorczyk, V. Mercat and P. E. Newstead, Stable bundles of rank 2 with 4 sections, *Int. J. Math.* to appear. arXiv:1006.1258v2.

5. I. Grzegorcyk and M. Teixidor i Bigas, Brill–Noether theory for stable vector bundles, in *Moduli spaces and vector bundles*, *London Mathematical Society Lecture Notes Series*, vol. 359 (Cambridge University Press, Cambridge, UK, 2009), 29–50.

6. H. Lange and P. E. Newstead, Clifford indices for vector bundles on curves, in *Affine flag manifolds and principal bundles* (Trends in Mathematics, Birkhäuser, Basel, 2010), 165–202.

7. H. Lange and P. E. Newstead, Generation of vector bundles computing Clifford indices, *Arch. Math.* **94** (2010), 529–537.

8. H. Lange and P. E. Newstead, Lower bounds for Clifford indices in rank three, *Math. Proc. Camb. Philos. Soc.* **150** (2011), 23–33, doi:10.1017/S0305004110000502.

9. H. Lange and P. E. Newstead, Further examples of stable bundles of rank 2 with 4 sections, *Pure Appl. Math. Q.* **7**(4) (2011), 1517–1528.

10. H. Lange and P. E. Newstead, Vector bundles of rank 2 computing Clifford indices. to appear. arXiv:1012.0469.

11. R. Lazarsfeld, Brill–Noether–Petri without degeneration, *J. Differ. Geom.* **23** (1986), 299–307.

12. V. Mercat, Clifford's theorem and higher rank vector bundles, *Int. J. Math.* **13** (2002), 785–796.

13. S. Mukai, Curves and symmetric spaces, *Proc. Jap. Acad.* **68**(ser. A) (1992), 7–10.

14. S. Mukai, Curves and symmetric spaces, *Ann. Math*. **173**(3) (2011), 1539–1558.

15. S. Mukai, Addendum to "Curves and symmetric spaces, II." Addendum RIMS-1395 (2010).

16. D. Mumford, Theta characteristics of an algebraic curve, *Ann. Scient. Éc. Norm. Sup., 4^e série, t.* **4** (1971), 181–192.

17. K. Paranjape and S. Ramanan, On the canonical ring of a curve, in *Algebraic geometry and commutative algebra*, vol. II (Hijikara H. et al., Editors) (Kinokuniya, Tokyo, 1988), 503–516.