J. Austral. Math. Soc. (*Series A*) 31 (1981), 15–19

ON SIMPLICES AND LATTICE POINTS

P. R. SCOTT

(Received 21 March 1980)

Communicated by A. J. van der Poorten

Abstract

Let S be a simplex in E^n which is homothetic to a given simplex S^*, which contains no point of the integral lattice in its interior, and which has maximal volume $V(S)$. We conjecture that $V(S) > n^n/n!$, and establish the conjecture for $n < 3$.

1980 *Mathematics subject classification* (*Amer. Math. Soc.*): primary 52 A 20, 52 A 43; secondary 10 E 05.

1. Introduction

The problem considered here is analogous to a problem proposed by Mordell [2], in which a lower bound is sought for the volumes of certain specified parallelopipeds centred at the origin. Mordell's conjecture is established for $n = 2, 3$ by Szekeres [4], [5] and Ko [1].

Two simplices in euclidean n-space, E^n, are said to be *homothetic* if they are similar and similarly placed. Let S^* be any given simplex. We say that S is a *maximal* simplex if S is homothetic to S^*, S contains no points of the integral lattice Γ_n in its interior, and $V(S)$ is maximal.

CONJECTURE. *If S is a maximal simplex in E^n, then*

$$V(S) > n^n/n!,$$

and this lower bound is best possible.

https://doi.org/10.1017/S1446788700018449 Published online by Cambridge University Press

We notice that this bound is attained for the simplex with vertices $\mathbf{0}, (n, 0, \ldots, 0), \ldots, (0, \ldots, 0, n)$. Since the interior of this simplex is defined by $x_i > 0$ $(1 \leqslant i \leqslant n)$, $\sum_{i=1}^n x_i < n$, it is clear that no point of Γ_n lies in the interior.

The conjecture is trivially true for $n = 1$. We shall establish it for $n = 2, 3$.

2. The question of existence

Let S be a maximal simplex in E^n. We ask if there exists a real *positive* number κ_n such that $V(S) \geqslant \kappa_n$. If the existence of κ_n can be established for each n, then the conjecture is easily proved. We show here that κ_2, κ_3 exist. However, as in Mordell's problem, for $n > 3$ the problem of existence seems to be intractable.

If S is a maximal simplex homothetic to a given simplex S^*, then clearly each $(n-1)$-dimensional face of S must contain a lattice point in its relative interior. For if not, the face can be translated outwards without introducing any lattice points into the interior of S, and S is not maximal.

The Case $n = 2$. Let A, B, C be lattice points on the edges of a maximal triangle S. Since A, B, C cannot be collinear, they form the vertices of a lattice triangle which has area not less than $\frac{1}{2}$, and which is contained in S. Hence $V(S) \geqslant \frac{1}{2}$, and we may take $\kappa_2 = \frac{1}{2}$.

The Case $n = 3$. Let A, B, C, D be lattice points in the relative interiors of the faces of a maximal tetrahedron S. If A, B, C, D are not coplanar, then they form the vertices of a lattice tetrahedron of volume not less than $\frac{1}{6}$, and we obtain $V(S) \geqslant \frac{1}{6}$.

Suppose then that A, B, C, D are coplanar. By suitable integral unimodular transformation of S (and S^*) we may assume these points lie in the xy-plane. Now S cannot lie in the region $|z| < \frac{1}{2}$, for then we could obtain a homothetic simplex S' of larger volume and containing no lattice points in its interior by translating S into the region $0 < z < 1$ and then enlarging. Hence we may assume without loss of generality that there is a point P of S in the plane $z = \frac{1}{2}$. Since S is convex, points A, B, C are not collinear, and A, B, C, P form the vertices of a proper tetrahedron contained in S. As the volume of this tetrahedron is not less than $\frac{1}{12}$, we have $V(S) \geqslant \frac{1}{12}$.

Hence we may take $\kappa_3 = \frac{1}{12}$. Alternatively, we can obtain $\kappa_3 \geqslant \frac{1}{3}$ by suitably dissecting the parallelopiped considered by Szekeres [5].

3. Two preliminary lemmas

We shall need the following two results.

LEMMA 1. *If the centre of gravity of each face of a simplex S in E^n is a point of Γ_n, then so is each vertex of S. Further, $V(S) \geqslant n^n/n!$.*

PROOF. Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{n+1}$ denote the vertices of S, and let $\mathbf{c}_i$ denote the centre of gravity of the $(n-1)$-dimensional face not containing $\mathbf{v}_i$ $(1 \leqslant i \leqslant n+1)$.

Then

$$n\mathbf{c}_i = \sum_{k=1}^{n+1} \mathbf{v}_k - \mathbf{v}_i \qquad (1 \leqslant i \leqslant n+1).$$

Solving these equations for $\mathbf{v}_i$ gives

$$\mathbf{v}_i = \sum_{k=1}^{n+1} \mathbf{c}_k - n\mathbf{c}_i \qquad (1 \leqslant i \leqslant n+1).$$

Hence if $\mathbf{c}_1, \ldots, \mathbf{c}_{n+1}$ are points of Γ_n, then so are the vertices $\mathbf{v}_1, \ldots, \mathbf{v}_{n+1}$. Further,

$$\mathbf{v}_i - \mathbf{v}_j = -n(\mathbf{c}_i - \mathbf{c}_j).$$

Hence the simplex S is homothetic to the lattice simplex L having vertices $\mathbf{c}_1, \ldots, \mathbf{c}_{n+1}$, and n times as large. Since $V(L) \geqslant 1/n!$, we deduce that $V(S) \geqslant n^n/n!$.

LEMMA 2. *Let T be an $(n-1)$-dimensional simplex in E^n having centre of gravity $\mathbf{t}$. Let F be an $(n-2)$-dimensional flat intersecting T and separating it into subsets U, W. Let T be rotated about F. Then*

(i) *U, W sweep out equal volumes if and only if $\mathbf{t}$ lies in F;*

(ii) *U sweeps out a larger volume than W if and only if F strictly separates $\mathbf{t}$ from W (in the hyperplane of T).*

PROOF. (i) Let U, W have centres of gravity $\mathbf{u}$, $\mathbf{w}$ respectively, and let $|U|$, $|W|$ denote the $(n-1)$-dimensional volumes of U, W. Without loss of generality, let the reference system be chosen such that

(a) the distance of each point $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ of T from F is given by $|x_1|$

(b) if $\mathbf{x} \in U$, $x_1 \geqslant 0$, and if $\mathbf{x} \in W$, $x_1 \leqslant 0$. Now since $\mathbf{t}$ is the centre of gravity of T,

$$(|U| + |W|) \cdot t_1 = |U| \cdot u_1 + |W| \cdot w_1.$$

Also, according to the extended version of Pappus' Theorem (see for example [3]), the volume of revolution generated by U is given by the product of $|U|$ and the length of the path of the centre of gravity $\mathbf{u}$. Hence if T is rotated about F through angle θ (> 0), the volume generated by U is $|U| \cdot u_1 \cdot \theta$. Similarly, noting that $w_1 < 0$, the volume generated by W is $|W| \cdot (-w_1) \cdot \theta$. These generated volumes are equal if and only if

$$|U| \cdot u_1 + |W| \cdot w_1 = 0.$$

But this is precisely the condition for $\mathbf{t}$ to lie on F.

(ii) Suppose now that $|U| \cdot u_1 \cdot \theta > |W| \cdot (-w_1) \cdot \theta$. This occurs when and only when $|U| \cdot u_1 + |W| \cdot w_1 > 0$. But this is the condition that $t_1 > 0$, and F strictly separates $\mathbf{t}$ from W as required.

4. Proof of the conjecture, assuming existence

Let us assume the existence of a maximal simplex S of smallest volume in E^n. We have seen that each $(n-1)$-dimensional face T of S contains at least one interior lattice point. Suppose T contains a lattice point in its relative interior, but not at its centre of gravity $\mathbf{t}$. Then we can choose a lattice point $\mathbf{p}$ interior to T, and an $(n-2)$-dimensional face R of T, such that $\mathbf{p}$ is closer to R than $\mathbf{t}$, and at least as close to R as any other lattice point interior to T. Choose an $(n-2)$-dimensional axis (flat) F through $\mathbf{p}$ and parallel to R. Now there exists a small rotation of T about F which introduces no new lattice points into the interior of S, and which by Lemma 2 decreases the volume of S. Hence S does not have smallest volume.

We deduce that each face of S contains just one lattice point at its centre of gravity. But now by Lemma 1, each vertex of S is a lattice point, and $V(S) \geqslant n^n/n!$.

Thus the conjecture has been established, providing a maximal simplex of smallest volume exists in E^n. It is certainly true for $n \leqslant 3$.

References

[1] C. Ko, 'Note on the lattice points in a parallelopiped', *J. London Math. Soc.* **12** (1936), 40–47.

[2] L. J. Mordell, 'Note on an arithmetical problem on linear forms', *J. London Math. Soc.* **12** (1936), 34–36.

[3] D. M. Y. Sommerville, *An introduction to the geometry of n dimensions* (Methuen, London, 1929).

[4] G. Szekeres, 'On a problem of the lattice plane', *J. London Math. Soc.* **12** (1937), 88–93.
[5] G. Szekeres, 'Note on lattice points within a parallelopiped', *J. London Math. Soc.* **12** (1937), 36–39.

Department of Pure Mathematics
University of Adelaide
Adelaide
South Australia

https://doi.org/10.1017/S1446788700018449 Published online by Cambridge University Press