A SAMPLING STUDY ON THE POWER FUNCTION OF THE χ^2 'INDEX OF DISPERSION' TEST

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(With 6 Figures in the Text)

1. STATEMENT OF THE PROBLEM

In many medical and biological counting problems it is known that under ideal conditions repeated counts should follow a Poisson distribution. This is true, for instance, of counts of bacteria in a number of equal volumes from a well-mixed suspension. In order to indicate whether counts show more variability than can usually be expected from a Poisson distribution, it is customary to use Fisher's χ^2-test of heterogeneity (Fisher, Thornton & Mackenzie, 1922; Fisher, 1954) based on the index of dispersion z, as defined in equation (2) below. It is of some interest to know how easily this test will detect real heterogeneity of various kinds, and this paper is concerned with the 'power function' of the z-test, that is, with the probability that the test indicates significant variability when heterogeneity is really present.

If $x_1, x_2, \ldots, x_n$ represent n independent observations from a Poisson distribution

$$e^{-\lambda}\frac{\lambda^x}{x!} \quad (x = 0, 1, \ldots) \tag{1}$$

with parameter equal to λ, the sampling distribution of the 'index of dispersion'

$$z = \sum_{i=1}^{n} (x_i - \bar{x})^2/\bar{x} \tag{2}$$

under the null hypothesis, is known to be approximately that of χ^2 with $n-1$ degrees of freedom. Here $\bar{x} = \sum_{i=1}^{n} x_i/n$ represents the sample mean. The nature of this approximation has been discussed by Hoel (1943) and in certain sampling experiments by Sukhatme (1938) and Lancaster (1952). Bateman (1950) has discussed the test with particular reference to the alternatives of a Neyman-type contagious distribution. Darwin (1957) considered an approximation to the power function of the z-test with alternatives of the form: (i) Thomas distribution, (ii) negative binomial, (iii) Neyman Type A-contagious.

This paper will be concerned with the approximate numerical evaluation by sampling or 'Monte Carlo' methods of the power function

$$\beta^* = P\{z \geqslant \chi^2_\alpha \mid H\}. \tag{3}$$

This is the probability that z attains or exceeds a significant value, where χ^2_α represents the tabular value of the χ^2-distribution with $n-1$ degrees of freedom and

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level of significance equal to α. No χ^2 correction for continuity has been used, however. With regard to the alternatives H, it is further assumed that

$$x_i\,(i = 1, 2, \ldots, n)$$

are drawn from the n independent Poisson distributions

$$\frac{e^{-\lambda_i}\lambda_i^{x_i}}{x_i\,!} \quad (x_i = 0, 1, \ldots). \tag{4}$$

Sukhatme (1937) has shown that the index z may be obtained as a likelihood ratio test of the hypothesis $H_0:\lambda_1 = \lambda_2 = \ldots = \lambda_n = \lambda$, and considered that under the null hypothesis H_0 the χ^2-distribution represents a satisfactory approximation to that of z if the population parameter λ is greater than or equal to 2.

2. COMPUTATIONAL PROCEDURES

From each of thirty-three distinct Poisson distributions with parameter $= \lambda_i$ ($i = 1, 2, \ldots, 33$; $0{\cdot}10 \leqslant \lambda_i \leqslant 10{\cdot}0$) 100 random observations x_{ij} ($i = 1, 2, \ldots, 33$; $j = 1, 2, \ldots, 100$) were drawn. The resulting sample discrete distribution functions were then checked by the 'goodness-of-fit' test for agreement with the original Poisson distribution for the separate λ values.

These samples, each consisting of one hundred observations, were formed into groups of samples of n, n taking in succession the values 2, 3, 4, 5, 7 and 10. For a fixed n, those samples were grouped together for which the parameters λ_1, λ_2, …, λ_n were such that $n\bar{\lambda}$ = constant. In particular, $\bar{\lambda}$ was assigned the integral values: $\bar{\lambda} = 2{\cdot}0, 3{\cdot}0, 4{\cdot}0, 5{\cdot}0$ for each of the n's ($n = 2, 3, 4, 5, 7, 10$), and, in addition: $\bar{\lambda} = 6{\cdot}0, 7{\cdot}0, 8{\cdot}0$ ($n = 2$). Table 1 shows the number of combinations used for each n and value. The situation in which $n = 3$ was a practical case of interest in some present counting experiments, and in particular 140 different realizations of this were computed.

Table 1. *Number of realizations of power function* (*samples of* 100 *each*)

	$\bar{\lambda} = 2{\cdot}0$	$\bar{\lambda} = 3{\cdot}0$	$\bar{\lambda} = 4{\cdot}0$	$\bar{\lambda} = 5{\cdot}0$	$\bar{\lambda} = 6{\cdot}0$	$\bar{\lambda} = 7{\cdot}0$	$\bar{\lambda} = 8{\cdot}0$	Total
$n = 2$	5	5	6	6	5	3	2	32
$n = 3$	30	34	33	17	15	8	3	140
$n = 4$	21	22	17	11	—	—	—	71
$n = 5$	104	15	6	8	—	—	—	133
$n = 7$	9	10	10	10	—	—	—	39
$n = 10$	10	10	9	10	—	—	—	39
								454

3. RESULTS

The empirical power function $\beta^* = \beta^*(\delta)$, or the proportion of values of z satisfying the inequality: $z \geqslant \chi^2_\alpha$ in (3) with levels of significance $\alpha = 0{\cdot}05$, $0{\cdot}01$, respectively, for fixed n and $\bar{\lambda}$ values, was then programmed on the IBM-650 from the successive Poisson samples. Details of the programming are available from the Research Computer Laboratory of the University of Washington.

In particular, the empirical power function $\beta^*(\delta)$ has been tabulated as a function of the 'distance' parameter

$$\delta = \sum_{i=1}^{n} (\lambda_i - \bar{\lambda})^2/\bar{\lambda} \tag{5}$$

between the successive samples in terms of their parameters λ_i.

Figs. 1–6. Results of sampling experiments.

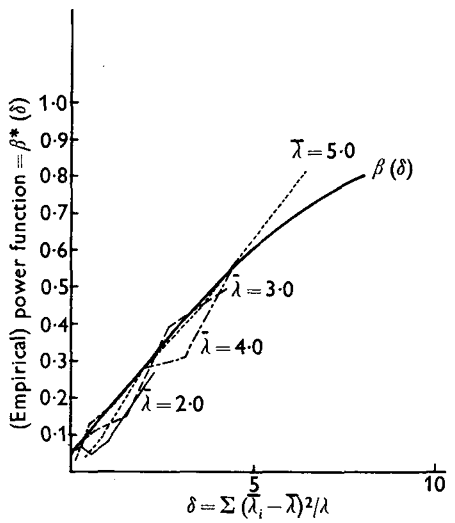

Fig. 1*a*. ($n = 2$)

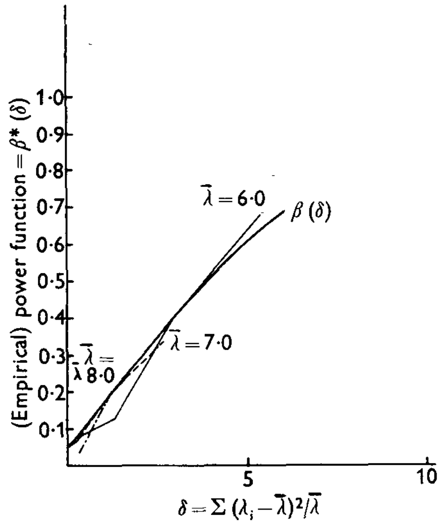

Fig. 1*b*. ($n = 2$)

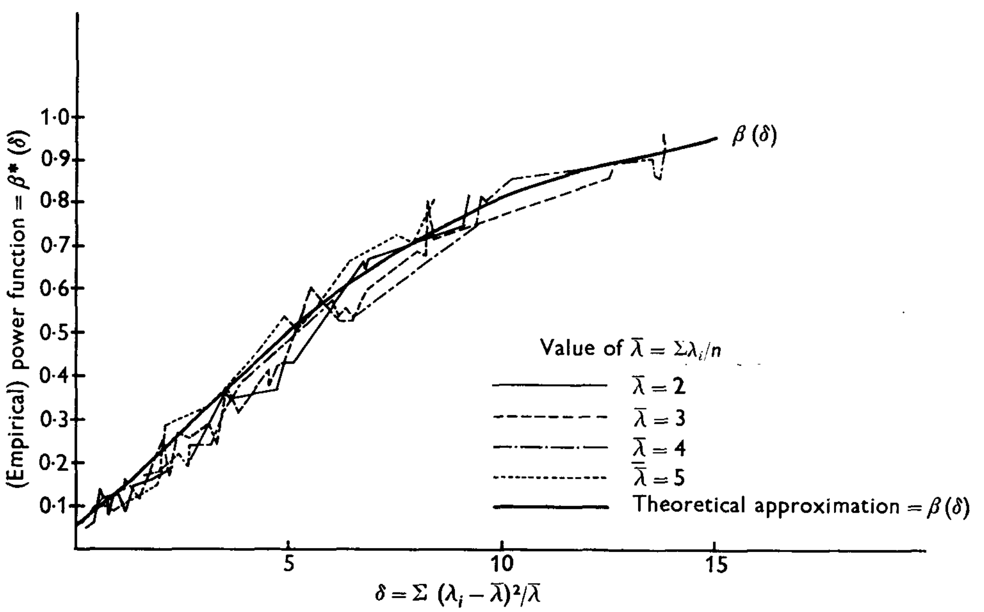

Fig. 2. ($n = 3$)

From Patnaik (1949), the value of the corresponding non-central χ^2 $(=\chi'^2_\nu)$ with $\nu = n-1$ degrees of freedom and parameter equal to $\frac{1}{2}\delta$, is

$$\beta(\delta) = \int_{\chi^2_\alpha}^{\infty} p(\chi'^2_\nu)\, d(\chi'^2_\nu)$$

$$= e^{-\frac{1}{2}\delta} \sum_{j=0}^{\infty} \frac{1}{j!} (\tfrac{1}{2}\delta)^j\, Q(\chi^2_\alpha | \nu + 2j), \tag{6}$$

where

$$Q(\chi^2_\alpha | r) = \frac{1}{\Gamma(\frac{1}{2}r)} \int_{\chi^2_\alpha}^{\infty} (\tfrac{1}{2}u)^{\frac{1}{2}r-1} e^{-\frac{1}{2}u}\, d(\tfrac{1}{2}u). \tag{7}$$

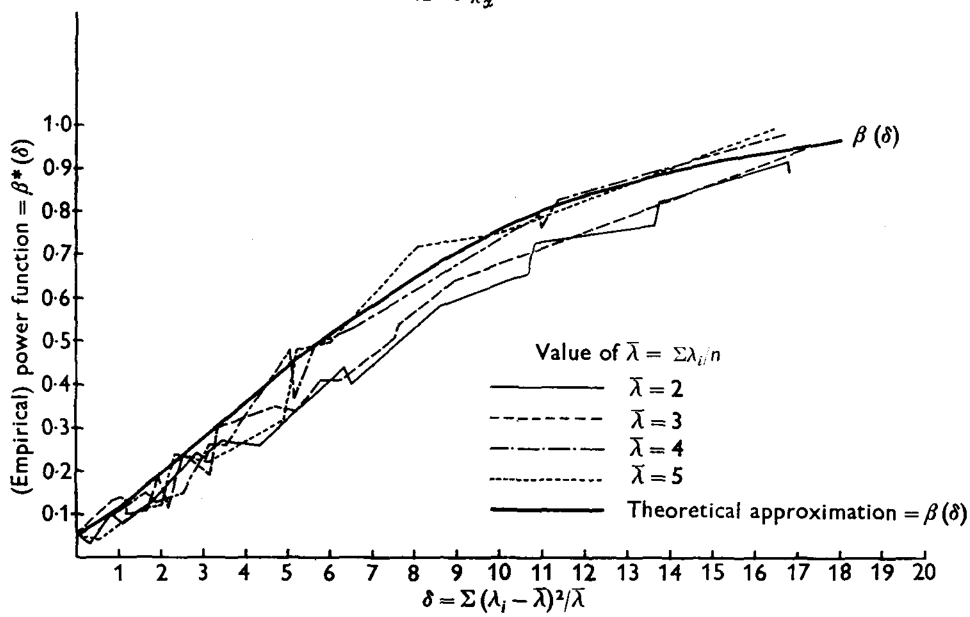

Fig. 3. $(n = 4)$

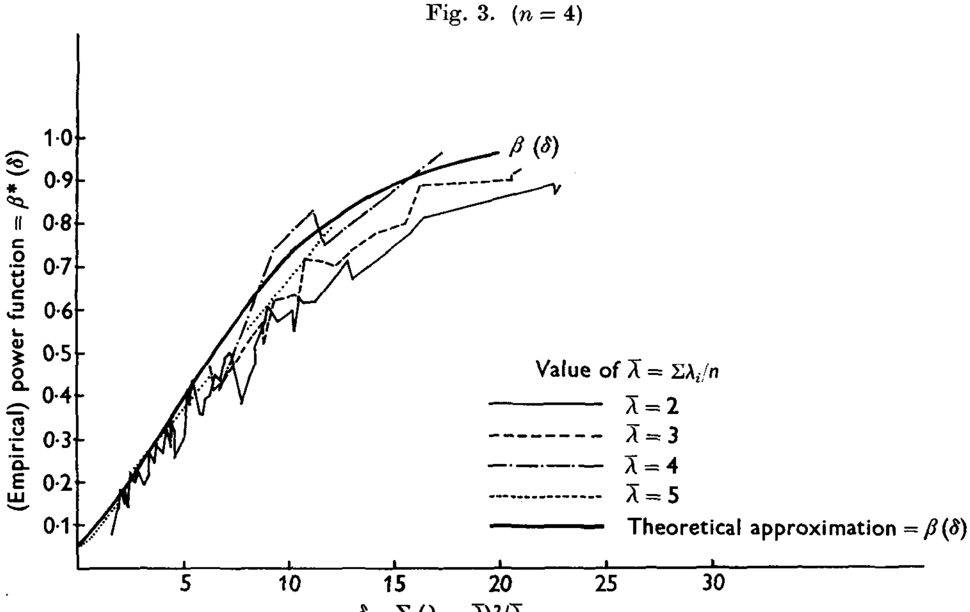

Fig. 4. $(n = 5)$

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In Figs. 1–6 $\beta(\delta)$ is represented by the solid heavy line for degrees of freedom, ν equal to 1, 2, 3, 4, 6, 9, respectively. It is known that the mean and variance of χ'^2_ν are: $\kappa_1 = \nu + \frac{1}{2}\delta$, $\kappa_2 = 2(\nu+\delta)$.

Some agreement of β and β^* may be noted in the figures generally for the values of n less than 5, within the limits of sampling variability of the non-central χ^2 for the $\alpha = 0{\cdot}05$ level of significance. However, the tendency of the non-central χ^2 to over-estimate the power function is noted generally for the higher n values. It appears

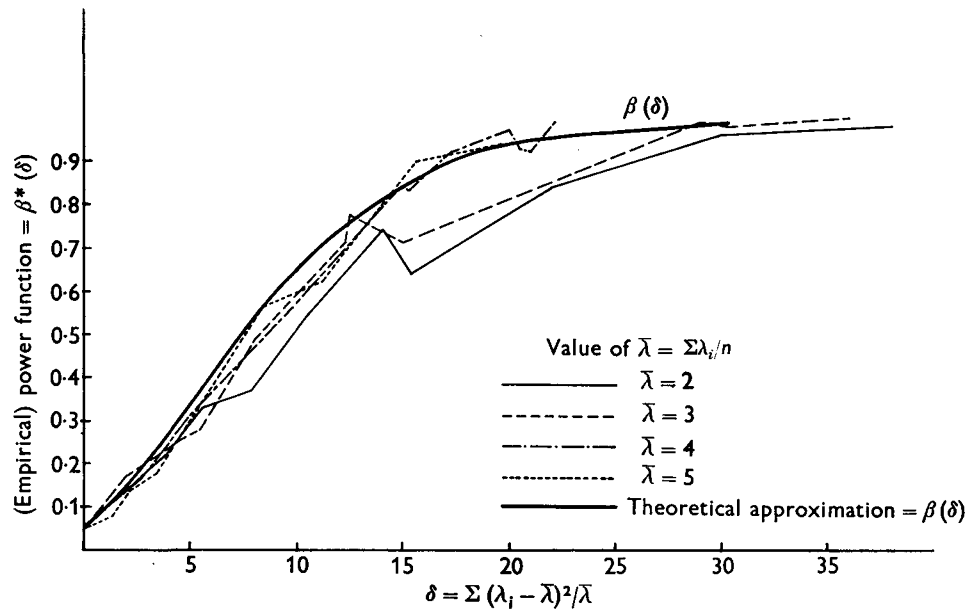

Fig. 5. ($n = 7$)

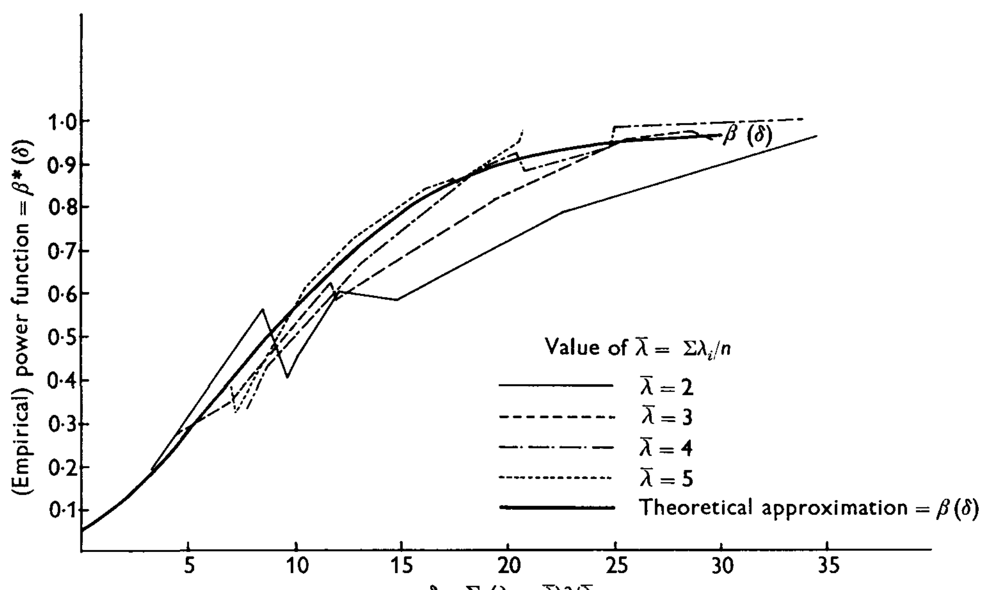

Fig. 6. ($n = 10$)

from the figures that the empirical power function depends on the values of $\bar{\lambda}$, and increases with the value of $\bar{\lambda}$. Similar results were found for the $\alpha = 0{\cdot}01$ level of significance, though these results are not reproduced due to limitations of space.

There are inevitably some intercorrelations of the results for successively increasing values of n in view of the fact that overlapping samples were used in the various combinations, as may be noted from the graphs with higher n values. The effect of this is difficult to ascertain, but could be measured in subsequent sampling experiments. In view of the elaborate initial checks of the separate Poisson samples and the machine checks of the individual z-values at all stages of the investigation, it is considered that the present realizations are essentially error free.

The 650 programme also provided for the computation of the mean and variance of the z-values for the combinations of n and $\bar{\lambda}$ listed in Table 1 for comparisons with the corresponding κ_1, κ_2 values for the non-central χ^2-distribution, and similar agreement was noted for the smaller values of n.

The preceding empirical results are thus intended to suggest the performance of the z-test for homogeneity of Poisson samples for small values of n particularly, in anticipation of the specific and rather involved exact distribution theory of z under the alternative hypotheses H.

SUMMARY

A total of 454 sampling experiments were performed on the groupings of 100 samples each from separate Poisson populations in order to establish estimates of frequency of rejection of the hypotheses of homogeneity by means of the z, or index of dispersion, test. Results of these artificial realizations are presented in Figs. 1–6, and compared with the corresponding non-central χ^2-distributions.

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