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COMPARISON OF MEASURES OF TOTALLY POSITIVE POLYNOMIALS

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Abstract

In this paper, explicit auxiliary functions are used to get upper and lower bounds for the Mahler measure of monic irreducible totally positive polynomials with integer coefficients. These bounds involve the length and the trace of the polynomial.

1. Introduction

Let $P = a_0x^d + a_1x^{d-1} + \cdots + a_d = a_0(x - \alpha_1)\cdots(x - \alpha_d)$ be a polynomial with complex coefficients.

We define:

- the *trace of P* as $\mathrm{trace}(P) = \sum_{i=1}^{d} \alpha_i$;
- the *length of P* as $\mathrm{L}(P) = \sum_{i=0}^{d} |a_i|$;
- the *Mahler measure of P* as $\mathrm{M}(P) = |a_0| \prod_{i=1}^{d} \max\{1, |\alpha_i|\}$.

We have the well-known inequality $2^{-d}\,\mathrm{L}(P) \le \mathrm{M}(P) \le \mathrm{L}(P)$. (For more details, see [Mi].)

Now we consider a polynomial P which is monic and totally positive (that is, its roots are all positive real numbers). In this case, $\mathrm{L}(P) = |P(-1)| = \prod_{i=1}^{d}(1 + \alpha_i)$. Then we have the basic inequality $\log \mathrm{L}(P) \le \mathrm{trace}(P)$.

In this paper, we prove the following results.

Theorem 1.1. *If P is a totally positive monic irreducible polynomial of degree d with integer coefficients, different from x, $x-1$, x^2-3x+1, $x^4-7x^3+13x^2-7x+1$, x^2-4x+1, $x^6-12x^5+44x^4-67x^3+44x^2-12x+1$ and $x^8-15x^7+83x^6-220x^5+303x^4-220x^3+83x^2-15x+1$, then*

$$\max\{2^{-d}\,\mathrm{L}(P), 1.058358^d\,\mathrm{L}(P)^{0.562454}\} \le \mathrm{M}(P) \le \min\{\mathrm{L}(P), 0.379128^d\,\mathrm{L}(P)^{1.803995}\}. \tag{1.1}$$

Theorem 1.2. *If P is a totally positive monic irreducible polynomial of degree d with integer coefficients, different from $x-1$, $x-2$, $x-3$, x^2-3x+1, x^3-5x^2+6x-1*

and $x^3 - 5x^2 + 5x + 1$*, then*

$$\log \mathrm{L}(P)^{1/d} \leq \min\left\{\frac{1}{d}\,\mathrm{trace}(P), 0.051012 + 0.472699\frac{1}{d}\,\mathrm{trace}(P)\right\}. \tag{1.2}$$

THEOREM 1.3. *If P is a totally positive monic irreducible polynomial of degree d with integer coefficients and with all roots in* $[0, 1000]$*, different from x and $x - 1$, then*

$$\log \mathrm{L}(P)^{1/d} \geq 0.801729 + 0.001990\frac{1}{d}\,\mathrm{trace}(P). \tag{1.3}$$

The proofs of these theorems use the principle of explicit auxiliary functions that was introduced by Smyth [Sm1]. The method is based on the fact that the resultant of two polynomials in $\mathbb{Z}[X]$ with no common roots is a nonzero integer.

For example, to obtain the lower bound in (1.1), we use the auxiliary function

$$f(x) = \log\max\{1, x\} - c_0\log(x+1) - \sum_{1\leq j\leq J} c_j \log|Q_j(x)| \geq m \quad \text{for } x > 0, \tag{1.4}$$

where the c_j are positive real numbers and the polynomials Q_j are nonzero elements of $\mathbb{Z}[X]$. Then

$$\sum_{i=1}^{d} f(\alpha_i) \geq md,$$

that is,

$$\log \mathrm{M}(P) \geq md + c_0 \log \mathrm{L}(P) + \sum_{1\leq j\leq J} c_j \log\left|\prod_{i=1}^{d} Q_j(\alpha_i)\right|.$$

We assume that P does not divide any Q_j. Then $\prod_{i=1}^{d} Q_j(\alpha_i)$ is a nonzero integer because it is the resultant of P and Q_j.

Therefore, if P does not divide any Q_j, then

$$\mathrm{M}(P) \geq e^{md}\,\mathrm{L}(P)^{c_0}.$$

To get the upper bound for $\mathrm{M}(P)$, we use the auxiliary function

$$f(x) = -\log\max\{1, x\} + c_0\log(x+1) - \sum_{1\leq j\leq J} c_j \log|Q_j(x)| \geq m \quad \text{for } x > 0.$$

The upper bound in (1.2) is obtained with the auxiliary function

$$f(x) = -\log(x+1) + c_0 x - \sum_{1\leq j\leq J} c_j \log|Q_j(x)| \geq m \quad \text{for } x > 0.$$

In general, it is not possible to get a lower bound for $\mathrm{L}(P)$ involving $\mathrm{trace}(P)$ with an auxiliary function. But, if we assume that the polynomial P has all its roots in an

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interval that is not too large, for instance [0, 1000], then we can get a result of the type (1.3) with the auxiliary function

$$f(x) = \log(x+1) - c_0 x - \sum_{1 \le j \le J} c_j \log |Q_j(x)| \ge m \quad \text{for } x \in [0, 1000].$$

Usually, the main problem is then to find a good list of polynomials Q_j which gives a value of m as large as possible. This is feasible by an inductive version of Wu's algorithm (for details, see [F2]) and it can happen that a great number of polynomials are useful (for example, 35 to get a lower bound for the trace; see [F2]). Here, we want to have only a few exceptional polynomials so that we stop the algorithm fairly quickly and thus we accept that our m is not the best possible, but is nonetheless sufficiently large to give good inequalities.

In Section 2 we explain how to construct the auxiliary function (1.4). The same method works for the other auxiliary functions. We also give a table of all polynomials involved in the different auxiliary functions and their coefficients. In Section 3, we give numerical examples for a particular family of polynomials. All the computations are done on a MacBook Pro Macintosh with the languages Pascal and Pari [Pari].

2. Construction of the explicit auxiliary function

2.1. Rewriting the auxiliary function. In the auxiliary function (1.4) we replace the numbers c_j by rational numbers.

So we may write

$$f(x) = \log \max\{1, x\} - c_0 \log(x+1) - \frac{t}{r} \log |Q(x)| \ge m \quad \text{for } x > 0, \tag{2.1}$$

where $Q \in \mathbb{Z}[X]$ is of degree r and t is a positive real number. We want to obtain a function f whose minimum m on $(0, \infty)$ is sufficiently large. Thus we seek a polynomial $Q \in \mathbb{Z}[X]$ such that

$$\sup_{x>0} |Q(x)|^{t/r} \frac{\max\{1, x\}}{(x+1)^{c_0}} \le e^{-m}.$$

If we suppose that t is fixed, we need to get an effective upper bound for the quantity

$$t_{\mathbb{Z},\varphi}([0, \infty)) = \liminf_{r \to +\infty} \inf_{\substack{P \in \mathbb{Z}[x] \\ \deg(P) = r}} \sup_{x>0} |P(x)|^{t/r} \varphi(x),$$

in which we use the weight $\varphi(x) = \max\{1, x\}/(x+1)^{c_0}$.

It is clear that this quantity is closely related to the usual integer transfinite diameter of an interval $I = [a, b]$, which is defined as

$$t_{\mathbb{Z}}(I) = \liminf_{n \to +\infty} \inf_{\substack{P \in \mathbb{Z}[x] \\ \deg(P) = n}} |P|_{\infty,I}^{1/n},$$

where $|P|_{\infty,I} = \sup_{t \in I} |P(t)|$ for all $P \in \mathbb{Z}[x]$.

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2.2. How to find the polynomials Q_j. Consider the auxiliary function

$$f(x) = \log \max\{1, x\} - c_0 \log(x+1) - \sum_{1 \le j \le J} c_j \log |Q_j(x)| \ge m \quad \text{for } x > 0.$$

The main idea is to find the polynomials Q_j by induction. We first optimise the auxiliary function $f_1 = \log \max\{1, x\} - c_0 \log(1+x) - c_1 \log x$. Then we take $t = c_0 \deg(x+1) + c_1 \deg(x)$. Suppose that we have $Q_1, \ldots, Q_J$ and an optimal function f for this set of polynomials in the form (2.1) with $t = \sum_{j=0}^{J} c_j \deg(Q_j)$. We seek a polynomial $R(x) = \sum_{l=0}^{k} a_l x^l \in \mathbb{Z}[x]$ of degree k ($k = 10$, for example) such that

$$\sup_{x \in I} |Q(x)R(x)|^{t/(r+k)} \frac{(1+x)^{c_0}}{\max\{1, x\}} \le e^{-m},$$

that is, such that

$$\sup_{x \in I} |Q(x)R(x)| \Big(\frac{(1+x)^{c_0}}{\max\{1, x\}} \Big)^{(r+k)/t}$$

is as small as possible. We apply LLL to the linear forms in the unknown coefficients a_l,

$$Q(x_i)R(x_i) \Big(\frac{(1+x_i)^{c_0}}{\max\{1, x_i\}} \Big)^{(r+k)/t}.$$

The numbers x_i are suitable points in $I = [0, 50]$ here, including the points where f has its least local minima. We get a polynomial R whose irreducible factors R_j are good candidates to enlarge the set of polynomials $(Q_1, \ldots, Q_J)$. We only keep the polynomials R_j which have a nonzero coefficient c_j in the new optimised auxiliary function f. After optimisation, some previous polynomials Q_j may have a zero coefficient and are removed.

2.3. Optimisation of the c_j. For the optimisation of the auxiliary function we use the semi-infinite linear programming method due to Smyth [Sm1]. We recall it briefly. We define by induction a sequence of finite sets X_n, $n \ge 0$, with $X_n \subset [0, \infty)$. We start with an arbitrary set of points X_0 of cardinality greater than J. At each step $n \ge 0$, we compute the best values for c_j by linear programming on the set X_n. We obtain a function f_n whose minimum $m_n = \min_{x \in X_n} f_n(x)$ is greater than $m'_n = \min_{x>0} f_n(x)$. We add to X_n the points of $[0, \infty)$ where f_n has a local minimum smaller than $m_n + \epsilon_n$, where $(\epsilon_n)_{n \ge 0}$ is a decreasing sequence of positive numbers tending to 0 when n is increasing and chosen such that the set X_n does not increase too quickly. We stop for instance when $m_n - m'_n < 10^{-6}$. If k steps are necessary, we take $m = m'_k$.

TABLE 1. Polynomials Q_j and their coefficients c_j, $1 \leq j \leq J$, the coefficient c_0 and the value of m for each inequality of Theorems 1.1, 1.2 and 1.3.

Theorem	Q_j	c_j	c_0	m
Theorem 1.1	Q_1 Q_2 Q_5	0.026208 0.271988 0.023069	0.562454	1.058358
left hand-side	Q_9 Q_{10} Q_{11}	0.008883 0.000895 0.003261		
Theorem 1.1	Q_1 Q_6	0.371232 0.030766	1.803995	0.379128
right hand-side				
Theorem 1.2	Q_2 Q_3 Q_4	0.134876 0.062529 0.004051	0.472699	−0.051012
	Q_5 Q_7 Q_8	0.046880 0.001968 0.018628		
Theorem 1.3	Q_1 Q_2	0.399387 0.196606	0.001990	0.801729

2.4. Numerical results. The polynomials used in the different auxiliary functions are:

$$\begin{aligned}
Q_1 &= x,\\
Q_2 &= x-1,\\
Q_3 &= x-2,\\
Q_4 &= x-3,\\
Q_5 &= x^2-3x+1,\\
Q_6 &= x^2-4x+1,\\
Q_7 &= x^3-5x^2+5x+1,\\
Q_8 &= x^3-5x^2+6x-1,\\
Q_9 &= x^4-7x^3+13x^2-7x+1,\\
Q_{10} &= x^6-12x^5+44x^4-67x^3+44x^2-12x+1,\\
Q_{11} &= x^8-15x^7+83x^6-220x^5+303x^4-220x^3+83x^2-15x+1.
\end{aligned}$$

Table 1 gives for each inequality of the different theorems the polynomials Q_j and their coefficients c_j, $1 \leq j \leq J$, the coefficient c_0 and the value of m.

3. Numerical example

The Gorshkov–Wirsing polynomials are defined as follows:

$$P_0(X) = X - 1 \quad \text{and} \quad P_n(X) = X^{\deg(P_{n-1})} P_{n-1}\Big(X + \frac{1}{X} - 2\Big) \quad \text{for } n \geq 1.$$

Smyth [Sm2] showed that the sequence $(\mathrm{M}(P_n)^{1/\deg(P_n)})_{n\geq 0}$ has a limit point $l = 1.727305\ldots$. The author [F3] proved that the sequence $(\mathrm{L}(P_n)^{1/\deg(P_n)})_{n\geq 0}$ has a limit point $l' = 2.376841\ldots$. It is easy to see that the sequence $((1/\deg(P_n))\,\mathrm{trace}(P_n))_{n\geq 0}$ has a limit point $l'' = 2$.

Previously, the author [F1] obtained, without explicit auxiliary functions, the following theorem.

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THEOREM 3.1. *Let P be a totally positive monic irreducible polynomial of degree d with integer coefficients and not divisible by x and x − 1. Then*

$$\left(\frac{1+\sqrt{5}}{2}\right)^d \left(\frac{\mathrm{L}(P)}{5^{d/2}}\right)^{(5-\sqrt{5})/2} \le \mathrm{M}(P) \le \left(\frac{1+\sqrt{5}}{2}\right)^d \left(\frac{\mathrm{L}(P)}{5^{d/2}}\right)^{\sqrt{5}}.$$

For the family of polynomials $(P_n)_{n\geq 0}$, this gives

$$1.687734 < \lim_{n\to\infty} \mathrm{M}(P_n)^{1/\deg(P_n)} = 1.727305\ldots < 1.854643,$$

whereas, by Theorem 1.1, we obtain the better inequalities

$$1.722326 < \lim_{n\to\infty} \mathrm{M}(P_n)^{1/\deg(P_n)} = 1.727305\ldots < 1.807488.$$

By Theorems 1.2 and 1.3,

$$0.851529 < \lim_{n\to\infty} \log \mathrm{L}(P_n)^{1/\deg(P_n)} = 0.8657723\ldots < 0.99641.$$

Thus, we see that for this particular family of polynomials the inequalities are quite good.

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