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TWISTED DOUBLING INTEGRALS FOR BRYLINSKI–DELIGNE EXTENSIONS OF CLASSICAL GROUPS

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Abstract We explain how to develop the twisted doubling integrals for Brylinski–Deligne extensions of connected classical groups. This gives a family of global integrals which represent Euler products for this class of nonlinear extensions.

Key words and phrases: Brylinski–Deligne extensions; twisted doubling integrals; tensor product L-functions

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1. Introduction

One of the goals in the Langlands program is to study analytic properties of automorphic L-functions. A basic approach to this problem is to find a global integral that represents the automorphic L-function one would like to study. There are some questions to be addressed in this approach. First, one needs to show that the global integral is Eulerian. For this, one usually uses certain multiplicity one results, such as uniqueness of Whittaker models. To establish the desired properties of the L-functions, one uses the corresponding properties of an Eisenstein series used in the global integral or some spectral theory results such as the Poisson summation formula. For reductive groups, the approach of global integrals is successful in several important cases. The purpose of this article is to describe a family of global integrals for a class of nonlinear covers of reductive groups.

1.1. Brylinski–Deligne covering groups

Let $\mathbf{G}$ be a connected reductive group over a number field F. In [1], Brylinski and Deligne considered the category of multiplicative $\mathbf{K}_2$-torsors on $\mathbf{G}$; these are extensions of $\mathbf{G}$ by the sheaf $\mathbf{K}_2$ of Quillen's K_2 group in the category of sheaves of groups on the big Zariski site of $\mathrm{Spec}(F)$:

$$1 \to \mathbf{K}_2 \to \overline{\mathbf{G}} \to \mathbf{G} \to 1.$$

Brylinski and Deligne gave an elegant and functorial classification of this category in terms of enhanced root-theoretic data, similar to the classification of split connected reductive groups by their root data.

We now assume that the base field F contains a full set of nth roots of unity. Then at every local place v, there is a functor from the category of multiplicative $\mathbf{K}_2$-torsors $\overline{\mathbf{G}}$ on $\mathbf{G}$ to the category of topological central extensions:

$$1 \to \mu_n \to \overline{G}_v \to G_v = \mathbf{G}(F_v) \to 1,$$

which glues to a central extension of the adelic group

$$1 \to \mu_n \to \overline{G}(\mathbb{A}) \to G(\mathbb{A}) \to 1.$$

The global extension is equipped with a natural splitting $G(F) \to \overline{G}(\mathbb{A})$. This naturally leads to the notion of automorphic forms on this class of groups. These topological central extensions may be considered of 'algebraic origin' and can be constructed using cocycles which are essentially algebraic in nature. This construction does not exhaust all topological central extensions, but it captures a sufficiently large class of such extensions, including all interesting examples which have been investigated so far.

1.2. A Langlands program for covering groups

There has been serious progress in extending the Langlands program to the Brylinski–Deligne extensions. We refer to [13] for a comprehensive discussion of the history of covering groups. From our point of view, there are several reasons to study automorphic forms on covering groups. The first is that a Langlands program for covering groups indeed provides supporting evidence to the Langlands program for linear groups. The second is to hope that we can relate automorphic forms on covering groups and linear groups and use this to gain new information regarding linear groups. Indeed, the development of the Langlands program already uses covering groups. A notable example is the Weil representations, defined on double covers of the symplectic groups. Another example is the Rankin–Selberg integral for the symmetric square L-function for GL_n, which uses the theta representations on double covers of general linear groups [2]. The theory of Weyl group multiple Dirichlet series, which is closely tied to Whittaker coefficients of Eisenstein series, has important consequences in analytic number theory.

A natural question is to test whether the myriad global integrals for various L-functions for linear groups have counterparts in the covering case. In the linear case, such a theory relies heavily on the uniqueness of certain models, in particular the Whittaker model. The failure of such multiplicity one results in the covering group case causes serious obstructions to some development for the theory, and it is fundamentally difficult to find integral representations for L-functions for covering groups.

Nevertheless, there are two recent constructions that set up the first step towards this area. The first is the calculation of constant terms of Eisenstein series on Brylinski–Deligne covering groups [10]. A consequence of this calculation is the meromorphic continuation of many interesting L-functions. Another example is a generalisation of the doubling method ([23, 7]) to covers of symplectic groups [16] (see also [6] for a brief description of the

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method). (Also note that the case of the double covers of symplectic groups was already considered in the literature; for example, see [9].) In [16], the unfolding argument and unramified calculation are carried out. As a consequence, the global integral represents the tensor product L-function for a cover of a symplectic group and a cover of a general linear group under certain assumptions. It also has the potential to establish a functional equation for the L-functions obtained.

1.3. The doubling integrals

The doubling integrals grew out of Rallis's work on the inner products of theta lifts – the Rallis inner product formula. This gives a global integral for the standard L-function of a classical group. As a global integral, the doubling integral [23] and its generalisations [7, 16] enjoy the following nice properties:

(1) It uses the matrix coefficient of a representation π of a classical group $G(\mathbb{A})$. As a consequence, this construction works for all cuspidal representations of $G(\mathbb{A})$. This is a special feature of the construction since most of other constructions only work for some but not all cuspidal representations.

(2) The global integral uses certain unique models for some degenerate representations. These representations can be viewed as the generalised Speh representations for covering groups. As we noted above, uniqueness of Whittaker models fails in general for covering groups. However, it is possible that for some 'degenerate' representations on certain covering groups, uniqueness of Whittaker models still holds. A typical example of this type is the theta representation [17]. Establishing the existence and verifying the expected properties are in fact the technical heart of the doubling method.

(3) One can write down a local version of these global integrals and use it to define local factors. Note that in the Langlands–Shahidi method for covering groups, one can define the local coefficients matrix instead of local coefficient, but it is not straightforward to have a definition of local factors. We refer the reader to [15] for recent progress.

1.4. Main results

The purpose of this article is to explain how to develop the twisted doubling integrals (see [7, 16]) for Brylinski–Deligne (BD) extensions of *connected classical groups*. Here, we use the conceptual description in [5], which works for all classical groups uniformly. A major part of this article is to explain the necessary (nontrivial) modifications in order to develop the twisted doubling integrals for covering groups.

We include almost all BD extensions for connected classical groups but have to exclude some cases of unitary groups for some technical reasons (see Remark 8.9). We can invoke some neat structures of BD extensions to minimise the use of cocycles as the construction is functorial in nature. Moreover, we only require the condition $\mu_n \subset F^\times$ (instead of $\mu_{2n} \subset F^\times$) in our construction. Once we have the correct setup, working with BD extensions

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allows us to study some basic properties of twisted doubling integrals just in the linear case. In particular, we carry out the unfolding part following the argument in [5].

The most important part in the setup is to find a suitable BD extension of a large classical group for the construction to work. This part is proved case by case in Section 10. This is relatively easy when the category of BD extensions is 'rigid'. There is additional complication in the non-simply connected and nonsplit case, with the case of $\mathbf{SO}_{2m}$ being the most sophisticated one.

As we noted above, another ingredient in the twisted doubling integrals is the construction of some 'Speh' type representations as the inducing data of some Siegel Eisenstein series. We believe that this is a very difficult question for covering groups, and we only discuss this briefly in Section 11. Indeed, following certain conjectures in [24] and [14], we suggest that one should construct such representations from representations of $\mathrm{GL}_k(\mathbb{A})$, instead of a cover of $\mathrm{GL}_k(\mathbb{A})$. As a consequence, we expect that the twisted doubling integrals represent the tensor product L-functions for $\overline{\mathbf{G}} \times \mathbf{GL}_k$. In the second factor, we only take a linear group instead of a cover of $\mathbf{GL}_k$. This is different from the L-functions obtained in [10] and [16]. It is an interesting question to relate these L-functions.

We also note that in [1], BD covers are only studied for connected reductive groups. So in this article we exclude the orthogonal group $\mathbf{O}_m$ and consider $\mathbf{SO}_m$ instead. (Here we use the term 'connected classical groups' for these groups. Note that we do not assume any condition on the splitness of the classical groups.) We leave other investigations such as unramified calculations, local and global theory as well as the construction of the generalised Speh representations to future work.

1.5. Structure of this article

The rest of this article is organised as follows. We first recall some preliminaries on Brylinski–Deligne covering groups in Sections 2–4. We recall some construction related to the classical groups in Section 5. Section 6 highlights a family of degenerate representations on covers of general linear groups. Even if we do not how to construct the generalised Speh representations for covering groups, we still know the properties that will be used in the global integrals. We then discuss the basic setup of the twisted doubling integrals in the linear case in Section 7 and discuss necessary assumptions for covering groups in Section 8. The long proofs in Section 8 are deferred to Section 10. We then introduce the global integral and prove a global identity in Section 9. We briefly discuss the construction of the generalised Speh representations and the L-functions obtained in Section 11.

1.6. Notation

We now give a list of notation that is commonly used in this article.

- For an algebraic group $\mathbf{G}$ over F, we sometimes write $G(F) = \mathbf{G}(F)$; if F is a local field, we sometimes write $G = G(F)$. If F is a global field, we usually write $[G]$ for $G(F)\backslash G(\mathbb{A})$ where $\mathbb{A}$ is the ring of adeles of F and $G_v = \mathbf{G}(F_v)$ for a local place v.

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- For an algebraic group $\mathbf{G}$ over F and a field extension L/F, we write $\mathbf{G}_L := \mathbf{G} \otimes_F L$ to be the base change of $\mathbf{G}$ to L.
- $J^{\diamondsuit}$: for a subgroup J of G, define $J^{\diamondsuit} = \{(g,g) \in G \times G \mid g \in J\}$.
- $\mathrm{Int}(g)$: for $g \in G$ and a subgroup $H \subset G$, the conjugation map induced by g is denoted $\mathrm{Int}(g) : H \mapsto gHg^{-1}$.
- For an abelian group A with multiplication, we write $[m]$ for the homomorphism $A \to A,\ x \mapsto x^m$.

2. Brylinski–Deligne extensions

Let F be a local or global field of characteristic 0. Let $\mathbf{G}$ be a connected reductive group over F. In this section, we discuss the Brylinski–Deligne extensions of $\mathbf{G}$. The main references for this section are [1] and [11].

2.1. Some structural facts

Let F_s denote a separable closure of F and $\Gamma = \mathrm{Gal}(F_s/F)$. Let $\mathbf{T}$ be a maximal F-torus of $\mathbf{G}$. Let

$$\{X,\ \Phi,;\ Y,\ \Phi^{\vee}\}$$

be the resulting (absolute) root system. Here X (respectively, Y) is the character lattice (respectively, cocharacter lattice) for $(\mathbf{G},\mathbf{T})$ over F_s. Both X and Y are naturally $\mathbb{Z}[\Gamma]$-modules. The group Γ acts on the root system Φ as well. Choose a set $\Delta \subset \Phi$ of simple roots from the set of roots Φ and $\Delta^{\vee}$ the corresponding simple coroots from $\Phi^{\vee}$. Write $Y^{sc} \subset Y$ for the sublattice generated by $\Phi^{\vee}$. Let $\mathbf{B} \subset \mathbf{G}_{F_s}$ be the Borel (F_s-)subgroup determined by Δ.

For each $\alpha \in \Phi$, one has the associated root subgroup $\mathbf{U}_\alpha \subset \mathbf{G}_{F_s}$ which is normalised by $\mathbf{T}_{F_s}$. We shall fix a Chevalley system of épinglage for $(\mathbf{G}_{F_s},\mathbf{T}_{F_s})$, so that for each $\alpha \in \Phi$, one has an F_s-isomorphism $x_\alpha : \mathbf{G}_a \simeq \mathbf{U}_\alpha$.

Denote by $W = W(\mathbf{G}) := N(\mathbf{T}_{F_s})/\mathbf{T}_{F_s}$ the (absolute) Weyl group of $(\mathbf{G},\mathbf{T})$, which we identify with the Weyl group of the coroot system.

2.2. Multiplicative $\mathbf{K}_2$-torsors

The reductive group $\mathbf{G}$ defines a sheaf of groups on the big Zariski site on $\mathrm{Spec}(F)$. Let $\mathbf{K}_2$ denote the sheaf of groups on $\mathrm{Spec}(F)$ associated to the $\mathbf{K}_2$-group in Quillen's K-theory. Then a multiplicative $\mathbf{K}_2$-torsor is an extension

$$1 \to \mathbf{K}_2 \to \overline{\mathbf{G}} \to \mathbf{G} \to 1$$

of sheaves of groups on $\mathrm{Spec}(F)$. We consider the category $\mathbf{CExt}(\mathbf{G},\mathbf{K}_2)$ of such extensions where the morphisms between objects are given by morphisms of extensions. Given two such central extensions, one can form the Baer sum: this equips $\mathbf{CExt}(\mathbf{G},\mathbf{K}_2)$ with the structure of a commutative Picard category.

In [1], Brylinski and Deligne made a deep study of $\mathbf{CExt}(\mathbf{G},\mathbf{K}_2)$ and obtained an elegant classification of this category when $\mathbf{G}$ is a connected reductive group. We first recall their results.

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2.3. Tori

Suppose that $\mathbf{T}$ is an F-torus, with cocharacter lattice $Y = \mathrm{Hom}(\mathbf{G}_m, \mathbf{T}_{F_s})$ and character lattice $X = \mathrm{Hom}(\mathbf{T}_{F_s}, \mathbf{G}_m)$.

Proposition 2.1. *Let $\mathbf{T}$ be a F-torus. The category $\mathbf{CExt}(\mathbf{T}, \mathbf{K}_2)$ is equivalent as a commutative Picard category (by an explicit functor) to the category whose objects are pairs $(Q, \mathcal{E})$, where*

- *Q is a Γ-invariant $\mathbb{Z}$-valued quadratic form on Y, with associated symmetric bilinear form*

$$B_Q(y_1, y_2) = Q(y_1 + y_2) - Q(y_1) - Q(y_2);$$

- *$\mathcal{E}$ is a Γ-equivariant central extension of groups*

$$1 \to F_s^\times \to \mathcal{E} \to Y \to 1$$

whose associated commutator map $[-.-] : Y \times Y \to F_s^\times$ is given by

$$[y_1, y_2] = (-1)^{B_Q(y_1, y_2)}.$$

The set of morphisms between $(Q, \mathcal{E})$ and $(Q', \mathcal{E}')$ is empty unless $Q = Q'$, in which case it is given by the set of isomorphism of Γ-equivariant extensions from $\mathcal{E}$ to $\mathcal{E}'$. The Picard structure is defined by

$$(Q, \mathcal{E}) + (Q', \mathcal{E}') = (Q + Q', \text{ Baer sum of } \mathcal{E} \text{ and } \mathcal{E}').$$

The isomorphism class of the extension $\mathcal{E}$ is completely determined by the commutator map and hence by the quadratic form Q. The extension $\mathcal{E}$ is obtained from $\overline{\mathbf{T}}$ as follows. Let $F_s((\tau))$ denote the field of Laurent series in the variable τ over F_s. Then one has

$$1 \to \mathbf{K}_2(F_s((\tau))) \to \overline{\mathbf{T}}(F_s((\tau))) \to \mathbf{T}(F_s((\tau))) = Y \otimes_{\mathbb{Z}} F_s((\tau))^\times \to 1.$$

The map $y \mapsto y(\tau)$ defines a group homomorphism $Y \to \mathbf{T}(F_s((\tau)))$. Pulling back by this morphism and pushing out by the residue map

$$\mathrm{Res} : \mathbf{K}_2(F_s((\tau))) \to \mathbf{K}_1(F_s) = F_s^\times$$

defined by

$$\mathrm{Res}(f, g) = (-1)^{v(f) \cdot v(g)} \cdot \left(\frac{f^{v(g)}}{g^{v(f)}}(0) \right),$$

one obtains the desired extension $\mathcal{E}$.

2.4. Simply connected groups

Suppose now that $\mathbf{G}$ is a simply connected semisimple group over F. Since $\mathbf{G}$ is simply connected, we have $Y^{sc} = Y$ and $\Delta^\vee$ is a basis for Y.

Proposition 2.2. *The category $\mathbf{CExt}(\mathbf{G}, \mathbf{K}_2)$ is equivalent to (as commutative Picard categories) to the category whose objects are Γ-invariant, W-invariant $\mathbb{Z}$-valued quadratic form Q on Y and whose only morphisms are the identity morphisms on each object.*

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We have the following ([27] Proposition 3.15).

Proposition 2.3. *There is a unique W-invariant quadratic form Q on Y, such that $Q(\alpha^\vee)=1$ for every short coroot $\alpha^\vee \in Y$ and every integer multiple of this quadratic form is Γ-invariant.*

As a result, whenever we are given such a quadratic form Q on Y, Q gives rise to a multiplicative $\mathbf{K}_2$-torsor $\overline{\mathbf{G}}_Q$ on $\mathbf{G}$, unique up to unique isomorphism, which may be pulled back to a multiplicative $\mathbf{K}_2$-torsor $\overline{\mathbf{T}}_Q$ on $\mathbf{T}$ and hence gives rise to an extension $\mathcal{E}_Q$ of Y by $F_s^\times$. The automorphism group of the extension $\mathcal{E}_Q$ is $\mathrm{Hom}_\Gamma(Y,F_s^\times)$. Following [1] Section 11 and [11] Subsection 2.4, one can rigidify $\mathcal{E}_Q$ by giving it an extra structure.

2.5. The Brylinski–Deligne liftings

We assume that $\mathbf{G}$ is simply connected. We also have a fixed Chevalley system of épinglage for $(\mathbf{G}_{F_s},\mathbf{T}_{F_s})$. In particular, for each $\alpha\in\Phi$ with associated root subgroup $\mathbf{U}_\alpha$, there is a fixed isomorphism over F_s:

$$x_\alpha : \mathbf{G}_a \to \mathbf{U}_\alpha \subset \mathbf{G}.$$

Indeed, one has an embedding over F_s

$$i_\alpha : \mathbf{SL}_2 \hookrightarrow \mathbf{G}$$

which restricts to $x_{\pm\alpha}$ on the upper and lower triangular subgroups of unipotent matrices. By [1], one has a canonical lifting

$$\tilde{x}_\alpha : \mathbf{G}_a \to \overline{\mathbf{U}}_\alpha \subset \overline{\mathbf{G}}.$$

For $t\in\mathbf{G}_m$, we set

$$n_\alpha(t) = x_\alpha(t)\cdot x_{-\alpha}(-t^{-1})\cdot x_\alpha(t) = i_\alpha\begin{pmatrix} 0 & t \\ -t^{-1} & 0\end{pmatrix} \in N(\mathbf{T}_Q)$$

and

$$\tilde{n}_\alpha(t) = \tilde{x}_\alpha(t)\cdot \tilde{x}_{-\alpha}(-t^{-1})\cdot \tilde{x}_\alpha(t) \in \overline{\mathbf{G}}_Q.$$

Then one has a map

$$s_\alpha : \mathbf{T}_\alpha := \alpha^\vee(\mathbf{G}_m) \to \overline{\mathbf{T}}_Q$$

given by

$$\alpha^\vee(t) \mapsto \tilde{n}_\alpha(t)\cdot\tilde{n}_\alpha(-1).$$

This is a section of $\overline{\mathbf{G}}_Q$ over $\overline{\mathbf{T}}_\alpha$ which is trivial at the identity element. This section is useful in describing the natural conjugation action of $N(\mathbf{T}_Q)$ on $\overline{\mathbf{T}}_Q$. By [1] Proposition 11.3, one has the nice formula

$$\tilde{n}_\alpha(1)\cdot\tilde{t}\cdot\tilde{n}_\alpha(1)^{-1} = \tilde{t}\cdot s_\alpha(\alpha^\vee(\alpha(t)^{-1})). \tag{1}$$

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We also use the following formula frequently ([10] (2)): for $t_1,t_2 \in \mathbf{G}_m$,

$$s_\alpha(\alpha^\vee(t_1)) \cdot s_\alpha(\alpha^\vee(t_2)) = s_\alpha(\alpha^\vee(t_1 t_2)) \cdot \{t_1,t_2\}^{Q(\alpha^\vee)}. \tag{2}$$

The collection of sections $\{s_\alpha : \alpha \in \Delta\}$ provides a collection of elements $s_\alpha(\alpha^\vee(a)) \in \mathbf{T}_Q$ with $a \in \mathbf{G}_m$ and $\overline{\mathbf{T}}_Q$ is generated by $\mathbf{K}_2$ and the collection of $s_\alpha(\alpha^\vee(a))$.

Taking points in $F((\tau))$, we have the element

$$s_\alpha(\alpha^\vee(\tau)) \in \overline{\mathbf{T}}_Q(F((\tau))),$$

which gives rise (via the construction of $\mathcal{E}_Q$) to an element

$$s_Q(\alpha^\vee) \in \mathcal{E}_Q.$$

Then we rigidify $\mathcal{E}_Q$ by equipping it with the set $\{s_Q(\alpha^\vee) \mid \alpha^\vee \in \Delta^\vee\}$: there is a unique automorphism of $\mathcal{E}_Q$ which fixes all of these elements.

In the following, we shall fix a choice of the data $(\overline{\mathbf{G}}_Q, \overline{\mathbf{T}}_Q, \mathcal{E}_Q)$ for each Γ-invariant, W-invariant quadratic form Q on Y when $\mathbf{G}$ is simply connected.

2.6. Weyl group action on Brylinski–Deligne liftings

Observe that $s_Q(\alpha^\vee)$ can be defined for every coroot (not necessarily simple coroot). We would like know how $s_Q(\alpha^\vee)$ behaves under the action of the Weyl group. Recall that for every $\alpha \in \Phi$, one can choose $w_\alpha := n_\alpha(1) \in \mathbf{G}(F_s)$ as a representative of the simple reflection $\mathbf{w}_\alpha$ corresponding to α. Let $\mathbf{w} = \mathbf{w}_{\alpha_1} \cdots \mathbf{w}_{\alpha_\ell}$ be a minimum decomposition of $\mathbf{w}$; we choose the following representative of $\mathbf{w}$:

$$w = w_{\alpha_1} \cdots w_{\alpha_\ell} \in \mathbf{G}(F_s).$$

This is independent of the minimum decomposition of $\mathbf{w}$. We choose $\tilde{w}_\alpha = \tilde{n}_\alpha(1)$ as a lift of w_α in $\mathbf{G}(F_s)$. In any case, the conjugation action of $\overline{\mathbf{G}}(F_s)$ on $\overline{\mathbf{G}}(F_s)$ descends to an action of $\mathbf{G}(F_s)$.

Fix a pair of roots α and β. Then we have a homomorphism

$$\mathrm{Int}(w_\alpha) \circ x_\beta = w_\alpha \cdot x_\beta \cdot w_\alpha^{-1} : \mathbf{G}_a \to \mathbf{U}_{\mathbf{w}_\alpha(\beta)}.$$

From [3] Subsection 3.2.2, there is a sign $\epsilon_{\alpha,\beta} \in \{\pm 1\}$ associated to the Chevalley system of épinglage such that

$$w_\alpha \cdot x_\beta(t) \cdot w_\alpha^{-1} = x_{\mathbf{w}_\alpha(\beta)}(\epsilon_{\alpha,\beta} t), \qquad t \in \mathbf{G}_m.$$

This implies that for $t \in \mathbf{G}_m$,

$$\tilde{w}_\alpha \cdot \tilde{x}_\beta(t) \cdot \tilde{w}_\alpha^{-1} = \tilde{x}_{\mathbf{w}_\alpha(\beta)}(\epsilon_{\alpha,\beta} t). \tag{3}$$

Lemma 2.4. *For $t \in \mathbf{G}_m$,*

$$\tilde{w}_\alpha \cdot s_\beta(\beta^\vee(t)) \cdot \tilde{w}_\alpha^{-1} = \{\epsilon_{\alpha,\beta}, t\}^{Q(\beta^\vee)} \cdot s_{\mathbf{w}_\alpha(\beta)}(\mathbf{w}_\alpha(\beta)^\vee(t)).$$

Proof. The calculation here is identical to [10] Corollary 7.4. For $t \in \mathbf{G}_m$, we have

$$\tilde{w}_\alpha \cdot \tilde{n}_\beta(t) \cdot \tilde{w}_\alpha^{-1} = (\tilde{w}_\alpha \cdot \tilde{x}_\beta(t) \cdot \tilde{w}_\alpha^{-1})(\tilde{w}_\alpha \cdot \tilde{x}_{-\beta}(-t^{-1}) \tilde{w}_\alpha^{-1})(\tilde{w}_\alpha \cdot \tilde{x}_\beta(t) \cdot \tilde{w}_\alpha^{-1}).$$

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By (3), this is

$$\tilde{x}_{\mathbf{w}_\alpha(\beta)}(\epsilon_{\alpha,\beta}t)\cdot\tilde{x}_{-\mathbf{w}_\alpha(\beta)}(-\epsilon_{\alpha,\beta}t^{-1})\cdot\tilde{x}_{\mathbf{w}_\alpha(\beta)}(\epsilon_{\alpha,\beta}t)=\tilde{n}_{\mathbf{w}_\alpha(\beta)}(\epsilon_{\alpha,\beta}t).$$

From this we deduce that for any $t\in\mathbf{G}_m$,

$$\begin{aligned}\tilde{w}_\alpha\cdot\tilde{n}_\beta(t)\cdot\tilde{n}_\beta(-1)\cdot\tilde{w}_\alpha^{-1}&=\tilde{n}_{\mathbf{w}_\alpha(\beta)}(\epsilon_{\alpha,\beta}t)\cdot\tilde{n}_{\mathbf{w}_\alpha(\beta)}(-\epsilon_{\alpha,\beta})\\&=\{\epsilon_{\alpha,\beta},t\}^{Q(\beta^\vee)}s_{\mathbf{w}_\alpha(\beta)}(\mathbf{w}_\alpha(\beta)^\vee(t)).\end{aligned}$$

The last equality follows from [10] (3). □

Lemma 2.5. *We have the following results.*

(1) *If* $\langle\alpha,\beta^\vee\rangle=-1$, *we have*

$$s_Q(\alpha^\vee+\beta^\vee)=s_Q(\beta^\vee)\cdot s_Q(\alpha^\vee)\cdot\epsilon_{\alpha,\beta}^{Q(\beta^\vee)}.$$

(2) *We have* $s_Q(-\alpha^\vee)\cdot s_Q(\alpha^\vee)=1$.

Proof. On the one hand, $\tilde{w}_\alpha\cdot s_\beta(\beta^\vee(\tau))\cdot\tilde{w}_\alpha^{-1}=s_{\mathbf{w}_\alpha(\beta)}(\mathbf{w}_\alpha(\beta)^\vee(\tau))\cdot\{\epsilon_{\alpha,\beta},\tau\}^{Q(\beta^\vee)}$. On the other hand, from (1), we obtain

$$\begin{aligned}&\tilde{w}_\alpha\cdot s_\beta(\beta^\vee(\tau))\cdot\tilde{w}_\alpha^{-1}=s_\beta(\beta^\vee(\tau))\cdot s_\alpha(\alpha^\vee(\alpha(\beta^\vee(\tau))^{-1}))\\=&s_\beta(\beta^\vee(\tau))\cdot s_\alpha(\alpha^\vee(\tau^{-\langle\alpha,\beta^\vee\rangle}))=s_\beta(\beta^\vee(\tau))\cdot s_\alpha(\alpha^\vee(\tau)).\end{aligned}$$

Observe that under the map $\mathbf{K}_2(F)\to\mathbf{K}_1(F)=F^\times$, $\mathrm{Res}\{\epsilon_{\alpha,\beta},\tau\}=\epsilon_{\alpha,\beta}$. Now the first statement follows.

The argument for the second statement is similar. We also need the following consequences of (2):

$$\begin{aligned}s_\alpha(\alpha^\vee(\tau))\cdot s_\alpha(\alpha^\vee(\tau^{-1}))=&\{\tau,\tau^{-1}\}^{Q(\alpha^\vee)}.\\s_\alpha(\alpha^\vee(\tau))\cdot s_\alpha(\alpha^\vee(\tau^{-2}))=&s_\alpha(\alpha^\vee(\tau^{-1}))\{\tau,\tau^{-2}\}^{Q(\alpha^\vee)}.\end{aligned}$$

Note that

$$\tilde{w}_\alpha\cdot s_\alpha(\alpha^\vee(\tau))\cdot\tilde{w}_\alpha^{-1}=s_{-\alpha}(-\alpha^\vee(\tau))\cdot\epsilon_{\alpha,-\alpha}^{Q(\alpha^\vee)}=s_{-\alpha}(-\alpha^\vee(\tau))\cdot(-1)^{Q(\alpha^\vee)}.$$

Again, by (1),

$$\tilde{w}_\alpha\cdot s_\alpha(\alpha^\vee(\tau))\cdot\tilde{w}_\alpha^{-1}=s_\alpha(\alpha^\vee(\tau))\cdot s_\alpha(\alpha^\vee(\tau^{-2}))=s_\alpha(\alpha^\vee(\tau^{-1}))\cdot\{\tau,\tau^{-2}\}^{Q(\alpha^\vee)}.$$

From these two equations, we deduce that

$$s_{-\alpha}(-\alpha^\vee(\tau))\cdot(-1)^{Q(\alpha^\vee)}=s_\alpha(\alpha^\vee(\tau^{-1}))\cdot\{\tau,\tau^{-2}\}^{Q(\alpha^\vee)}.$$

We now have

$$s_{-\alpha}(-\alpha^\vee(\tau))\cdot s_\alpha(\alpha^\vee(\tau))=\{\tau,\tau^{-1}\}^{Q(\alpha^\vee)}\cdot(-1)^{Q(\alpha^\vee)}\cdot\{\tau,\tau^{-2}\}^{Q(\alpha^\vee)}.$$

From this we deduce that

$$s_Q(\alpha^\vee)\cdot s_Q(-\alpha^\vee)=1.$$

□

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We write $\alpha^\vee$ as a sum of simple coroots:

$$\alpha^\vee = \alpha_{i_1}^\vee + \cdots + \alpha_{i_n}^\vee.$$

We say that this expression has property $(*)$ if the following holds:

$$(*): \alpha_{i_1}^\vee,\ \alpha_{i_1}^\vee + \alpha_2^\vee,\ \cdots,\ \alpha_{i_1}^\vee + \cdots + \alpha_{i_n}^\vee \in \Phi^\vee. \tag{4}$$

Lemma 2.6. *Assume that* $\epsilon_{\alpha,\beta}^{Q(\beta^\vee)} = 1$ *and* $\langle \alpha, \beta^\vee \rangle = -1$ *for all* α, β. *If we write* $\alpha^\vee$ *as a sum of simple coroots such that property* $(*)$ *holds, then*

$$s_Q(\alpha^\vee) = s_Q(\alpha_{i_n}^\vee) \cdots s_Q(\alpha_{i_2}^\vee) \cdot s_Q(\alpha_{i_1}^\vee).$$

Proof. This follows from Lemma 2.5 and property $(*)$ by induction. □

2.7. General reductive groups

Let $\mathbf{G}$ be a connected reductive group over F, with a fixed Chevalley system of épinglage for $(\mathbf{G}, \mathbf{T})$. Let $X^{sc} \subset X \otimes_{\mathbb{Z}} \mathbb{Q}$ be the dual lattice of Y^{sc}. Then the quadruple $(X^{sc}, \Delta, Y^{sc}, Y)$ is the root system of the simply connected cover $\mathbf{G}^{sc}$ of the derived group $\mathbf{G}^{der}$ of $\mathbf{G}$ and one has a natural map

$$q: \mathbf{G}^{sc} \to \mathbf{G}^{der} \to \mathbf{G}.$$

Let $\mathbf{T}^{sc}$ be the preimage of $\mathbf{T}$ in $\mathbf{G}^{sc}$. It is a maximal F-torus of $\mathbf{G}^{sc}$ with cocharacter group $Y^{sc} \subset Y$, so that one has a commutative diagram

$$\begin{array}{ccc} \mathbf{T}^{sc} & \longrightarrow & \mathbf{G}^{sc} \\ \downarrow & & \downarrow \\ \mathbf{T} & \longrightarrow & \mathbf{G} \end{array}$$

The restriction $Q^{sc} := Q|_{Y^{sc}}$ gives an element $\overline{\mathbf{G}}^{sc} \in \mathbf{CExt}(\mathbf{G}^{sc}, \mathbf{K}_2)$. It also gives the extension $\mathcal{E}^{sc}_{Q|_{Y^{sc}}}$. For simplicity, we just write $\mathcal{E}_{Q^{sc}}$ with no confusion caused.

Theorem 2.7. *The category* $\mathbf{CExt}(\mathbf{G}, \mathbf{K}_2)$ *is equivalent to the category* $\mathbf{BD}(\mathbf{G}, \mathbf{T})$ *whose objects are triples* $(Q, \mathcal{E}, f)$, *where*

- $Q: Y \to \mathbb{Z}$ *is a* Γ*-invariant* W*-invariant quadratic form;*
- $\mathcal{E}$ *is a* Γ*-equivariant extension of* Y *by* $F_s^\times$ *with commutator map* $[y_1, y_2] = (-1)^{B_Q(y_1, y_2)}$;
- f *is a* Γ*-equivariant morphism from* $\mathcal{E}_{Q^{sc}}$ *to* $\mathcal{E}$ *such that the following diagram commutes:*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & F_s^\times & \longrightarrow & \mathcal{E}_{Q^{sc}} & \longrightarrow & Y^{sc} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & F_s^\times & \longrightarrow & \mathcal{E} & \longrightarrow & Y & \longrightarrow & 1 \end{array} \tag{5}$$

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The set of morphisms from $(Q,\mathcal{E},f)$ *to* $(Q',\mathcal{E}',f')$ *is empty unless* $Q=Q'$*, in which case it consists of* Γ*-equivariant isomorphisms of extensions* $\phi:\mathcal{E}\to\mathcal{E}'$ *such that* $f=f'\circ\phi$*.*

3. Topological covering groups

We now pass from the algebro-geometric world of multiplicative $\mathbf{K}_2$-torsors to the world of topological central extensions. We first assume that F is a local field. If F is non-Archimedean, let $\mathcal{O}$ denote its ring of integers and let p be the residue characteristic.

3.1. BD covering groups

Start with a multiplicative $\mathbf{K}_2$-torsor $\overline{\mathbf{G}}$ on $\mathbf{G}$, with associated BD data $(Q,\mathcal{E},f)$. Since $H^1(F,\mathbf{K}_2)=0$, by taking F-points, we obtain a short exact sequence of abstract groups

$$1\to\mathbf{K}_2(F)\to\overline{\mathbf{G}}(F)\to G=\mathbf{G}(F)\to 1.$$

Let $\mu(F)$ denote the set of roots of unity contained in the local field $F\neq\mathbb{C}$; when $F=\mathbb{C}$, we let $\mu(F)$ be the trivial group. Then the Hilbert symbol gives a map

$$(-,-)_F:\mathbf{K}_2(F)\to\mu(F).$$

For any n dividing $\#\mu(F)$, one has the nth Hilbert symbol

$$(-,-)_n:(-,-)_F^{\#\mu(F)/n}:\mathbf{K}_2(F)\to\mu_n(F).$$

By pushing out the exact sequence via the Hilbert symbol $\mathbf{K}_2(F)\to\mu_n(F)$, we obtain an exact sequence of locally compact topological groups

$$1\to\mu_n(F)\to\overline{G}\to G\to 1.$$

We call this the BD covering group associated to the BD data $(Q,\mathcal{E},f,n)$.

3.2. Unipotent subgroups

Let $\mathcal{N}_G$ be the set of all unipotent elements of G. Because a BD extension is uniquely split over any unipotent subgroup, one has unique splittings

$$\tilde{x}_\alpha:F\to\overline{U}_\alpha \text{ for each } \alpha\in\Phi.$$

Indeed, as shown in [22] Appendix I and [18] Proposition 2.2.1, there is a unique section

$$i:\mathcal{N}_G\to\overline{G}$$

satisfying the following:

- for each unipotent subgroup $\mathbf{U}\subset\mathbf{G}$, the restriction of i to $U=\mathbf{U}(F)$ is a group homomorphism;
- the map i is G-equivariant.

3.3. Tori

The following result is a consequence of [1] Proposition 3.13.

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Proposition 3.1. *Let L be any field containing F over which $\mathbf{T}$ splits. Let $\overline{\mathbf{T}}(L)$ be the resulting central extension*

$$1 \to \mathbf{K}_2(L) \to \overline{\mathbf{T}}(L) \to \mathbf{T}(L) \to 1.$$

Then the commutator of this extension satisfies

$$\mathrm{Comm}(y_1(u_1), y_2(u_2)) = \{u_1, u_2\}^{B_Q(y_1, y_2)},$$

for all $y_1, y_2 \in \mathrm{Hom}(\mathbf{G}_m, \mathbf{T}_L)$ and all $u_1, u_2 \in L^\times$.

We would like to note the following useful observation. If $\mathbf{T} = \mathbf{T}_1 \times \mathbf{T}_2$, then $Y = Y_1 \oplus Y_2$. We have the following consequence.

Lemma 3.2. *If $B_Q(y_1, y_2) = 0$ for all $y_1 \in Y_1, y_2 \in Y_2$, then $\overline{T}_1$ and $\overline{T}_2$ commute in $\overline{T}$.*

Proof. This is an immediate consequence of Proposition 3.1. □

Let $\mathbf{G} = \mathbf{G}_1 \times \mathbf{G}_2$ with maximal torus $\mathbf{T} = \mathbf{T}_1 \times \mathbf{T}_2$. Then there is a corresponding decomposition of cocharacter lattice $Y = Y_1 \oplus Y_2$. Let $\overline{\mathbf{G}} \in \mathbf{CExt}(\mathbf{G}, \mathbf{K}_2)$. This gives $\overline{\mathbf{G}}_i \in \mathbf{CExt}(\mathbf{G}_i, \mathbf{K}_2)$ for $i = 1, 2$. We have inclusions $\overline{G}_i \to \overline{G}, i = 1, 2$.

Lemma 3.3. *If $B_Q(y_1, y_2) = 0$ for all $y_1 \in Y_1, y_2 \in Y_2$, then $\overline{G}_1$ and $\overline{G}_2$ commute in $\overline{G}$.*

Proof. The group $\overline{G}$ is generated by $\overline{T}$ and $\mathcal{N}_G$. We only have to verify the following:

(1) $\overline{T}_1$ and $\overline{T}_2$ commute;
(2) $\overline{T}_1$ and $\mathcal{N}_{G_2}$ commute;
(3) $\overline{T}_2$ and $\mathcal{N}_{G_1}$ commute;
(4) $\mathcal{N}_{G_1}$ and $\mathcal{N}_{G_2}$ commute;

The first statement is simply Lemma 3.2. The second follows from the fact that T_1 and $\mathcal{N}_{G_2}$ commute in G and the unipotent section is G-equivariant. The rest is similar. □

3.4. The tame case

We now discuss the splitting of maximal compact subgroups at unramified places.

Let F be a non-Archimedean field with ring of integers $\mathcal{O}$. Suppose that the group $\mathbf{G}$ has an integral model $\underline{G}$ over $\mathcal{O}$. The $\mathbf{K}_2$-extension $\overline{\mathbf{G}}$ might not be defined over $\mathcal{O}$. If it is, then there is a natural splitting of $\overline{G}$ over $K = \mathbf{G}(\mathcal{O})$.

The $\mathbf{K}_2$-extension $\overline{\mathbf{G}}$ yields a short exact sequence

$$1 \to \mathbf{K}_2(F) \to \overline{\mathbf{G}}(F) \to \mathbf{G}(F) \to 1.$$

If $p \nmid n$, then this 'tameness' gives an exact sequence

$$1 \to \mathbf{K}_2(\mathcal{O}) \to \mathbf{K}_2(F) \xrightarrow{\mathrm{Hilb}_n} \mu_n \to 1.$$

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This gives a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbf{K}_2(\mathcal{O}) & \longrightarrow & \overline{\mathbf{G}}(\mathcal{O}) & \longrightarrow & \mathbf{G}(\mathcal{O}) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mu_n & \longrightarrow & \overline{G} & \longrightarrow & G & \longrightarrow & 1 \end{array}.$$

Thus, the central extension $1 \to \mu_n \to \overline{G} \to G \to 1$ is endowed with a splitting over the hyperspecial maximal compact subgroup $\mathbf{G}(\mathcal{O})$.

3.5. Adelic BD covering

In this section, F is a global field. For a place v of F, we write F_v for the completion of F at v.

Starting with a BD extension $\overline{\mathbf{G}}$ over $\mathrm{Spec}(F)$ and a positive integer n such that $|\mu_n(F)| = n$, Brylinski and Deligne showed using results of [20] that one inherits the following data:

- for each place v of k, a local BD covering group $\overline{G}_v$ of degree n;
- for almost all v, a splitting $s_v : \mathbf{G}(\mathcal{O}_v) \to \overline{G}_v$;
- a restricted direct product $\prod_v' \overline{G}_v$ with respect to the family of subgroups $s_v(\mathbf{G}(\mathcal{O}_v))$, from which one can define

$$\overline{G}(\mathbb{A}) := \prod_v{}' \overline{G}_v / \{(\zeta_v) \in \oplus_v \mu_n(k_v) : \prod_v \zeta_v = 1\},$$

  which gives a topological central extension

$$1 \to \mu_n(F) \to \overline{G}(\mathbb{A}) \to G(\mathbb{A}) \to 1,$$

  called the adelic or global BD covering group;
- a natural inclusion

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mu_n(F_v) & \longrightarrow & \overline{G}_v & \longrightarrow & \mathbf{G}(F_v) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mu_n(F) & \longrightarrow & \overline{G}(\mathbb{A}) & \longrightarrow & \mathbf{G}(\mathbb{A}) & \longrightarrow & 1 \end{array}$$

  for each place v of k;
- a natural splitting

$$i : \mathbf{G}(F) \to \overline{G}(\mathbb{A}),$$

  which allows one to consider the space of automorphic forms on $\overline{G}(\mathbb{A})$.

In this article, we fix an embedding $\epsilon : \mu_n \to \mathbb{C}^\times$. We say a representation π of $\overline{G}(\mathbb{A})$ is ϵ-genuine if μ_n acts via ϵ.

We briefly recall how the splitting i is obtained. Let $X = \mathrm{Spec}(\mathcal{O}_F)$. Let S_1 be a finite set of finite places of F. We assume that S_1 is large enough so that the conclusion of [1]

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Lemma 10.6 holds. Write $S = S_1 \cup \{\text{infinite places}\}$. This gives a central extension

$$1 \to \mathrm{H}^0(X - S_1, \mathbf{K}_2) \to E_1 \to G(X - S_1) \to 1.$$

For v a place of F, it maps to the local central extension

$$1 \to \mu_n \to \overline{G}_v \to G_v \to 1.$$

For an unramified place v, the map factors through a central extension

$$1 \to \mathbf{K}_2(\mathcal{O}_v) \to \overline{\mathbf{G}(\mathcal{O}_v)} \to \mathbf{G}(\mathcal{O}_v) \to 1.$$

If $p \nmid n$, the exact sequence

$$1 \to \mathbf{K}_2(\mathcal{O}_v) \to \mathbf{K}_2(F_v) \to F_v^{\times} \to 1$$

shows that $\mathbf{K}_2(\mathcal{O}_v)$ maps trivially to μ_n. We obtain a trivialisation of $\overline{G}_v$ over $\mathbf{G}(\mathcal{O}_v)$.

We now have a commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathrm{H}^0(X - S_1, \mathbf{K}_2) & \longrightarrow & E_1 & \longrightarrow & G(X - S_1) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \prod_{v \in S} \mu_v & \longrightarrow & \prod_{v \notin S} \overline{G}_v \times \prod_{v \in X - S_1} \mathbf{G}(\mathcal{O}_v) & \to & \prod_{v \notin S} G_v \times \prod_{v \in X - S_1} \mathbf{G}(\mathcal{O}_v) & \longrightarrow & 1
\end{array}$$

provided that for all v in $X - S_1$, $p \nmid n$. This holds for S_1 is large enough. The first vertical map, composed with the reciprocity map $\prod \zeta_v$ with values in μ_n, vanishes. We hence obtain

$$\begin{array}{ccccccccc}
 & & & & & & G(X - S_1) & & \\
 & & & & & \swarrow & \downarrow & & \\
1 & \longrightarrow & \mu_n & \longrightarrow & \overline{\prod_{v \notin S} G_v} \times \prod_{v \in X - S_1} \mathbf{G}(\mathcal{O}_v) & \longrightarrow & \prod_{v \notin S} G_v \times \prod_{v \in X - S_1} \mathbf{G}(\mathcal{O}_v) & \longrightarrow & 1
\end{array}$$

Taking direct limit over S gives the desired natural splitting:

$$\begin{array}{ccccccccc}
 & & & & & & G(F) & & \\
 & & & & & \swarrow & \downarrow & & \\
1 & \longrightarrow & \mu_n & \longrightarrow & \overline{G}(\mathbb{A}) & \longrightarrow & G(\mathbb{A}) & \longrightarrow & 1
\end{array}$$

4. Pullback, pushout and Baer sum

We now discuss several constructions that give new exact sequences: pullback, pushout and the Baer sum. In this section, we would like to describe these constructions in terms of the BD data.

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4.1. Pushout

We now recall the definition of pushout.

Definition 4.1. For a central extension

$$1 \to A \xrightarrow{i} E \xrightarrow{p} G \to 1$$

and a homomorphism $f : A \to B$ of abelian groups, we define

$$f_*(E) := (B \times E)/\langle (f(a), i(a)^{-1}) : a \in A \rangle.$$

The maps $B \to f_*(E), b \mapsto (b,1)$ and $f_*(E) \to G, (b,e) \mapsto p(e)$ define an exact sequence

$$1 \to B \to f_*(E) \to G \to 1.$$

This exact sequence is called the pushout by f.

4.2. Baer sum

Another method to construct new exact sequences is the Baer sum. In this article, we only consider the Baer sum of n copies of an exact sequence.

Consider an exact sequence

$$1 \to A \to E \to G \to 1 \tag{6}$$

with A abelian. By taking the direct sum of n copies of the exact sequence (6), we obtain

$$1 \to \oplus_{i=1}^n A \to \oplus_{i=1}^n E \to \oplus_{i=1}^n G \to 1.$$

By pushing out the exact sequence via the product map

$$\mathrm{pr} : \prod_{i=1}^n A \to A, \qquad (x_i) \mapsto \prod_{i=1}^n x_i,$$

we obtain an exact sequence:

$$1 \to A \to \mathrm{pr}_*(\oplus_{i=1}^n E) \to \oplus_{i=1}^n G \to 1.$$

Now we pull back this exact sequence via the diagonal map

$$d : G \to \oplus_{i=1}^n G, \qquad x \mapsto (x, \cdots, x)$$

to obtain

$$1 \to A \to d^*(p_* \oplus_{i=1}^n E) \to G \to 1.$$

This exact sequence is the Baer sum of n copies of (6).

We now claim that this is also the same as pushing out (6) by the map $[n] : A \to A$, $x \mapsto x^n$.

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Lemma 4.2. *The following commutative diagram gives an isomorphism of exact sequences:*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & A & \longrightarrow & [n]_*(E) & \longrightarrow & G & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & A & \longrightarrow & d^*(p_* \oplus_{i=1}^n E) & \longrightarrow & G & \longrightarrow & 1 \end{array}$$

Proof. Recall that $[n]_*(E) = (A \times E)/\langle x^n, i^{-1}(x) \mid x \in A\rangle$ and

$$p_*(\oplus_{i=1}^n E) = A \times (\oplus_{i=1}^n E)/\langle \prod_{i=1}^n x_i, (i^{-1}(x_1), \cdots, i^{-1}(x_n)) \mid (x_1, \cdots, x_n) \in \oplus_{i=1}^n A\rangle.$$

We now define $[n]_*(E) \to d^*(p_* \oplus_{i=1}^n E)$ by

$$(a, e) \mapsto (a, (e, \cdots, e)).$$

It is straightforward to check that this is well-defined and is an isomorphism of exact sequences. □

4.3. Functoriality of pullback

Let $f : \mathbf{G} \to \mathbf{H}$ be a morphism of connected reductive groups. Let

$$1 \to \mathbf{K}_2 \to \overline{\mathbf{H}} \to \mathbf{H} \to 1$$

be a multiplicative $\mathbf{K}_2$-torsor on $\mathbf{H}$. By pulling back via f, we obtain a multiplicative $\mathbf{K}_2$-torsor on $\mathbf{G}$:

$$1 \to \mathbf{K}_2 \to f^*(\overline{\mathbf{H}}) \to \mathbf{G} \to 1.$$

For ease of notation, let us write $\overline{\mathbf{G}} = f^*(\overline{\mathbf{H}})$. Thus, this fits into a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbf{K}_2 & \longrightarrow & \overline{\mathbf{G}} & \longrightarrow & \mathbf{G} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathbf{K}_2 & \longrightarrow & \overline{\mathbf{H}} & \longrightarrow & \mathbf{H} & \longrightarrow & 1 \end{array}$$

and gives a functor

$$\mathbf{CExt}(\mathbf{H}, \mathbf{K}_2) \to \mathbf{CExt}(\mathbf{G}, \mathbf{K}_2). \tag{7}$$

At every local place v, the pullback determines the following data:

- At every local place v, we have a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mu_n & \longrightarrow & \overline{G}_v & \longrightarrow & G_v & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mu_n & \longrightarrow & \overline{H}_v & \longrightarrow & H_v & \longrightarrow & 1 \end{array}$$

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- The commutative diagram is compatible with the lift of unipotent elements. In other words, the following diagram commutes:

$$\begin{array}{ccc} \mathcal{N}_{G_v} & \longrightarrow & \overline{G}_v \\ \downarrow & & \downarrow \\ \mathcal{N}_{H_v} & \longrightarrow & \overline{H}_v \end{array}$$

- In the tame case, the commutative diagram is compatible with the natural lift of maximal compact subgroups:

$$\begin{array}{ccc} \mathbf{G}(\mathcal{O}_v) & \longrightarrow & \overline{G}_v \\ \downarrow & & \downarrow \\ \mathbf{H}(\mathcal{O}_v) & \longrightarrow & \overline{H}_v \end{array}$$

We now move to the global setup. So from now on, F is a global field. The local homomorphisms glue to

$$\prod_v \overline{G}_v \to \prod_v \overline{H}_v.$$

As $f_v(s_v(\mathbf{G}(\mathcal{O}_v))) \subset s_v(\mathbf{H}(\mathcal{O}_v))$, we obtain a homomorphism

$$\prod_v{}' \overline{G}_v \to \prod_v{}' \overline{H}_v \to \overline{H}(\mathbb{A}).$$

This map factors through

$$f_{\mathbb{A}} : \overline{G}(\mathbb{A}) \to \overline{H}(\mathbb{A}).$$

From the construction of the natural splitting $H(F) \to \overline{H}(\mathbb{A})$, it is not hard to check that this is compatible with the splitting over rational points. In other words, the diagram

$$\begin{array}{ccc} \mathbf{G}(F) & \longrightarrow & \overline{G}(\mathbb{A}) \\ \downarrow & & \downarrow \\ \mathbf{H}(F) & \longrightarrow & \overline{H}(\mathbb{A}) \end{array}$$

commutes.

We now describe the functor (7) in terms of the BD data. Let $\mathbf{T}_G$ and $\mathbf{T}_H$ be maximal F-tori of $\mathbf{G}$ and $\mathbf{H}$, respectively. We assume that $f(\mathbf{T}_G) \subset \mathbf{T}_H$. The map $\mathbf{T}_G \to \mathbf{T}_H$

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induces a map $Y_G \to Y_H$, which gives $\mathcal{E}_G$ as the pullback of $\mathcal{E}_H$ via $Y_G \to Y_H$:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & F_s^\times & \longrightarrow & \mathcal{E}_G & \longrightarrow & Y_G & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & F_s^\times & \longrightarrow & \mathcal{E}_H & \longrightarrow & Y_H & \longrightarrow & 1 \end{array}. \tag{8}$$

The homomorphism $\mathbf{G} \to \mathbf{H}$ also determines a map $\mathbf{G}^{sc} \to \mathbf{H}^{sc}$. This gives a map $\mathcal{E}_{\mathbf{G}^{sc}} \to \mathcal{E}_{\mathbf{H}^{sc}}$. It is easy to check that the image of $\mathcal{E}_{\mathbf{G}^{sc}} \to \mathcal{E}_{\mathbf{H}^{sc}} \to \mathcal{E}_H$ agrees with the image of $\mathcal{E}_G \to \mathcal{E}_H$, which gives a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & F_s^\times & \longrightarrow & \mathcal{E}_{\mathbf{G}^{sc}} & \longrightarrow & Y_G^{sc} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & F_s^\times & \longrightarrow & \mathcal{E}_G & \longrightarrow & Y_G & \longrightarrow & 1 \end{array}. \tag{9}$$

Proposition 4.3. *With notation as above, the functor (7) can be described in terms of BD data:*

$$\mathbf{BD}(\mathbf{H},\mathbf{T}_H) \to \mathbf{BD}(\mathbf{G},\mathbf{T}_G), \qquad (Q_H,\mathcal{E}_H,f_H) \mapsto (Q_G,\mathcal{E}_G,f_G),$$

where

- $Q_G = Q_H|_{Y_G}$;
- $\mathcal{E}_G$ *is given by the top row of (8);*
- f_G *is given by the commutative diagram in (9).*

Proof. The quadratic form is determined by the commutator map on $\overline{\mathbf{T}}_G$. The other two invariants follow from their construction from $\overline{\mathbf{G}}$ directly. □

4.4. Functoriality of pushout

The pushout action is functorial, so it can be glued to a construction of sheaves. Let $f \in \mathrm{End}(\mathbf{K}_2)$. Then for a multiplicative $\mathbf{K}_2$-torsor $\overline{\mathbf{G}}$, one can push it out via f to obtain a new multiplicative $\mathbf{K}_2$-torsor. In this article, we consider the following map:

$$[m] : \mathbf{K}_2 \to \mathbf{K}_2, \qquad x \mapsto x^m$$

for an integer m. We have a natural map $\overline{\mathbf{G}} \to \overline{\mathbf{G}}^\natural$ which fits into the commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbf{K}_2 & \longrightarrow & \overline{\mathbf{G}} & \longrightarrow & \mathbf{G} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathbf{K}_2 & \longrightarrow & \overline{\mathbf{G}}^\natural & \longrightarrow & \mathbf{G} & \longrightarrow & 1 \end{array}$$

This defines a functor

$$\mathbf{CExt}(\mathbf{G},\mathbf{K}_2) \to \mathbf{CExt}(\mathbf{G},\mathbf{K}_2).$$

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At every local place v, we obtain the following:

- We have the following commutative diagram:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mu_n & \longrightarrow & \overline{G}_v & \longrightarrow & G_v & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mu_n & \longrightarrow & \overline{G}_v^{\natural} & \longrightarrow & G_v & \longrightarrow & 1 \end{array}.$$

  where the first vertical map is $x \mapsto x^m$.
- the map $\mathcal{N}_{G_v} \to \overline{G}_v \to \overline{G}_v^{\natural}$ is the canonical unipotent section for $\overline{G}_v^{\natural}$;
- in the tame case, the map $\mathbf{G}(\mathcal{O}_v) \to \overline{G}_v \to \overline{G}_v^{\natural}$ is the natural splitting of maximal compact subgroups.

Globally, we can glue the local maps to obtain a global map. The map $G(F) \to \overline{G}(\mathbb{A}) \to \overline{G}^{\natural}(\mathbb{A})$ is the natural splitting for the multiplicative $\mathbf{K}_2$-torsor $\overline{\mathbf{G}}^{\natural}$.

We now describe the functor $\mathbf{CExt}(\mathbf{G},\mathbf{K}_2) \to \mathbf{CExt}(\mathbf{G},\mathbf{K}_2)$ in terms of BD data.

Proposition 4.4. *The functor*

$$\mathbf{CExt}(\mathbf{G},\mathbf{K}_2) \to \mathbf{CExt}(\mathbf{G},\mathbf{K}_2), \qquad \overline{\mathbf{G}} \mapsto \overline{\mathbf{G}}^{\natural}$$

in terms of the BD data is given by

$$\mathbf{BD}(\mathbf{G},\mathbf{T}) \mapsto \mathbf{BD}(\mathbf{G},\mathbf{T}), \qquad (Q,\mathcal{E},f) \mapsto (Q^{\natural},\mathcal{E}^{\natural},f^{\natural}),$$

where

- $Q^{\natural} = mQ$;
- $\mathcal{E}^{\natural}$ *is obtained from* $\mathcal{E}$ *by pushing out via the map* $[m] : F_s^{\times} \to F_s^{\times}, x \mapsto x^m$;
- $f^{\natural}$ *is obtained by pushing out the commutative diagram in (5) via the map* $[m]$.

Proof. The functoriality of Baer multiples $\overline{\mathbf{G}} \mapsto \overline{\mathbf{G}}^{\natural}$ can be found in [28] Theorem 2.2. Indeed, the quadratic form $Q^{\natural}$ is determined by the multiplication map on $\overline{\mathbf{T}}^{\natural}$. The other two invariants again follow from the construction directly. □

Observe that if $m \equiv -1 \mod n$, then

$$[m] : \mathbf{K}_2 \to \mathbf{K}_2, \qquad \zeta \mapsto \zeta^m$$

becomes $\mu_n \to \mu_n, \zeta \mapsto \zeta^{-1}$ after taking the Hilbert symbol. Thus,

$$\overline{G}_v^{\natural} = (\mu_n \times \overline{G}_v)/\langle(\zeta,\zeta) : \zeta \in \mu_n\rangle.$$

We have a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mu_n & \longrightarrow & \overline{G}_v & \longrightarrow & G_v & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mu_n & \longrightarrow & \overline{G}_v^{\natural} & \longrightarrow & G_v & \longrightarrow & 1 \end{array},$$

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where the first vertical map is the inverse map and the second vertical map $\overline{G}_v \to \overline{G}_v^{\natural}$ is a group isomorphism. This works globally as well.

5. Doubling variables

We now review some definitions related to classical groups that are used in the twisted doubling integrals. Here we use the conceptual description in [5].

5.1. Classical groups

For the definition of classical groups, we follow the setup in [29].

By an involution of an algebra D whose centre E contains F, we mean an arbitrary anti-automorphism ρ of D of order 2 under which F is the fixed subfield of E. We denote the restriction of ρ to E also by ρ. We take a couple of (D,ρ) belonging to the following five types:

(a) $D = E = F$ and ρ is the identity map;

(b) D is a division quaternion algebra over $E = F$ and ρ is the main involution of D;

(c) D is a division algebra central over a quadratic extension E of F and ρ generates $\mathrm{Gal}(E/F)$;

(d) $D = \mathrm{M}_2(E)$, $E = F$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{\rho} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$;

(e) $D = \mathbf{D} \oplus \mathbf{D}^{\mathrm{op}}$, $E = F \oplus F$ and $(x,y)^{\rho} = (y,x)$, where $\mathbf{D}$ is a division algebra central over F and $\mathbf{D}^{\mathrm{op}}$ is its opposite algebra.

If F is local, we fix a nontrivial additive character ψ_F of F; if F is global, we fix a nontrivial additive character ψ_F of $F\backslash\mathbb{A}$. If $E = F$, we set $\psi = \psi_F$; if E/F is an étale quadratic algebra, we set $\psi = \psi_F \circ \mathrm{tr}_{E/F}$. The global version is defined similarly. If x is a square matrix with coordinates in D, then $\nu(x) \in E$ and $\tau(x) \in E$ stand for its reduced norm and reduced trace to the centre E of D.

The rank of D as a module over E is a square of a natural number which will be denoted by d. We assume D to be division if F is a number field, so that D is of type (d) (respectively (e)) will appear in our later discussion as a localisation of a global D of type (b) (respectively (c)).

Let ϵ be either 1 or -1. We fix once and for all of the triple (D,ρ,ϵ).

Let W be a free left D-module of rank m. By an ϵ-skew Hermitian space we mean a structure $\mathcal{W} = (W,\langle\ ,\ \rangle)$, where $\langle\ ,\ \rangle$ is an ϵ-skew Hermitian form on W; that is, an F-bilinear map $\langle\ ,\ \rangle : W \times W \to D$ such that

$$\langle x,y\rangle^{\rho} = -\epsilon\langle y,x\rangle, \qquad \langle ax,by\rangle = a\langle x,y\rangle b^{\rho},\ (a,b \in D;\ x,y \in W).$$

Such a form is called nondegenerate if $\langle x,W\rangle = 0$ implies that $x = 0$. We assume that $\langle\ ,\ \rangle$ is nondegenerate.

We denote the ring of all D-linear endomorphisms of W by $\mathrm{End}_D(W)$ and set $\mathbf{GL}_D(W) = \mathrm{End}_D(W)^{\times}$. Note that $\mathbf{GL}_D(W)$ acts on W on the right. We sometimes

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write $\mathbf{GL}_{W;D}$ for $\mathbf{GL}_D(W)$ for ease of notation. Let

$$\mathbf{G} = \{g \in \mathbf{GL}_D(W) : \langle xg, yg\rangle = \langle x,y\rangle \text{ for all } x,y \in W\}$$

be the unitary group of $(W,\langle\ ,\ \rangle)$, which is a reductive algebraic group defined over F. It is important to realise that $\mathbf{G}$ always comes together with a space W and a form $\langle\ ,\ \rangle$. We usually just speak of $\mathbf{G}$ and the data $\mathcal{W} = (W,\langle\ ,\ \rangle)$ will be implicitly understood. We write $\mathbf{G} = \mathbf{G}(\mathcal{W})$ when the dependence of $\mathbf{G}$ on $\mathcal{W}$ needs to be stressed.

5.2. Doubling homomorphism

Let $\mathcal{W} = (W,\langle\ ,\ \rangle)$ be one of the ϵ-skew Hermitian forms described above. Let k be a fixed positive integer. We would like to define the following in this section:

$$(\mathbf{G}, \mathbf{G}^{\square,k}, \iota, \mathbf{P}, \mathbf{N}^{\bullet}_{\mathcal{W},k}, \psi^{\bullet}_{\mathcal{W},k}). \tag{10}$$

Put $W^{\square,k} = W^{\oplus 2k}$. We usually write

$$W^{\square,k} = W_{1,+} \oplus W_{2,+} \oplus \cdots \oplus W_{k,+} \oplus W_{k,-} \oplus \cdots \oplus W_{2,-} \oplus W_{1,-}$$

to distinguish the copies of W in $W^{\square,k}$. We write an element in $W^{\square,k}$ as

$$(\boldsymbol{x};\boldsymbol{y}) = (x_1,\cdots,x_k;y_k,\cdots,y_1), \qquad x_i \in W_{i,+},\ y_i \in W_{i,-}.$$

Define an ϵ-skew Hermitian form $\langle\ ,\ \rangle^{\square,k}$ on $W^{\square,k}$ by

$$\langle(\boldsymbol{x};\boldsymbol{y}),(\boldsymbol{x}';\boldsymbol{y}')\rangle^{\square,k} = \sum_{i=1}^{k}(\langle x_i,x_i'\rangle - \langle y_i,y_i'\rangle) \qquad (x_i,x_i' \in W_{i,+}; y_i,y_i' \in W_{i,-}).$$

Let $\mathbf{G}^{\square,k}$ denote the unitary group of $(W^{\square,k},\langle\ ,\ \rangle^{\square,k})$.

For $W^{\square} = W_+ \oplus W_-$, let

$$W^{\nabla} = \{(x,-x) \in W_+ \oplus W_- : x \in W\}$$

be the graph of minus the identity map from W to W and

$$W^{\Delta} = \{(x,x) \in W_+ \oplus W_- : x \in W\}$$

be the graph of the identity map. Given $x \in W$, we write

$$x^{\Delta} = (x,x) \in W^{\Delta} \text{ and } x^{\nabla} = (x,-x) \in W^{\nabla}.$$

We have the following observations:

(1) For each i, $W_i^{\square} = W_{i,+} \oplus W_{i,-} = W_i^{\Delta} + W_i^{\nabla}$. Both W_i^{Δ} and W_i^{∇} are totally isotropic in $W^{\square,k}$.

(2) The space W^{Δ} is isomorphic to W as vector spaces via

$$W^{\Delta} \simeq W, \qquad (x,x) \mapsto x.$$

The space W^{∇} is identified with W via $(x,-x) \mapsto 2x$. Thus, we can view $\mathbf{G}(\mathcal{W})$ as a subgroup of $\mathbf{GL}_D(W^{\Delta})$ or $\mathbf{GL}_D(W^{\nabla})$ and identify $\mathrm{Hom}_D(W_i^{\nabla},W_j^{\nabla})$ with $\mathrm{End}_D(W)$.

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Define

$$W^{\Delta,k} = W_1^{\Delta} \oplus \cdots \oplus W_k^{\Delta}, \qquad W^{\nabla,k} = W_1^{\nabla} \oplus \cdots \oplus W_k^{\nabla}.$$

Both spaces are totally isotropic in $W^{\square,k}$ and $W^{\square,k} = W^{\Delta,k} + W^{\nabla,k}$. This is a complete polarisation of $W^{\square,k}$. Unless otherwise specified, we write $\mathbf{P} = \mathbf{P}(W^{\Delta,k})$.

We first construct a Fourier coefficient for the group $\mathbf{G}^{\square,k}$. We choose the following flag of totally isotropic subspaces in $W^{\square,k}$:

$$0 \subset W_k^{\nabla} \subset W_{k-1}^{\nabla} \oplus W_k^{\nabla} \subset \cdots \subset W_2^{\nabla} \oplus \cdots \oplus W_k^{\nabla}. \tag{11}$$

Let $\mathbf{P}^{\bullet}_{\mathcal{W},k} = \mathbf{M}^{\bullet}_{\mathcal{W},k} \cdot \mathbf{N}^{\bullet}_{\mathcal{W},k}$ be the corresponding parabolic subgroup. Then

$$\mathbf{M}^{\bullet}_{\mathcal{W},k} \simeq \mathbf{GL}_D(W_k^{\nabla}) \times \cdots \times \mathbf{GL}_D(W_2^{\nabla}) \times \mathbf{G}(\mathcal{W}_1^{\square}).$$

The character is defined on the group $\mathbf{N}^{\bullet}_{\mathcal{W},k}$.

We reindex the flag in (11) as

$$0 \subset Y_1 \subset \cdots \subset Y_{k-1}$$

and extend it to

$$0 \subset Y_1 \subset \cdots \subset Y_{k-1} \subset Y_{k-1}^{\perp} \subset \cdots \subset Y_1^{\perp} \subset W^{\square,k}.$$

Note that except for $Y_{k-1}^{\perp}/Y_{k-1} = W_1^{\square}$, the quotient between two successive terms is isomorphic to either W^{Δ} or W^{∇}. For convenience, we write $Y_0 = 0$ and $Y_k := Y_{k-1}^{\perp}$.

To describe the character of $\mathbf{N}^{\bullet}_{\mathcal{W},k}(F)\backslash\mathbf{N}^{\bullet}_{\mathcal{W},k}(\mathbb{A})$, we have to specify elements

$$A_i \in \mathrm{Hom}_D(Y_i/Y_{i-1}, Y_{i+1}/Y_i) \simeq \mathrm{End}_D(W^{\nabla}), \qquad i = 1, \cdots, k-2,$$

and

$$A_{k-1} \in \mathrm{Hom}_D(Y_{k-1}/Y_{k-2}, Y_{k-1}^{\perp}/Y_{k-1}) \simeq \mathrm{Hom}_D(W^{\nabla}, W^{\square}).$$

We choose $A_1, \cdots, A_{k-2}$ to be the identity map in $\mathrm{End}_D(W^{\nabla})$. The map

$$Y_{k-1}/Y_{k-2} \xrightarrow{A_{k-1}} Y_{k-1}^{\perp}/Y_{k-1} \xrightarrow{A_k} Y_{k-2}^{\perp}/Y_{k-1}^{\perp}$$

is translated from

$$W^{\nabla} \to W_+ \oplus W_- \to W^{\Delta}, \qquad x^{\nabla} \mapsto (2x, 0) \mapsto 2x^{\Delta}.$$

Note that $A_k \circ A_{k-1}$ is an isomorphism.

An element $u \in \mathbf{N}^{\bullet}_{\mathcal{W},k}(F)\backslash\mathbf{N}^{\bullet}_{\mathcal{W},k}(\mathbb{A})$ induces

$$u_i : (Y_{i+1}/Y_i) \otimes (F\backslash\mathbb{A}) \to (Y_i/Y_{i-1}) \otimes (F\backslash\mathbb{A}).$$

Then we define

$$\psi^{\bullet}_{\mathcal{W},k} : \mathbf{N}^{\bullet}_{\mathcal{W},k}(F)\backslash\mathbf{N}^{\bullet}_{\mathcal{W},k}(\mathbb{A}) \to \mathbb{C}, \qquad u \mapsto \psi\left(\sum_{i=1}^{k-1} \tau(u_i \circ A_i)\right).$$

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Given $(g_1,g_2)\in\mathbf{G}\times\mathbf{G}$, we define its action on $W_{1,+}\oplus W_{2,+}\oplus\cdots\oplus W_{k,+}\oplus W_{k,-}\oplus\cdots\oplus W_{2,-}\oplus W_{1,-}$ via

$$(x_1,\cdots,x_k;y_k,\cdots,y_2,y_1)(g_1,g_2)=(x_1g_1,\cdots,x_kg_1;y_kg_1,\cdots,y_2g_1,y_1g_2).$$

This extends to an action of $\mathbf{G}\times\mathbf{G}$ on $W^{\square,k}$ and gives a map

$$\iota=\iota_k:\mathbf{G}\times\mathbf{G}\to\mathbf{G}^{\square,k}.$$

It is in fact a homomorphism and, in particular, the images of these two copies of $\mathbf{G}$ commute in $\mathbf{G}^{\square,k}$. It is straightforward to check that $\iota(G\times G)$ lies in the stabiliser of $\psi^{\bullet}_{\mathcal{W},k}$ in $\mathbf{G}^{\square,k}$.

Recall that for a subgroup J of G, we define $J^{\diamondsuit}=\{(g,g)\in G\times G\mid g\in J\}$. We have the following results from [5].

Lemma 5.1 ([5] Lemma 5.1). *We have* $\iota(\mathbf{G}\times\mathbf{G})\cap\mathbf{P}=\iota(\mathbf{G}^{\diamondsuit})$.

Lemma 5.2 ([5] Lemma 5.2). *The modular quasicharacter* $\delta_{\iota(\mathbf{G}\times\mathbf{G});\mathbf{N}^{\bullet}_{\mathcal{W},k}}(\iota(g_1,g_2))=1$ *for any* $g_1,g_2\in\mathbf{G}$.

5.3. The case of special orthogonal groups

We now discuss the case of special orthogonal groups. Since the group $\mathbf{O}(W)$ is disconnected, to consider multiplicative $\mathbf{K}_2$-torsors, it would be better to consider its connected component $\mathbf{SO}(W)$. We now explain the modifications in order to develop the twisted doubling integrals. For the group $\mathbf{O}(W)$ and a fixed positive integer k, we have defined a list of input $(\mathbf{G},\mathbf{G}^{\square},\iota,\mathbf{P},\mathbf{N}^{\bullet}_{\mathcal{W},k},\psi^{\bullet}_{\mathcal{W},k})$. We now explain how to define it for the group $\mathbf{SO}(W)$.

The doubling homomorphism $\iota:\mathbf{O}(W)\times\mathbf{O}(W)\to\mathbf{O}(W^{\square,k})$ restricts to

$$\iota:\mathbf{SO}(W)\times\mathbf{SO}(W)\to\mathbf{SO}(W^{\square,k}).$$

Note that $\mathbf{P},\mathbf{N}^{\bullet}_{\mathcal{W},k}\subset\mathbf{SO}(W^{\square,k})$. Thus, we can still use $\psi^{\bullet}_{\mathcal{W},k}$ and $\mathbf{N}^{\bullet}_{\mathcal{W},k}$ to define a Fourier coefficient of an automorphic form on $\mathbf{SO}(W^{\square,k})(\mathbb{A})$. Thus, we take

$$(\mathbf{SO}(W),\mathbf{SO}(W^{\square,k}),\iota,\mathbf{P},\mathbf{N}^{\bullet}_{\mathcal{W},k},\psi^{\bullet}_{\mathcal{W},k})$$

to be the input in the case of special orthogonal groups. To unify our discussion, if $\mathbf{G}=\mathbf{SO}(W)$, then we take $\mathbf{G}^{\square,k}=\mathbf{SO}(W^{\square,k})$.

The case of inner forms of orthogonal groups can be treated similarly.

6. Degenerate representations

The purpose of this section is to discuss a family of representations that are used in the global zeta integrals. These can be viewed as the analog of the generalised Speh representations in the covering group setup. As indicated in [5], in order to prove the global identity, one only has to use information on Fourier coefficients of these representations. We will discuss the conjectural construction of such representations in Section 11.

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6.1. Degenerate Whittaker models

We first recall the definition of degenerate Whittaker models. One can attach a degenerate Whittaker model for a Whittaker pair (see [12, 21]). Given an admissible representation π, an important question is to find the largest nilpotent orbits that support degenerate Whittaker models for π. Locally, this determination is related to other nilpotent invariants such as the wavefront set. We refer the reader to the introduction of [12] for a comprehensive account of discussion.

In this article, we consider only a subclass of degenerate Whittaker models for $\mathrm{GL}_{n,D}$. As explained in [5] Subsection 2.4, this is sufficient for determining nilpotent invariants and fits into our examples later. As unipotent subgroups split canonically over covering groups, these notions transfer from the linear case to the covering group case automatically. In the following, we only define these in the linear case.

Let $R = F$ if F is a local field and $R = \mathbb{A}/F$ if F is a number field. Fix a nontrivial additive character $\psi_F : R \to \mathbb{C}^\times$. Let D be a central division algebra as in Subsection 5.1. Let W be a free left D-module of rank m and consider the group $\mathbf{GL}_D(W)$. Recall that we sometimes write $\mathbf{GL}_{W;D}$ for $\mathbf{GL}_D(W)$ for ease of notation. Let

$$\mathcal{Y} : 0 \subset Y_1 \subset Y_2 \subset \cdots \subset Y_k \subset W$$

be a flag of distinct subspaces of W. We sometimes write $Y_0 = \{0\}$ and $Y_{k+1} = W$ for convenience. The stabiliser of $\mathcal{Y}$ is a parabolic subgroup $\mathbf{P}(\mathcal{Y}) = \mathbf{M}(\mathcal{Y}) \cdot \mathbf{N}(\mathcal{Y})$ with Levi component $\mathbf{M}(\mathcal{Y})$. Then as algebraic groups,

$$\mathbf{N}(\mathcal{Y})^{ab} \cong \prod_{i=1}^{k} \mathrm{Hom}_D(Y_{i+1}/Y_i, Y_i/Y_{i-1}), \qquad u \mapsto (u_i)_{i=1}^k.$$

To give a character of $\mathbf{N}(\mathcal{Y})(R)$, we specify an element in

$$\mathcal{A} = (A_1, \cdots, A_k) \in \prod_{i=1}^{k} \mathrm{Hom}_D(Y_i/Y_{i-1}, Y_{i+1}/Y_i).$$

More concretely, given such an $\mathcal{A}$, we define a character $\psi_{\mathcal{A}}$ of $N(\mathcal{Y})(R)$ by

$$\psi_{\mathcal{A}}(u) = \psi\left(\sum_{i=1}^{k} \tau(u_i \circ A_i)\right).$$

Here, $\psi = \psi_F \circ \mathrm{tr}_{E/F}$.

Assume now we have a pair $(\mathbf{N}(\mathcal{Y}), \psi_{\mathcal{A}})$. Globally, for an irreducible automorphic representation π of $\overline{\mathrm{GL}}_{W;D}(\mathbb{A})$, we define the $(N(\mathcal{Y}), \psi_{\mathcal{A}})$-Fourier coefficient of $\phi \in \pi$ as

$$\phi^{N(\mathcal{Y}), \psi_{\mathcal{A}}}(g) = \int\limits_{[N(\mathcal{Y})]} \phi(ug)\psi_{\mathcal{A}}(u)\, du.$$

Locally, we consider the space $\mathrm{Hom}_{N(\mathcal{Y})(F)}(\pi, \psi_{\mathcal{A}})$ of $(N(\mathcal{Y}), \psi_{\mathcal{A}})$-functional for an admissible representation π of $\overline{\mathrm{GL}}_{W;D}(F)$.

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6.2. Representations of type $(k,m)_D$

The purpose of this section is to introduce the notion of representations of type $(k,m)_D$, both locally and globally. These representations are supported on a suitable nilpotent orbit and admit unique models of degenerate type. In the linear case and when D is a field, the generalised Speh representations are examples of such representations.

In this section, we assume that $\dim_D W = km$.

Definition 6.1. We say a pair $(N(\mathcal{Y}),\psi_{\mathcal{A}})$ is in the orbit $(k^m)_D$ if $\mathcal{Y}$ is of the form

$$0 \subset Y_1 \subset \cdots \subset Y_{k-1} \subset W$$

and for $i = 1,\cdots,k-1$, $\dim_D Y_i = mi$ and A_i is an isomorphism.

The stabiliser of a coefficient in the orbit $(k^m)_D$ is isomorphic to $\mathrm{GL}_{m,D}$.

Definition 6.2. We say a pair $(N(\mathcal{Y}),\psi_{\mathcal{A}})$ lies in an orbit higher than $(k^m)_D$ if

$$A_{i+k-1} \circ \cdots \circ A_i \neq 0$$

for some i.

Note that this implies that there are at least k proper subspaces in the flag $\mathcal{Y}$.

Definition 6.3. We say a representation θ of a local group $\overline{\mathrm{GL}}_{W;D}$ is of type $(k,m)_D$ if the following two conditions hold:

(1) For a pair $(N(\mathcal{Y}),\psi_{\mathcal{A}})$ that lies in the orbit $(k^m)_D$,

$$\dim \mathrm{Hom}_{N(\mathcal{Y})}(\theta,\psi_{\mathcal{A}}) = 1.$$

(2) For any pair $(N(\mathcal{Y}),\psi_{\mathcal{A}})$ that lies in an orbit higher than $(k^m)_D$,

$$\dim \mathrm{Hom}_{N(\mathcal{Y})}(\theta,\psi_{\mathcal{A}}) = 0.$$

Remark 6.4. By Frobenius reciprocity, $\mathrm{Hom}_{N(\mathcal{Y})}(\theta,\psi_{\mathcal{A}}) \simeq \mathrm{Hom}_{\overline{\mathrm{GL}}_{W;D}}(\theta, \mathrm{Ind}_{N(\mathcal{Y})}^{\overline{\mathrm{GL}}_{W;D}}(\psi_{\mathcal{A}}))$. An element in the latter space is called a $(N(\mathcal{Y}),\psi_{\mathcal{A}})$-model for θ. For a representation θ of type $(k,m)_D$, we write $\mathrm{Wh}_{N(\mathcal{Y}),\psi_{\mathcal{A}}}(\theta)$ for the image of a nonzero map in $\mathrm{Hom}_{\overline{\mathrm{GL}}_{W;D}}(\theta, \mathrm{Ind}_{N(\mathcal{Y})}^{\overline{\mathrm{GL}}_{W;D}}(\psi_{\mathcal{A}}))$.

Definition 6.5. We say an irreducible automorphic representation θ of $\overline{\mathrm{GL}}_{W;D}(\mathbb{A})$ is of type $(k,m)_D$ if the following conditions hold:

(1) The representation supports a nonzero $(N(\mathcal{Y}),\psi_{\mathcal{A}})$-Fourier coefficient such that the pair is in the orbit $(k^m)_D$.

(2) For any pair $(N(\mathcal{Y}),\psi_{\mathcal{A}})$ that lies in an orbit higher than $(k^m)_D$, the $(N(\mathcal{Y}),\psi_A)$-Fourier coefficient vanishes identically.

(3) The local component θ_v is a representation of type $(k,m)_D$ for every place v.

We also say that the nilpotent orbit attached to θ is $(k^m)_D$ if only parts (1) and (2) hold.

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6.3. Invariance under stabiliser

This section is a straightforward adaption of [5] Subsection 2.4.2 to the case of covering groups. We collect necessary results but omit the proofs.

We now assume that $\dim_D W = km$ and the representation θ of $\overline{\mathrm{GL}}_{W;D}$ is of type $(k,m)_D$. We consider the following situation:

$$0 \subset Y_1 \subset \cdots \subset Y_{k-1} \subset W$$

such that $\dim_D Y_i = mi$, $A_2, \cdots, A_{k-1}$ are isomorphisms and the rank of A_1 is $a > 0$ (which might not be of full rank).

The Fourier coefficients defined by such a pair enjoy an extra invariance property. We start with the case of $a = m$. Recall that the stabiliser $\mathrm{St}_{\mathcal{A}}$ of a pair $(N(\mathcal{Y}),\psi_{\mathcal{A}})$ that lies in the orbit $(k^m)_D$ is isomorphic to $\mathrm{GL}_{m,D}$. We start with the local version.

Lemma 6.6 ([5] Lemma 2.14). *Let θ be an irreducible ϵ-genuine admissible representation of $\overline{\mathrm{GL}}_{W;D}(F)$ that is of type $(k,m)_D$.*

(1) *The stabiliser $\overline{\mathrm{St}}_{\mathcal{A}}$ acts on $\mathrm{Hom}_{N(\mathcal{Y})}(\theta,\psi_{\mathcal{A}})$ via an ϵ-genuine character $\overline{\chi}_\theta : \overline{\mathrm{St}}_{\mathcal{A}}(F) \to \mathbb{C}^\times$.*

(2) *For $f \in \mathrm{Wh}_{N(\mathcal{Y}),\psi_{\mathcal{A}}}(\theta)$,*

$$f(gh) = \overline{\chi}_\theta(g) f(h)$$

for $g \in \overline{\mathrm{St}}_{\mathcal{A}}(F)$ and $h \in \overline{\mathrm{GL}}_{W;D}(F)$.

Here is the global version.

Lemma 6.7 ([5] Lemma 2.15). *Let θ be an irreducible unitary ϵ-genuine automorphic representation of $\mathrm{GL}_{W;D}(\mathbb{A})$. Then there is a character $\overline{\chi}_\theta : \overline{\mathrm{St}}_{\mathcal{A}}(F\backslash\mathbb{A}) \to \mathbb{C}^\times$ such that, for any $\phi \in \theta$,*

$$\phi^{N(\mathcal{Y}),\psi_{\mathcal{A}}}(gh) = \overline{\chi}_\theta(g)\phi^{N(\mathcal{Y}),\psi_{\mathcal{A}}}(h)$$

for any $g \in \overline{\mathrm{St}}_{\mathcal{A}}(F\backslash\mathbb{A})$ and $h \in \overline{\mathrm{GL}}_{W;D}(\mathbb{A})$.

We now consider the case $a < m$. Define $S_{\mathcal{A}}$ to be the subgroup of $M(\mathcal{Y})$:

$$N(\mathrm{Ker}(A_1)) \times \{1\} \times \cdots \times \{1\} \subset \mathrm{GL}(Y_1) \times \mathrm{GL}(Y_2/Y_1) \times \cdots \times \mathrm{GL}(W/Y_{k-1}).$$

Here, $N(\mathrm{Ker}(A_1))$ is the unipotent radical of the parabolic subgroup of $\mathrm{GL}(Y_1)$ stabilising $\mathrm{Ker}(A_1)$. Then the unipotent group $S_{\mathcal{A}}$ is in the stabiliser of the pair $(N(\mathcal{Y}),\psi_{\mathcal{A}})$. (Note that $S_{\mathcal{A}}$ is not the full stabiliser.) The Fourier coefficient $\phi^{N(\mathcal{Y}),\psi_{\mathcal{A}}}(g)$ is left-invariant under $[S_{\mathcal{A}}]$.

Proposition 6.8 ([5] Proposition 2.17). *For $\phi \in \theta$,*

$$\phi^{N(\mathcal{Y}),\psi_{\mathcal{A}}}(gh) = \phi^{N(\mathcal{Y}),\psi_{\mathcal{A}}}(h)$$

for any $g \in S_{\mathcal{A}}(F\backslash\mathbb{A})$ and $h \in \overline{\mathrm{GL}}_{W;D}(\mathbb{A})$.

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Remark 6.9. To further develop the local and global theory of the twisted doubling integrals, we need finer properties of these representations. For instance, multiplicativity of γ-factors corresponds to the properties of representations of type $(k,m)_D$ with respect to parabolic induction. As we do not require these properties in the present article, we leave them to a future article.

7. Basic setup in the linear case

We now review the basic setup of the twisted doubling integrals from [5] Section 6. Write $\mathbf{P}=\mathbf{M}\cdot\mathbf{N}$. Then $P(F)\backslash G^{\square,k}(F)$ can be identified with the flag variety $\Omega(W^{\square,k})$ of maximal totally isotropic subspaces of $W^{\square,k}$. (In the case of special orthogonal groups, this corresponds to a subset of $\Omega(W^{\square,k})$.) The identification is given by $\gamma\mapsto W^{\Delta,k}\gamma$. We write $L=W^{\Delta,k}\gamma$.

We define the following subset of $G^{\square,k}(F)$:

$$\Omega_1=\{\gamma\in G^{\square,k}(F):\psi^{\bullet}_{\mathcal{W},k}|_{[\gamma^{-1}N\gamma\cap N^{\bullet}_{\mathcal{W},k}]}\neq 1\}.$$

If $\gamma\in G^{\square,k}(F)-\Omega_1$, then $\psi^{\bullet}_{\mathcal{W},k}|_{[\gamma^{-1}N\gamma\cap N^{\bullet}_{\mathcal{W},k}]}=1$. The character $\psi^{\bullet}_{\mathcal{W},k}$ induces a character on a unipotent subgroup of $[\gamma^{-1}N\gamma\backslash\gamma^{-1}P\gamma]$ which is isomorphic to a general linear group over D. It is defined by the following pair:

$$([\gamma^{-1}N\gamma\cap N^{\bullet}_{\mathcal{W},k}\backslash\gamma^{-1}P\gamma\cap N^{\bullet}_{\mathcal{W},k}],\psi^{\bullet}_{\mathcal{W},k}). \tag{12}$$

We define

$$\Omega_2=\{\gamma\in G^{\square,k}(F)-\Omega_1:(12)\text{ is given by a pair that lies in an orbit higher than }(k^m)_D\}.$$

Both Ω_1 and Ω_2 are double cosets in $P(F)\backslash G^{\square,k}(F)/N^{\bullet}_{\mathcal{W},k}(F)$ and we have a nice geometric interpretation of $\Omega_1\cup\Omega_2$.

Lemma 7.1 ([5] Subsection 6.1). *With notation as above,*

$$\Omega_1\cup\Omega_2=\{\gamma\in G^{\square,k}(F):L\cap Y_{k-1}=\{0\}\}.$$

Proof. The only new case is the case of special orthogonal groups, which follows from the case of orthogonal cases. □

Let $\tilde{G}^{\square,k}(F)=G^{\square,k}(F)-(\Omega_1\cup\Omega_2)$ and consider $P(F)\backslash\tilde{G}^{\square,k}(F)/N^{\bullet}_{\mathcal{W},k}(F)$. It is stable under the right action of $\iota(G\times G)(F)$.

The results in [5] Subsection 6.2 can be summarised as follows:

Proposition 7.2 ([5] Subsection 6.2). *We have the following:*

(1) *The double coset $P(F)\backslash\tilde{G}^{\square,k}(F)/\iota(G\times G)N^{\bullet}_{\mathcal{W},k}(F)$ is finite.*

(2) *For an $\iota(G\times G)(F)$-orbit in $P(F)\backslash\tilde{G}^{\square,k}(F)/\iota(G\times G)N^{\bullet}_{\mathcal{W},k}(F)$, one can choose a representative γ such that*
   - *The Fourier coefficient in (12) is of the form studied in Subsection 6.3. The value a can be determined explicitly by γ. If $a<m$, let S_λ denote the subgroup $S_{\mathcal{A}}$ in Subsection 6.3.*

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(3) *The stabiliser R_- in $\{1\}\times G$ of each representative in (2) contains the unipotent radical N_- of a parabolic subgroup of $\{1\}\times G$ as a normal subgroup.*

(4) *If $a<m$, then N^- is nontrivial and the projection of $\{1\}\times N^-$ to M is a subgroup of S_λ. In this case we say that this orbit is negligible.*

(5) *If N^- is trivial, we call this orbit the main orbit. The representative of the main orbit can be chosen to be the identity element. The stabiliser of the identity element in $\iota(G\times G)(F)$ is $P(F)\cap\iota(G\times G)(F)=\iota(G^\diamondsuit)(F)$.*

Proof. The only case to check is the case of special orthogonal groups or its inner forms. It can be deduced from the orthogonal group case by noting that there is a bijection between

$$P(F)\backslash\tilde{\mathrm{O}}(W^{\square,k})(F)/\iota(\mathrm{O}(W)\times\mathrm{O}(W))N^\bullet_{\mathcal{W},k}(F)$$

and

$$P(F)\backslash\tilde{\mathrm{SO}}(W^{\square,k})(F)/\iota(\mathrm{SO}(W)\times\mathrm{SO}(W))N^\bullet_{\mathcal{W},k}(F).$$

□

8. Assumptions in the covering group case

We now discuss the necessary modifications in the case of covering groups. In order to keep the length of this section reasonable, we defer some of the proofs to Section 10.

From now on, we consider $\mathbf{G}$ to be one of the following groups:

(1) $\mathbf{Sp}(W)$ or its inner forms;

(2) $\mathbf{SO}(W)$ with $\dim W$ even or its inner forms;

(3) $\mathbf{SO}(W)$ with $\dim W$ odd and $\dim W\geq 3$;

(4) $\mathbf{U}(W)$.

For each group in the list and a fixed integer k, we have a list of input $(\mathbf{G},\mathbf{G}^{\square,k},\iota,\mathbf{P},\mathbf{N}^\bullet_{\mathcal{W},k},\psi^\bullet_{\mathcal{W},k})$ from the previous section.

Remark 8.1. The group $\mathbf{SO}_1$ is trivial, so all of the results in this section are trivial in this case. We will exclude this case. (We still need to consider this case locally in order to discuss multiplicativity, for example.)

Let n be a fixed positive integer. Let $\overline{\mathbf{G}}\in\mathbf{CExt}(\mathbf{G},\mathbf{K}_2)$ which is classified by the BD data $(Q,\mathcal{E},f)$ given a choice of a maximal F-torus $\mathbf{T}$. In the case of unitary groups, $\mathbf{U}(W)_{F_s}\simeq\mathbf{GL}_{dm,F_s}$. We also assume that quadratic form Q in the BD data is decomposable. (This means that if we write $Y=Y_1\oplus Y_2$ according to $\mathbf{T}=\mathbf{T}_1\times\mathbf{T}_2$, then Q can be written as the direct sum of $Q|_{Y_1}$ and $Q|_{Y_2}$.) This assumption will greatly simplify the situation. We will discuss more on this in Remark 8.9.

Lemma 8.2. *Assume that D is a field and $\dim_D W>1$. Then $2\mid Q(\alpha^\vee)$ where $\alpha^\vee$ is a coroot in the Siegel parabolic subgroup of $\mathbf{G}$.*

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This result will be proved case by case in Section 10. In the unitary case, this result is not true without the decomposable assumption.

Let $n_Q = n/\gcd(n, Q(\alpha^\vee))$ where $\alpha^\vee$ is a coroot in the Siegel parabolic subgroup of $\mathbf{G}_{F_s}$. In the case $\mathbf{G} = \mathbf{U}_1$, the quadratic form is $Q(e_1^\vee) = a$. We define $n_Q = n/\gcd(n, 2a)$. Without loss of generality, we assume that n_Q is either n or $n/2$.

Let $\overline{\mathbf{G}^{\square, kn_Q}} \in \mathbf{CExt}(\mathbf{G}^{\square, kn_Q}, \mathbf{K}_2)$. The pullback of $\overline{\mathbf{G}^{\square, kn_Q}}$ via the doubling homomorphism

$$\iota : \mathbf{G} \times \mathbf{G} \to \mathbf{G}^{\square, kn_Q}$$

gives two multiplicative $\mathbf{K}_2$-torsors on $\mathbf{G}$:

$$1 \to \mathbf{K}_2 \to \overline{\mathbf{G}}_- \to \iota(1 \times \mathbf{G}) \to 1$$

and

$$1 \to \mathbf{K}_2 \to \overline{\mathbf{G}}_+ \to \iota(\mathbf{G} \times 1) \to 1.$$

A priori, these two multiplicative $\mathbf{K}_2$-torsors may not be isomorphic. In fact, a simple calculation on the quadratic form suggests that they are not isomorphic unless $kn_Q = 1$. In any case, what we need is the following result.

Theorem 8.3. *For $\overline{\mathbf{G}} \in \mathbf{CExt}(\mathbf{G}, \mathbf{K}_2)$, there exists $\overline{\mathbf{G}^{\square, kn_Q}} \in \mathbf{CExt}(\mathbf{G}^{\square, kn_Q}, \mathbf{K}_2)$ such that $\overline{\mathbf{G}}_- \simeq \overline{\mathbf{G}}$ and $\overline{\mathbf{G}}_+$ is the pushout of $\overline{\mathbf{G}}$ by the endomorphism $[2kn_Q - 1] : \mathbf{K}_2 \to \mathbf{K}_2$.*

This theorem is proved as a special case of Corollary 10.6.

Let $\overline{\mathbf{G}^{\square, kn_Q}}$ be a multiplicative $\mathbf{K}_2$-torsor given by the theorem. We simply write $\overline{\mathbf{G}} = \overline{\mathbf{G}}_-$ and $\overline{\mathbf{G}}^\natural = \overline{\mathbf{G}}_+$. Then these two extensions fit into the following commutative diagram:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbf{K}_2 & \longrightarrow & \overline{\mathbf{G}} & \longrightarrow & \iota(1 \times \mathbf{G}) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathbf{K}_2 & \longrightarrow & \overline{\mathbf{G}}^\natural & \longrightarrow & \iota(\mathbf{G} \times 1) & \longrightarrow & 1 \end{array}$$

Here, the leftmost vertical map is $[2kn_Q - 1] : \mathbf{K}_2 \to \mathbf{K}_2$. We fix a morphism $\overline{\mathbf{G}} \to \overline{\mathbf{G}}^\natural$ once and for all.

The decomposable assumption in the unitary case significantly simplifies our argument since the following result is true. This will be proved in Proposition 10.7.

Lemma 8.4. *The multiplicative $\mathbf{K}_2$-torsors $\overline{\mathbf{G}}^\natural$ and $\overline{\mathbf{G}}$ commute in $\overline{\mathbf{G}^{\square, kn_Q}}$.*

A consequence of this lemma is that we have a doubling homomorphism of multiplicative $\mathbf{K}_2$-torsors (instead of a map of sets):

$$\overline{\mathbf{G}}^\natural \times \overline{\mathbf{G}} \to \overline{\mathbf{G}^{\square, kn_Q}}.$$

By composing it with the fixed morphism $\overline{\mathbf{G}} \to \overline{\mathbf{G}}^\natural$, we obtain a homomorphism

$$\iota : \overline{\mathbf{G}} \times \overline{\mathbf{G}} \to \overline{\mathbf{G}^{\square, kn_Q}},$$

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which fits into the following commutative diagram:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbf{K}_2 \times \mathbf{K}_2 & \longrightarrow & \overline{\mathbf{G}} \times \overline{\mathbf{G}} & \longrightarrow & \mathbf{G} \times \mathbf{G} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathbf{K}_2 & \longrightarrow & \overline{\mathbf{G}^{\square, kn_Q}} & \longrightarrow & \mathbf{G}^{\square, kn_Q} & \longrightarrow & 1 \end{array}.$$

Here the first vertical map is given by

$$\mathbf{K}_2 \times \mathbf{K}_2 \to \mathbf{K}_2, \qquad (x,y) \mapsto x^{2kn_Q - 1} y.$$

8.1. The local doubling homomorphism

We now discuss the local and global consequences using results from Section 4. The homomorphism

$$\iota : \overline{G}_v^{\natural} \times \overline{G}_v \to \overline{G_v^{\square, kn_Q}}$$

is a lift of the local doubling homomorphism $\iota : G_v \times G_v \to G_v^{\square, kn_Q}$. Here are the consequences:

- The images of $\overline{G}_v$ and $\overline{G}_v^{\natural}$ commute in $\overline{G_v^{\square, kn_Q}}$.
- We have a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mu_n \times \mu_n & \longrightarrow & \overline{G}_v \times \overline{G}_v & \longrightarrow & G_v \times G_v & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mu_n & \longrightarrow & \overline{G_v^{\square, kn_Q}} & \longrightarrow & G_v^{\square, kn_Q} & \longrightarrow & 1 \end{array}$$

  where the first vertical map is given by

$$\mu_n \times \mu_n \to \mu_n, \qquad (\zeta_1, \zeta_2) \mapsto \zeta_1^{-1} \zeta_2.$$

Lemma 8.5. *We have the following:.*

(1) *For a unipotent subgroup* $\mathbf{U}$ *of* $\mathbf{G}$*,*

$$i_u(\iota(g_1, g_2)) = \iota(i_u(g_1), i_u(g_2))$$

for $g_1, g_2 \in \mathbf{U}(F_v)$*.*

(2) *Let* F *be a non-Archimedean field. Assume that* $p \nmid n$*. We have*

$$s_v(\iota(g_1, g_2)) = \iota(s_v(g_1), s_v(g_2))$$

for $g_1, g_2 \in \mathbf{G}(\mathcal{O}_v)$*.*

Proof. These are simply consequences of the results in Section 4. □

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8.2. Splitting over the diagonal copy

We now discuss an important consequence of Lemma 8.4. We write $\overline{(G\times G)}_v$ to be the pullback of $\overline{G_v^{\square,kn_Q}}$ via $\iota: G_v\times G_v\to G_v^{\square,kn_Q}$.

Lemma 8.6. *With the above assumptions, there is a natural splitting* $G_v^{\diamondsuit}\to\overline{(G\times G)}_v$.

Proof. We know that the homomorphism

$$\overline{G}_v\times\overline{G}_v\to\overline{(G\times G)}_v\to\overline{G_v^{\square,kn_Q}}$$

restricts to $\mu_n\times\mu_n\to\mu_n,(\zeta_1,\zeta_2)\mapsto\zeta_1^{-1}\zeta_2$. Thus, we obtain

$$\overline{(G\times G)}_v=(\overline{G}_v\times\overline{G}_v)/\mu_n^{\diamondsuit}.$$

This implies that the image of $\overline{G}_v^{\diamondsuit}\subset\overline{G}_v\times\overline{G}_v$ in $\overline{(G\times G)}_v$ is $\overline{G}_v^{\diamondsuit}/\mu_n^{\diamondsuit}\simeq G_v^{\diamondsuit}$. Thus, we have a natural splitting $G_v^{\diamondsuit}\to\overline{(G\times G)}_v$. □

8.3. The global doubling homomorphism

The local doubling homomorphism glues to

$$\iota_{\mathbb{A}}:\prod_v\overline{G}_v\times\prod_v\overline{G}_v\to\prod_v\overline{G_v^{\square,kn_Q}}.$$

As $\iota_v(s_v(\mathbf{G}(\mathcal{O}_v))\times s_v(\mathbf{G}(\mathcal{O}_v)))\subset s_v(\mathbf{G}^{\square,kn_Q}(\mathcal{O}_v))$ for almost all v, we obtain a homomorphism

$$\iota_{\mathbb{A}}:\prod_v{}'\,\overline{G}_v\times\prod_v{}'\,\overline{G}_v\to\prod_v{}'\,\overline{G_v^{\square,kn_Q}}\to\overline{G^{\square,kn_Q}}(\mathbb{A}).$$

This map factors through

$$\iota_{\mathbb{A}}:\overline{G}(\mathbb{A})\times\overline{G}(\mathbb{A})\to\overline{G^{\square,kn_Q}}(\mathbb{A}).$$

To summarise, we have obtained a global doubling homomorphism so that

- the restriction to $\mu_n\times\mu_n$ is given by
$$\iota_{\mathbb{A}}:\mu_n\times\mu_n\to\mu_n,\qquad(\zeta_1,\zeta_2)\mapsto\zeta_1^{-1}\zeta_2.$$
- this map is a lift of the linear doubling homomorphism $G(\mathbb{A})\times G(\mathbb{A})\to G^{\square,kn_Q}(\mathbb{A})$.
- this map is also compatible with the local doubling homomorphism.

The section over the rational points is also compatible with the doubling homomorphism.

Proposition 8.7. *We have a commutative diagram*

$$\begin{array}{ccc} G(F)\times G(F) & \longrightarrow & \overline{G}(\mathbb{A})\times\overline{G}(\mathbb{A}) \\ \downarrow & & \downarrow \\ G^{\square,kn_Q}(F) & \longrightarrow & \overline{G^{\square,kn_Q}}(\mathbb{A}) \end{array}.$$

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Proof. This is again a consequence of the results in Section 4. □

We can also check that the cover splits over $\iota_{\mathbb{A}}(G(\mathbb{A})^{\diamondsuit})$.

Lemma 8.8. *There is a natural map* $G(\mathbb{A}) \to \overline{(G \times G)}(\mathbb{A}) \to \overline{G^{\square, kn_Q}}(\mathbb{A})$.

Proof. This is trivial from the local result Lemma 8.8. □

Remark 8.9. An important consequence of Lemma 8.4 is that we have $\overline{G} \times \overline{G}/\mu_n^{\diamondsuit} \simeq \overline{G \times G}$. Given an ϵ-genuine representation π and an ϵ^{-1}-genuine representation π', their tensor product descends to a representation of $\overline{G \times G}$.

We briefly explain what would happen when Lemma 8.4 does not hold. We no longer have a homomorphism $\overline{\mathbf{G}} \times \overline{\mathbf{G}} \to \overline{\mathbf{G} \times \mathbf{G}}$. Thus, given two automorphic representations of $\overline{G}(\mathbb{A})$, one cannot directly construct an automorphic representation of $\overline{G \times G}(\mathbb{A})$. Instead, one has to use a version of metaplectic tensor product, as in [19, 25, 26]. In this article, we do not plan to treat such cases. Moreover, the diagonal copy might not split.

9. The twisted doubling integrals

We can now present the global twisted doubling integral.

Notation: we use notation $[\overline{G}]$ for $G(F)\backslash\overline{G}(\mathbb{A})$ for a multiplicative $\mathbf{K}_2$-torsor of $\mathbf{G}$. For a unipotent subgroup $\mathbf{N} \subset \mathbf{G}$, let $[N] = \mathbf{N}(F)\backslash\mathbf{N}(\mathbb{A})$. Recall that we always identify $[N]$ with the subgroup $i_u([N])$ of $[\overline{G}]$.

9.1. Petersson inner product

Let π be an irreducible ϵ-genuine cuspidal automorphic representation of $\overline{G}(\mathbb{A})$ realised on a space $V_\pi \subset L^2_\epsilon(G(F)\backslash\overline{G}(\mathbb{A}))$, where we fix an embedding $\pi \hookrightarrow V_\pi \subset \mathcal{A}(\overline{G}(\mathbb{A}))$. The contragredient representation $\pi^\vee$ is ϵ^{-1}-genuine and is realised on the complex conjugate $\overline{V_\pi}$ of V_π. The Petersson pairing $\mathcal{P} = \mathcal{P}_\pi : V_\pi \boxtimes \overline{V_\pi} \to \mathbb{C}$ is defined by

$$\mathcal{P}_\pi(\xi_1 \boxtimes \xi_2) = \frac{1}{n} \int\limits_{G(F)\backslash\overline{G}(\mathbb{A})} \xi_1(g)\xi_2(g)\, dg.$$

The integrand is trivial on both μ_n and $G(F)$ and the pairing is $\overline{G}(\mathbb{A})$-invariant.

The Petersson inner product admits a slightly different formula. The function $\xi_1 \boxtimes \xi_2 \in \pi \boxtimes \pi^\vee$ is a function on $\overline{G}(\mathbb{A}) \times \overline{G}(\mathbb{A})$ which is trivial on $\mu_n^{\diamondsuit}$. Thus, this descends to a function on $\overline{G \times G}(\mathbb{A})$, which will be denoted as $\overline{\xi_1 \boxtimes \xi_2}$. By Lemma 8.8, the image of $\overline{G}(\mathbb{A})$ under $\overline{G}(\mathbb{A}) \times \overline{G}(\mathbb{A}) \to \overline{G \times G}(\mathbb{A})$ is $G(\mathbb{A})$. It is easy to check that

$$\mathcal{P}_\pi(\xi_1 \boxtimes \xi_2) = \int\limits_{G(F)\backslash G(\mathbb{A})} \overline{\xi_1 \boxtimes \xi_2}((g,g))\, dg.$$

9.2. Metaplectic restricted tensor product

We now recall the notion of meteplectic restricted tensor product. Notation: $\pi = \tilde{\otimes}'_v \pi_v$. We can view π as a representation of $\prod'_v \overline{G}_v$ via the projection map $\prod'_v \overline{G}_v \to \overline{G}(\mathbb{A})$. The

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space of the representation stays the same. Thus, we can write π as a restricted tensor product $\pi = \otimes_v' \pi_v$, where π_v is an admissible representation of $\overline{G}_v$.

We fix isomorphisms $V_\pi \simeq \tilde{\otimes}_v' \pi_v$ and $\overline{V_\pi} \simeq \tilde{\otimes}_v' \pi_v^\vee$.

Lemma 9.1. *We can choose a standard local pairing $\mathcal{P}_{\pi_v} : \pi_v \boxtimes \pi_v^\vee \to \mathbb{C}$ at every local place v in order that*

$$\mathcal{P}_\pi(\xi) = \prod_v \mathcal{P}_{\pi_v}(\xi_v)$$

for all $\xi = \otimes_v \xi_v \in V_\pi \boxtimes \overline{V_\pi}$, where $\mathcal{P}_{\pi_v}(\xi_{0,v}) = 1$ for almost all $s_v(K_v) \times s_v(K_v)$-invariant vectors $\xi_{0,v} \in \pi_v \boxtimes \pi_v^\vee$ used to define the restricted tensor products.

Proof. Let π_v be an ϵ-genuine representation of $\overline{G}_v$. The action on $\pi^\vee$ is given via

$$\langle \xi_v, \pi_v^\vee(g)\xi_v^\vee \rangle = \langle \pi_v(g^{-1})\xi_v, \xi_v^\vee \rangle.$$

Thus, $\pi_v^\vee$ is an ϵ^{-1}-genuine representation of $\overline{G}_v$. Recall that $\mathrm{Hom}_{\overline{G}_v}(\pi_v, \pi_v) = \mathbb{C}$. This implies that $\mathrm{Hom}_{\overline{G}_v}(\pi_v \boxtimes \pi_v^\vee, \mathbb{C}) = \mathbb{C}$.

The Petersson inner product defines a $\overline{G}(\mathbb{A})$-equivariant pairing on global representations

$$\pi \boxtimes \pi^\vee \to \mathbb{C}.$$

This (abstract) pairing is also $\prod_v' \overline{G}_v$-equivariant. One can argue as in the linear case to prove the result (see [4] Subsection 3.5 for an analogous argument). □

9.3. Eisenstein series

We now describe the Eisenstein series that appear in the global construction. Let θ be an irreducible unitary automorphic representation of $\overline{\mathrm{GL}}_{kmn_Q,D}(\mathbb{A})$ of type $(kn_Q, m)_D$. We define the normalised global induced representation $I(s,\theta) = \mathrm{Ind}_{\overline{P}(\mathbb{A})}^{\overline{G^{\square,kmn_Q}}(\mathbb{A})}(\theta \cdot \nu^s)$. Here ν is defined as $\overline{P}(\mathbb{A}) \to P(\mathbb{A}) \to M(\mathbb{A}) \to \mathbb{C}^\times$.

For any holomorphic section $\tilde{\phi}^{(s)}$ of $I(s,\theta)$, we write $\phi^{(s)}(g) = \tilde{\phi}^{(s)}(g;1)$ to be the value at the identity. We form the associated Eisenstein series $E(\phi^{(s)})$ on $G^{\square,kmn_Q}(F)\backslash\overline{G^{\square,kmn_Q}}(\mathbb{A})$ by

$$E(\phi^{(s)})(g) = \sum_{\gamma \in P(F)\backslash G^{\square,kmn_Q}(F)} \phi^{(s)}(\gamma g).$$

The Eisenstein series converges for $\Re s \gg 0$. By the theory of Eisenstein series, it can be continued to a meromorphic function in s on all of $\mathbb{C}$ satisfying a functional equation.

We now discuss the extra invariance property in this setup. Notation: we write $N_{\mathcal{W}}^\bullet = N_{\mathcal{W},kn_Q}^\bullet \subset G^{\square,kmn_Q}$ and $\psi_{\mathcal{W}}^\bullet = \psi_{\mathcal{W},kn_Q}^\bullet$ for ease of notation.

Let

$$f^{(s)}(g) = f_\phi^{(s)}(g) := \int\limits_{[N_{\mathcal{W}}^\bullet \cap P]} \phi^{(s)}(ug)\psi_{\mathcal{W}}^\bullet(u)\, du = \int\limits_{[N_{\mathcal{W}}^\bullet \cap P]} \tilde{\phi}^{(s)}(g;u)\psi_{\mathcal{W}}^\bullet(u)\, du.$$

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The pair $(\mathbf{N}^{\bullet}_{\mathcal{W}} \cap \mathbf{P}, \psi^{\bullet}_{\mathcal{W}}|_{[N^{\bullet}_{\mathcal{W}} \cap P]})$ gives a Fourier coefficient in the orbit $((kn_Q)^m)_D$ and $\iota(G^{\diamondsuit})$ lies in the stabiliser. For fixed g, the function $u \mapsto \phi^{(s)}(ug)$ on $(N^{\bullet}_{\mathcal{W}} \cap P)(\mathbb{A})$ is an element of $\theta \cdot \nu^s$. Thus, one can view $f^{(s)}(g)$ as a $(kn_Q, m)_D$-coefficient of an element in $\theta \cdot \nu^s$.

Lemma 9.2. *There exists a character $\chi_\theta : [G] \to \mathbb{C}^\times$ such that for $\iota(g,g) \in \iota(G(\mathbb{A})^{\diamondsuit})$ and $h \in \overline{G^{\square, kmn_Q}}(\mathbb{A})$,*

$$f^{(s)}(\iota(g,g)h) = \chi_\theta(\nu(g)) f^{(s)}(h).$$

Proof. Recall that the homomorphism

$$\overline{G}^{\diamondsuit}(\mathbb{A}) \to \overline{G}(\mathbb{A}) \times \overline{G}(\mathbb{A}) \to \overline{G^{\square, kmn_Q}}(\mathbb{A})$$

is trivial on μ_n and thus descends to

$$G^{\diamondsuit}(\mathbb{A}) \to \overline{G^{\square, kmn_Q}}(\mathbb{A}).$$

This indeed gives a homomorphism

$$G^{\diamondsuit}(\mathbb{A}) \to \overline{P}(\mathbb{A}) \to \overline{M}(\mathbb{A}).$$

The image of $G^{\diamondsuit}(\mathbb{A})$ lies in the stabiliser of the Fourier coefficient. Thus, the Fourier coefficient is left-equivariant under a character χ_θ; in other words,

$$f^{(s)}(\iota(g,g)h) = \chi_\theta(\nu(g)) f^{(s)}(h),$$

for $\iota(g,g) \in \iota(G(\mathbb{A})^{\diamondsuit})$ and $h \in \overline{G^{\square, kmn_Q}}(\mathbb{A})$. □

9.4. The global integral

We view χ_θ as a character of $\overline{G}(\mathbb{A})$ via $\overline{G}(\mathbb{A}) \to G(\mathbb{A}) \xrightarrow{\chi_\theta} \mathbb{C}^\times$, which we still denote as χ_θ. We define the global integral to be

$$Z(\xi_1 \boxtimes \xi_2, \phi^{(s)}) = \frac{1}{n^2} \int\limits_{[\overline{G} \times \overline{G}]} \chi_\theta(\nu(g_2))^{-1} \xi_1(g_1) \xi_2(g_2) \times \int\limits_{[N^{\bullet}_{\mathcal{W}}]} E(\phi^{(s)})(u \cdot \iota(g_1, g_2)) \psi^{\bullet}_{\mathcal{W}}(u)\, du\, dg_1\, dg_2.$$

Since $\iota_{\mathbb{A}}(G(F) \times G(F)) \subset G^{\square, kmn_Q}(F)$, this integral is well-defined.

Since the two cusp forms are rapidly decreasing on $G(F)\backslash\overline{G}(\mathbb{A})$ and the Eisenstein series is only of moderate growth, we see that the integral converges absolutely for all s away from the poles of the Eisenstein series and is hence meromorphic in s.

Remark 9.3. One can easily check that the integrand as a function of $\overline{G}(\mathbb{A}) \times \overline{G}(\mathbb{A})$, is trivial on $\mu_n \times \mu_n$ and thus can be viewed as a function on $G(\mathbb{A}) \times G(\mathbb{A})$. The factor $1/n^2$ will be cancelled out if we write $Z(\xi_1 \boxtimes \xi_2, \phi^{(s)})$ as an integral over $G(\mathbb{A}) \times G(\mathbb{A})$.

Remark 9.4. Observe that the function $\xi_1 \boxtimes \xi_2$ is trivial on μ_n and thus descends to a function on $\overline{G \times G}$. Such functions generate an irreducible cuspidal representation of

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$\overline{G\times G}(\mathbb{A})$, which we denote by $\overline{\pi\boxtimes\pi^\vee}$. In the proof of the main global identity below, we only use the fact that $\overline{\pi\boxtimes\pi^\vee}$ is cuspidal.

9.5. Main global identity

We now state the main global identity.

Theorem 9.5. *When* $\Re s\gg 0$, $Z(\xi_1\boxtimes\xi_2,\phi^{(s)})$ *equals*

$$\frac{1}{n^2}\int\limits_{G^\diamond(F)\backslash(\overline{G}\times\overline{G})(\mathbb{A})}\chi_\theta(\nu(g_2))^{-1}\xi_1(g_1)\xi_2(g_2)\times\int\limits_{(N_{\mathcal{W}}^\bullet\cap P)(F)\backslash N_{\mathcal{W}}^\bullet(\mathbb{A})}\phi^{(s)}(u\cdot\iota(g_1,g_2))\psi_{\mathcal{W}}^\bullet(u)\,du\,dg_1\,dg_2.$$

Proof. The proof is similar to that of [5]. We will give a sketch here.

When $\Re s\gg 0$, the global integral becomes

$$Z(\xi_1\boxtimes\xi_2,\phi^{(s)})=\frac{1}{n^2}\int\limits_{[\overline{G}\times\overline{G}]}\chi_\theta(\nu(g_2))^{-1}\xi_1(g_1)\xi_2(g_2)\times\int\limits_{[N_{\mathcal{W}}^\bullet]}\sum_{\gamma\in P(F)\backslash G^{\square,kmn_Q}(F)}\phi^{(s)}(\gamma u\cdot\iota(g_1,g_2))\psi_{\mathcal{W}}^\bullet(u)\,du\,dg_1\,dg_2.$$

We can rewrite the integral as a sum over $P(F)\backslash G^{\square,kmn_Q}(F)/\iota(G\times G)N_{\mathcal{W}}^\bullet(F)$. Our goal is to show that only the double coset $P(F)\iota(G\times G)N_{\mathcal{W}}^\bullet(F)$ supports nonzero contribution.

We first calculate the Fourier coefficient and deduce that

$$\begin{aligned}&\int\limits_{[N_{\mathcal{W}}^\bullet]}\sum_{\gamma\in P(F)\backslash G^{\square,kmn_Q}(F)}\phi^{(s)}(\gamma u\cdot\iota(g_1,g_2))\psi_{\mathcal{W}}^\bullet(u)\,du\\&=\int\limits_{[N_{\mathcal{W}}^\bullet]}\sum_{\gamma\in P(F)\backslash G^{\square,kmn_Q}(F)/N_{\mathcal{W}}^\bullet(F)}\sum_{\gamma'\in\gamma^{-1}P(F)\gamma\cap N(F)\backslash N_{\mathcal{W}}^\bullet(F)}\phi^{(s)}(\gamma\gamma' u\cdot\iota(g_1,g_2))\psi_{\mathcal{W}}^\bullet(u)\,du\\&=\sum_{\gamma\in P(F)\backslash G^{\square,kmn_Q}(F)/N_{\mathcal{W}}^\bullet(F)}\int\limits_{(N_{\mathcal{W}}^\bullet\cap\gamma^{-1}P\gamma)(F)\backslash N_{\mathcal{W}}^\bullet(\mathbb{A})}\phi^{(s)}(\gamma u\cdot\iota(g_1,g_2))\psi_{\mathcal{W}}^\bullet(u)\,du.\end{aligned}\tag{13}$$

For each $\gamma\in P(F)\backslash G^{\square,kmn_Q}(F)/N_{\mathcal{W}}^\bullet(F)$ and $h\in\overline{G^{\square,kmn_Q}}(\mathbb{A})$, we write

$$I_\gamma(h)=\int\limits_{(N_{\mathcal{W}}^\bullet\cap\gamma^{-1}P\gamma)(F)\backslash N_{\mathcal{W}}^\bullet(\mathbb{A})}\phi^{(s)}(\gamma uh)\psi_{\mathcal{W}}^\bullet(u)\,du,$$

$$J_\gamma(h)=\int\limits_{[N_{\mathcal{W}}^\bullet\cap\gamma^{-1}P\gamma]}\phi^{(s)}(\gamma uh)\psi_{\mathcal{W}}^\bullet(u)\,du.$$

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Thus,

$$I_\gamma(h) = \int\limits_{(N^\bullet_{\mathcal{W}} \cap \gamma^{-1} P \gamma \backslash N^\bullet_{\mathcal{W}})(\mathbb{A})} J_\gamma(uh) \psi^\bullet_{\mathcal{W}}(u)\, du.$$

Recall that in Section 7 we defined two subsets Ω_1 and Ω_2 of $G^{\square, kmn_Q}$. If $\gamma \in \Omega_1$, then $J_\gamma(h)$ contains the following inner integral:

$$\int\limits_{[N^\bullet_{\mathcal{W}} \cap \gamma^{-1} N \gamma]} \phi^{(s)}(\gamma uh) \psi^\bullet_{\mathcal{W}}(u)\, du. \tag{14}$$

Note that $\phi^{(s)}$ is left invariant under $[N]$. Thus, as a function on $[N^\bullet_{\mathcal{W}} \cap \gamma^{-1} N \gamma]$, $\phi^{(s)}(\gamma uh)$ is a constant function. According to the definition of Ω_1, the restriction of $\psi^\bullet_{\mathcal{W}}(u)$ to $[N^\bullet_{\mathcal{W}} \cap \gamma^{-1} N \gamma]$ is a nontrivial character. Therefore, (14) vanishes and so do $J_\gamma(h)$ and $I_\gamma(h)$.

If $\gamma \in \Omega_2$, then (14) is constant as a function on $[N^\bullet_{\mathcal{W}} \cap \gamma^{-1} N \gamma]$ and $J_\gamma(h)$ becomes a Fourier coefficient of θ which is given by a pair that lies in an orbit higher than $((kn_Q)^m)_D$. We now have $J_\gamma(h) = 0$ since θ is a representation of type $(kn_Q, m)_D$. Thus, $I_\gamma(h) = 0$ as well.

Therefore, we have shown that (13) equals

$$\sum_{\gamma \in P(F) \backslash \tilde{G}^{\square, kmn_Q}(F) / N^\bullet_{\mathcal{W}}(F)} I_\gamma(\iota(g_1, g_2))$$

and, therefore,

$$Z(\xi_1 \boxtimes \xi_2, \phi^{(s)}) = \frac{1}{n^2} \int\limits_{[\overline{G} \times \overline{G}]} \chi_\theta(\nu(g_2))^{-1} \xi_1(g_1) \xi_2(g_2) \sum_{\gamma \in P(F) \backslash \tilde{G}^{\square, kmn_Q}(F) / N^\bullet_{\mathcal{W}}(F)} I_\gamma(\iota(g_1, g_2))\, dg_1\, dg_2.$$

We now break the sum and exchange it with integration again. This shows that the above equation equals

$$\frac{1}{n^2} \sum_{\gamma \in P(F) \backslash \tilde{G}^{\square, kmn_Q}(F) / \iota(G \times G) N^\bullet_{\mathcal{W}}(F)} K_\gamma,$$

where

$$K_\gamma = \int\limits_{\iota^{-1}(\gamma P \gamma^{-1} \cap \iota(G \times G))(F) \backslash (\overline{G} \times \overline{G})(\mathbb{A})} \chi_\theta(\nu(g_1))^{-1} \xi_1(g_1) \xi_2(g_2) I_\gamma(\iota(g_1, g_2))\, dg_1\, dg_2.$$

We now apply results in Proposition 7.2. The double coset $P(F) \backslash \tilde{G}^{\square, kmn_Q}(F) / \iota(G \times G) N^\bullet_{\mathcal{W}}(F)$ is finite and the only open coset is $P(F) \iota(G \times G) N^\bullet_{\mathcal{W}}(F)$. For negligible double cosets, by Propositions 7.2 and 6.8, we know

$$J_\gamma(\iota(1, g_2)h) = J_\gamma(h)$$

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for $g_2 \in N_-(\mathbb{A})$ and $h \in \overline{G^{\square,kmn_Q}}(\mathbb{A})$. The proof as in [5] Proposition 6.7 shows that

$$I_\gamma(\iota(1,g_2)h) = I_\gamma(h)$$

for $g_2 \in N_-(\mathbb{A})$ and $h \in \overline{G^{\square,kmn_Q}}(\mathbb{A})$. Here we need to use Lemma 5.2.

We conclude that for a negligible double coset, the contribution K_γ contains the inner integral

$$\int\limits_{[N_-]} \xi_2(ug_2)\ du.$$

This is zero since $\pi^\vee$ is cuspidal. Thus, only the main orbit has nonzero contribution. We have arrived at

$$Z(\xi_1 \boxtimes \xi_2, \phi^{(s)}) = \frac{1}{n^2} \int\limits_{G^\diamond(F)\backslash(\overline{G}\times\overline{G})(\mathbb{A})} \chi_\theta(\nu(g_2))^{-1}\xi_1(g_1)\xi_2(g_2)I_1(\iota(g_1,g_2))\ dg_1\ dg_2.$$

This proves the result. □

9.6. Euler product

We can indeed rewrite $Z(\xi_1 \boxtimes \xi_2, \phi^{(s)})$ as an Euler product using uniqueness of $(kn_Q,m)_D$-models for θ.

Let $\mathbf{N}^\circ_{\mathcal{W}} = \mathbf{N}^\bullet_{\mathcal{W}} \cap \mathbf{N}(W^{\nabla,kn_Q})$. Then for any $h \in \overline{G^{\square,kmn_Q}}(\mathbb{A})$,

$$\begin{aligned}
&\int\limits_{(N^\bullet_{\mathcal{W}}\cap P)(F)\backslash N^\bullet_{\mathcal{W}}(\mathbb{A})} \phi^{(s)}(u\cdot\iota(g_1,g_2)h)\psi^\bullet_{\mathcal{W}}(u)\ du \\
&= \int\limits_{N^\circ_{\mathcal{W}}(\mathbb{A})} \int\limits_{[N^\bullet_{\mathcal{W}}\cap P]} \phi^{(s)}(uu'\cdot\iota(g_1,g_2)h)\psi^\bullet_{\mathcal{W}}(uu')\ du\ du' \\
&= \int\limits_{N^\circ_{\mathcal{W}}(\mathbb{A})} f^{(s)}(u'\cdot\iota(g_1,g_2)h)\psi^\bullet_{\mathcal{W}}(u')\ du' \\
&= \int\limits_{N^\circ_{\mathcal{W}}(\mathbb{A})} f^{(s)}(\iota(g_2,g_2)\cdot u'\cdot\iota(g_2^{-1}g_1,1)h)\psi^\bullet_{\mathcal{W}}(u')\ du' \\
&= \chi_\theta(\nu(g_2)) \int\limits_{N^\circ_{\mathcal{W}}(\mathbb{A})} f^{(s)}(u\cdot\iota(g_2^{-1}g_1,1)h)\psi^\bullet_{\mathcal{W}}(u)\ du.
\end{aligned}$$

Observe that we use change of variable and Lemma 5.2 in the third equality and Lemma 9.2 in the last equality. We also use the fact that the canonical lift of $N^\bullet_{\mathcal{W}}(\mathbb{A})$ is

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$G^{\square,kmn_Q}(\mathbb{A})$-equivariant. As a consequence, we can write $Z(\xi_1\boxtimes\xi_2,\phi^{(s)})$ as

$$\begin{aligned}
&\frac{1}{n^2}\int\limits_{G^\diamond(F)\backslash(\overline{G}\times\overline{G})(\mathbb{A})}\xi_1(g_1)\xi_2(g_2)\int\limits_{N^\circ_{\mathcal{W}}(\mathbb{A})}f^{(s)}(u\cdot\iota(g_2^{-1}g_1,1))\psi^\bullet_{\mathcal{W}}(u)\,du\,dg_2\,dg_1\\
&=\frac{1}{n^2}\int\limits_{G^\diamond(F)\backslash(\overline{G}\times\overline{G})(\mathbb{A})}\xi_1(g_2g_1)\xi_2(g_2)\int\limits_{N^\circ_{\mathcal{W}}(\mathbb{A})}f^{(s)}(u\cdot\iota(g_1,1))\psi^\bullet_{\mathcal{W}}(u)\,du\,dg_2\,dg_1\\
&=\frac{1}{n^2}\int\limits_{\overline{G}(\mathbb{A})}\int\limits_{[\overline{G}]}\xi_1(g_2g_1)\xi_2(g_2)\int\limits_{N^\circ_{\mathcal{W}}(\mathbb{A})}f^{(s)}(u\cdot\iota(g_1,1))\psi^\bullet_{\mathcal{W}}(u)\,du\,dg_2\,dg_1\\
&=\frac{1}{n}\int\limits_{\overline{G}(\mathbb{A})}\mathcal{P}(\pi(g)\xi_1\boxtimes\xi_2)\int\limits_{N^\circ_{\mathcal{W}}(\mathbb{A})}f^{(s)}(u\cdot\iota(g,1))\psi^\bullet_{\mathcal{W}}(u)\,du\,dg.
\end{aligned}$$

For decomposable data, it follows from uniqueness of $(kn_Q,m)_D$-models for θ_v that

$$f^{(s)}(g)=\prod_v f_v^{(s)}(g_v).$$

If, furthermore, $\xi_i=\otimes_v\xi_{i,v}$, then

$$Z(\xi_1\boxtimes\xi_2,\phi^{(s)})=\prod_v Z_v(\xi_{1,v}\boxtimes\xi_{2,v},f_v^{(s)}),$$

where

$$Z_v(\xi_{1,v}\boxtimes\xi_{2,v},f_v^{(s)})=\frac{1}{n}\int\limits_{\overline{G}_v}\mathcal{P}_v(\pi(g)\xi_{1,v}\boxtimes\xi_{2,v})\int\limits_{N^\circ_{\mathcal{W}}}f_v^{(s)}(u\cdot\iota(g,1))\psi^\bullet_{\mathcal{W}}(u)\,du\,dg.$$

10. BD data of pullback

The goal in this section is to prove the unproven results in Section 8.

10.1. Results

Let $\mathbf{G}$ be one of the following groups:

(1) $\mathbf{Sp}(W)$ or its inner forms;
(2) $\mathbf{SO}(W)$ with $\dim W$ even or its inner forms;
(3) $\mathbf{SO}(W)$ with $\dim W$ odd and $\dim W\geq 3$;
(4) $\mathbf{U}(W)$.

Observe that over F_s, we have to consider the following groups: $\mathbf{Sp}_{2m}$,$\mathbf{SO}_{2m}$,$\mathbf{SO}_{2m+1}$ and $\mathbf{GL}_m$.

Let $\overline{\mathbf{G}}\in\mathbf{CExt}(\mathbf{G},\mathbf{K}_2)$ with BD data $(Q,\mathcal{E},f)$. For simplicity, we write $\mathbf{G}^\square:=\mathbf{G}^{\square,1}$. Recall that in the unitary group case, we assume that the quadratic form Q in the BD data is decomposable.

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Definition 10.1. We define the subcategory

$$\mathbf{CExt}(\mathbf{G},\mathbf{K}_2)^{\Delta} \subset \mathbf{CExt}(\mathbf{G},\mathbf{K}_2) \times \mathbf{CExt}(\mathbf{G},\mathbf{K}_2)$$

as follows: a pair $(\overline{\mathbf{G}},\overline{\mathbf{G}}')$ is in the subcategory if and only if $\overline{\mathbf{G}}'$ is isomorphic to $\overline{\mathbf{G}}$.

Definition 10.2. We define the subcategory

$$\mathbf{CExt}(\mathbf{G},\mathbf{K}_2)^{\natural} \subset \mathbf{CExt}(\mathbf{G},\mathbf{K}_2) \times \mathbf{CExt}(\mathbf{G},\mathbf{K}_2)$$

as follows: a pair $(\overline{\mathbf{G}}',\overline{\mathbf{G}})$ is in the subcategory if and only if $\overline{\mathbf{G}}'$ is isomorphic to the pushout of $\overline{\mathbf{G}}$ by the map $[2k-1]$.

Definition 10.3. We define the subcategory

$$\mathbf{CExt}(\mathbf{G}^{\square},\mathbf{K}_2)^{\Delta}$$

of $\mathbf{CExt}(\mathbf{G}^{\square},\mathbf{K}_2) \times \cdots \times \mathbf{CExt}(\mathbf{G}^{\square},\mathbf{K}_2)$ as follows: an object $(\overline{\mathbf{G}_1^{\square}},\cdots,\overline{\mathbf{G}_k^{\square}})$ is in the subcategory if and only if the multiplicative $\mathbf{K}_2$-torsors $\overline{\mathbf{G}_1^{\square}},\cdots,\overline{\mathbf{G}_k^{\square}}$ are isomorphic.

Proposition 10.4. *Let $\overline{\mathbf{G}^{\square}} \in \mathbf{CExt}(\mathbf{G}^{\square},\mathbf{K}_2)$. The pullback of $\overline{\mathbf{G}^{\square}}$ via the doubling homomorphism $\iota : \mathbf{G} \times \mathbf{G} \to \mathbf{G}^{\square}$ gives two multiplicative $\mathbf{K}_2$-torsors on $\mathbf{G}$:*

$$1 \to \mathbf{K}_2 \to \overline{\mathbf{G}}_1 \to \iota(1 \times \mathbf{G}) \to 1$$

and

$$1 \to \mathbf{K}_2 \to \overline{\mathbf{G}}_2 \to \iota(\mathbf{G} \times 1) \to 1.$$

Then the resulting functor

$$\mathbf{CExt}(\mathbf{G}^{\square},\mathbf{K}_2) \to \mathbf{CExt}(\mathbf{G},\mathbf{K}_2) \times \mathbf{CExt}(\mathbf{G},\mathbf{K}_2)$$

is essentially surjective on $\mathbf{CExt}(\mathbf{G},\mathbf{K}_2)^{\Delta}$.

Proposition 10.5. *The pullback via the homomorphism $\mathbf{G}^{\square} \times \cdots \times \mathbf{G}^{\square} \to \mathbf{G}^{\square,k}$ gives a functor*

$$\mathbf{CExt}(\mathbf{G}^{\square,k},\mathbf{K}_2) \to \mathbf{CExt}(\mathbf{G}^{\square},\mathbf{K}_2) \times \cdots \times \mathbf{CExt}(\mathbf{G}^{\square},\mathbf{K}_2).$$

Here, both $\mathbf{G}^{\square}$ and $\mathbf{CExt}(\mathbf{G}^{\square},\mathbf{K}_2)$ appear k times. This functor is essentially surjective on $\mathbf{CExt}(\mathbf{G}^{\square},\mathbf{K}_2)^{\Delta}$.

With the above two propositions, we deduce the following fact.

Corollary 10.6. *The functor induced by the doubling isomorphism*

$$\mathbf{CExt}(\mathbf{G}^{\square,k},\mathbf{K}_2) \to \mathbf{CExt}(\mathbf{G},\mathbf{K}_2) \times \mathbf{CExt}(\mathbf{G},\mathbf{K}_2)$$

is essentially surjective on $\mathbf{CExt}(\mathbf{G} \times \mathbf{G},\mathbf{K}_2)^{\natural}$.

Proof. We first consider

$$\mathbf{G} \times \mathbf{G} \to \mathbf{G} \times \cdots \times \mathbf{G}, \qquad (g_1,g_2) \mapsto (g_1,g_2,g_1,g_1,\cdots,g_1,g_1).$$

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Here the target has $2k$ copies of $\mathbf{G}$. Then we can write $\iota : \mathbf{G} \times \mathbf{G} \to \mathbf{G}^{\square,k}$ as the composition of

$$\mathbf{G} \times \mathbf{G} \to \mathbf{G} \times \cdots \times \mathbf{G} \to \mathbf{G}^{\square} \times \cdots \times \mathbf{G}^{\square} \to \mathbf{G}^{\square,k}. \tag{15}$$

By Propositions 10.4 and 10.5, we can find $\overline{\mathbf{G}^{\square,k}}$ so that its pullbacks to each $\mathbf{G}$ under the map

$$\mathbf{G} \times \cdots \times \mathbf{G} \to \mathbf{G}^{\square,k}$$

are all isomorphic to $\overline{\mathbf{G}}$. Thus, $\overline{\mathbf{G}^{\square,k}}$ pulls back to $\overline{\mathbf{G}}$ for the second copy of $\mathbf{G}$ in $\iota : \mathbf{G} \times \mathbf{G} \to \mathbf{G}^{\square,k}$.

We now consider the pullback to the first copy. Then (15) restricts to

$$\mathbf{G} \times \{1\} \to \mathbf{G} \times \{1\} \times \mathbf{G} \times \mathbf{G} \times \cdots \times \mathbf{G} \to \mathbf{G}^{\square} \times \mathbf{G}^{\square} \times \cdots \times \mathbf{G}^{\square} \to \mathbf{G}^{\square,k}.$$

We temporarily focus on the second map. By Lemma 3.3, the copies of $\overline{\mathbf{G}}$ commute in $\overline{\mathbf{G}^{\square,k}}$, which gives a homomorphism

$$\overline{\mathbf{G}} \times \{1\} \times \overline{\mathbf{G}} \times \overline{\mathbf{G}} \cdots \times \overline{\mathbf{G}} \to \overline{\mathbf{G}^{\square,k}}.$$

The multiplicative $\mathbf{K}_2$-torsor $\overline{\mathbf{G} \times \{1\} \times \mathbf{G} \times \mathbf{G} \times \cdots \times \mathbf{G}}$ obtained by pulling pack along

$$\mathbf{G} \times \{1\} \times \mathbf{G} \times \mathbf{G} \times \cdots \times \mathbf{G} \to \mathbf{G}^{\square,k}$$

is isomorphic to the pushout of $\overline{\mathbf{G}} \times \{1\} \times \overline{\mathbf{G}} \times \overline{\mathbf{G}} \times \cdots \times \overline{\mathbf{G}}$ via the product map $\mathbf{K}_2 \times \cdots \times \mathbf{K}_2 \to \mathbf{K}_2$.

Finally, to obtain the pullback to the first copy of $\mathbf{G}$ from $\overline{\mathbf{G}^{\square,k}}$, we need to pull back $\overline{\mathbf{G} \times \{1\} \times \mathbf{G} \times \mathbf{G} \times \cdots \times \mathbf{G}}$ via the diagonal map

$$\mathbf{G} \to \mathbf{G} \times \{1\} \times \mathbf{G} \times \mathbf{G} \times \cdots \times \mathbf{G}, \qquad g_1 \mapsto (g_1, 1, g_1, g_1, \cdots, g_1).$$

It follows from the definition that $\overline{\mathbf{G}}^{\natural}$ is the Baer sum of $2k-1$ copies of $\overline{\mathbf{G}}$. □

Proposition 10.7. *In the case of unitary groups, we assume that the quadratic form Q is decomposable. For the multiplicative $\mathbf{K}_2$-torsor $\overline{\mathbf{G}^{\square,k}}$ on $\mathbf{G}^{\square,k}$ given by Corollary 10.6, $\overline{\mathbf{G}}$ and $\overline{\mathbf{G}}^{\natural}$ commute in $\overline{\mathbf{G}^{\square,k}}$.*

Proof. This follows from Lemma 3.3 and a simple calculation on the quadratic form. See also the details in all of the cases. □

The proof of Proposition 10.5 is similar to (and easier than) Proposition 10.4. So we will only give the details in the latter case. The proof of Proposition 10.4 will be given case by case in the rest of this section.

10.2. Strategy of the proofs

In this section, we explain the strategy of the proof of Proposition 10.4 and set up some notation that is commonly used.

Let $\mathbf{T}$ be a maximal F-torus of $\mathbf{G}$. Then $\iota(\mathbf{T} \times \mathbf{T}) \subset \iota(\mathbf{G} \times \mathbf{G})$ is an F-torus of $\mathbf{G}^{\square}$. Let $\mathbf{T}^{\square} \supset \iota(\mathbf{T} \times \mathbf{T})$ be a maximal F-torus of $\mathbf{G}^{\square}$. Note that $\mathbf{T}^{\square} = \iota(\mathbf{T} \times \mathbf{T})$ except in the case

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of odd $\mathbf{SO}(W)$. Observe that this is not a maximal F-torus in $\mathbf{P}(W^{\Delta})$ but only up to a conjugation over F_s. We still use this nonsplit torus even if $\mathbf{G}^{\square}$ might be a split group over F.

The torus $\mathbf{T}$ splits over F_s. Let Y be the cocharacter lattice of $\mathbf{T}$ over F_s. Let $Y^{\square}$ be the cocharacter lattice of $\mathbf{T}^{\square}$ over F_s. Then $Y^{\square} \supset Y \oplus Y$. We first fix a Chevalley system of pinning for $(\mathbf{G}_{F_s}, \mathbf{T}_{F_s})$ and then choose one for $(\mathbf{G}^{\square}_{F_s}, \mathbf{T}^{\square}_{F_s})$ which is compatible with $\iota : \mathbf{G} \times \mathbf{G} \to \mathbf{G}^{\square}$.

Notation: the BD data for $\overline{\mathbf{G}}$ are denoted $(Q, \mathcal{E}, f)$; to distinguish the BD data for the two copies of $\overline{\mathbf{G}}$, we use $(Q_+, \mathcal{E}_+, f_+)$ and $(Q_-, \mathcal{E}_-, f_-)$ when needed. The BD data for $\overline{\mathbf{G}}^{\square}$ are denoted $(Q^{\square}, \mathcal{E}^{\square}, f^{\square})$.

Definition 10.8. We define the subcategory

$$\mathbf{BD}(\mathbf{G},\mathbf{T})^{\Delta} \subset \mathbf{BD}(\mathbf{G},\mathbf{T}) \times \mathbf{BD}(\mathbf{G},\mathbf{T})$$

as follows: the pair of triples $(Q, \mathcal{E}, f)$ and $(Q', \mathcal{E}', f')$ is in the subcategory if and only if $(Q, \mathcal{E}, f)$ is isomorphic to $(Q', \mathcal{E}', f')$.

It is easy to see that Proposition 10.4 is equivalent to the following result.

Proposition 10.9. *The functor induced by pulling back via* $\mathbf{G} \times \mathbf{G} \to \mathbf{G}^{\square}$

$$\mathbf{BD}(\mathbf{G}^{\square}, \mathbf{T}^{\square}) \to \mathbf{BD}(\mathbf{G},\mathbf{T}) \times \mathbf{BD}(\mathbf{G},\mathbf{T})$$

is essentially surjective on $\mathbf{BD}(\mathbf{G},\mathbf{T})^{\Delta}$.

We will prove this result case by case.

10.3. A useful lemma

Given an exact sequence

$$1 \to F_s^{\times} \to \mathcal{E} \to Y \to 1,$$

we can push out the direct sum of two copies via the product map $\mathrm{pr} : F_s^{\times} \times F_s^{\times} \to F_s^{\times}$ to obtain

$$1 \to F_s^{\times} \to \mathrm{pr}_*(\mathcal{E} \oplus \mathcal{E}) \to Y \oplus Y \to 1.$$

We now give a useful criterion to compare elements in $\mathrm{pr}_*(\mathcal{E} \oplus \mathcal{E})$. Recall that

$$\mathrm{pr}_*(\mathcal{E} \oplus \mathcal{E}) = F_s^{\times} \times (\mathcal{E} \oplus \mathcal{E}) / \langle (x_1 x_2, x_1^{-1}, x_2^{-1}) : (x_1, x_2) \in F_s^{\times} \times F_s^{\times} \rangle.$$

Define

$$\mathrm{mul} : F_s^{\times} \times (\mathcal{E} \oplus \mathcal{E}) \to \mathcal{E}, \qquad (x, e_1, e_2) \mapsto x e_1 e_2.$$

This gives a well-defined map $\mathrm{mul} : \mathrm{pr}_*(\mathcal{E} \oplus \mathcal{E}) \to \mathcal{E}$.

Lemma 10.10. *Let* (x, e_1, e_2) *and* (x', e_1', e_2') *be two elements in* $F_s^{\times} \times (\mathcal{E} \oplus \mathcal{E})$. *They have the same image in* $\mathrm{pr}_*(\mathcal{E} \oplus \mathcal{E})$ *if and only if* (e_1, e_2) *and* (e_1', e_2') *have the same image in* $Y \oplus Y$ *and* $\mathrm{mul}(x, e_1, e_2) = \mathrm{mul}(x', e_1', e_2')$.

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Proof. The 'only if' part is trivial. We now prove the 'if' part. If (e_1,e_2) and (e_1',e_2') have the same image in $Y\oplus Y$, then $(e_1',e_2')=(e_1y_1,e_2y_2)$ for some $y_1,y_2\in F_s^\times$. The condition $\mathrm{mul}(x,e_1,e_2)=\mathrm{mul}(x',e_1',e_2')$ implies that $x=x'y_1y_2$. This shows that $(x',e_1',e_2')=(xy_1^{-1}y_2^{-1},e_1y_1,e_2y_2)$. The proof is complete. □

10.4. Symplectic groups

We now discuss the case of symplectic groups or their inner forms. This is probably the easiest case since $\mathbf{G}$ is simply connected. A multiplicative $\mathbf{K}_2$-torsor $\overline{\mathbf{G}}$ on $\mathbf{G}$ is determined by a Galois invariant Weyl group invariant quadratic form Q on Y. In [27] Proposition 3.15, it is shown that for every integer a, there is a unique such quadratic form on Y such that its value on a short coroot is a. In other words, we have an equivalence of categories

$$\mathbf{CExt}(\mathbf{G},\mathbf{K}_2)\to\mathbf{BD}(\mathbf{G},\mathbf{T})\to\mathbb{Z},$$

where the last functor sends a quadratic form to its value on a short coroot.

The torus $\mathbf{T}\times\mathbf{T}$ is a maximal F-torus of $\mathbf{G}^\square$, so $\mathbf{T}^\square=\mathbf{T}\times\mathbf{T}$. The cocharacter lattice of $\mathbf{T}^\square$ over F_s is $Y^\square=Y\oplus Y$. The functor

$$\mathbf{CExt}(\mathbf{G}^\square,\mathbf{K}_2)\to\mathbf{CExt}(\mathbf{G},\mathbf{K}_2)\times\mathbf{CExt}(\mathbf{G},\mathbf{K}_2)$$

can be described in terms of

$$\mathbf{BD}(\mathbf{G}^\square,\mathbf{T}^\square)\to\mathbf{BD}(\mathbf{G},\mathbf{T})\times\mathbf{BD}(\mathbf{G},\mathbf{T}).$$

In terms of $\mathbb{Z}\to\mathbb{Z}\times\mathbb{Z}$, it is simply $a\mapsto(a,a)$. Proposition 10.9 follows trivially.

10.5. Special even orthogonal groups and inner forms

We now consider the case of special even orthogonal groups. We start with some basic results.

10.5.1. Preparation. Let $n=(d\cdot\dim_D W)/2$. (The results in this section only involve multiplicative $\mathbf{K}_2$-torsors. This n has no relation with the degree of the cover. It will not cause any confusion.) We choose a standard basis of $Y=\mathbb{Z}^n=\mathrm{Span}\{e_1^\vee,\cdots,e_n^\vee\}$ so that the root lattice is given by

$$Y^{sc}=\mathrm{Span}\{\alpha_1^\vee,\cdots,\alpha_n^\vee\}=\mathrm{Span}\{e_1^\vee-e_2^\vee,\cdots,e_{n-1}^\vee-e_n^\vee,e_{n-1}^\vee+e_n^\vee\}.$$

Note that Y^{sc} is a sublattice of Y with index 2. A W-invariant quadratic form Q is determined by its value on a coroot. Let $Q(\alpha_1^\vee)=a$.

Lemma 10.11. *We have* $2\mid a$.

Proof. We know that $Q(\alpha_{n-1}^\vee)=Q(\alpha_n^\vee)=a$. Let $Q(e_n^\vee)=b\in\mathbb{Z}$. As $Q(2e_n^\vee)=Q(\alpha_{n-1}^\vee)+Q(\alpha_n^\vee)$, we have $4b=2a$. This implies $2\mid a$. □

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The torus $\mathbf{T}^{\square} = \mathbf{T} \times \mathbf{T}$ is a maximal F-torus of $\mathbf{G}^{\square}$. Its cocharacter lattice over F_s is $Y^{\square} = Y_+ \oplus Y_-$. We write

$$\begin{aligned} Y_+^{sc} &= \mathrm{Span}\{e_1^\vee - e_2^\vee, \cdots, e_{n-1}^\vee - e_n^\vee, e_{n-1}^\vee + e_n^\vee\} \\ Y_-^{sc} &= \mathrm{Span}\{e_{n+1}^\vee - e_{n+2}^\vee, \cdots, e_{2n-1}^\vee - e_{2n}^\vee, e_{2n-1}^\vee + e_{2n}^\vee\} \\ Y^{\square,sc} &= \mathrm{Span}\{e_1^\vee - e_2^\vee, \cdots, e_{2n-1}^\vee - e_{2n}^\vee, e_{2n-1}^\vee + e_{2n}^\vee\}. \end{aligned}$$

Thus, $Y_+^{sc} \oplus Y_-^{sc}$ is a subgroup of $Y^{\square,sc}$ of index 2 and $e_1^\vee - e_{n+1}^\vee \notin Y_+^{sc} \oplus Y_-^{sc}$.

Let $Q^{\square}$ be a W-invariant quadratic form on $Y^{\square}$.

Lemma 10.12. *The restriction of $Q^{\square}$ to $Y_+ \oplus Y_-$ is a direct sum of two quadratic forms $Q_+ \oplus Q_-$ with $Q_+ = Q_-$. In particular, we have*

$$B_{Q^{\square}}((y_+,0),(0,y_-)) = 0$$

for $y_+ \in Y_+$ and $y_- \in Y_-$.

Conversely, for a W-invariant quadratic form Q on Y, there is a unique W-invariant quadratic form $Q^{\square}$ which restricts to $Q_+ \oplus Q_-$ on $Y_+ \oplus Y_-$.

Proof. The proof is straightforward. □

10.5.2. Construction of BD data. Let $\overline{\mathbf{G}}$ be a multiplicative $\mathbf{K}_2$-torsor on $\mathbf{G}$ with BD data $(Q,\mathcal{E},f)$. We use it to construct a Galois equivariant triple $(Q^{\square},\mathcal{E}^{\square},f^{\square})$.

Let Q be a Galois invariant W-invariant quadratic form on Y. Define $Q^{\square} = Q \oplus Q$ to be a quadratic form on $Y^{\square} = Y \oplus Y$. This is a W-invariant quadratic form and thus Galois invariant from the proof of [27] Proposition 3.15.

Second, from the exact sequence for $\mathcal{E}$, we form the exact sequence

$$1 \to F_s^\times \oplus F_s^\times \to \mathcal{E} \oplus \mathcal{E} \to Y \oplus Y = Y^{\square} \to 1.$$

We push it out via the product map $\mathrm{pr} : F_s^\times \oplus F_s^\times \to F_s^\times$ to obtain

$$\begin{array}{ccccccccc} 1 & \longrightarrow & F_s^\times \oplus F_s^\times & \longrightarrow & \mathcal{E} \oplus \mathcal{E} & \longrightarrow & Y \oplus Y & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & F_s^\times & \longrightarrow & \mathrm{pr}_*(\mathcal{E} \oplus \mathcal{E}) & \longrightarrow & Y^{\square} & \longrightarrow & 1 \end{array}$$

The commutator map of the bottom exact sequence is given by $[y_1,y_2] = (-1)^{B_{Q^{\square}}(y_1,y_2)}$. Set $\mathcal{E}^{\square} := \mathrm{pr}_*(\mathcal{E} \oplus \mathcal{E})$. The bottom exact sequence will be the second BD invariant for $\mathbf{G}^{\square}$.

We also have a natural map $\mathrm{pr}_*(f \oplus f) : \mathrm{pr}_*(\mathcal{E}_{Q^{sc}} \oplus \mathcal{E}_{Q^{sc}}) \to \mathrm{pr}_*(\mathcal{E} \oplus \mathcal{E})$ which fits into

$$\begin{array}{ccccccccc} 1 & \longrightarrow & F_s^\times & \longrightarrow & \mathrm{pr}_*(\mathcal{E}_{Q^{sc}} \oplus \mathcal{E}_{Q^{sc}}) & \longrightarrow & Y^{sc} \oplus Y^{sc} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & F_s^\times & \longrightarrow & \mathrm{pr}_*(\mathcal{E} \oplus \mathcal{E}) & \longrightarrow & Y^{\square} & \longrightarrow & 1 \end{array}$$

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Let $\mathbf{G}^{sc} = \mathbf{Spin}_{2n} \to \mathbf{G} = \mathbf{SO}_{2n}$. The multiplicative $\mathbf{K}_2$-torsor of $\mathbf{Spin}_{2n}$ is determined by a quadratic form $Y = Y^{sc}$, which is determined by its values on any simple coroots. From the map

$$\mathbf{Spin}_{2n} \times \mathbf{Spin}_{2n} \to \mathbf{Spin}_{4n},$$

we obtain the following commutative diagram:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & F_s^\times \oplus F_s^\times & \longrightarrow & \mathcal{E}_{Q^{sc}} \oplus \mathcal{E}_{Q^{sc}} & \longrightarrow & Y_+^{sc} \oplus Y_-^{sc} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & F_s^\times & \longrightarrow & \mathcal{E}_{Q^{\square,\,sc}} & \longrightarrow & Y^{\square,\,sc} & \longrightarrow & 1 \end{array},$$

where the first vertical map is given by multiplication. By our choice of Chevelley system of épinglage, under the second vertical map, the image of $s_{Q_\pm^{sc}}(\alpha^\vee)$ is $s_{Q^{\square,\,sc}}(\alpha^\vee)$ for a root α in $Y_\pm^{sc}$. This commutative diagram factors through pushing out by $\mathrm{pr} : F_s^\times \oplus F_s^\times \to F_s^\times$ and we have the following commutative diagram:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & F_s^\times & \longrightarrow & \mathrm{pr}_*(\mathcal{E}_{Q^{sc}} \oplus \mathcal{E}_{Q^{sc}}) & \longrightarrow & Y_+^{sc} \oplus Y_-^{sc} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & F_s^\times & \longrightarrow & \mathcal{E}_{Q^{\square,\,sc}} & \longrightarrow & Y^{\square,\,sc} & \longrightarrow & 1 \end{array}.$$

We now want to construct the third BD invariant. That is, we need to construct a map

$$\begin{array}{ccccccccc} 1 & \longrightarrow & F_s^\times & \longrightarrow & \mathcal{E}_{Q^{\square,\,sc}} & \longrightarrow & Y^{\square,\,sc} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & F_s^\times & \longrightarrow & \mathrm{pr}_*(\mathcal{E} \oplus \mathcal{E}) & \longrightarrow & Y^{\square} & \longrightarrow & 1 \end{array}$$

in which the middle map $f^\square$ extends $\mathrm{pr}_*(\mathcal{E}_{Q^{sc}} \oplus \mathcal{E}_{Q^{sc}}) \to \mathrm{pr}_*(\mathcal{E} \oplus \mathcal{E})$. We also use the notation f_+ to denote the map

$$\mathcal{E}_{Q^{sc}} \xrightarrow{e \mapsto (e,1)} \mathcal{E}_{Q^{sc}} \oplus \mathcal{E}_{Q^{sc}} \to \mathrm{pr}_*(\mathcal{E}_{Q^{sc}} \oplus \mathcal{E}_{Q^{sc}}) \to \mathrm{pr}_*(\mathcal{E} \oplus \mathcal{E}).$$

Similarly, we define f_-.

The map $f^\square$ is determined by its images on $\{s_{Q^{\square,\,sc}}(\alpha^\vee) \mid \alpha^\vee \in \Delta^\vee_{\mathbf{G}^\square}\}$. Since we require that

$$f^\square(s_{Q^{\square,\,sc}}(\alpha^\vee)) = f_\pm(s_{Q_\pm^{sc}}(\alpha^\vee))$$

for a root α in $Y_\pm^{sc}$, this trivially determines $f^\square$ by $f_+ \oplus f_-$ except $f^\square(s_{Q^{\square,\,sc}}(e_n^\vee - e_{n+1}^\vee))$. We have to choose this value so that

$$f^\square(s_{Q^{\square,\,sc}}(e_{n-1}^\vee + e_n^\vee)) = f_+(s_{Q^{sc}}(e_{n-1}^\vee + e_n^\vee)).$$

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We find that it would be slightly more convenient to work with the following setup: we choose $f^{\square}(s_{Q^{\square,sc}}(e_1^\vee - e_{n+1}^\vee))$ so that

$$f^{\square}(s_{Q^{\square,sc}}(e_1^\vee + e_2^\vee)) = f_+(s_{Q^{sc}}(e_1^\vee + e_2^\vee)).$$

Lemma 10.13. *If we choose $f^{\square}(s_{Q^{\square,sc}}(e_1^\vee - e_{n+1}^\vee))$ as the unique element such that its projection to $Y \oplus Y$ is $e_1^\vee - e_{n+1}^\vee$ and $\mathrm{mul}(f^{\square}(s_{Q^{\square,sc}}(e_1^\vee - e_{n+1}^\vee))) = 1$, then $f^{\square}(s_{Q^{\square,sc}}(e_1^\vee + e_2^\vee)) = f_+(s_{Q^{sc}}(e_1^\vee + e_2^\vee))$. In other words, $f^{\square}$ is an extension of $f \oplus f$.*

Proof. Recall that

$$e_1^\vee + e_2^\vee = (e_2^\vee - e_1^\vee) + (e_1^\vee - e_{n+1}^\vee) + (e_{n+1}^\vee - e_{n+2}^\vee) + (e_{n+1}^\vee + e_{n+2}^\vee) + (e_1^\vee - e_{n+1}^\vee).$$

This decomposition satisfies the condition $(*)$ in (4). By Lemma 2.6 (note that $2 \mid Q(\alpha^\vee)$),

$$\begin{aligned} s_{Q^{\square,sc}}(e_1^\vee + e_2^\vee) = {} & s_{Q^{\square,sc}}(e_1^\vee - e_{n+1}^\vee) s_{Q^{\square,sc}}(e_{n+1}^\vee + e_{n+2}^\vee) s_{Q^{\square,sc}}(e_{n+1}^\vee - e_{n+2}^\vee) \\ & \times s_{Q^{\square,sc}}(e_1^\vee - e_{n+1}^\vee) s_{Q^{\square,sc}}(e_2^\vee - e_1^\vee). \end{aligned}$$

This implies that $f^{\square}(s_{Q^{\square,sc}}(e_1^\vee + e_2^\vee)) =$

$$\begin{aligned} & f^{\square}(s_{Q^{\square,sc}}(e_1^\vee - e_{n+1}^\vee)) f^{\square}(s_{Q^{\square,sc}}(e_{n+1}^\vee + e_{n+2}^\vee)) f^{\square}(s_{Q^{\square,sc}}(e_{n+1}^\vee - e_{n+2}^\vee)) \\ & \times f^{\square}(s_{Q^{\square,sc}}(e_1^\vee - e_{n+1}^\vee)) f^{\square}(s_{Q^{\square,sc}}(e_2^\vee - e_1^\vee)) \end{aligned}$$

or

$$\begin{aligned} & f^{\square}(s_{Q^{\square,sc}}(e_1^\vee - e_{n+1}^\vee)) f_-(s_{Q^{sc}}(e_{n+1}^\vee + e_{n+2}^\vee)) f_-(s_{Q^{sc}}(e_{n+1}^\vee - e_{n+2}^\vee)) \\ & \times f^{\square}(s_{Q^{\square,sc}}(e_1^\vee - e_{n+1}^\vee)) f_+(s_{Q^{sc}}(e_2^\vee - e_1^\vee)). \end{aligned}$$

We now calculate its image under mul. We observe that

$$\mathrm{mul}\left(\frac{f_-(s_{Q^{sc}}(e_{n+1}^\vee - e_{n+2}^\vee)) f_-(s_{Q^{sc}}(e_{n+1}^\vee + e_{n+2}^\vee))}{f_+(s_{Q^{sc}}(e_1^\vee - e_2^\vee)) f_+(s_{Q^{sc}}(e_1^\vee + e_2^\vee))}\right) = 1.$$

From our choice of $f^{\square}(s_{Q^{\square,sc}}(e_1^\vee - e_{n+1}^\vee))$,

$$\mathrm{mul}(f^{\square}(s_{Q^{\square,sc}}(e_1^\vee - e_{n+1}^\vee))) = 1.$$

Using these two facts, together with Lemma 2.5 part (2), we deduce that

$$\mathrm{mul}(f^{\square}(s_{Q^{\square,sc}}(e_1^\vee + e_2^\vee))) = \mathrm{mul}(f_+(s_{Q^{sc}}(e_1^\vee + e_2^\vee))).$$

By Lemma 10.10, this shows that

$$f^{\square}(s_{Q^{\square,sc}}(e_1^\vee + e_2^\vee)) = f_+(s_{Q^{sc}}(e_1^\vee + e_2^\vee)).$$

This completes the proof. □

Remark 10.14. It seems that the choice of $f^{\square}(s_{Q^{\square,sc}}(e_1^\vee - e_{n+1}^\vee))$ is quite delicate, but it is not hard to see that this is almost the only choice. In the next section, we will show that $f^{\square}$ is also Galois invariant by carefully analyzing the Galois action on $\mathcal{E}$.

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10.5.3. Digression on the Chevalley system. To show that $f^{\square}$ is Γ-equivariant, we need to have some understanding of the Galois action on $\mathcal{E}_{Q^{\square,sc}}$. The Galois action on $\mathcal{E}_{Q^{\square,sc}}$ comes from the (possibly nonsplit) maximal F-torus $\mathbf{T}^{\square,sc}$. We still denote this action as σ. We now would like to understand $\sigma(s_{Q^{\square,sc}}(e_1^\vee - e_{n+1}^\vee))$ for $\sigma \in \Gamma$.

We start with some general facts about reductive groups over F_s. Let $\mathrm{Aut}(\mathbf{G}_{F_s})$ be the automorphism group of $\mathbf{G}_{F_s}$. Let $\mathrm{Inn}(\mathbf{G}_{F_s})$ denote the subgroup of inner automorphisms. Let $\tau \in \mathrm{Aut}(\mathbf{G}_{F_s})$. Let $(\mathbf{B},\mathbf{T})$ be a choice of Borel subgroup and maximal torus which gives a based root datum $(X,\Phi,\Delta;Y,\Phi^\vee,\Delta^\vee)$. Then there exists $g_\tau \in \mathbf{G}(F_s)$ such that $\mathrm{Int}(g_\tau)(\tau\mathbf{B}) = \mathbf{B}$ and $\mathrm{Int}(g_\tau)(\tau\mathbf{T}_{F_s}) = \mathbf{T}_{F_s}$. This induces an automorphism of Δ. There is a split exact sequence

$$1 \to \mathrm{Inn}(\mathbf{G}_{F_s}) \to \mathrm{Aut}(\mathbf{G}_{F_s}) \to \mathrm{Aut}(\Delta) \to 1.$$

A splitting of this exact sequence is determined by a choice of $x_\alpha : \mathbf{G}_a \simeq \mathbf{U}_\alpha$ for $\alpha \in \Delta$.

We first would like to understand the action of Γ on the root subgroup $x_{e_1-e_{n+1}} : \mathbf{G}_m \simeq \mathbf{U}_{e_1-e_{n+1}} \to \mathbf{G}^{\square,sc}$. Note that $\mathbf{G}^{\square,sc} \to \mathbf{G}^{\square}$ restricts to an isomorphism on the root subgroup $\mathbf{U}_{e_1-e_{n+1}}$. We will use the same notation for both root subgroups. It is sufficient to understand the action of Γ on $x_{e_1-e_{n+1}} : \mathbf{G}_m \simeq \mathbf{U}_{e_1-e_{n+1}} \to \mathbf{G}^{\square}$.

To have good control on this, we have to relate the Galois action on $\mathbf{G} \times \mathbf{G}$ and $\mathbf{G}^{\square}$. (We can also argue directly for $\mathbf{G}^{sc} \times \mathbf{G}^{sc}$ and $\mathbf{G}^{\square,sc}$.)

Recall that $\mathbf{G}$ is a subgroup of $\mathbf{GL}_{W;D}$. Fix an isomorphism $D \otimes_F F_s \simeq \mathrm{M}_2(F_s)$. Put $z = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in \mathrm{M}_2(F_s)$ and set $W_z := z(W \otimes_F F_s)$. The restriction $g \mapsto g|_{W_z}$ gives an isomorphism of $\mathbf{GL}_{W;D}(F_s)$ onto the group $\mathbf{GL}_{W_z}(F_s)$.

Let $\langle\ ,\ \rangle_z$ be the restriction of $\langle\ ,\ \rangle$ on W_z. Then $\langle\ ,\ \rangle_z$ is an F_s-bilinear mapping with value in the 1-dimensional F_s-vector space zDz^ρ and it is nondegenerate and has the opposite symmetry as $\langle\ ,\ \rangle$ under interchange of the two variables. The restriction $g \mapsto g|_{W_z}$ gives an isomorphism of $\mathbf{G}_{F_s}$ onto the group $\mathbf{G}_z := \mathbf{G}(W_z, \langle\ ,\ \rangle_z)$. Let $\mathbf{T}_z$ be the image of $\mathbf{T}$ in $\mathbf{G}_z$.

To summarise, we can identify the commutative diagram

$$\begin{array}{ccc} \mathbf{G}_{F_s} \times \mathbf{G}_{F_s} & \longrightarrow & \mathbf{G}_{F_s}^{\square} \\ \downarrow & & \downarrow \\ \mathbf{GL}_{W;D,F_s} \times \mathbf{GL}_{W;D,F_s} & \longrightarrow & \mathbf{GL}_{W^{\square};D,F_s} \end{array}$$

with

$$\begin{array}{ccc} \mathbf{G}_z \times \mathbf{G}_z & \longrightarrow & \mathbf{G}_z^{\square} \\ \downarrow & & \downarrow \\ \mathbf{GL}_{W_z;F_s} \times \mathbf{GL}_{W_z;F_s} & \longrightarrow & \mathbf{GL}_{W_z^{\square};F_s} \end{array}$$

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Let $\mathbb{G}$ be the F-split group with a maximal split torus $\mathbb{T}$ so that the root system of $(\mathbb{G},\mathbb{T})$ is the same as the root system of $(\mathbf{G}_{F_s},\mathbf{T}_{F_s})$. The group $\mathbf{GL}_{W_z;F_s}$ is split and has an F-structure $\mathbf{GL}_{W_z;F}$. We realise $\mathbb{G}$ as a subgroup of $\mathbf{GL}_{W_z;F}$. Then there exists $h \in \mathbf{GL}_{W_z}(F_s)$ such that

$$h \cdot \mathbf{G}_z(F_s) \cdot h^{-1} = \mathbb{G}(F_s), \qquad h \cdot \mathbf{T}_z(F_s) \cdot h^{-1} = \mathbb{T}(F_s).$$

From this we can transfer the based root datum for $(\mathbf{G},\mathbf{T})$ to a based root datum of $(\mathbb{G},\mathbb{T})$. Thus, we obtain a basis of $Y_{\mathbb{G}}$ from a basis of Y. We write it as $(\mathbb{e}_1,\cdots,\mathbb{e}_n)$.

The action of Γ on $(\mathbf{G},\mathbf{T})$ gives an action of Γ on $(\mathbb{G},\mathbb{T})$:

$$\mathbb{G}(F_s) \to \mathbb{G}(F_s), \qquad g \mapsto \tilde{\sigma}(g) := \mathrm{Int}(h) \circ \sigma \circ \mathrm{Int}(h^{-1})(g).$$

The map $\sigma^\dagger := \sigma^{-1} \circ (\mathrm{Int}(h) \circ \sigma \circ \mathrm{Int}(h^{-1})) \in \mathrm{Aut}(\mathbb{G}_{F_s})$. It is easy to see that $\sigma^\dagger = \mathrm{Int}(\sigma(h)h^{-1})$. The element $\sigma^\dagger$ induces an action on the root system of $(\mathbb{G},\mathbb{T})$, which we again denote as $\sigma^\dagger$. Then there exists a lift w_σ of a Weyl group element $\mathbf{w}_\sigma$ of $(\mathbb{G},\mathbb{T})$ such that $\mathbf{w}_\sigma(\sigma^\dagger(\Delta)) = \Delta$.

A Chevalley system for $(\mathbf{G}_z,\mathbf{T}_z)$ can also be translated to a Chevalley system for $(\mathbb{G},\mathbb{T})$. The action of σ on the root system of $(\mathbf{G}_z,\mathbf{T}_z)$ and the pinning can be read from the action of $\sigma^\dagger$ on $(\mathbb{G},\mathbb{T})$.

We write the action down more explicitly. Let $y : \mathbf{G}_m \to \mathbf{T}_z$ be a cocharacter of $\mathbf{T}_z$. Then the action of σ on y is given by

$$\mathbf{G}_m \to \mathbf{G}_m \to \mathbf{T}_z \to \mathbf{T}_z, \qquad t \mapsto \sigma^{-1}(t) \mapsto y(\sigma^{-1}(t)) \mapsto \sigma(y(\sigma^{-1}(t))).$$

Let $x_\alpha : \mathbf{G}_a \to \mathbf{U}$ be a root subgroup. Then the action of σ on α is given by

$$\mathbf{G}_a \to \mathbf{G}_a \to \mathbf{U}_{F_s} \to \mathbf{U}_{F_s}, \qquad t \mapsto \sigma^{-1}(t) \mapsto x_\alpha(\sigma^{-1}(t)) \mapsto \sigma(x_\alpha(\sigma^{-1}(t))).$$

When transferring this action to $(\mathbb{G},\mathbb{T})$, these actions are given by the following:

- $y : \mathbf{G}_m \to \mathbb{T}$ is sent to the element $\sigma^\dagger(y)$, defined as
$$\mathbf{G}_m \to \mathbb{T}, \qquad t \mapsto \tilde{\sigma}(y(\sigma^{-1}(t))).$$
- $x_\alpha : \mathbf{G}_a \to \mathbb{U}$ is sent to $x_{\sigma^\dagger(\alpha)}$, defined as
$$\mathbf{G}_a \to \mathbb{U}, \qquad t \mapsto \tilde{\sigma}(x_\alpha(\sigma^{-1}(t))).$$

In this way, a based root datum for $(\mathbf{G},\mathbf{T})$ determines one for $(\mathbb{G},\mathbb{T})$.

The group $\mathbb{G}$ is the connected component of some orthogonal group $\mathbb{O}_{2n}$. It is not hard to check that $\sigma(h)h^{-1} \in \mathbb{O}_{2n}(F_s)$. As a consequence, the automorphism of the Dynkin diagram given by $w_\sigma \circ \sigma^\dagger$ is either the identity or the isomorphism permuting $\mathbb{e}_{n-1} - \mathbb{e}_n$ and $\mathbb{e}_{n-1} + \mathbb{e}_n$. (In other words, triality does not appear in the case of D_4.) In either case, this isomorphism can be realised by the conjugation given by a lift τ_σ of a Weyl group element in $\mathbf{GL}_{W_z}$.

We now have two different pinnings for $(\mathbb{G},\mathbb{T})$. The first is $x_\alpha : \mathbf{G}_a \to \mathbb{U}$ which is translated from $(\mathbf{G},\mathbf{T})$. The other is $\mathrm{Int}(\tau_\sigma w_\sigma) \circ x_{\sigma^\dagger(\alpha)}$. They might not be the same.

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But we can choose $t_\sigma \in \mathbb{T}(F_s)$ such that

$$\mathrm{Int}(t_\sigma \tau_\sigma w_\sigma) \circ x_{\sigma^\dagger(\alpha)} = x_\alpha, \qquad \alpha \in \Delta.$$

This implies that $\sigma^\dagger = \mathrm{Int}(w_\sigma^{-1}\tau_\sigma^{-1}t_\sigma^{-1})$.

10.5.4. The action on $\mathbf{G}^\square$. We now have that

$$\iota(h,h)\cdot \mathbf{T}^\square(F_s)\cdot \iota(h,h)^{-1} \subset \iota(h,h)\cdot \mathbf{G}^\square(F_s)\cdot \iota(h,h)^{-1}$$

is $\mathbb{T}^\square(F_s) \subset \mathbb{G}^\square(F_s)$ for a maximal split torus $\mathbb{T}^\square$ inside a split group $\mathbb{G}^\square$. As in the case of $\mathbf{G}$, here we realise $\mathbb{G}^\square(F_s) \subset \mathbf{GL}_{W_z^\square}(F_s)$. We can read the Galois action on $(\mathbf{G}^\square, \mathbf{T}^\square)$ from

$$\sigma^\dagger = \mathrm{Int}(\iota(\sigma(h)h^{-1}, \sigma(h)h^{-1})) \in \mathrm{Aut}(\mathbb{G}^\square(F_s)).$$

This map preserves $\mathbb{T}^\square(F_s)$. From our discussion above, we know that

$$\mathrm{Int}(\iota(\sigma(h)h^{-1}, \sigma(h)h^{-1})) = \mathrm{Int}(\iota(w_\sigma^{-1}\tau_\sigma^{-1}t_\sigma^{-1}, w_\sigma^{-1}\tau_\sigma^{-1}t_\sigma^{-1})).$$

Lemma 10.15. *For any $\sigma \in \Gamma$,*

$$\sigma^\dagger \circ x_{\mathbb{e}_1 - \mathbb{e}_{n+1}} = \mathrm{Int}(\iota(w_\sigma^{-1}, w_\sigma^{-1})) \circ x_{\mathbb{e}_1 - \mathbb{e}_{n+1}}.$$

Proof. It is easy to check that $\mathrm{Int}(\iota(t_\sigma^{-1}, t_\sigma^{-1}))$ acts trivially on $x_{\mathbb{e}_1 - \mathbb{e}_{n+1}}$. Moreover, $\mathrm{Int}(\iota(\tau_\sigma^{-1}, \tau_\sigma^{-1}))$ acts trivially on $x_{\mathbb{e}_1 - \mathbb{e}_{n+1}}$ as well. This completes the proof. □

We translate the above lemma back to the case of $(\mathbf{G}, \mathbf{T})$. We deduce that $\sigma(x_{e_1 - e_{n+1}}) = \tilde{w}_\sigma \circ x_{e_1 - e_{n+1}}$ for some lift $\tilde{w}_\sigma$ of a Weyl group element $\tilde{\mathbf{w}}_\sigma$ for $(\mathbf{G}^\square, \mathbf{T}^\square)$. In other words, the Galois action on $e_1 - e_{n+1}$ is the same as the action by some Weyl group element.

Lemma 10.16. *For any $\sigma \in \Gamma$,*

$$\sigma(s_{Q^{\square,sc}}(e_1^\vee - e_{n+1}^\vee)) = \tilde{w}_\sigma \cdot s_{Q^{\square,sc}}(e_1^\vee - e_{n+1}^\vee)\cdot \tilde{w}_\sigma^{-1} = s_{Q^{\square,sc}}(\tilde{\mathbf{w}}_\sigma(e_1^\vee - e_{n+1}^\vee)).$$

Proof. This follows from Lemma 2.4 and the discussion above. Note that from Lemma 10.11, we always have $\epsilon_{\alpha,\beta}^{Q(\beta^\vee)} = 1$. □

Lemma 10.17. *For $\sigma \in \Gamma$, there exists i and a sign such that $\sigma(e_1 - e_{n+1})$ is of the form $\pm(e_i - e_{n+i})$.*

Proof. Suppose $\sigma(e_1) = \sum_j a_j e_j$ for some $a_j \in \mathbb{Z}$. Since the action of σ is the same for both copies of $\mathbf{G}$, we have $\sigma(e_{n+1}) = \sum_j a_j e_{n+j}$. We know

$$\sigma(e_1 - e_{n+1}) = \sum_j a_j(e_j - e_{n+j})$$

must be a root of $\mathbf{G}^\square$. Thus, it must be of the form $\pm(e_i - e_{n+i})$. □

Lemma 10.18. *If $\sigma(e_1 - e_{n+1}) = e_i - e_{n+i}$ for some i, then*

$$\sigma(s_{Q^{\square,sc}}(e_1^\vee - e_{n+1}^\vee))/s_{Q^{\square,sc}}(e_1^\vee - e_{n+1}^\vee) \in \mathrm{pr}_*(\mathcal{E}_{Q^{sc}} \oplus \mathcal{E}_{Q^{sc}}),$$

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and

$$\mathrm{mul}(\sigma(s_{Q^{\square,sc}}(e_1^\vee - e_{n+1}^\vee))/s_{Q^{\square,sc}}(e_1^\vee - e_{n+1}^\vee)) = 1.$$

Proof. The first statement is straightforward. Using

$$(e_i^\vee - e_{n+i}^\vee) + (e_1^\vee - e_i^\vee) = (e_1^\vee - e_{n+1}^\vee) + (e_{n+1}^\vee - e_{n+i}^\vee)$$

and Lemma 2.6, we have

$$s_{Q^{\square,sc}}(e_i^\vee - e_{n+i}^\vee) \cdot s_{Q^{\square,sc}}(e_1^\vee - e_i^\vee) = s_{Q^{\square,sc}}(e_1^\vee - e_{n+1}^\vee) \cdot s_{Q^{\square,sc}}(e_{n+1}^\vee - e_{n+i}^\vee).$$

Now the result follows the fact that $\mathrm{mul}(s_{Q^{\square,sc}}(e_1^\vee - e_i^\vee)/s_{Q^{\square,sc}}(e_{n+1}^\vee - e_{n+i}^\vee)) = 1$. □

10.5.5. Galois equivariance. We are now ready to prove that $f^\square$ is Galois equivariant.

Proposition 10.19. *We have that*

$$f^\square(\sigma(s_{Q^{\square,sc}}(\alpha^\vee))) = \sigma(f^\square(s_{Q^{\square,sc}}(\alpha^\vee))) \tag{16}$$

for $\alpha^\vee \in \Delta^\vee_{\mathbf{G}^\square}$ *and* $\sigma \in \Gamma$.

Proof. The only nontrivial case is $\alpha^\vee = e_1^\vee - e_{n+1}^\vee$.

Both sides in (16) project to $\sigma(\alpha^\vee)$. Thus, to show (16), it suffices to show that they are the same under the map mul. It is easy to show that for any $\sigma \in \Gamma$,

$$\mathrm{mul}(\sigma(f^\square(s_{Q^{\square,sc}}(e_1^\vee - e_{n+1}^\vee))) = 1.$$

We now calculate $\mathrm{mul}(f^\square(\sigma(s_{Q^{\square,sc}}(e_1^\vee - e_{n+1}^\vee))))$. We have three cases to consider. First, if $\sigma(e_1^\vee - e_{n+1}^\vee) = \tilde{\mathbf{w}}_\sigma(e_1^\vee - e_{n+1}^\vee) = -(e_1^\vee - e_{n+1}^\vee)$, then by Lemmas 2.5 and 10.16,

$$\begin{aligned}\sigma(s_{Q^{\square,sc}}(e_1^\vee - e_{n+1}^\vee)) &= s_{Q^{\square,sc}}(\tilde{\mathbf{w}}_\sigma(e_1^\vee - e_{n+1}^\vee)) \\ &= s_{Q^{\square,sc}}(-(e_1^\vee - e_{n+1}^\vee)) = s_{Q^{\square,sc}}(e_1^\vee - e_{n+1}^\vee)^{-1},\end{aligned}$$

and this implies that $\mathrm{mul}(f^\square(\sigma(s_{Q^{\square,sc}}(e_1^\vee - e_{n+1}^\vee)))) = 1$.

We now assume that $\sigma(e_1 - e_{n+1}) = e_i - e_{n+i}$ for some i. Then by Lemma 10.18 and our choice of $f^\square(s_{Q^{\square,sc}}(e_1^\vee - e_{n+1}^\vee))$,

$$\mathrm{mul}(f^\square(\sigma(s_{Q^{\square,sc}}(e_1^\vee - e_{n+1}^\vee)))) = \mathrm{mul}(f^\square(s_{Q^{\square,sc}}(e_1^\vee - e_{n+1}^\vee))) = 1.$$

Finally, we have to consider the case $\sigma(e_1 - e_{n+1}) = -(e_i - e_{n+i})$ for some $i \neq 1$. This can be proved by combining arguments in the previous two cases. This completes the proof. □

10.6. Unitary groups

We now consider the unitary group case. Recall that $\mathbf{G}_{F_s} = \mathbf{GL}_{n,F_s}$. We choose a standard basis of $\mathbf{T}_{F_s}$ so that $Y = \mathrm{Span}\{e_1^\vee, \cdots, e_n^\vee\}$ with the following set of simple roots:

$$\{e_1^\vee - e_2^\vee, \cdots, e_{n-1}^\vee - e_n^\vee\}.$$

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Let Q be a Weyl invariant quadratic form on Y. Then Q is determined by the following two integers p and q:

$$B_Q(e_i^\vee, e_i^\vee) = 2p, \qquad B_Q(e_i^\vee, e_j^\vee) = q \text{ for } i \neq j.$$

Then for any coroot $\alpha^\vee$, $Q(\alpha^\vee) = 2p - q$. Since we assume that Q is decomposable, we have $q = 0$.

The group $\mathbf{G}^\square$ has a maximal F-torus $\mathbf{T}^\square := \mathbf{T} \times \mathbf{T}$. The cocharacter lattice of $\mathbf{T}^\square$ over F_s is $Y^\square = Y \oplus Y$. We choose standard basis so that $Y_+ = \mathrm{Span}\{e_1^\vee, \cdots, e_n^\vee\}$ and $Y_- = \mathrm{Span}\{e_{n+1}^\vee, \cdots, e_{2n}^\vee\}$. The choice of simple roots is given as above. We choose the following set of simple roots for $\mathbf{G}^\square$:

$$\{e_1^\vee - e_2^\vee, \cdots, e_{2n-1}^\vee - e_{2n}^\vee\}.$$

Since we assume that $q = 0$, $Q^\square$ is a direct sum $Q \oplus Q$ on $Y \oplus Y$.

10.6.1. Construction of BD data. Let $(Q, \mathcal{E}, f)$ be the Galois equivariant BD data for $\overline{\mathbf{G}}$. We now construct a Galois invariant BD data $(Q^\square, \mathcal{E}^\square, f^\square)$ for $\mathbf{G}^\square$.

Given a quadratic form Q on Y which is determined by an integer p as above, we define $Q^\square := Q \oplus Q$. The quadratic form is Weyl invariant and Galois invariant.

Remark 10.20. If $q \neq 0$, the quadratic form $Q \oplus Q$ on $Y \oplus Y$ is not W-invariant. We have to choose a different one. The argument in the rest of section will require some modification in order to handle this case.

We can again take the direct sum of two copies of $\mathcal{E}$ and push it out via the multiplication map to obtain

$$1 \to F_s^\times \to \mathrm{pr}_*(\mathcal{E} \oplus \mathcal{E}) \to Y^\square \to 1.$$

The commutator is $[y_1, y_2] = (-1)^{B_{Q^\square}(y_1, y_2)}$. We take $\mathcal{E}^\square = \mathrm{pr}_*(\mathcal{E} \oplus \mathcal{E})$, and this is the second BD invariant.

We can now proceed as in the case of even orthogonal groups. We will not repeat the definitions of these notations here. We now have to define $f^\square : \mathcal{E}_{Q^{\square, sc}} \to \mathrm{pr}_*(\mathcal{E} \oplus \mathcal{E})$ such that its composition with the map $\mathrm{pr}_*(\mathcal{E}_{Q^{sc}} \oplus \mathcal{E}_{Q^{sc}}) \to \mathcal{E}_{Q^{\square, sc}}$ gives

$$f_+ \oplus f_- : \mathrm{pr}_*(\mathcal{E}_{Q^{sc}} \oplus \mathcal{E}_{Q^{sc}}) \to \mathrm{pr}_*(\mathcal{E} \oplus \mathcal{E}).$$

The map is already determined on the image of $\mathrm{pr}_*(\mathcal{E}_{Q^{sc}} \oplus \mathcal{E}_{Q^{sc}})$ in $\mathcal{E}_{Q^{\square, sc}}$. One only has to determine $f^\square(s_{Q^{\square, sc}}(e_n^\vee - e_{n+1}^\vee))$ so that the map is Galois equivariant.

10.6.2. Galois equivariance. We can verify Galois equivariance using the argument in Subsection 10.5.5. Here we give another proof using Hilbert's Theorem 90. We would like to show that there exists $f^\square(s_{Q^{\square, sc}}(e_n^\vee - e_{n+1}^\vee))$ such that

$$f^\square(\sigma(s_{Q^{\square, sc}}(e_n^\vee - e_{n+1}^\vee))) = \sigma(f^\square(s_{Q^{\square, sc}}(e_n^\vee - e_{n+1}^\vee)))$$

for all $\sigma \in \Gamma$.

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For ease of notation, we write $a = s_{Q^{\square},sc}(e_n^\vee - e_{n+1}^\vee)$. We take an arbitrary $f^{\square}$ and define a function $c : \Gamma \to F_s^\times$ as follows:

$$c(\sigma) = \frac{f^{\square}(\sigma(a))}{\sigma(f^{\square}(a))}.$$

We show that c is a 1-cocycle. In other words, we prove the following result.

Lemma 10.21. *For any $\sigma_1, \sigma_2 \in \Gamma$,*

$$c(\sigma_1\sigma_2) = f(\sigma_1) \cdot \sigma_1(f(\sigma_2)).$$

Proof. We write

$$c(\sigma_1\sigma_2) = \frac{f^{\square}(\sigma_1\sigma_2(a))}{\sigma_1\sigma_2(f^{\square}(a))} = \frac{f^{\square}(\sigma_1\sigma_2(a))}{\sigma_1(f^{\square}(\sigma_2(a)))} \cdot \frac{\sigma_1(f^{\square}(\sigma_2(a)))}{\sigma_1\sigma_2(f^{\square}(a))}.$$

It suffices to show that

$$\frac{f^{\square}(\sigma_1\sigma_2(a))}{\sigma_1(f^{\square}(\sigma_2(a)))} = \frac{f^{\square}(\sigma_1(a))}{\sigma_1(f^{\square}(a))}$$

or

$$\frac{f^{\square}(\sigma_1\sigma_2(a))}{f^{\square}(\sigma_1(a))} = \frac{\sigma_1(f^{\square}(\sigma_2(a)))}{\sigma_1(f^{\square}(a))}.$$

Note that $\sigma_2(a)/a$ projects to $Y^{sc} \oplus Y^{sc}$. Thus, the left-hand side is

$$f^{\square}(\sigma_1(\sigma_2(a)/a)) = \sigma_1(f^{\square}(\sigma_2(a)/a)).$$

This proves the result. □

Hilbert's Theorem 90 says that $H^1(\Gamma, F_s^\times) = 1$. In other words, a 1-cocycle must be a coboundary. This means that there exists $x \in F_s^\times$ such that $c(\sigma) = \sigma(x)/x$.

We now define

$$\tilde{f}^{\square}(a) = f^{\square}(a)x.$$

Then

$$\tilde{f}^{\square}(\sigma(a)) = f^{\square}(\sigma(a))x = \sigma(f^{\square}(a))c(\sigma) \cdot x = \sigma(f^{\square}(a))\sigma(x) = \sigma(\tilde{f}^{\square}(a)).$$

This implies that $\tilde{f}^{\square}$ is Γ-equivariant.

10.7. Special odd orthogonal groups

We now treat the case of $\mathbf{G} = \mathbf{SO}_{2n+1}$. We first begin with some discussion of the BD data. We can write $Y = \mathrm{Span}\{e_1^\vee, \cdots, e_n^\vee\}$ and let

$$\{e_1^\vee - e_2^\vee, \cdots, e_{n-1}^\vee - e_n^\vee, 2e_n^\vee\}$$

be the coroots of $\mathbf{SO}_{2n+1}$. A W-invariant quadratic form Q on Y is determined by its value on a short coroot. Let $Q(\alpha_1^\vee) = a$. As in the even orthogonal case, we can similarly prove the following.

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Lemma 10.22. *We have that* $2 \mid a$.

Observe that $\mathbf{T}\times\mathbf{T}$ is an F-torus in $\mathbf{G}^{\square}$ but not a maximal torus. Let $\mathbf{T}^{\square}\supset\mathbf{T}\times\mathbf{T}$ be a maximal F-torus of $\mathbf{G}^{\square}$.

We write $Y_+ = \mathrm{Span}\{e_1^\vee,\cdots,e_n^\vee\}$, $Y_- = \mathrm{Span}\{e_{n+1}^\vee,\cdots,e_{2n}^\vee\}$. We have

$$\begin{aligned} Y_+^{sc} &= \mathrm{Span}\{e_1^\vee - e_2^\vee,\cdots,e_{n-1}^\vee - e_n^\vee, 2e_n^\vee\}\\ Y_-^{sc} &= \mathrm{Span}\{e_{n+1}^\vee - e_{n+2}^\vee,\cdots,e_{2n-1}^\vee - e_{2n}^\vee, 2e_{2n}^\vee\}\\ Y^{\square,sc} &= \mathrm{Span}\{e_1^\vee - e_2^\vee,\cdots,e_{2n}^\vee - e_{2n+1}^\vee, e_{2n}^\vee + e_{2n+1}^\vee\}. \end{aligned}$$

Let $Q^{\square}$ be W-invariant quadratic form on $Y^{\square}$.

Lemma 10.23. *The restriction of* $Q^{\square}$ *to* $Y_+\oplus Y_-$ *is a direct sum of two quadratic forms* $Q_+\oplus Q_-$. *And we have* $Q_+ = Q_-$.

Conversely, given $Q = Q_+ = Q_-$, *there is a unique* W*-invariant quadratic form* $Q^{\square}$ *which restricts to* $Q_+\oplus Q_-$ *on* $Y_+\oplus Y_-$.

Proof. This is straightforward. □

Let $\overline{\mathbf{G}}$ be a multiplicative $\mathbf{K}_2$-torsor on $\mathbf{G}$ with BD data $(Q,\mathcal{E},f)$. We now construct a BD data $(Q^{\square},\mathcal{E}^{\square},f^{\square})$ for $\mathbf{G}^{\square}$. We only explain the difference in this case but will not repeat all of the details.

The construction of $Q^{\square}$ is straightforward since such a quadratic form is determined by its values on its short coroot.

We now define the second BD invariant. Note that $Y^{\square}/Y\oplus Y = \mathbb{Z}\cdot e_{2n+1}^\vee$. Consider

$$\mathcal{E}\oplus\mathcal{E}\oplus(F_s^\times\times\mathbb{Z})$$

with the following multiplication:

$$(e_1,e_2,(x,a))\cdot(e_1',e_2',(x',a')) := (e_1e_1',e_2e_2',(xx'(-1)^{B_{Q^{\square}}((y_1,y_2),a'\cdot e_{2n+1}^\vee)},a+a')).$$

Here, (y_1,y_2) is the image of (e_1,e_2) under $\mathcal{E}\to Y$. This defines an exact sequence

$$1\to F_s^\times\oplus F_s^\times\oplus F_s^\times\to\mathcal{E}\oplus\mathcal{E}\oplus(F_s^\times\times\mathbb{Z})\to Y^{\square} = Y\oplus Y\oplus\mathbb{Z}\cdot e_{2n+1}^\vee\to 1.$$

Pushing out by the product map $\mathrm{pr}: F_s^\times\oplus F_s^\times\oplus F_s^\times\to F_s^\times$ gives

$$1\to F_s^\times\to\mathcal{E}^{\square}\to Y^{\square}\to 1.$$

One can verify that the commutator map is given by $(-1)^{B_{Q^{\square}}(y_1,y_2)}$. We also have a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & F_s^\times & \longrightarrow & \mathrm{pr}_*(\mathcal{E}_{Q^{sc}}\oplus\mathcal{E}_{Q^{sc}}) & \longrightarrow & Y^{sc}\oplus Y^{sc} & \longrightarrow & 1\\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & F_s^\times & \longrightarrow & \mathcal{E}^{\square} & \longrightarrow & Y^{\square} & \longrightarrow & 1 \end{array}$$

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To construct the third BD invariant, one has to construct $f^{\square}: \mathcal{E}_{Q^{\square,sc}} \to \mathcal{E}^{\square}$ which extends the map $\mathrm{pr}_*(\mathcal{E}_{Q^{sc}} \oplus \mathcal{E}_{Q^{sc}}) \to \mathcal{E}^{\square}$. We only have to specify the values

$$f^{\square}(s_{Q^{\square,sc}}(e_1^{\vee} - e_{n+1}^{\vee})) \text{ and } f^{\square}(s_{Q^{\square,sc}}(e_{2n}^{\vee} - e_{2n+1}^{\vee}))$$

so that $f^{\square}$ is Galois equivariant. To choose the first value, we use the argument presented in the $\mathbf{SO}_{2n}$ case. The choice for the second value is identical to the unitary case. This completes the proof.

11. *L*-functions

So far we only give a global zeta integral which represents an Euler product but have not said anything regarding the L-functions obtained from the twisted doubling integrals. The construction relies on the construction of representations of type $(k,m)_D$. In the linear case, a good source of such representations is the generalised Speh representations. Here we present a conjectural picture. Further investigations are necessary in order to gain a complete understanding of the local and global theory.

We make a couple of assumptions to simplify the situation in our discussion. We assume that $D = F$, so that only the group $\mathbf{GL}_m$ will be involved. We also assume that the quadratic form Q appearing in the BD data is decomposable. So we will not have any problems regarding parabolic induction. As we pointed out earlier, to treat the nondecomposable case, we need to have a suitable version of 'metaplectic tensor product'.

Fix an integer p. This determines a W-invariant decomposable quadratic form on the cocharacter lattice for every $\mathbf{GL}_m$. Let $n_Q = n/\gcd(n, Q(\alpha^{\vee}))$ for any $\alpha^{\vee} \in \Delta^{\vee}$ if $m \geq 2$ and $n_Q = n/\gcd(n, 2a)$ for $a = Q(e_1^{\vee})$ and $m = 1$. The twisted doubling integral relies on the following construction of the inducing data in the Eisenstein series. To be more precise, we would like to have

$$\tau \in \mathrm{Irr}^u_{cusp}(\mathrm{GL}_k(\mathbb{A})) \to \theta^{(n)}(\tau, m) \in \mathrm{Irr}^u(\overline{\mathrm{GL}}^{(n)}_{kmn_Q}(\mathbb{A})). \tag{17}$$

For every local place v, the local analog is given by

$$\tau_v \in \mathrm{Irr}^u_{gen}(\mathrm{GL}_{k,v}) \to \theta^{(n)}(\tau_v, m) \in \mathrm{Irr}^u(\overline{\mathrm{GL}}^{(n)}_{kmn_Q,v}).$$

Here, the superscript u means that only unitary representations are considered and the subscript *gen* means generic representations.

We expect the following list of properties:

- The construction is local-to-global compatible: $\theta^{(n)}(\tau, m) = \otimes'_v \theta^{(n)}(\tau_v, m)$ if $\tau = \otimes'_v \tau_v$.
- The representation $\theta^{(n)}(\tau, m)$ is of type (kn_Q, m).
- For all k, n and the multiplicative $\mathbf{K}_2$-torsor on $\mathbf{GL}_k$ determined by the integer p as above, assume that there exists a 'Shimura-type lift' from $\mathrm{Irr}(\overline{\mathrm{GL}}^{(n)}_k(\mathbb{A})) \to \mathrm{Irr}(\mathrm{GL}_k(\mathbb{A}))$ which is also local to global compatible. If τ does not lie in the image of the Shimura lift from $\overline{\mathrm{GL}}^{(n')}_k(\mathbb{A})$ to $\mathrm{GL}_k(\mathbb{A})$ for any $n' \mid n$, then the lift is cuspidal. If it does, then this can be constructed using residues of Eisenstein series.

This is also discussed in [24] and [14].

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Remark 11.1. We also expect more properties of this construction to be valid. Such properties are motivated by the local and global theories of the twisted doubling integrals. For instance, this construction should satisfy a multiplicativity with respect to the parabolic induction. This will be used in order to establish the multiplicativity of γ-factors.

Remark 11.2. Instead of (17), one might consider

$$\tau \in \mathrm{Irr}^{u}_{cusp}(\overline{\mathrm{GL}}_k(\mathbb{A})) \to \theta^{(n)}(\tau,m) \in \mathrm{Irr}^{u}(\overline{\mathrm{GL}}^{(n)}_{kmn_Q}(\mathbb{A})),$$

given by residues of Eisenstein series. However, the orbit of $\theta^{(n)}(\tau,m)$ might not be of (kn_Q,m) due to the existence of cuspidal theta representations. We refer the reader to [8] Subsection 3.2 for some discussion on this matter.

As a consequence, we expect that the twisted doubling integrals give an integral representation for the tensor product L-function for $\overline{\mathbf{G}} \times \mathbf{GL}_k$. (In the unitary case, it would be $\overline{\mathbf{G}} \times \mathrm{Res}_{E/F}(\mathbf{GL}_k)$.) Our formulation is slightly different from [16]. Also note that in [10] the Langlands–Shahidi type L-functions appear in the constant terms of Eisenstein series, in which the tensor product L-function for $\overline{\mathbf{G}} \times \overline{\mathbf{GL}}_k$ is obtained for split classical groups $\mathbf{G}$, among many other interesting L-functions. It would be interesting to relate the L-functions obtained from these constructions.

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References

[1] J.-L. Brylinski and P. Deligne, Central extensions of reductive groups by $\mathbf{K}_2$, *Publ. Math. Inst. Hautes Études Sci.* **94** (2001), 5–85.

[2] D. Bump and D. Ginzburg, Symmetric square L-functions on GL(r), *Ann. Math. (2)* **136**(1) (1992), 137–205.

[3] F. Bruhat and J. Tits, Groupes réductifs sur un corps local. II. Schémas en groupes. Existence d'une donnée radicielle valuée, *Inst. Hautes Études Sci. Publ. Math.* **60** (1984), 197–376.

[4] D. Bump, *Automorphic Forms and Representations*, Vol. **55** of *Cambridge Studies in Advanced Mathematics* (Cambridge University Press, Cambridge, 1997).

[5] Y. Cai, Twisted doubling integrals for classical groups, *Math. Z.* **297**(3–4) (2021), 1075–1104.

[6] Y. Cai, S. Friedberg, D. Ginzburg and E. Kaplan, Doubling constructions for covering groups and tensor product L-functions, Preprint, 2016, arXiv:1601.08240.

[7] Y. Cai, S. Friedberg, D. Ginzburg and E. Kaplan, Doubling constructions and tensor product L-functions: the linear case, *Invent. Math.* **217**(3) (2019), 985–1068.

[8] S. Friedberg and D. Ginzburg, On the genericity of Eisenstein series and their residues for covers of GL_m, *Int. Math. Res. Not. IMRN* **4** (2017), 1000–1012.

[9] W. T. Gan, Doubling zeta integrals and local factors for metaplectic groups, *Nagoya Math. J.* **208** (2012), 67–95.

[10] F. Gao, The Langlands-Shahidi L-functions for Brylinski-Deligne extensions, *Amer. J. Math.* **140**(1) (2018), 83–137.

[11] W. T. Gan and F. Gao, The Langlands-Weissman program for Brylinski-Deligne extensions, *Astérisque* **398** (2018), 187–275.

[12] R. Gomez, D. Gourevitch and S. Sahi, Generalized and degenerate Whittaker models, *Compos. Math.* **153**(2) (2017), 223–256.

[13] W. T. Gan, F. Gao and M. H. Weissman, *L-groups and the Langlands program for covering groups: a historical introduction*, *Astérisque* **398** (2018), 1–31.

[14] D. Ginzburg, Tensor product L-functions on metaplectic covering groups of GL_r, Preprint, 2019, arXiv:1908.07720.

[15] F. Gao, F. Shahidi and D. Szpruch, Local coefficients and gamma factors for principal series of covering groups, *Mem. Amer. Math. Soc.* to appear, arXiv:1902.02686.

[16] E. Kaplan, Doubling constructions and tensor product L-functions: coverings of the symplectic group, Reprint, 2019, arXiv:1902.00880.

[17] D. A. Kazhdan and S. J. Patterson, Metaplectic forms, *Inst. Hautes Études Sci. Publ. Math.* **59** (1984), 35–142.

[18] W.-W. Li, La formule des traces pour les revêtements de groupes réductifs connexes. I. Le développement géométrique fin, *J. Reine Angew. Math.* **686** (2014), 37–109.

[19] P. Mezo, Metaplectic tensor products for irreducible representations, *Pacific J. Math.* **215**(1) (2004), 85–96.

[20] C. C. Moore, Group extensions of p-adic and adelic linear groups, *Inst. Hautes Études Sci. Publ. Math.* **35** (1968), 157–222.

[21] C. Mœglin and J.-L. Waldspurger, Modèles de Whittaker dégénérés pour des groupes p-adiques, *Math. Z.* **196**(3) (1987), 427–452.

[22] C. Mœglin and J.-L. Waldspurger, *Spectral Decomposition and Eisenstein Series*, Vol. **113** of *Cambridge Tracts in Mathematics* (Cambridge University Press, Cambridge, 1995). Une paraphrase de l'Écriture [A paraphrase of Scripture].

[23] I. Piatetski-Shapiro and S. Rallis, *L-Functions for the Classical Groups*, Vol. **1254** of *Lecture Notes in Math.* (Springer, New York, 1987).

[24] T. Suzuki, Distinguished representations of metaplectic groups, *Amer. J. Math.* **120**(4) (1998), 723–755.

[25] S. Takeda, Metaplectic tensor products for automorphic representation of $\widetilde{\mathrm{GL}}(r)$, *Canad. J. Math.* **68**(1) (2016), 179–240.

[26] S. Takeda, Remarks on metaplectic tensor products for covers of GL_r, *Pacific J. Math.* **290**(1) (2017), 199–230.

[27] M. H. Weissman, Managing metaplectiphobia: covering p-adic groups, in *Harmonic Analysis on Reductive, p-Adic Groups*, Vol. **543** of *Contemp. Math.* (American Mathematical Society, Providence, RI, 2011), 237–277.

[28] M. H. Weissman, Covering groups and their integral models, *Trans. Amer. Math. Soc.* **368**(5) (2016), 3695–3725.

[29] S. Yamana, L-functions and theta correspondence for classical groups, *Invent. Math.* **196**(3) (2014), 651–732.