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Measure and Integral[1]

By W. W. ROGOSINSKI.

1. It is now nearly half a century since H. Lebesgue, whose obituary the reader may have seen in *Nature* not so long ago, created his theory of the integral which since then has superseded in modern analysis the classical conception due to B. Riemann. It is, I think, regrettable that knowledge of the Lebesgue integral seems to be still largely confined to the research worker. There is nothing unduly abstract or unnatural in this theory, nor anything in the proofs which would be too difficult for a good honours student to grasp. If the aim of university education be the teaching of general ideas and methods rather than that of technicalities, then the modern notion of the integral should not be omitted from the mathematical syllabus. It is the purpose of this purely expository note to sketch the build up of both the Riemann and the Lebesgue integral on the common geometrical basis of "measure" and thus to make evident to the uninitiated reader the striking advantages of the new integral.

2. We shall confine ourselves to the integral of a function $y = f(x)$ of one variable; the generalisation to more variables will appear obvious. Suppose first that $f(x)$ is continuous and non-negative in the interval $\langle a, b \rangle$, i.e. for $a \leqq x \leqq b$. Then, with the classical definition of the *definite integral*,

[1] This is a slightly revised version of a lecture given to the Edinburgh Mathematical Society on May 5th, 1945.

$$\int_a^b f(x)\,dx = A, \tag{2.1}$$

where A is the "area" of the region above the x-axis $y = 0$, below the curve $y = f(x)$, and between the lines $x = a$ and $x = b$. In fact, this is the fundamental application of the integral calculus. Also (Figure 1) the corresponding area $A(t)$ between $x = a$ and $x = t$, where $a \leqq t \leqq b$, is a differentiable function of t, and its derivative satisfies

$$A'(t) = f(t). \tag{2.2}$$

From this it follows that

$$A(t) = \Phi(t) - \Phi(a), \tag{2.3}$$

where $\Phi(x)$ is any *primitive function* (indefinite integral) of $f(x)$, i.e. a function such that

$$\Phi'(x) = f(x). \tag{2.4}$$

In this way the problem of finding the area A is reduced to that of "integrating" $f(x)$. Integration being the operation inverse to differentiation, the *Calculus* thus becomes available. All this, of course, is familiar.

3. In the relation (2.1) it is assumed that we know what an area is. Now we have, of course, a more or less clear intuition of area. But this is not good enough. Mathematics being a part of Logic, or perhaps *vice versa*, its language requires exact definitions. An intuitive preconception can lead to a mathematical notion, but is not equivalent to it. Once the mathematical definition has been decided upon, intuition should serve only as a guide. The problem of area is as old as mathematics itself. The ingenious methods of "exhaustion," that is approximation to an area through polygons, used by Archimedes and the later Greek geometers, are well known. We shall see that the same ideas lead to the modern definition.

Now the integral of a continuous function may be defined analytically as the limit of its Riemann sums (see (6.3)), and one could then use (2.1) to define the above area A. Clearly, this would be highly artificial. It is rather the other way round that Riemann's method becomes intelligible, i.e. as a method of exhaustion for A. Again, one might try to use (2.3) for the definition of A. But then it is not at all obvious, though true, that every continuous function $f(x)$ possesses a primitive function $\Phi(x)$, as in (2.4). Finally, if we wish to throw off the restriction as to continuous functions, when the ordinate set becomes less simple, it is even more evident that we should first obtain an efficient definition of a general area and apply this, in turn, to define the definite integral by means of (2.1).

4. Any definition of area should agree with the intuitive preconception in its fundamentals. Let E be any plane set of points. Its "area" $A = A(E)$ should satisfy the following postulates:

(i) *A is non-negative.*

(ii) *If E_1 is contained in E, in symbols, if $E_1 \subset E$, then $A(E_1) \leqq A(E)$.*

(iii) *A is additive. That is, the area of a finite sum of non-overlapping sets E_k is equal to the sum of the areas $A(E_k)$.*

(iv) *For polygons the area has its "elementary" value.*

(v) *Congruent sets have equal area.*

Any such definition of area will lead, by (2.1), to a corresponding definition of the definite integral of a non-negative function $f(x)$ over $\langle a, b \rangle$. Here the set E, the *ordinate set* of f, consists of the points $a \leqq x \leqq b$, $0 \leqq y \leqq f(x)$ in the x, y-plane. Similarly, any definition of "volume" will lead to a definition of a definite integral of a function $z = f(x, y)$, and so on.

If f is not non-negative, the integral is defined by

$$\int_a^b f(x)\, dx = \int_a^b f_+(x)\, dx - \int_a^b |f_-(x)|\, dx, \tag{4.1}$$

where $f_+(x) = f(x)$ when $f(x) \geqq 0$, and $f_+(x) = 0$ otherwise, and where $f_-(x)$ has a similar meaning. It is assumed that both integrals (areas) in (4.1) are finite. In other words, our definition of integral implies *absolute* integrability, that is the integrability of $|f(x)|$. The same applies to integrals of more than one variable.

5. Riemann's classical definition of the integral by approximation sums (1854) is really equivalent to the definition of the area of an ordinate set by exhaustion. The corresponding general theory of the area, however, was developed only considerably later by G. Peano (1887) and C. Jordan (1892).

We shall use the word *content* for the Peano-Jordan area, indicating the content of the set E by $c(E)$. We restrict ourselves to plane sets, but it will be obvious how to proceed in other dimensions. For linear sets the content will be the "length," in space the "volume," and so on in higher dimensions.

We call a set $x_1 \leqq x \leqq x_2$, $y_1 \leqq y \leqq y_2$ an interval I. Naturally we put

(5.1) $$c(I) = (x_2 - x_1)(y_2 - y_1)$$

as its content. We permit $x_2 = x_1$ and $y_2 = y_1$, when $c(I) = 0$. So a single point will have the content nought. Next, let $P = I_1 + I_2 + \ldots + I_n$ be a finite sum of " separate " intervals, i.e. intervals which have at most frontier points in common. In accordance with postulates 4, (iii) and (iv) we define

(5.2) $$c(P) = c(I_1) + c(I_2) + \ldots + c(I_n).$$

It is easy to prove that this definition is independent of how we write P as sum of intervals. These " primary " sets P form the material by means of which the general content is to be defined. (See Figure 2.)

Now let E be any *bounded* set, that is any set contained in some interval. We attach to it two finite numbers.

(i). We wish to approach E as closely as possible by primary sets from outside. So we define as the *outer content* of E

(5.3) $$\bar{c}(E) = l.b.\, c(P) \qquad (E \subset P),$$

the lower bound ($l.b.$) to be taken with respect to all primary sets P containing E.

(ii). Similarly, approaching E from inside, we define as its *inner content*

(5.4) $$\underline{c}(E) = u.b.\, c(P) \qquad (P \subset E),$$

where the upper bound ($u.b.$) is to be taken with respect to all sets P contained in E. Clearly

(5.5) $$0 \leqq \underline{c}(E) \leqq \bar{c}(E).$$

The definition of *the Peano-Jordan content* is now: The bounded set E has a content $c(E)$ if, and only if,

(5.6) $$\underline{c}(E) = \bar{c}(E) \qquad [= c(E)].$$

This definition satisfies all the postulates in 4. The proofs are quite simple, as one would expect considering the genuine simplicity of the procedure.

Not every bounded set has a content. Consider, for instance, the set R of all " rational " points in a given interval I, that is the set of all points (x, y) in I for which both x and y are rational. Now R is " everywhere dense " in I. It is, therefore, obvious that I itself is the " smallest " primary set containing R, while there is no primary set with positive content contained in R. Hence $\underline{c}(R) = 0$ and $\bar{c}(R) = c(I)$.

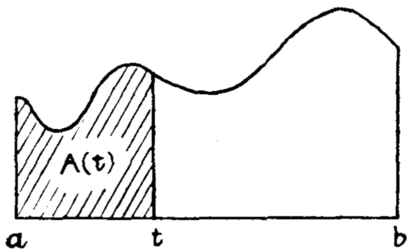

Figure 1.

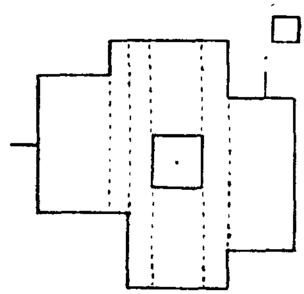

Figure 2: A finite primary set.

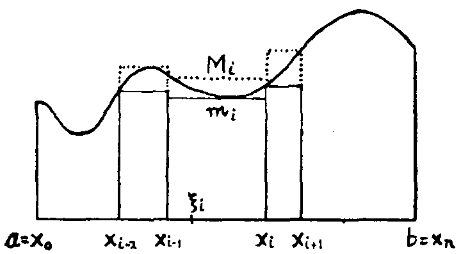

Figure 3: Riemann division.

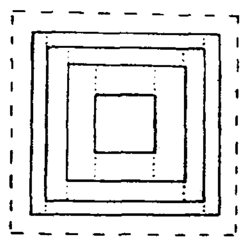

Figure 4:*
An 'open' square S as primary set.

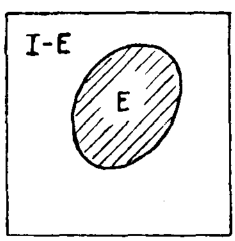

Figure 5.

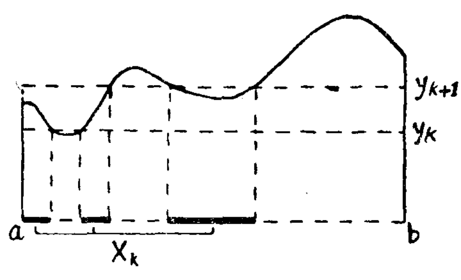

Figure 6: Lebesgue division.

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* The inner closed squares of which the first four are drawn in Figure 4 approach S from inside. The open square S is the sum of this sequence of closed squares. The dotted lines indicate a representation of S as a sum of contiguous intervals.

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6. Let E be the ordinate set corresponding to the non-negative and *bounded* function $f(x)$ defined in $\langle a, b\rangle$. (See Figure 3.) The two numbers

$$(6.1) \qquad \underline{c}(E) = \underline{\int_a^b} f(x)\,dx, \qquad \bar{c}(E) = \overline{\int_a^b} f(x)\,dx$$

are known respectively as the lower and upper Riemann integrals of $f(x)$ (first introduced by G. Darboux, 1875). $f(x)$ is integrable, in Riemann's sense, if, and only if, E has a content, in which case

$$(6.2) \qquad c(E) = \int_a^b f(x)\,dx.$$

Riemann's original definition is

$$(6.3) \qquad \int_a^b f(x)\,dx = \lim \sum_1^n (x_i - x_{i-1})\, f(\xi_i),$$

where the "limit" indicates that $n \to \infty$ and that the "maximum length" of the intervals $\langle x_{i-1}, x_i\rangle$ of the "division" of $\langle a, b\rangle$ should tend to nought, ξ_i being an arbitrary point in $\langle x_{i-1}, x_i\rangle$. Now

$$(6.4) \quad s = \sum_1^n (x_i - x_{i-1})\, m_i \leqq \sum_1^n (x_i - x_{i-1})\, f(\xi_i) \leqq \sum_1^n (x_i - x_{i-1})\, M_i = S,$$

where m_i and M_i denote the lower and upper bounds of the values of $f(x)$ in $\langle x_{i-1}, x_i\rangle$, respectively. Given the division, the sum s is apparently the content of the largest primary set built "over the division" and contained in E, while S is the content of the smallest such primary set containing E. It follows that the integrals (6.1) are equal to the upper bound of the s of all possible divisions of $\langle a, b\rangle$ and to the lower bound of all S, respectively. Since ξ_i is arbitrary, the existence of the limit (6.3) implies that these two integrals coincide, and *vice versa*. Hence the two definitions (6.2) and (6.3) are equivalent.

The definition (6.2) is surely the more natural one. But it should be kept in mind that, in order to obtain the familiar properties of the integral, the Riemann sums are indispensable as analytical tools.

To give an example of a bounded function that is not Riemann integrable, take

$$(6.5) \qquad \chi(x) = \begin{cases} 1 \\ 0 \end{cases} \text{ when } x \text{ is } \begin{cases} \text{rational} \\ \text{irrational} \end{cases}, \; 0 \leqq x \leqq 1.$$

Evidently, every s is 0 and every S is 1, so that

(6.6) $$\underline{\int_0^1} \chi(x)\,dx = 0; \quad \overline{\int_0^1} \chi(x)\,dx = 1.$$

$\chi(x)$ is the "characteristic" function of the (linear) set R of rational points in $\langle 0, 1\rangle$. The inner length (inner linear content) of R is 0, the outer length is 1.

7. The above definition of area and integral appears very natural and satisfactory. The restriction to bounded sets and functions is easily removed in the familiar way by introducing "improper" (Cauchy-) integrals. There are, however, considerable deficiencies inherent which appear whenever limiting processes are encountered.

Let $E_1 \subset E_2 \subset \ldots . E_k \subset \ldots .$ be an increasing sequence of uniformly bounded sets and let E be its limit. One would wish that E should have a content when the E_k have. This need not be the case. The set R in 5 provides a simple example. It is well known that R is enumerable, i.e. that the points P of R can be arranged as a sequence P_n. If we now take as E_k the set of the first k points P_n, then $c(E_k) = 0$, while the limit R has no content. Similarly, the limit of a decreasing sequence of sets $E_1 \supset E_2 \supset \ldots . E_k \supset \ldots .$, that is the set of points belonging to all E_k, need have no content when the E_k have.

As for the integral, let $f(x) = \lim\limits_{k \to \infty} f_k(x)$ where the $f_k(x)$ are integrable over $\langle a, b\rangle$. It is of the highest importance in analysis to have practicable and far reaching tests for the legitimacy of the operation

(7.1) $$\lim_{k \to \infty} \int_a^b f_k(x)\,dx = \int_a^b \lim_{k \to \infty} f_k(x)\,dx = \int_a^b f(x)\,dx,$$

that is of the interchange of the symbols $\int$ and lim. In particular, the limit $f(x)$ should be integrable. This need not be so, even when $f(x)$ is bounded. Consider the rational points in $\langle 0, 1\rangle$ arranged as a sequence x_n. If $f_k(x) = 1$ when $x = x_n$ for $n \leqq k$, and $f_k(x) = 0$ otherwise, then evidently $\int_0^1 f_k(x)\,dx = 0$. But $\lim\limits_{k \to \infty} f_k(x) = \chi(x)$, where $\chi(x)$ is the non-integrable function (6.5). This example cannot be put aside as too artificial. For $\chi(x)$ may be generated by limiting processes from continuous functions. In fact,

(7.2) $$\chi(x) = \lim_{n \to \infty} \{ \lim_{m \to \infty} (\cos n!\,\pi x)^{2m} \}.$$

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It is familiar that (7.1) is valid when the $f_k(x)$ converge uniformly. This is the only practicable test for the Riemann integral and it involves a condition far too stringent for many applications. There is a more general test due to Arzelà (1885): (7.1) is valid whenever the $f_k(x)$ are uniformly bounded provided that $f(x)$ be integrable. The latter condition is awkward since there is no simple criterion for the integrability of the limit.

The important relation (2.2), i.e.

$$\frac{d}{dt}\int_a^t f(x)\,dx = f(t), \tag{7.3}$$

is correct wherever f is continuous. But it is difficult to say more at this stage. The converse of (7.3), that is

$$\int_a^t f'(x)\,dx = f(t) - f(a) \tag{7.4}$$

need not be true even if $f'(x)$ exists throughout and is bounded. It is valid when $f'(x)$ is also integrable; again the same awkward condition as above.

8. The deficiencies inherent in Peano-Jordan's content and Riemann's integral were soon recognised. The first decisive steps to overcome them were taken by E. Borel (1898), but it was his pupil H. Lebesgue (1902) who developed these ideas and gave the complete theory of the new measure and integral. It should be mentioned that, at about the same time and independent of Lebesgue, W. H. Young and G. Vitali arrived at similar theories (1904). But it is the Lebesgue form of these results which has proved to be the most satisfactory.

It is to-day fairly easy to say where the trouble in the definition of the area arises. It is the postulate 4, (iii) of additivity which, by its restriction to finite sums of sets, kept the range of primary sets, fundamental for the definition, too narrow and thus prevented the smooth application to infinite sums and other limiting sets. We replace, therefore, this postulate by a more potent one which in itself refers to infinite sums:

(iii*): *A is fully additive; i.e. the area of a finite or infinite sum of non-overlapping sets E_k is equal to the sum of the areas $A(E_k)$.*

On first thought it might appear curious that, by merely asking more in the one postulate (iii), we should obtain better results. But it is easy to see why this is so. The new postulate forces our hands to choose a much wider range of "primary" sets, and so we may hope to approach a given set E more closely by these sets.

The Lebesgue area of a set E, to be defined now, is usually called the *measure* $m(E)$. For linear sets the measure will be the "length," in space the "volume," and so on.

We start, of course, by putting $m(I) = c(I)$, as in (5.1). Next, we take as "primary" sets (*cf.* Figure 4) any finite or infinite sum

$$P = I_1 + I_2 + \dots + I_k + \dots \tag{8.1}$$

of an at most enumerable sequence of *separate* intervals I_k, and we define, to satisfy postulates (iii*) and 4, (iv),

$$m(P) = m(I_1) + m(I_2) + \dots + m(I_k) + \dots . \tag{8.2}$$

The sum on the right hand side may diverge when $m(P) = \infty$. The representation (8.1) of P is, of course, not unique. But it is not difficult to prove that the definition (8.2) is independent of the choice of the representation. It should be noted that any enumerable set, for instance the set R in 5, is a primary set of measure 0, the I_k being points in this case.

Now let E be any set, not necessarily bounded. The definition of the *outer measure* $\bar{m}(E)$ is, in analogy to (5.3),

$$\bar{m}(E) = l.b.\ m(P) \qquad (E \subset P). \tag{8.3}$$

$\bar{m}(E)$ may be infinite. Since the range of primary sets is wider than in the Peano-Jordan case, we have $\bar{m}(E) \leqq \bar{c}(E)$, if E is bounded. The definition of the inner measure $\underline{m}(E)$ is slightly different from (5.4). First let E be bounded and $E \subset I$, say. Then we put (Figure 5)

$$\underline{m}(E) = m(I) - \bar{m}(I - E); \tag{8.4}$$

that is, instead of approaching E from inside we approach the "complement" $I - E$ from outside and take the appropriate difference.[1] If E is not bounded, let E_n be the set of points of E inside the interval $|x| \leqq n, |y| \leqq n$. We then define $\underline{m}(E) = \lim\limits_{n \to \infty} \underline{m}(E_n)$.

$\underline{m}(E)$ may be infinite. It follows easily from (8.4) that $\underline{c}(E) \leqq \underline{m}(E)$ if E is bounded, and that then

$$0 \leqq \underline{c}(E) \leqq \underline{m}(E) \leqq \bar{m}(E) \leqq \bar{c}(E). \tag{8.5}$$

Finally, a set E is *measurable* and has a measure $m(E)$, if

$$\underline{m}(E) = \bar{m}(E) \qquad [= m(E)]. \tag{8.6}$$

[1] A closer approach is effected in this way. (8.4) holds also for the content.

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$m(E)$ can be infinite; but it may, of course, be finite for an unbounded set. It follows from (8.5) that $m(E) = c(E)$ whenever the content exists. The converse is not true as the example of the above set R shows. Hence the measure is more effective than the content. There are, however, sets, though rather "artificial" ones, that are not measurable.

9. The definition of the Lebesgue integral is now obvious. Let E be the ordinate set of a non-negative (not necessarily bounded) function $f(x)$ defined in $\langle a, b\rangle$. We have

$$(9.1) \qquad \underline{m}(E) = \underline{\int_a^b} f(x)\, dx, \quad \bar{m}(E) = \overline{\int_a^b} f(x)\, dx,$$

the lower and the upper Lebesgue integrals. The function $f(x)$ is *integrable* if these two integrals are *finite and coincide*, and then

$$(9.2) \qquad m(E) = \int_a^b f(x)\, dx.$$

Since $m(E) = c(E)$ whenever the content exists, Riemann integrability implies Lebesgue integrability, and the two integrals have then the same value. The converse is not true. It is easy to see that the set E, corresponding to the function $\chi(x)$ in (6.5), has the Lebesgue measure 0, so that the Lebesgue integral of $\chi(x)$ is 0. But $\chi(x)$ is not Riemann integrable.

If $f(x)$ is not non-negative, its integral is defined by (4.1).

10. The Lebesgue measure is perfect as regards all the usual operations with sets. In particular, E is measurable and

$$(10.1) \qquad m(E) = \lim_{k \to \infty} m(E_k),$$

whenever E is the limit of an increasing or decreasing sequence of measurable sets E_k of finite measure. This follows easily from the definition of the measure. As a consequence $f(x)$ is integrable and

$$(10.2) \qquad \int_a^b f(x)\, dx = \lim_{k \to \infty} \int_a^b f_k(x),$$

whenever $f(x)$ is the limit of an increasing or decreasing sequence of non-negative integrable functions $f_k(x)$, provided that the limit on the right hand side is finite. From this is easily derived the following test of Lebesgue: The formula (10.2) is valid, for the Lebesgue integral, if the $f_k(x)$ are integrable and "dominated" convergent, i.e. convergent and $|f_k(x)| \leqq \Phi(x)$ where $\Phi(x)$ is integrable. This contains Arzelà's test in 7, but without the awkward condition that the limit $f(x)$ should be integrable. Lebesgue's test is one of the most important tools in analysis.

As for (7.3), it holds where f is continuous. But we can say now that it holds "*almost everywhere*" in $\langle a, b\rangle$, that is everywhere except perhaps for a set of points x of linear measure zero. Its converse (7.4) holds whenever $f'(x)$ exists and is bounded in $\langle a, b\rangle$. But, even for the Lebesgue integral, (7.4) need not be true if we know only the existence and integrability of $f'(x)$. This fact has given rise to further important generalisations of the integral (Denjoy, Perron).

As we have pointed out in 4, Lebesgue integrability of $f(x)$ implies that of $|f(x)|$. The function $y = x^{-1} \sin(x^{-1})$ is neither Riemann nor Lebesgue integrable over $\langle 0, 1\rangle$ though its Cauchy integral exists. It is clear how to cover such integrals in both the Riemann and the Lebesgue theory.

11. All the familiar properties and rules for the Riemann integral, for instance the mean value theorems and the rules of substitution and integration by parts, remain valid for the Lebesgue integral. To derive these, certain approximation sums, similar to the Riemann sums, are required.[1] (See Figure 6.)

Let $y_k < y_{k+1}$. It can be shown that for an integrable $f(x)$ the set X_k of all points x in $\langle a, b\rangle$ for which $y_k \leqq f(x) < y_{k+1}$ must have a linear measure μ_k, whatever the choice of y_k and y_{k+1}. Functions with this property are called *measurable*. The class of measurable functions in $\langle a, b\rangle$ has the important property that all the fundamental operations on measurable functions lead again to measurable functions. In particular, the limit $f(x)$ of a sequence of measurable functions is itself measurable.

Now let $f(x)$ be measurable, and let $y_k < y_{k+1}$ for $-\infty < k < \infty$ be any "division" of the y-axis such that $y_{-k} \to -\infty$, $y_k \to \infty$ as $k \to \infty$. Then $f(x)$ is integrable and

$$(11) \qquad \int_a^b f(x)\,dx = \lim \sum_{-\infty}^{\infty} \eta_k \mu_k,$$

provided the "limit" on the right hand side exists and is finite.[2] Here the limit indicates that Max $(y_{k+1} - y_k)$ tends to nought, η_k being an arbitrary value with $y_k \leqq \eta_k < y_{k+1}$. This is the analogue of

[1] Compare E. C. Titchmarsh, The Theory of Functions, 2nd Ed. (Oxford, 1939), Chapter X.

[2] The limit always exists but need not be finite, if $f(x)$ is measurable and non-negative.

Riemann's formula (6.3). The Lebesgue sums use a "horizontal" division of the y-values, while the Riemann sums have a "vertical" division of the x-axis.

The following characterisation of the Riemann integrable functions will throw a final light on our subject: A function $f(x)$ is integrable in the Riemann sense if, and only if, it is measurable and bounded in $\langle a, b\rangle$, and if the set of its points of discontinuity has the linear Lebesgue measure zero.

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Finite projective geometry.

By A. G. WALKER.

1. The following description of the projective geometry of a finite number of points in 2-space is almost certainly known to those acquainted with projective geometry or with modern algebra. The object of this brief account is to show how certain finite systems can be presented in a form easily understood by students, and how they provide simple but instructive examples of fundamental ideas and "constructions." The fact that these examples belong to a geometry which is essentially non-Euclidean has great teaching value to those students who are apt to confuse projective geometry with the "method of projection" in Euclidean geometry. The underlying algebra is described briefly in § 4, but an understanding of this is not essential to the geometry. This algebraic work may, however, be of interest to those to whom Galois fields are fairly new.

2. The axioms generally adopted for projective geometry of 2-space are nine in number, being the three axioms of incidence (there is one and only one line passing through two points, and two lines have a point in common), the three axioms of extension (there is at least one line and at least three points on every line, and not all points lie on the same line), the axiom of perspective triangles (Desargues' theorem), the projective axiom (Pappus' theorem), and the diagonal axiom (the three diagonal points of a quadrangle are not