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THE GENERATING GRAPH OF INFINITE ABELIAN GROUPS

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Abstract

For a group G, let $\Gamma(G)$ denote the graph defined on the elements of G in such a way that two distinct vertices are connected by an edge if and only if they generate G. Let $\Gamma^*(G)$ be the subgraph of $\Gamma(G)$ that is induced by all the vertices of $\Gamma(G)$ that are not isolated. We prove that if G is a 2-generated noncyclic abelian group, then $\Gamma^*(G)$ is connected. Moreover, $\operatorname{diam}(\Gamma^*(G)) = 2$ if the torsion subgroup of G is nontrivial and $\operatorname{diam}(\Gamma^*(G)) = \infty$ otherwise. If F is the free group of rank 2, then $\Gamma^*(F)$ is connected and we deduce from $\operatorname{diam}(\Gamma^*(\mathbb{Z} \times \mathbb{Z})) = \infty$ that $\operatorname{diam}(\Gamma^*(F)) = \infty$.

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1. Introduction

For any group G, the generating graph associated to G, written $\Gamma(G)$, is the graph where the vertices are the elements of G and we draw an edge between g_1 and g_2 if G is generated by g_1 and g_2. If G is not 2-generated, then there will be no edge in this graph. Thus, it is natural to assume that G is 2-generated.

There could be many isolated vertices in this graph. All of the elements in the Frattini subgroup will be isolated vertices, but we can also find isolated vertices outside the Frattini subgroup (for example, the nontrivial elements of the Klein subgroup are isolated vertices in $\Gamma(\mathrm{Sym}(4))$).

Let $\Gamma^*(G)$ be the subgraph of $\Gamma(G)$ that is induced by all the vertices that are not isolated. In [2], it is proved that if G is a 2-generated finite soluble group, then $\Gamma^*(G)$ is connected. Later, in [4], the second author investigated the diameter of such a graph proving that if G is a finite soluble group, then $\operatorname{diam}(\Gamma^*(G)) \leq 3$. This bound is best possible, although $\operatorname{diam}(\Gamma^*(G)) \leq 2$ in some relevant cases. In particular, $\operatorname{diam}(\Gamma^*(G)) \leq 2$ whenever G is a finite abelian group.

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In this paper we analyse the graph $\Gamma^*(G)$ when G is an infinite 2-generated abelian group. If G is an infinite cyclic group, then the graph $\Gamma(G)$ coincides with $\Gamma^*(G)$ and it is connected with diameter 2 (indeed, any vertex of $\Gamma(G)$ is adjacent to a vertex corresponding to a generator of G). So, we may assume that G is noncyclic. There are two different cases. First we consider the situation when the torsion subgroup $T(G)$ of G is nontrivial. This means that $G = \mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ for some integer $n \geq 2$. In this case we will see that an argument relying on Dirichlet's theorem on arithmetic progressions allows us to prove that $\Gamma^*(G)$ is connected and $\mathrm{diam}(\Gamma^*(G)) = 2$. The situation is more intriguing when $T(G)$ is trivial, that is, when $G \cong \mathbb{Z} \times \mathbb{Z}$. In this latter case we will deduce that $\Gamma^*(G)$ is a connected graph using the fact that every 2×2 invertible matrix over $\mathbb{Z}$ is a product of elementary matrices [6, Theorem 14.2]. However, although every element $A \in \mathrm{GL}(2, \mathbb{Z})$ can be written as a product of elementary matrices, the number of elementary matrices required depends on the entries of A and it is not bounded. As explained for example in [1], the reason for this is related to the fact that arbitrarily many divisions with remainder may be required when any (general) Euclidean algorithm is used to find the greatest common divisor of two integers. This leads us to conjecture that the diameter of $\Gamma^*(\mathbb{Z} \times \mathbb{Z})$ could be infinite. We will confirm this conjecture proving that the fact that division chains in $\mathbb{Z}$ can be arbitrarily long (as happens for example when two consecutive Fibonacci numbers are considered) implies that the distance between two vertices of $\Gamma^*(\mathbb{Z} \times \mathbb{Z})$ can also be arbitrarily large. Summarising, we obtain the following result.

Theorem 1.1. *Let G be a 2-generated abelian group. Then $\Gamma^*(G)$ is connected. Moreover, if either G is cyclic or the torsion subgroup of G is nontrivial, then* $\mathrm{diam}(\Gamma^*(G)) = 2$, *while* $\mathrm{diam}(\Gamma^*(\mathbb{Z} \times \mathbb{Z})) = \infty$.

Finally, let F be the free group of rank 2. The fact that every automorphism of F can be expressed as a product of elementary Nielsen transformations can be used to prove that the graph $\Gamma^*(F)$ is connected (see Theorem 4.2). Moreover, we will deduce from $\mathrm{diam}(\Gamma^*(\mathbb{Z} \times \mathbb{Z}) = \infty$ that the graph $\Gamma^*(F)$ cannot have a finite diameter (see Corollary 4.4).

2. Abelian groups with a nontrivial torsion subgroup

If G is a 2-generated finite abelian group, then $\Gamma^*(G)$ is connected with diameter at most 2. This can be deduced from [4, Corollary 3], but a short alternative proof can also be given. Indeed, we can write $G = P_1 \times \cdots \times P_t$ as a direct product of its Sylow subgroups. It is easy to see that $\Gamma^*(G)$ is the cross product of the graphs $\Gamma^*(P_1), \ldots, \Gamma^*(P_t)$, so it suffices to prove that the graph $\Gamma^*(G)$ is connected with diameter at most 2 in the particular case when G is a 2-generated abelian p-group. On the other hand, $\Gamma^*(G) = \Gamma^*(G/\operatorname{Frat}(G))$, so we may assume that G is an elementary abelian p-group of rank at most 2. The conclusion follows from the fact that given two nonzero vectors v_1, v_2 of a two-dimensional vector space, there always exists a third vector w such that $\langle v_1, w\rangle = \langle v_2, w\rangle$.

In the remaining part of this section, we assume that G is an infinite 2-generated abelian group and that the torsion subgroup of G is nontrivial. This means that there exists $n \geq 2$ such that $G = \mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$. In what follows, we denote by $\bar{x}$ the image of an integer x in $\mathbb{Z}/n\mathbb{Z}$ under the natural epimorphism.

LEMMA 2.1. *Let $G = \mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ and $a, b, c, d \in \mathbb{Z}$. The vertices $(a, \bar{b})$ and $(c, \bar{d})$ are adjacent in $\Gamma(G)$ if and only if the following two conditions hold:*

(i) $\gcd(a, c) = 1$;

(ii) $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *is coprime to n.*

PROOF. Note that $(a, \bar{b})$ and $(c, \bar{d})$ generate G if and only if $(a, b), (c, d)$ and $(0, n)$ generate $\mathbb{Z} \times \mathbb{Z}$. The latter condition is equivalent to saying that the invariant factors of the matrix

$$A = \begin{pmatrix} a & b \\ c & d \\ 0 & n \end{pmatrix}$$

are invertible elements of $\mathbb{Z}$, that is:

(1) $\gcd(a, b, c, d, 0, n) = 1$;

(2) $\gcd\left(\det\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \det\begin{pmatrix} a & b \\ 0 & n \end{pmatrix}, \det\begin{pmatrix} c & d \\ 0 & n \end{pmatrix}\right) = 1.$

Now $\gcd(a, b)$ divides $\gcd(ad - bc, an, bn)$, so condition (2) implies $\gcd(a, b) = 1$. From $\gcd(a, b) = 1$, we have $\gcd(an, bn) = n$ and consequently

$$\gcd\left(\det\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \det\begin{pmatrix} a & b \\ 0 & n \end{pmatrix}, \det\begin{pmatrix} c & d \\ 0 & n \end{pmatrix}\right) = \gcd\left(\det\begin{pmatrix} a & b \\ c & d \end{pmatrix}, n\right). \qquad \square$$

THEOREM 2.2. *Let $G = \mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ for some fixed $n \geq 2$. Then $\Gamma^*(G)$ is connected and* $\mathrm{diam}(\Gamma^*(G)) = 2$.

PROOF. Let $(a, \bar{b})$ and $(c, \bar{d})$ be two nonisolated vertices in $\Gamma(G)$ and consider the group $H = \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$. As observed above, $\mathrm{diam}(\Gamma^*(H)) = 2$, so taking the corresponding nonisolated vertices $(\bar{a}, \bar{b})$ and $(\bar{c}, \bar{d})$ in $\Gamma(H)$, there exists a pair (x, y) of positive integers such that

$$\det\begin{pmatrix} a & b \\ x & y \end{pmatrix} \quad \text{and} \quad \det\begin{pmatrix} c & d \\ x & y \end{pmatrix}$$

are both coprime to n. Next we show that there exists $t \in \mathbb{Z}$ such that $(x + tn, \bar{y})$ is adjacent simultaneously to $(a, \bar{b})$ and $(c, \bar{d})$ in $\Gamma(G)$. By Lemma 2.1, it is enough to find $t \in \mathbb{Z}$ satisfying the conditions

$$\gcd(a, x + tn) = 1 = \gcd(c, x + tn). \tag{2.1}$$

Indeed, by construction, both determinants

$$\det\begin{pmatrix} a & b \\ x + tn & y \end{pmatrix} \quad \text{and} \quad \det\begin{pmatrix} c & d \\ x + tn & y \end{pmatrix}$$

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are coprime to n. Let $u = \gcd(x, n)$. Since $\gcd(a, u)$ divides both n and $\det\left(\begin{smallmatrix} a & b \\ x & y \end{smallmatrix}\right)$, it follows that $\gcd(a, u) = 1$. Similarly, $\gcd(b, u) = 1$. Now there exist positive integers x^* and m such that $x = ux^*$ and $n = um$. Observe that $\gcd(x^*, m) = 1$ and so, by Dirichlet's theorem on arithmetic progressions, the set $\{x^* + tm \mid t \in \mathbb{Z}\}$ contains infinitely many primes. Take $t \in \mathbb{Z}$ such that $x^* + tm = p$ is a prime greater than a and c. Then $up = d(x^* + tm) = x + tn$ is the desired element satisfying both conditions in (2.1). □

3. The abelian free group of rank 2

In this section, we analyse the case where G is a 2-generated noncyclic torsion-free abelian group, that is, $G = \mathbb{Z} \times \mathbb{Z}$. First of all, note that (a, b) and (x, y) form a generating pair for G if and only if $\det\left(\begin{smallmatrix} a & b \\ x & y \end{smallmatrix}\right) = \pm 1$, so the edges of $\Gamma^*(G)$ are in correspondence with the elements of $\mathrm{GL}(2, \mathbb{Z})$, up to a swap obtained by multiplying by the matrix $J = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ on the left. Indeed, since the graph $\Gamma^*(G)$ is undirected, any generating pair (a, b) and (x, y) of G can be represented equivalently by either the matrix $A = \left(\begin{smallmatrix} a & b \\ x & y \end{smallmatrix}\right)$ or the matrix $A^* = \left(\begin{smallmatrix} x & y \\ a & b \end{smallmatrix}\right) = JA$.

Lemma 3.1. *Let (a, b) be a vertex of $\Gamma(G)$. Then (a, b) is nonisolated if and only if a and b are coprime.*

Proof. A pair (a, b) is a nonisolated vertex of $\Gamma(G)$ if and only if there exists a pair of integers (x, y) such that $\det\left(\begin{smallmatrix} a & b \\ x & y \end{smallmatrix}\right) = \pm 1$, that is, there exists an integer solution (x, y) either for the equation $ay - bx = 1$ or for the equation $ay - bx = -1$. Since this holds if and only if $\gcd(a, b)$ divides ± 1, the lemma is proved. □

Let R be a commutative ring. An elementary matrix is an element of $\mathrm{GL}(n, R)$ obtained by applying elementary transformations to the identity matrix I_n. There exist three different kinds of elementary matrices, corresponding respectively to three different types of elementary row transformations (or, equivalently, column transformations):

- transpositions P_{ij}, with $i \neq j$, obtained from I_n by exchanging row i and row j;
- dilations $D_i(u)$ obtained from I_n by multiplying row i by the unit $u \in U(R)$;
- transvections $T_{ij}(r)$, with $i \neq j$ and $r \in R$, obtained from I_n by adding r times row j to row i.

Lemma 3.2. *Assume that $\left(\begin{smallmatrix} a_1 & a_2 \\ b_1 & b_2 \end{smallmatrix}\right), \left(\begin{smallmatrix} c_1 & c_2 \\ d_1 & d_2 \end{smallmatrix}\right) \in \mathrm{GL}(2, \mathbb{Z})$. If there exists an elementary matrix X in $\mathrm{GL}(2, \mathbb{Z})$ such that*

$$\begin{pmatrix} c_1 & c_2 \\ d_1 & d_2 \end{pmatrix} = X \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix},$$

then $\alpha := (a_1, a_2)$, $\beta := (b_1, b_2)$, $\gamma := (c_1, c_2)$ and $\delta := (d_1, d_2)$ belong to the same connected component of $\Gamma(\mathbb{Z} \times \mathbb{Z})$.

Proof. The statement is clearly true if X is either a transposition or a dilation. If $X = \left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right)$, then $\gamma = \alpha + t\beta$, $\delta = \beta$ and $(\alpha, \beta, \alpha + t\beta)$ is a path in $\Gamma(\mathbb{Z} \times \mathbb{Z})$. Similarly, if $X = \left(\begin{smallmatrix} 1 & 0 \\ t & 1 \end{smallmatrix}\right)$, then $\gamma = \alpha$, $\delta = t\alpha + \beta$ and $(\beta, \alpha, t\alpha + \beta)$ is a path in $\Gamma(\mathbb{Z} \times \mathbb{Z})$. □

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Theorem 3.3. *Let $G = \mathbb{Z} \times \mathbb{Z}$. Then the graph $\Gamma^*(G)$ is connected.*

Proof. Assume $\alpha = (x_1, x_2)$ is a nonisolated vertex of $\Gamma(G)$. There exists $\beta = (y_1, y_2)$ such that $G = \langle \alpha, \beta \rangle$. This means that $A = \left(\begin{smallmatrix} x_1 & x_2 \\ y_1 & y_2 \end{smallmatrix}\right) \in \mathrm{GL}(2, \mathbb{Z})$, so by [6, Theorem 14.2] there exist t elementary matrices $J_1, \ldots, J_t$ such that $A = J_1 \cdots J_t$. By iterated applications of Lemma 3.2, for every $r \in \{1, \ldots, t\}$, the rows of the matrix $J_r \cdots J_t$ belong to the connected component of $\Gamma(G)$ containing $(1, 0)$ and $(0, 1)$. In particular, $(1, 0)$ and (x_1, x_2) are in the same connected component. □

For every nonisolated vertex (x, y) of $\Gamma(\mathbb{Z} \times \mathbb{Z})$, denote by $\mathcal{N}(x, y)$ the neighbourhood of (x, y) in $\Gamma(\mathbb{Z} \times \mathbb{Z})$. Fix $(x_0, y_0) \in \mathcal{N}(x, y)$. The matrices in $\mathrm{GL}(2, \mathbb{Z})$ having (x, y) as second row are precisely those of the form

$$\begin{pmatrix} kx + x_0 & ky + y_0 \\ x & y \end{pmatrix}, \quad \begin{pmatrix} kx - x_0 & ky - y_0 \\ x & y \end{pmatrix} \quad \text{for } k \in \mathbb{Z},$$

so $\mathcal{N}(x, y) = \{(kx + x_0, ky + y_0), (kx - x_0, ky - y_0) \mid k \in \mathbb{Z}\}$. In particular,

$$\mathcal{N}(0, 1) = \{(\pm 1, k) \mid k \in \mathbb{Z}\}.$$

Lemma 3.4. *Let (a, b) be a vertex in $\Gamma(\mathbb{Z} \times \mathbb{Z})$ at distance d from $(0, 1)$ and let (x, y) be a vertex adjacent to (a, b) and having distance $d - 1$ from $(0, 1)$. Then there exist $q_1, \ldots, q_d \in \mathbb{Z}$ and $\eta_1, \eta_2 \in \{0, 1\}$ such that*

$$\begin{pmatrix} x & y \\ a & b \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & q_d \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & q_2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & q_1 \end{pmatrix} (-1)^{\eta_1} E^{\eta_2},$$

where $E = \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

Proof. For any integer α, set

$$T_\alpha = \begin{pmatrix} 0 & 1 \\ 1 & \alpha \end{pmatrix}, \quad T^*_\alpha = \begin{pmatrix} 0 & 1 \\ -1 & \alpha \end{pmatrix}.$$

We have $T_\alpha E = T^*_\alpha$.

We argue by induction on d. If the vertex (a, b) is at distance 1 from $(0, 1)$, then (a, b) is of type $(\pm 1, \alpha)$, where α is an arbitrary integer. In particular, as desired,

$$\begin{pmatrix} 0 & 1 \\ 1 & \alpha \end{pmatrix} = T_\alpha I_2, \quad \begin{pmatrix} 0 & 1 \\ -1 & \alpha \end{pmatrix} = T_\alpha E.$$

Assume that the statement is true for any vertex at distance $d - 1$ from $(0, 1)$. We can write

$$\begin{pmatrix} x & y \\ a & b \end{pmatrix} = T \begin{pmatrix} \tilde{x} & \tilde{y} \\ x & y \end{pmatrix},$$

where $(\tilde{x}, \tilde{y})$ is a vertex adjacent to (x, y) at distance $d - 2$ from $(0, 1)$ and either $T = T_\alpha$ or $T = T^*_\alpha$ for some $\alpha \in \mathbb{Z}$. By the inductive hypothesis, there exist $q_1, \ldots, q_{d-1} \in \mathbb{Z}$ and $\tilde{\eta}_1, \tilde{\eta}_2 \in \{0, 1\}$ such that

$$\begin{pmatrix} x & y \\ a & b \end{pmatrix} = T T_{q_{d-1}} \cdots T_{q_1} A$$

with $A = (-1)^{\tilde{\eta}_1} E^{\tilde{\eta}_2}$. If $T = T_\alpha$, we are done. If $T = T_\alpha^*$, then

$$T_\alpha^* T_{q_{d-1}} \cdots T_{q_1} A = T_\alpha^* E(ET_{q_{d-1}}E) \cdots (ET_{q_1}E)EA = T_\alpha T_{-q_{d-1}} \cdots T_{-q_1}(-1)^{d-1}EA.$$

This completes the proof. □

Let R be an integral domain and pick two nonzero elements $a, b \in R$. Following [6, Section 14], we say that (a, b) belongs to a Euclidean chain of length n if there is a sequence of divisions

$$r_i = q_{i+1} r_{i+1} + r_{i+2}, \quad r_i, q_i \in R, \quad 0 \le i \le n-1,$$

such that $a = r_0, b = r_1, r_n \neq 0$ and $r_{n+1} = 0$.

Lemma 3.5. *If the vertex (a, b) of $\Gamma^*(\mathbb{Z} \times \mathbb{Z})$ has distance d from $(0, 1)$, then (b, a) belongs to a Euclidean chain of length at most d.*

Proof. Let (a, b) be a nonisolated vertex of $\Gamma^*(G)$ at distance d from $(0, 1)$. By Lemma 3.4, there exist integers $q_1, \ldots, q_d$ and $\eta_1, \eta_2 \in \{0, 1\}$ such that

$$\begin{pmatrix} * & * \\ a & b \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & q_d \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & q_2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & q_1 \end{pmatrix} (-1)^{\eta_1} E^{\eta_2}. \tag{3.1}$$

Observe that, for any integers α, β,

$$\begin{pmatrix} 0 & 1 \\ 1 & \alpha \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & \beta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}.$$

Let t be the largest even integer with $t \le d$. From the previous observation, we can rewrite the decomposition in (3.1) as

$$\begin{pmatrix} * & * \\ a & b \end{pmatrix} E^{\eta_2} \begin{pmatrix} 1 & -q_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -q_2 & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & -q_{t-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -q_t & 1 \end{pmatrix} = (-1)^{\eta_1} A \tag{3.2}$$

with

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{if } t \text{ is even, } A = \begin{pmatrix} 0 & 1 \\ 1 & q_d \end{pmatrix} \text{ otherwise.}$$

Define

$$r_0 = b, \quad r_1 = (-1)^{\eta_2} a \quad \text{and} \quad r_{i+2} = r_i - q_{i+1} r_{i+1} \quad \text{for } 0 \le i \le t-1.$$

It follows from (3.2) that $r_{t+1} = 0$ if d is even and $r_{t+1} = (-1)^{\eta_1}$ if d is odd. In the latter case $0 = r_t - q r_{t+1}$, with $q = (-1)^{\eta_1} r_t$, so in both cases $(b, (-1)^{\eta_2} a)$ (and consequently (b, a)) belongs to a Euclidean chain of length at most d. □

Corollary 3.6. *The diameter of $\Gamma^*(\mathbb{Z} \times \mathbb{Z})$ is not finite.*

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Proof. In [3], Lazard proved that the fastest method to obtain the greatest common divisor (gcd) of two integers by means of a succession of generalised divisions consists in taking, at each step, the division $a = bq + r$ with $|r| \leq |b|/2$. In particular, consider the series F_n of the Fibonacci numbers. The shortest Euclidean chain for the pair (F_{2n+1}, F_{2n}) is

$$\begin{aligned} F_{2n+1} &= 2F_{2n} - F_{2n-2}, \\ F_{2n} &= 3F_{2n-2} - F_{2n-4}, \\ F_{2n-2} &= 3F_{2n-4} - F_{2n-6}, \\ &\dots \\ F_4 &= 3F_2 - F_0. \end{aligned}$$

If follows from Lemma 3.5 that the distance in the graph $\Gamma^*(\mathbb{Z} \times \mathbb{Z})$ of the vertex (F_{2n}, F_{2n+1}) from $(0, 1)$ is at least $n + 1$, so the diameter of $\Gamma^*(\mathbb{Z} \times \mathbb{Z})$ is not finite. □

4. The free group of rank 2

In this section, F denotes the free group of rank 2. Recall (see, for example, [5, Theorem 3.2]) that the automorphism group of the free group with ordered basis x_1, x_2 is generated by the elementary Nielsen transformations:

(1) switch x_1 and x_2;
(2) replace x_1 with x_1^{-1} and fix x_2;
(3) replace x_1 with $x_1 \cdot x_2$ and fix x_2.

Lemma 4.1. *Let $F = \langle a_1, a_2 \rangle = \langle b_1, b_2 \rangle$. If there exists an elementary Nielsen transformation γ such that $a_1^\gamma = b_1$ and $a_2^\gamma = b_2$, then a_1 and b_1 (and consequently a_2 and b_2) belong to the same connected component of $\Gamma(F)$.*

Proof. The statement is clearly true if γ is of type (1) or (2). In the third case $b_1 = a_1 \cdot a_2$, so $\langle b_1, a_1 \rangle = \langle a_1, a_2 \rangle = F$. □

Theorem 4.2. *The graph $\Gamma^*(F)$ is connected.*

Proof. Fix x_1, x_2 such that $F = \langle x_1, x_2 \rangle$ and let a be a nonisolated vertex of $\Gamma(F)$. There exists b such that $F = \langle a, b \rangle$ and the map sending x_1 to a and x_2 to b extends to an epimorphism γ of F to itself. Since F is Hopfian (see, for example, [5, Theorem 2.13]), $\gamma \in \mathrm{Aut}(F)$ with $x_1^\gamma = a$ and $x_2^\gamma = b$. Thus, there exist t elementary Nielsen transformations $\gamma_1, \dots, \gamma_t$ such that $\gamma = \gamma_1 \cdots \gamma_t$. By Lemma 4.1, $x_1, x_1^{\gamma_1}$ and, for every $r \in \{2, \dots, t-1\}$, the two elements $x_1^{\gamma_1 \cdots \gamma_{r-1}}$ and $x_1^{\gamma_1 \cdots \gamma_r}$ belong to the same connected component of $\Gamma(F)$. This implies that x_1 and $x_1^\gamma = a$ are in the same connected component. □

Lemma 4.3. *Let $f \in F$. If fF' is a nonisolated vertex of $\Gamma(F/F')$, then there exists a nonisolated vertex f^* of $\Gamma(F)$ such that $f^*F' = fF'$.*

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PROOF. Let $\overline{F} = F/F'$ and, for every $x \in F$, let $\overline{x} = xF'$. Since F' is a characteristic subgroup of F, there exists a natural epimorphism from $\mathrm{Aut}(F) \to \mathrm{Aut}(\overline{F}) \cong \mathrm{GL}(2, \mathbb{Z})$ sending any automorphism γ of F to the induced automorphism $\overline{\gamma}$ of $\overline{F}$. Notice that $\overline{x^\gamma} = \overline{x}^{\overline{\gamma}}$ for every $x \in F$. Fix a generating pair (a, b) of F. Since $\overline{a}$ and $\overline{f}$ are nonisolated vertices of $\overline{F} \cong \mathbb{Z} \times \mathbb{Z}$, there exists $\alpha \in \mathrm{Aut}(\overline{F})$ such that $\overline{f} = \overline{a}^\alpha$. Let $\gamma \in \mathrm{Aut}(F)$ be such that $\overline{\gamma} = \alpha$ and let $f^* = a^\gamma$. Since $\langle f^*, b^\gamma \rangle = F$, the vertex f^* is nonisolated in $\Gamma(F)$. Moreover, $\overline{f^*} = \overline{a^\gamma} = \overline{a}^{\overline{\gamma}} = \overline{a}^\alpha = \overline{f}$. □

COROLLARY 4.4. *The diameter of* $\Gamma^*(F)$ *is not finite.*

PROOF. Let d be any positive integer. By Corollary 3.6, there exist two nonisolated vertices α and β of $\Gamma(F/F')$ such that the distance between α and β in $\Gamma(F/F')$ is at least d. By Lemma 4.3, there exist two nonisolated vertices a and b of $\Gamma(F)$ such that $\alpha = aF'$ and $\beta = bF'$. We claim that the distance between a in b in $\Gamma(F)$ is at least d. Indeed, assume that $x_0 = a, x_1, \ldots, x_{n-1}, x_n = b$ is a path in $\Gamma(F)$ joining a and b. Then $\alpha = x_0F', x_1F', \ldots, x_{n-1}F', x_nF' = \beta$ is a path in $\Gamma(F/F')$ joining α and β and necessarily we have $n \geq d$. □

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