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SOME GLOBAL EXISTENCE RESULTS ON LOCALLY FINITE GRAPHS

SHOUDONG MAN and GUOQING ZHANG

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Abstract

Let $G = (V, E)$ be a locally finite graph with the vertex set V and the edge set E, where both V and E are infinite sets. By dividing the graph G into a sequence of finite subgraphs, the existence of a sequence of local solutions to several equations involving the p-Laplacian and the poly-Laplacian systems is confirmed on each subgraph, and the global existence for each equation on graph G is derived by the convergence of these local solutions. Such results extend the recent work of Grigor'yan, Lin and Yang [*J. Differential Equations*, **261** (2016), 4924–4943; *Rev. Mat. Complut.*, **35** (2022), 791–813]. The method in this paper also provides an idea for investigating similar problems on infinite graphs.

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1. Introduction

Partial differential equations play an important role in dynamics, mathematical physics, engineering, geometry and the other sciences. For examples, readers are referred to [1, 2, 8, 12] and the references therein. As they model discrete systems, it is important to study such equations on graphs.

In recent years, Grigor'yan, Lin and Yang systematically raised and studied several partial differential equations involving Yamabe equations, Kazdan–Warner equations and Schrödinger equations on graphs [4–6]. They first established the Sobolev spaces and the functional framework on graphs. As a consequence, variational methods are applied to solve partial differential equations on graphs, that is, to find critical points of various functionals. In particular, in [4], they derived the Sobolev embedding theorems

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on graphs, that is, if $G = (V, E)$ is a locally finite graph and $\Omega \subset V$ is a bounded domain, then

$$W_0^{1,s}(\Omega) \hookrightarrow L^\gamma(\Omega) \quad \text{for } s > 1 \text{ and } 1 \le \gamma \le +\infty,$$

while if $G = (V, E)$ is a finite graph, then

$$W^{1,s}(V) \hookrightarrow L^\gamma(V) \quad \text{for } s > 1 \text{ and } 1 \le \gamma \le +\infty.$$

They observed that the Sobolev embedding theorems on graphs are quite different from those on Euclidean space, and thus they could assume different growth conditions on the nonlinear terms f and g on graphs. They also observed that the Sobolev space they established is pre-compact. By these crucial observations, they studied the following p-Laplacian equation:

$$\begin{cases} -\Delta_p u = f(x, u) & \text{in } \Omega^\circ, \\ u \ge 0 \ \text{ in } \ \Omega^\circ, u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1-1}$$

on a locally finite graph and obtained a local solution. They also studied the following equation on a locally finite graph:

$$\begin{cases} \mathcal{L}_{m.p} u = f(x, u) & \text{in } \Omega^\circ, \\ |\nabla^j u| = 0, 0 \le j \le m-1 & \text{on } \partial\Omega, \end{cases} \tag{1-2}$$

and obtained a nontrivial local solution. In addition, they studied several equations on a finite graph and obtained global solutions. For more details, we refer readers to [4]. In [11], Lin and Yang studied several Laplacian equations involving the Schrödinger equation, the mean field equation and the Yamabe equation on a locally finite graph $G = (V, E)$, and obtained global solutions in Sobolev space $W^{1,2}(V)$ or its subspace $\mathcal{H}$ by using calculus of variations, where

$$\mathcal{H} = \Big\{u \in W^{1,2}(V) : \int_V (|\nabla u|^2 + hu^2)\, d\mu < \infty\Big\}.$$

For more results about differential equations on graphs, we refer readers to [3, 7, 9, 10], for example.

In this paper, we give a different division of graph $G = (V, E)$ from that in [11]. We divide the locally finite graph $G = (V, E)$ into a sequence of finite subgraphs

$$G_k = (V_k, E_k) \quad \text{where } k = 1, 2, 3, \ldots, \text{ and } V = \bigcup_{k=1}^{\infty} (V_k \cup \partial V_k)$$

(please see (2-3) for details), while in [11], the graph G is divided into a sequence of balls centred at a fixed point O in V. We investigate several equations on each subgraph, and obtain a sequence of local solutions to each equation. At last, we derive the global existence of nontrivial solutions to each equation on graph G through the convergence of these local solutions. We extend the local existence results of [4, Problems 1 and 2] to global existence results on locally finite graphs, and also extend the results for $W^{1,2}(V)$ in [11] to space $W^{m,p}(V)$, where $p \ge 2$, $p \in \mathbb{R}$, $m \ge 1$ and m is an integer.

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This paper is organized as follows. In Section 2, we give some preliminary results on graphs and state our main results. In Section 3, we prove our main results.

2. Preliminaries and main results

Let $G = (V, E)$ denote a graph where V is the vertex set and E is the edge set. Let $x \sim y$ represent that vertex x is adjacent to vertex y, and (x, y) denote an edge in E connecting vertices x and y. Assume that $\chi_{xy} = \chi_{yx} > 0$, where χ_{xy} is the edge weight. A graph G is called connected if for any vertices $x, y \in V$, there exists a sequence $\{x_i\}_{i=0}^n$ that satisfies

$$x = x_0 \sim x_1 \sim x_2 \sim \cdots \sim x_n = y.$$

The degree of vertex x, denoted by $\vartheta(x)$, is the number of edges connected to x. If for every vertex x of V the number of edges connected to x is finite, we say that G is a locally finite graph. The finite measure $\vartheta(x) = \sum_{y\sim x} \chi_{xy}$.

In this paper, let $G = (V, E)$ denote a connected graph. Thus, graph $G = (V, E)$ has no isolated vertices.

From [4], for any function $u : V \to \mathbb{R}$, the $\vartheta(x)$-Laplacian of u is defined as

$$\Delta u(x) = \frac{1}{\vartheta(x)} \sum_{y\sim x} \chi_{xy}(u(y) - u(x)).$$

The associated gradient form is written as

$$\begin{aligned}\Gamma(u, v)(x) &= \frac{1}{2}\{\Delta(u(x)v(x)) - u(x)\Delta v(x) - v(x)\Delta u(x)\}\\ &= \frac{1}{2\vartheta(x)} \sum_{y\sim x} \chi_{xy}(u(y) - u(x))(v(y) - v(x)).\end{aligned}$$

The length of the gradient for u is denoted by

$$|\nabla u|(x) = \sqrt{\Gamma(u, u)(x)} = \left(\frac{1}{2\vartheta(x)} \sum_{y\sim x} \chi_{xy}(u(y) - u(x))^2\right)^{1/2}.$$

The length of the m-order gradient of u is written as

$$|\nabla^m u|(x) = \begin{cases}|\nabla \Delta^{(m-1)/2} u| & \text{when } m \text{ is odd,}\\ |\Delta^{m/2} u| & \text{when } m \text{ is even,}\end{cases}$$

where $|\Delta^{m/2} u|$ is the usual absolute value of the function $\Delta^{m/2} u$. To compare with the Euclidean setting, the integral of a function $u : V \to \mathbb{R}$ is defined as

$$\int_V u \, d\vartheta = \sum_{x\in V} \vartheta(x) u(x).$$

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Let $\mathcal{L}_{m,p}u$ be defined in the distributional sense: for any function ϕ, there holds

$$\int_V (\mathcal{L}_{m,p}u)\phi\, d\vartheta = \begin{cases} \int_V |\nabla^m u|^{p-2}\Gamma(\Delta^{(m-1)/2}u, \Delta^{(m-1)/2}\phi)\, d\vartheta & \text{when } m \text{ is odd,} \\ \int_V |\nabla^m u|^{p-2}\Delta^{m/2}u\Delta^{m/2}\phi\, d\vartheta & \text{when } m \text{ is even.} \end{cases}$$

In particular, the poly-Laplacian $(-\Delta)^m u$ can be defined as

$$(-\Delta)^m u = \mathcal{L}_{m,2}u.$$

The p-Laplacian of $u : V \to \mathbb{R}$, namely $\Delta_p u$, is defined in the distributional sense as

$$\Delta_p u(x) = \frac{1}{2\vartheta(x)} \sum_{y\sim x} \chi_{xy}(|\nabla u|^{p-2}(y) + |\nabla u|^{p-2}(x))(u(y) - u(x)). \tag{2-1}$$

Let $G = (V, E)$ denote a locally finite graph and $\Omega \subset V$. For any integer $m \geq 1$ and any $p > 1$, $W^{m,p}(\Omega)$ is defined as a space of all functions $u : \Omega \to \mathbb{R}$ with the norm

$$\|u\|_{W^{m,p}(\Omega)} = \Big(\sum_{k=0}^{m} \int_\Omega |\nabla^k u|^p\, d\vartheta \Big)^{1/p} < +\infty.$$

Denote $C_0^m(\Omega)$ as a set of all functions $u : \Omega \to \mathbb{R}$ with $u = |\nabla u| = \cdots = |\nabla^{m-1}u| = 0$ on $\partial\Omega$. Here, $W_0^{m,p}(\Omega)$ is denoted as the completion of $C_0^m(\Omega)$ with the norm

$$\|u\|_{W_0^{m,p}(\Omega)} = \Big(\int_\Omega |\nabla^m u|^p\, d\vartheta \Big)^{1/p}.$$

Moreover, for any $s > 0$, $L^s(\Omega)$ denotes a linear space with the norm

$$\|u\|_{L^s(\Omega)} = \Big(\int_\Omega |u|^s\, d\vartheta \Big)^{1/s}.$$

Additionally, $L^\infty(\Omega)$ means

$$\|u\|_{L^\infty(\Omega)} = \sup_{x\in\Omega} |u(x)| < \infty.$$

Obviously, $W_0^{m,p}(\Omega)$ and $L^s(\Omega)$ are two Banach spaces, and we have the following famous Sobolev embedding theorem derived by Grigor'yan, Lin and Yang.

THEOREM A [4, Theorem 7]. *Let $G = (V, E)$ be a locally finite graph and Ω be a bounded domain of V with $\Omega^0 \neq \emptyset$. Let m be any positive integer and $p > 1$. Then, $W_0^{m,p}(\Omega)$ is embedded in $L^q(\Omega)$ for all $1 \leq q \leq +\infty$, that is, there exists a constant C_0 depending only on p, m and Ω such that for all $u \in W_0^{m,p}(\Omega)$,*

$$\Big(\int_\Omega |u|^q\, d\vartheta \Big)^{1/q} \leq C_0 \Big(\int_\Omega |\nabla^m u|^p\, d\vartheta \Big)^{1/p}.$$

In particular, denoting $\vartheta_0 = \min_{x\in\Omega} \vartheta(x)$, there holds

$$\|u\|_{L^\infty(\Omega)} \leq \frac{C_0}{\vartheta_0} \|u\|_{W_0^{m,p}(\Omega)}. \tag{2-2}$$

Moreover, $W_0^{m,p}(\Omega)$ is pre-compact, namely, if $\{u_k\}$ is bounded in $W_0^{m,p}(\Omega)$, then up to a subsequence, still denoted by $\{u_k\}$, there exists some $u \in W_0^{m,p}(\Omega)$ such that $u_k \to u$ in $W_0^{m,p}(\Omega)$.

Now, we state our main results.

First, we define a sequence of subgraphs of $G = (V, E)$ denoted by

$$G_k = (V_k, E_k), \tag{2-3}$$

where $k = 1, 2, 3, \ldots$, V_k is a finite vertex set, E_k is a finite edge set, $V_k \subset V$ and $E_k \subset E$, such that

$$V_i \cap V_j = \emptyset \quad \text{when } i \neq j \text{ and } V = \bigcup_{i=1}^{\infty} (V_i \cup \partial V_i).$$

For any finite set $\mathfrak{R} \subset V$, $W^{m,p}(\mathfrak{R})$ is a Sobolev space including all functions $u : \mathfrak{R} \to \mathbb{R}$ with the norm

$$\|u\|_{W^{m,p}(\mathfrak{R})} = \bigg(\int_{\mathfrak{R}} (|\nabla^m u|^p + |u|^p)\, d\vartheta \bigg)^{1/p}.$$

For any $q > 0$, $p > 1$ and integer $m \geq 1$, we have the below Sobolev embedding theorem involving the two spaces $W^{m,p}(\mathfrak{R})$ and $L^q(\mathfrak{R})$.

THEOREM 1. *Let $G = (V, E)$ be a connected and locally finite graph. Suppose that the measure $\vartheta(x) \geq \vartheta_0 > 0$ for all $x \in V$, where ϑ_0 is a constant. Let $m \geq 1$ be any integer, $p > 1$ and $q > 0$. Then, for any finite set $\mathfrak{R} \subset V$,*

$$\|u\|_{L^q(\mathfrak{R})} \leq C_* \|u\|_{W^{m,p}(\mathfrak{R})},$$

where

$$C_* = \vartheta_0^{-1/p} (\operatorname{vol}(\mathfrak{R}))^{1/q} \quad \text{and} \quad \operatorname{vol}(\mathfrak{R}) = \sum_{x \in \mathfrak{R}} \vartheta(x).$$

In particular,

$$\|u(x)\|_{L^\infty(\mathfrak{R})} \leq \vartheta_0^{-1/p} \|u(x)\|_{W^{m,p}(\mathfrak{R})}. \tag{2-4}$$

Moreover, by [4, Theorem 8], *$W^{m,p}(\mathfrak{R})$ is pre-compact, namely, if $\{u_k\}$ is bounded in $W^{m,p}(\mathfrak{R})$, then up to a subsequence, still denoted by $\{u_k\}$, there exists some $u \in W^{m,p}(\mathfrak{R})$ such that $u_k \to u$ in $W^{m,p}(\mathfrak{R})$.*

REMARK 2. Let $\mathbb{D}_k = W_0^{m,2}(V_k) \cap W_0^{n,2}(V_k)$ be the space with the norm

$$\|u\|_{\mathbb{D}_k} = \|u\|_{W_0^{m,2}(V_k)} + \|u\|_{W_0^{n,2}(V_k)}.$$

Take a sequence of functions $(\hat{u}_j) \in \mathbb{D}_k$. If there exist

$$u_k \in W_0^{m,2}(V_k) \quad \text{and} \quad u_k \in W_0^{n,2}(V_k)$$

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such that

$$\|\hat{u}_j - u_k\|_{W_0^{m,2}(V_k)} \to 0 \quad \text{and} \quad \|\hat{u}_j - u_k\|_{W_0^{n,2}(V_k)} \to 0 \quad \text{as } j \to \infty,$$

we can easily get that

$$\|\hat{u}_j - u_k\|_{\mathbb{D}_k} = \|\hat{u}_j - u_k\|_{W_0^{m,2}(V_k)} + \|\hat{u}_j - u_k\|_{W_0^{n,2}(V_k)} \to 0 \quad \text{as } j \to \infty,$$

and thus $\hat{u}_j \to u_k$ in $\mathbb{D}_k$ as $j \to \infty$. In addition, if we use $\mathfrak{R}$ instead of V_k and set $\mathbb{D}_k = W^{m,2}(\mathfrak{R}) \cap W^{n,2}(\mathfrak{R})$, we can get a similar result.

Now, we obtain the following global existence result for a class of semi-linear elliptic equations involving the p-Laplacian, which can be used in wavelets and dimension reductions for high-dimensional data because p is a tunable parameter.

THEOREM 3. *Let $G = (V, E)$ be a connected and locally finite graph. Suppose that $\vartheta(x) \geq \vartheta_0 > 0$ for all $x \in V$, where ϑ_0 is a positive constant. Assume $K(x) \in L^{s/(s-1)}(V)$, where $s > 1$. The function $f(x, u) \in C(V \times \mathbb{R}, \mathbb{R})$ satisfies*

$$uf(x, u) \geq 0 \quad \textit{for all } (x, u) \in V \times \mathbb{R} \tag{2-5}$$

and

$$|f(x, u)| \leq a(x) + b|u|^{s-1}, \ K(x) - f(x, u) \not\equiv 0 \quad \textit{for all } (x, u) \in V \times \mathbb{R},$$

where $a(x) \in L^{s/(s-1)}(V)$, $b > 0$ and $s > 1$. Then, for any $p > 2$, there exists a nontrivial solution to the following problem:

$$-\Delta_p u = K(x) - f(x, u) \quad \textit{in } V. \tag{2-6}$$

Moreover, if

$$f(x, u) \leq K(x) \quad \text{for all } (x, u) \in V \times \mathbb{R}, \tag{2-7}$$

there exists a positive solution to (2-6).

Now, we extend the global existence result for p-Laplacian to Laplacian systems. The next result is about a poly-Laplacian system on a locally finite graph G.

THEOREM 4. *Let $G = (V, E)$ be a connected and locally finite graph. Suppose that $\vartheta(x) \geq \vartheta_0 > 0$ for all $x \in V$, where ϑ_0 is a positive constant. For $i = 1, 2$, assume that*

$$\lambda_i > 0, \varphi_i(x) \in L^1(V) \quad \textit{and} \quad \varphi_i(x) > 0 \quad \textit{for all } x \in V.$$

Then for any integers $m, n > 1$, there exists a solution to the following problem:

$$(-\Delta)^m u + (-\Delta)^n u = -\lambda_1 \frac{\varphi_1(x)e^{-u}}{\int_V \varphi_1(x)e^{-u}\, d\vartheta} + \lambda_2 \frac{\varphi_2(x)e^{u}}{\int_V \varphi_2(x)e^{u}\, d\vartheta} \tag{2-8}$$

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in V. Moreover, if

$$\lambda_1 \frac{\varphi_1(x)}{\int_V \varphi_1(x)\,d\vartheta} - \lambda_2 \frac{\varphi_2(x)}{\int_V \varphi_2(x)\,d\vartheta} \not\equiv 0 \quad \text{for all } x \in V,$$

there exists a nontrivial solution to problem (2-8).

Now, we consider another division of graph $G = (V, E)$. By [11], for any fixed $O \in V$, the distance between x and O, denoted by $\rho(x) = \rho(x, O)$, is the minimum number of edges connecting them. Thus, for any integer $k \geq 1$, we define a ball centred at O with radius k by

$$B_k = B_k(O) = \{x \in V \mid \rho(x) < k\}. \tag{2-9}$$

The boundary of B_k can be defined as

$$\partial B_k = \{x \in V \mid \rho(x) = k\}.$$

REMARK 5. Set $\mathbb{X}_k = W_0^{m,p}(B_k) \times W_0^{n,q}(B_k)$ and define the norm on $\mathbb{X}_k$ as

$$\|(u, v)\|_{\mathbb{X}_k} = \max\{\|u\|_{W_0^{m,p}(B_k)}, \|v\|_{W_0^{n,q}(B_k)}\}.$$

Take a sequence of functions $(\hat{u}_j, \hat{v}_j) \in \mathbb{X}_k$. If there exist

$$u_k \in W_0^{m,p}(B_k) \quad \text{and} \quad v_k \in W_0^{n,q}(B_k)$$

such that

$$\|\hat{u}_j - u_k\|_{W_0^{m,p}(B_k)} \to 0 \quad \text{and} \quad \|\hat{v}_j - v_k\|_{W_0^{n,q}(B_k)} \to 0 \quad \text{as } j \to \infty,$$

we can easily get that

$$\|(\hat{u}_j, \hat{v}_j) - (u_k, v_k)\|_{\mathbb{X}_k} = \max\{\|\hat{u}_j - u_k\|_{W_0^{m,p}(B_k)}, \|\hat{v}_j - v_k\|_{W_0^{n,q}(B_k)}\} \to 0 \quad \text{as } j \to \infty.$$

Thus,

$$(\hat{u}_j, \hat{v}_j) \to (u_k, v_k) \text{ in } \mathbb{X}_k \quad \text{as } j \to \infty.$$

In addition, if we use $\mathfrak{R}$ instead of B_k and set $\mathbb{X}_k = W^{m,p}(\mathfrak{R}) \times W^{n,q}(\mathfrak{R})$, we can get a similar result.

Next, we study the system (2-10), where each equation can be viewed as one type of Kazdan–Warner equation when $u = v$. The Kazdan–Warner equation has very important applications in geometry. By the division of graph G, we can obtain a global solution to the system on graph G.

THEOREM 6. *Let $G = (V, E)$ be a connected and locally finite graph. Suppose that $\vartheta(x) \geq \vartheta_0 > 0$ for all $x \in V$, where ϑ_0 is a positive constant. Assume that*

$$K(x) > 0, \quad \kappa(x) > 0, \quad K(x) \not\equiv \kappa(x), \quad K(x) \in L^{s/(s-1)}$$

and

$$\kappa(x) \in L^{s/(s-1)} \quad \text{for all } x \in V \text{ and } s > 1.$$

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Then, for any integer $m, n \geq 2$ and $p, q > 2$, there exists a nontrivial solution to the following problem:

$$\begin{cases} \mathcal{L}_{m,p} u = K(x)e^{-u-v} - \kappa(x) & \text{in } V, \\ \mathcal{L}_{n,q} v = \kappa(x)e^{-u-v} - K(x) & \text{in } V. \end{cases} \tag{2-10}$$

3. Proof of main results

First, we prove the Sobolev embedding theorem in Theorem 1. For the proof in the case of $W_0^{1,2}(V)$ and $L^p(V)$, where $p > 0$.

PROOF OF THEOREM 1. Since for all $x \in \mathfrak{R}$,

$$\begin{aligned} \|u(x)\|^p_{W^{m,p}(\mathfrak{R})} &= \int_{\mathfrak{R}} (|\nabla^m u(x)|^p + |u(x)|^p)\, d\vartheta \\ &\geq \sum_{x \in \mathfrak{R}} \vartheta(x)|u(x)|^p \\ &\geq \sum_{x \in \mathfrak{R}} \vartheta_0 |u(x)|^p, \end{aligned}$$

we get

$$|u(x)| \leq \vartheta_0^{-1/p} \|u(x)\|_{W^{m,p}(\mathfrak{R})}. \tag{3-1}$$

It is easy to see that (3-1) implies

$$\|u(x)\|_{L^\infty(\mathfrak{R})} \leq \vartheta_0^{-1/p} \|u(x)\|_{W^{m,p}(\mathfrak{R})}. \tag{3-2}$$

For all $1 \leq q < +\infty$, by (3-2),

$$\begin{aligned} \|u(x)\|_{L^q(\mathfrak{R})} &= \Big(\sum_{x \in \mathfrak{R}} \vartheta(x)|u(x)|^q \Big)^{1/q} \\ &\leq \vartheta_0^{-1/p} (\mathrm{vol}(\mathfrak{R}))^{1/q} \|u(x)\|_{W^{m,p}(\mathfrak{R})}, \end{aligned}$$

where $\mathrm{vol}(\mathfrak{R}) = \sum_{x \in \mathfrak{R}} \vartheta(x)$.

Thus, we complete this proof. □

Now, we prove the existence results. The method can be viewed as a variational method from local existence to global existence.

PROOF OF THEOREM 3. Let $G_k = (V_k, E_k)$ be a subgraph defined as in (2-3). Define the functional $J_{k(K)} : W_0^{1,p}(V_k) \to \mathbb{R}$ by

$$J_{k(K)}(u) = \frac{1}{p} \int_{V_k} |\nabla u|^p\, d\vartheta - \int_{V_k} K(x) u\, d\vartheta + \int_{V_k} F(x, u)\, d\vartheta, \tag{3-3}$$

where

$$F(x, u) = \int_0^u f(x, t)\, dt \quad \text{and} \quad F(x, 0) = 0.$$

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From (2-5),

$$F(x,u) \geq 0 \quad \text{for all } (x,u) \in V \times \mathbb{R}. \tag{3-4}$$

It is easy to see that

$$J_{k(K)} \in C^1(W_0^{1,p}(V_k), \mathbb{R}).$$

Set

$$\Theta_k = \inf_{u \in W_0^{1,p}(V_k)} J_{k(K)}(u).$$

Obviously,

$$\Theta_k \leq J_{k(K)}(0) = 0. \tag{3-5}$$

To proceed, the proof is divided into three steps.

Step 1. For any integer $k \geq 1$, $J_{k(K)}(u)$ is bounded from below in $W_0^{1,p}(V_k)$.

By the Hölder inequality, Theorem A and the Young inequality, for any $s > 1$,

$$\begin{aligned}
\int_{V_k} Ku\,d\vartheta &\leq \|u\|_{L^s(V_k)} \|K\|_{L^{s/(s-1)}(V)} \\
&\leq C_0 \|u\|_{W_0^{1,p}(V_k)} \|K\|_{L^{s/(s-1)}(V)} \\
&= \frac{1}{2}\|u\|_{W_0^{1,p}(V_k)} \cdot 2C_0 \|K\|_{L^{s/(s-1)}(V)} \\
&\leq \frac{1}{2p}\|u\|^p_{W_0^{1,p}(V_k)} + \frac{p-1}{2p}(2C_0)^{p/(p-1)} \|K\|^{p/(p-1)}_{L^{s/(s-1)}(V)}.
\end{aligned} \tag{3-6}$$

By (3-3), (3-4) and (3-6),

$$J_{k(K)}(u) \geq \frac{1}{2p}\|u\|^p_{W_0^{1,p}(V_k)} - \frac{p-1}{2p}(2C_0)^{p/(p-1)} \|K\|^{p/(p-1)}_{L^{s/(s-1)}(V)} \tag{3-7}$$

$$\geq -\frac{p-1}{2p}(2C_0)^{p/(p-1)} \|K\|^{p/(p-1)}_{L^{s/(s-1)}(V)}. \tag{3-8}$$

Therefore, we have that $J_{k(K)}(u)$ is bounded from below in $W_0^{1,p}(V_k)$.

Step 2. For any integer $k \geq 1$, there exists a function $u_k \in W_0^{1,p}(V_k)$ such that

$$J_{k(K)}(u_k) = \Theta_k = \inf_{u \in W_0^{1,p}(V_k)} J_{k(K)}(u).$$

Moreover, u_k satisfies the Euler–Lagrange equation

$$\begin{cases} -\Delta_p u_k = K(x) - f(x,u_k) & \text{in } V_k, \\ u_k = 0 & \text{on } \partial V_k. \end{cases} \tag{3-9}$$

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Obviously, by (3-5) and (3-8),

$$-\frac{p-1}{2p}(2C_0)^{p/(p-1)}\|K\|_{L^{s/(s-1)}(V)}^{p/(p-1)} \le \Theta_k \le 0. \tag{3-10}$$

By (3-10), we get that Θ_k is a bounded sequence of numbers.

Now fix a positive integer k. We take a sequence of functions $(\hat{u}_j) \in W_0^{1,p}(V_k)$ such that

$$J_{k(K)}(\hat{u}_j) \to \Theta_k \quad \text{as } j \to \infty.$$

Thus, by (3-7),

$$\|\hat{u}_j\|_{W_0^{1,p}(V_k)} \le M,$$

where M is a positive constant. Therefore, we get that $\{\hat{u}_j\}$ is bounded in $W_0^{1,p}(V_k)$. By Theorem A, we have that up to a subsequence, $\hat{u}_j$ converges to some function $u_k \in W_0^{1,p}(V_k)$. Clearly,

$$J_{k(K)}(u_k) = \Theta_k = \inf_{u \in W_0^{1,p}(V_k)} J_{k(K)}(u).$$

Thus, u_k satisfies the Euler–Lagrange equation (3-9).

Step 3. There exist $\tilde{u} : V \to \mathbb{R}$ and a subsequence of $\{u_k\}$, still denoted by $\{u_k\}$, such that $\{u_k\}$ converges to $\tilde{u}$ uniformly in V, that is, $\tilde{u}$ is a solution to (2-6). Moreover, $\tilde{u} > 0$ for all $x \in V$.

By (3-8) and (3-10),

$$\|u_k\|_{W_0^{1,p}(V_k)}^p \le M_1, \tag{3-11}$$

where M_1 is a positive constant independent of k. By Theorem A and (3-11),

$$\|u_k\|_{L^\infty(V_k)} \le \frac{C_0}{\vartheta_0}\|u_k\|_{W_0^{1,p}(V_k)} \le M_2, \tag{3-12}$$

where M_2 is a positive constant independent of k.

It is easy to see that $\{u_k\}$ can be viewed as a sequence defined on V with $u_k = 0$ on $V \setminus V_k$, that is,

$$u_k(x) = \begin{cases} u_k(x) & \text{for } x \in V_k, \\ 0 & \text{for } x \in V \setminus V_k. \end{cases} \tag{3-13}$$

Therefore, by (3-12) and (3-13), we get that $\{u_k\}$ is uniformly bounded in V. Thus, there exists a subsequence of $\{u_k\}$, still denoted by $\{u_k\}$, such that $\{u_k\}$ converges to $\tilde{u}$ uniformly in V, that is,

$$\lim_{k\to\infty} u_k(x) = \tilde{u}(x).$$

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For any fixed $x \in V$, letting $k \to \infty$ in (3-9),

$$-\Delta_p \tilde{u}(x) = K(x) - f(x, \tilde{u}(x)) \quad \text{in } V. \tag{3-14}$$

Thus, $\tilde{u}(x)$ is a solution to (2-6). Since $K(x) - f(x,u) \not\equiv 0$, we get $\tilde{u} \not\equiv 0$.

Now, we prove that $\tilde{u} > 0$ if (2-7) holds. We set

$$\tilde{u}^+ = \max\{\tilde{u}(x), 0\} \quad \text{and} \quad \tilde{u}^- = \min\{\tilde{u}(x), 0\}.$$

It is easy to see that $\tilde{u} = \tilde{u}^+ + \tilde{u}^-$. Noting that

$$|\nabla \tilde{u}^-|^2 \leq \Gamma(\tilde{u}^-) + \Gamma(\tilde{u}^-, \tilde{u}^+) = \Gamma(\tilde{u}^-, \tilde{u}),$$

since $K(x) - f(x,u) \geq 0$,

$$\int_V |\nabla \tilde{u}^-|^p \, d\vartheta \leq -\int_V \tilde{u}^- \Delta_p \tilde{u} \, d\vartheta = \int_V \tilde{u}^- (K(x) - f(x, \tilde{u})) \, d\vartheta \leq 0.$$

So,

$$\tilde{u}^- \equiv 0 \quad \text{and thus } \tilde{u} = \tilde{u}^+ + \tilde{u}^- \geq 0 \text{ for all } x \in V.$$

Next, we prove that $\tilde{u} > 0$ for all $x \in V$. Suppose not. Then there would exist a point $x^* \in V$ such that

$$\tilde{u}(x^*) = 0 = \min_{x \in V} \tilde{u}(x).$$

Thus, by (2-1), we get $\Delta_p \tilde{u}(x^*) > 0$. Then, by (3-14),

$$-\Delta_p \tilde{u}(x^*) = K(x^*) - f(x^*, \tilde{u}(x^*)) < 0,$$

which contradicts (2-7). Therefore, we have that $\tilde{u}(x) > 0$ for all $x \in V$, and the proof is completed. □

PROOF OF THEOREM 4. Let $G_k = (V_k, E_k)$ be a subgraph defined as in (2-3). By Remark 2, set $\mathbb{D}_k = W_0^{m,2}(V_k) \cap W_0^{n,2}(V_k)$ with the norm

$$\|u\|_{\mathbb{D}_k} = \|u\|_{W_0^{m,2}(V_k)} + \|u\|_{W_0^{n,2}(V_k)}.$$

Define the functional $J_{k(Q)} : \mathbb{D}_k \to \mathbb{R}$ by

$$\begin{aligned} J_{k(Q)}(u) = {} & \frac{1}{2} \int_{V_k} |\nabla^m u|^2 \, d\vartheta + \frac{1}{2} \int_{V_k} |\nabla^n u|^2 \, d\vartheta \\ & - \lambda_1 \log \int_{V_k} \varphi_1(x) e^{-u} \, d\vartheta - \lambda_2 \log \int_{V_k} \varphi_2(x) e^{u} \, d\vartheta. \end{aligned} \tag{3-15}$$

Obviously, $J_{k(Q)} \in C^1(\mathbb{D}_k, \mathbb{R})$.

Since $\varphi_1(x) > 0$ for all $x \in V$, there exists some $x_0 \in V_k$ such that

$$\vartheta_0 \varphi_1(x_0) \leq \vartheta(x_0) \varphi_1(x_0) \leq \int_{V_k} \varphi_1(x) \, d\vartheta = \sum_{x \in V_k} \vartheta(x) \varphi_1(x). \tag{3-16}$$

Similarly,

$$\vartheta_0\varphi_2(x_0) \leq \int_{V_k} \varphi_2(x)\, d\vartheta. \tag{3-17}$$

Set $\Theta_k = \inf_{u\in\mathbb{D}_k} J_{k(Q)}(u)$. By (3-16) and (3-17),

$$\Theta_k \leq J_{k(Q)}(0) \leq -\lambda_1 \log(\vartheta_0\varphi_1(x_0)) - \lambda_2 \log(\vartheta_0\varphi_2(x_0)). \tag{3-18}$$

To proceed, the proof is divided into three steps.

Step 1. For any integer $k \geq 1$, $J_{k(Q)}(u)$ is bounded from below in $\mathbb{D}_k$.

By Cauchy's inequality and (2-2), for any $\varepsilon > 0$,

$$e^u \leq e^{u^2/4\varepsilon\|u\|^2_{\mathbb{D}_k} + \varepsilon\|u\|^2_{\mathbb{D}_k}} \leq e^{C_0^2/4\varepsilon\vartheta_0^2 + \varepsilon\|u\|^2_{\mathbb{D}_k}},$$
$$e^{-u} \leq e^{(-u)^2/4\varepsilon\|u\|^p_{\mathbb{D}_k} + \varepsilon\|u\|^2_{\mathbb{D}_k}} \leq e^{C_0^2/4\varepsilon\vartheta_0^2 + \varepsilon\|u\|^2_{\mathbb{D}_k}}.$$

Thus,

$$\log \int_{V_k} \varphi_1(x)e^{-u}\, d\vartheta \leq \log\|\varphi_1\|_{L^1(V)} + \frac{C_0^2}{4\varepsilon\vartheta_0^2} + \varepsilon\|u\|^2_{\mathbb{D}_k}, \tag{3-19}$$

$$\log \int_{V_k} \varphi_2(x)e^{u}\, d\vartheta \leq \log\|\varphi_2\|_{L^1(V)} + \frac{C_0^2}{4\varepsilon\vartheta_0^2} + \varepsilon\|u\|^2_{\mathbb{D}_k}. \tag{3-20}$$

By (3-15), (3-19) and (3-20),

$$J_{k(Q)}(u) \geq \Big(\frac{1}{2} - \varepsilon\Big)\|u\|^2_{\mathbb{D}_k} - \log(\|\varphi_1\|_{L^1(V)} \cdot \|\varphi_2\|_{L^1(V)}) - \frac{C_0^2}{2\varepsilon\vartheta_0^2}.$$

Choosing $\varepsilon = \frac{1}{4}$, for any $u \in \mathbb{D}_k$,

$$J_{k(Q)}(u) \geq \frac{1}{4}\|u\|^2_{\mathbb{D}_k} - \log(\|\varphi_1\|_{L^1(V)} \cdot \|\varphi_2\|_{L^1(V)}) - \frac{C_0^2}{2\varepsilon\vartheta_0^2} \tag{3-21}$$

$$\geq -\log(\|\varphi_1\|_{L^1(V)} \cdot \|\varphi_2\|_{L^1(V)}) - \frac{C_0^2}{2\varepsilon\vartheta_0^2}. \tag{3-22}$$

Therefore, we have that $J_{k(Q)}(u)$ is bounded from below in $\mathbb{D}_k$.

Step 2. For any integer $k \geq 1$, there exists a function $u_k \in \mathbb{D}_k$ such that

$$J_{k(Q)}(u_k) = \Theta_k = \inf_{u\in\mathbb{D}_k} J_{k(Q)}(u).$$

Moreover, u_k satisfies the Euler–Lagrange equation:

$$(-\Delta)^m u_k + (-\Delta)^n u_k = -\lambda_1 \frac{\varphi_1(x)e^{-u_k}}{\int_V \varphi_1(x)e^{-u_k}\, d\vartheta} + \lambda_2 \frac{\varphi_2(x)e^{u_k}}{\int_V \varphi_2(x)e^{u_k}\, d\vartheta} \quad \text{in } V. \tag{3-23}$$

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Obviously, by (3-18) and (3-22),

$$-\log(\|\varphi_1\|_{L^1(V)}\cdot\|\varphi_2\|_{L^1(V)})-\frac{C_0^2}{2\varepsilon\vartheta_0^2}\le\Theta_k\le-\lambda_1\log\left(\vartheta_0\varphi_1(x_0)\right)-\lambda_2\log\left(\vartheta_0\varphi_2(x_0)\right). \tag{3-24}$$

By (3-24), we get that Θ_k is a bounded sequence of numbers.

Now fix a positive integer k. We take a sequence of functions $\hat{u}_j\in\mathbb{D}_k$ such that

$$J_{k(Q)}(\hat{u}_j)\to\Theta_k\quad\text{as } j\to\infty.$$

Thus,

$$\|\hat{u}_j\|_{\mathbb{D}_k}\le M,$$

where M is a positive constant. Therefore, we get that $\{\hat{u}_j\}$ is bounded in $\mathbb{D}_k$. By Theorem A and Remark 2, we have that up to a subsequence, $\hat{u}_j$ converges to some function $u_k\in\mathbb{D}_k$. Clearly,

$$J_{k(Q)}(u_k)=\Theta_k=\inf_{u\in\mathbb{D}_k}J_{k(Q)}(u).$$

Thus, u_k satisfies the Euler–Lagrange system (3-23).

Step 3. There exist $\acute{u}:V\to\mathbb{R}$ and a subsequence of $\{u_k\}$, still denoted by $\{u_k\}$, such that $\{u_k\}$ converges to $\acute{u}$, that is, $\acute{u}$ is a solution to (2-8).

Since Θ_k is bounded, by (3-21),

$$\|u_k\|^2_{\mathbb{D}_k}\le M_1, \tag{3-25}$$

where M_1 is a positive constant independent of k.

By Theorem A and (3-25), for any $u_k\in\mathbb{D}_k$,

$$\|u_k\|_{L^\infty(V_k)}\le\frac{C_0}{\vartheta_0}\|u_k\|_{\mathbb{D}_k}\le M_2, \tag{3-26}$$

where M_2 is a positive constant independent of k.

Let $\{u_k\}$ be extended as a sequence defined on V with $u_k=0$ on $V\setminus V_k$, that is,

$$u_k(x)=\begin{cases}u_k(x) & \text{for } x\in V_k,\\ 0 & \text{for } x\in V\setminus V_k.\end{cases} \tag{3-27}$$

Therefore, by (3-26) and (3-27), we get that $\{u_k\}$ is uniformly bounded in V. Thus, there exists a subsequence of $\{u_k\}$, still denoted by $\{u_k\}$, such that $\{u_k\}$ converges to $\acute{u}$ uniformly in V, that is,

$$\lim_{k\to\infty}u_k(x)=\acute{u}(x).$$

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For any fixed $x \in V$, letting $k \to \infty$ in (3-23),

$$(-\Delta)^m \acute{u} + (-\Delta)^n \acute{u} = -\lambda_1 \frac{\varphi_1(x)e^{-\acute{u}}}{\int_V \varphi_1(x)e^{-\acute{u}}\, d\vartheta} + \lambda_2 \frac{\varphi_2(x)e^{\acute{u}}}{\int_V \varphi_2(x)e^{\acute{u}}\, d\vartheta} \quad \text{in } V. \tag{3-28}$$

Thus, $\acute{u}(x)$ is a solution to (2-8).

Moreover, if

$$\lambda_1 \frac{\varphi_1(x)}{\int_V \varphi_1(x)\, d\vartheta} - \lambda_2 \frac{\varphi_2(x)}{\int_V \varphi_2(x)\, d\vartheta} \not\equiv 0 \quad \text{for all } x \in V,$$

by (3-28), we get that $\acute{u}(x) \not\equiv 0$. Therefore, $\acute{u}(x)$ is a nontrivial solution to (2-8). □

Proof of Theorem 6. Let B_k be defined as in (2-9). Set

$$\mathbb{X}_k = W_0^{m,p}(B_k) \times W_0^{n,q}(B_k)$$

with the norm

$$\|(u, v)\|_{\mathbb{X}_k} = \max\{\|u\|_{W_0^{m,p}(B_k)}, \|v\|_{W_0^{n,q}(B_k)}\}.$$

Define the functional $J_{k(\kappa)} : \mathbb{X}_k \to \mathbb{R}$ by

$$\begin{aligned} J_{k(\kappa)}(u, v) =& \frac{1}{p}\int_{B_k} |\nabla^m u|^p\, d\vartheta + \frac{1}{q}\int_{B_k} |\nabla^n v|^q\, d\vartheta + \int_{B_k} K(x)e^{-u-v}\, d\vartheta \\ &+ \int_{B_k} \kappa(x)e^{-u-v}\, d\vartheta + \int_{B_k} \kappa(x)u\, d\vartheta + \int_{B_k} K(x)v\, d\vartheta. \end{aligned} \tag{3-29}$$

Set

$$\Theta_k = \inf_{(u,v)\in\mathbb{X}_k} J_{k(\kappa)}(u, v).$$

Obviously,

$$\Theta_k \le J_{k(\kappa)}(0, 0) = \int_{B_k} K(x)\, d\vartheta + \int_{B_k} \kappa(x)\, d\vartheta \le \int_V K(x)\, d\vartheta + \int_V \kappa(x)\, d\vartheta. \tag{3-30}$$

To proceed, the proof is divided into four steps.

Step 1. For any positive integer k, $J_{k(\kappa)}$ is bounded from below in $\mathbb{X}_k$.

Noting the fact that $e^{-t} > -t + 1$ for all $t \in \mathbb{R}$, since $K(x), \kappa(x) > 0$, by (3-29),

$$\begin{aligned} J_{k(\kappa)}(u, v) \ge& \frac{1}{p}\int_{B_k} |\nabla^m u|^p\, d\vartheta + \frac{1}{q}\int_{B_k} |\nabla^n v|^q\, d\vartheta \\ &- \int_{B_k} Ku\, d\vartheta - \int_{B_k} \kappa v\, d\vartheta + \int_{B_k} (K + \kappa)\, d\vartheta. \end{aligned} \tag{3-31}$$

By the Hölder inequality, Theorem A and the Young inequality,

$$\begin{aligned}\int_{B_k} Ku\,d\vartheta &\le \|u\|_{L^s(B_k)}\|K\|_{L^{s/(s-1)}(V)}\\ &\le C_0\|u\|_{W_0^{m,p}(B_k)}\|K\|_{L^{s/(s-1)}(V)}\\ &= \frac{1}{2}\|u\|_{W_0^{m,p}(B_k)}\cdot 2C_0\|K\|_{L^{s/(s-1)}(V)}\\ &\le \frac{1}{2p}\|u\|^p_{W_0^{m,p}(B_k)} + \frac{p-1}{2p}(2C_0)^{p/(p-1)}\|K\|^{p/(p-1)}_{L^{s/(s-1)}(V)}. \end{aligned}\tag{3-32}$$

Similarly,

$$\int_{B_k} \kappa v\,d\vartheta \le \frac{1}{2q}\|v\|^q_{W_0^{n,q}(B_k)} + \frac{q-1}{2q}(2C_0)^{q/(q-1)}\|\kappa\|^{q/(q-1)}_{L^{\gamma/(\gamma-1)}(V)}. \tag{3-33}$$

By (3-31), (3-32) and (3-33),

$$\begin{aligned}J_{k(\kappa)}(u,v) &\ge \frac{1}{2p}\|u\|^p_{W_0^{m,p}(B_k)} + \frac{1}{2q}\|v\|^q_{W_0^{n,q}(B_k)} - \frac{p-1}{2p}(2C_0)^{p/(p-1)}\|K\|^{p/(p-1)}_{L^{s/(s-1)}(V)}\\ &\quad - \frac{q-1}{2q}(2C_0)^{q/(q-1)}\|\kappa\|^{q/(q-1)}_{L^{\gamma/(\gamma-1)}(V)} \qquad (3\text{-}34)\\ &\ge -\frac{p-1}{2p}(2C_0)^{p/(p-1)}\|K\|^{p/(p-1)}_{L^{s/(s-1)}(V)} - \frac{q-1}{2q}(2C_0)^{q/(q-1)}\|\kappa\|^{q/(q-1)}_{L^{\gamma/(\gamma-1)}(V)}. \qquad (3\text{-}35)\end{aligned}$$

Therefore, for any positive integer k, $J_{k(\kappa)}$ is bounded from below in $\mathbb{X}_k$.

Step 2. For any positive integer k, there exists $(u_k, v_k) \in \mathbb{X}_k$ such that

$$J_{k(\kappa)}(u_k, v_k) = \Theta_k = \inf_{(u,v)\in\mathbb{X}_k} J_{k(\kappa)}(u,v).$$

Moreover, (u_k, v_k) satisfies the Euler–Lagrange system:

$$\begin{cases}\mathcal{L}_{m,p}u_k = K(x)e^{-u_k-v_k} - \kappa(x) & \text{in } V_k,\\ \mathcal{L}_{n,q}v_k = \kappa(x)e^{-u_k-v_k} - K(x) & \text{in } V_k,\\ u_k = v_k = 0 & \text{on } \partial V_k.\end{cases} \tag{3-36}$$

Obviously, by (3-30) and (3-35), there holds

$$\begin{cases}\Theta_k \ge -\dfrac{p-1}{2p}(2C_0)^{p/(p-1)}\|K\|^{p/(p-1)}_{L^{s/(s-1)}(V)} - \dfrac{q-1}{2q}(2C_0)^{q/(q-1)}\|\kappa\|^{q/(q-1)}_{L^{\gamma/(\gamma-1)}(V)},\\ \Theta_k \le \int_V K(x)\,d\vartheta + \int_V \kappa(x)\,d\vartheta.\end{cases} \tag{3-37}$$

By (3-37), we know that Θ_k is a bounded sequence of numbers.

Now we fix a positive integer k and take a sequence of functions $\{(\hat{u}_j, \hat{v}_j)\} \subset \mathbb{X}_k$ satisfying

$$J_{k(\kappa)}(\hat{u}_j, \hat{v}_j) \to \Theta_k \quad \text{as } j \to \infty. \tag{3-38}$$

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It follows from (3-34) and (3-38) that $\{\hat{u}_j\}$ and $\{\hat{v}_j\}$ are bounded in $W_0^{m,p}(B_k)$ and $W_0^{n,q}(B_k)$, respectively. By Theorem A, we get that there exist

$$u_k \in W_0^{m,p}(B_k) \quad \text{and} \quad v_k \in W_0^{n,q}(B_k)$$

such that

$$\|\hat{u}_j - u_k\| \to 0 \quad \text{and} \quad \|\hat{v}_j - v_k\| \to 0 \quad \text{as } j \to \infty.$$

By Remark 5, for $(u_k, v_k) \in \mathbb{X}_k$, up to a subsequence, $\{(\hat{u}_j, \hat{v}_j)\}$ converges to (u_k, v_k). Clearly,

$$J_{k(\kappa)}(u_k, v_k) = \Theta_k = \inf_{(u,v)\in\mathbb{X}_k} J_{k(\kappa)}(u, v).$$

Thus, (u_k, v_k) satisfies the Euler–Lagrange system (3-36).

Step 3. For any finite set $\mathfrak{R} \subset V$, (u_k, v_k) is uniformly bounded in $W_0^{m,p}(\mathfrak{R}) \times W_0^{n,q}(\mathfrak{R})$.

By (3-34) and (3-38),

$$\|u_k\|^p_{W_0^{m,p}(B_k)} \le M_1, \tag{3-39}$$

$$\|v_k\|^q_{W_0^{n,q}(B_k)} \le M_1, \tag{3-40}$$

where M_1 is a positive constant independent of k. When k is large enough, we get that $\mathfrak{R} \subset V_k$. By (2-4), (3-39) and (3-40),

$$\|u_k\|_{L^\infty(\mathfrak{R})} \le M_2 \quad \text{and} \quad \|v_k\|_{L^\infty(\mathfrak{R})} \le M_2,$$

where M_2 is a positive constant independent of k. Therefore, $\{(u_k, v_k)\}$ is uniformly bounded in $W_0^{m,p}(\mathfrak{R}) \times W_0^{n,q}(\mathfrak{R})$.

Step 4. There exist $(u^*, v^*) : V \times V \to \mathbb{R}$ and a subsequence of $\{(u_k, v_k)\}$, still denoted by $\{(u_k, v_k)\}$, such that $\{(u_k, v_k)\}$ converges to (u^*, v^*), that is, (u^*, v^*) is a solution to (2-10).

By Step 3, we have that $\{(u_k, v_k)\}$ is uniformly bounded in B_1. Thus, there exists a subsequence of $\{(u_k, v_k)\}$, denoted by $\{(u_{1k}, v_{1k})\}$, and functions (u_1^*, v_1^*) such that $(u_{1k}, v_{1k}) \to (u_1^*, v_1^*)$ in B_1. By Step 3 again, $\{(u_{1k}, v_{1k})\}$ is uniformly bounded in B_2. Then there exists a subsequence of $\{(u_{1k}, v_{1k})\}$, denoted by $\{(u_{2k}, v_{2k})\}$, and functions (u_2^*, v_2^*) such that $(u_{2k}, v_{2k}) \to (u_2^*, v_2^*)$ in B_2. Obviously, $(u_1^*, v_1^*) = (u_2^*, v_2^*)$ in B_1. Repeating this process, we can find a diagonal subsequence $\{(u_{kk}, v_{kk})\}$, which is still denoted by $\{(u_k, v_k)\}$, and functions $(u^*, v^*) : V \times V \to \mathbb{R}$ such that for any finite set $\mathfrak{R} \subset V$, $(u_k, v_k) \to (u^*, v^*)$ in $\mathfrak{R}$. For any fixed $x \in V$, let $k \to \infty$ in (3-36), so

$$\begin{cases} \mathcal{L}_{m,p} u^* = K(x) e^{-u^* - v^*} - \kappa(x) & \text{in } V, \\ \mathcal{L}_{n,q} v^* = \kappa(x) e^{-u^* - v^*} - K(x) & \text{in } V. \end{cases}$$

Since $K(x) \not\equiv \kappa(x)$, we get $(u^*, v^*) \not\equiv (0, 0)$. Therefore, (u^*, v^*) is a nontrivial solution to (2-10). □

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SHOUDONG MAN, College of Science and Technology,
Tianjin University of Finance and Economics, Zhujiang Road,
Tianjin 300222, PR China
e-mail: shoudongmantj@tjufe.edu.cn

GUOQING ZHANG, College of Science,
University of Shanghai for Science and Technology, Jungong Road,
Shanghai 200093, PR China
e-mail: shzhangguoqing@126.com

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