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MODULARITY OF THE LATTICE OF CONGRUENCES OF A REGULAR ω-SEMIGROUP*

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Dedicated to Professor C. Tibiletti Marchionna on her 70th birthday

In this paper a characterization of the regular ω-semigroups whose congruence lattice is modular is given. The characterization obtained for such semigroups generalizes the one given by Munn for bisimple ω-semigroups and completes a result of Baird dealing with the modularity of the sublattice of the congruence lattice of a simple regular ω-semigroup consisting of congruences which are either idempotent separating or group congruences.

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Introduction

A regular ω-semigroup S is a regular semigroup whose set of idempotents $E(S)$, or shortly E, forms an ω-chain

$$e_0 > e_1 > \cdots > e_n > \cdots$$

under the natural order defined on E by the rule $e \geqq f$ if and only if $ef = f = fe$. The congruences on this interesting class of inverse semigroups have been described (see, e.g., [2] and [9]), but this description has been not much used for the investigation of the congruence lattice $L(S)$.

As regards the modularity of $L(S)$, Munn [8] characterized bisimple ω-semigroups whose lattice of congruences is modular, observing that the congruences on S reduce to idempotent separating and group congruences. Baird [1] studied simple ω-semigroups and gave a necessary and sufficient condition for the sublattice of $L(S)$ consisting of those congruences which are either idempotent separating or group congruences to be a modular lattice.

This paper completes the previous results giving the characterization of regular ω-semigroups whose congruences lattice is modular.

As usual σ denotes the least group congruence, H and D the Green's relations, ι the identity congruence on S and N the set of non-negative integers.

The undefined terminology and notation can be found in [10].

1. Modularity of the lattice of congruences on simple regular ω-semigroups

This section deals with the characterization of simple regular ω-semigroups whose

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lattice of congruence is modular. (A lattice L is *modular* if for every $a,b,c\in L$ with $a\leqq b$, $a\cup c=b\cup c$ and $a\cap c=b\cap c$ imply $a=b$, or equivalently if we have $(a\cup c)\cap b=a\cup(c\cap b)$). Two authors, Kocin [5] and Munn [7], gave structure theorems for simple regular ω-semigroups. The one given by Munn is the following:

Theorem A. *Let d be a positive integer and let $\{G_i \mid i=0,\dots,d-1\}$ be a set of d pairwise disjoint groups. Let γ_{d-i} be a homomorphism of G_{d-1} into G_0 and if $d>1$, let γ_i be a homomorphism of G_i into G_{i+1} $(i=0,\dots,d-2)$. For every $n\in\mathbb{N}$ let $\gamma_n=\gamma_{n(\bmod d)}$ ($n(\bmod d)$) denote the integer equivalent to n modulo d, belonging to $\mathbb{N}$ and less than d). For $m,n\in\mathbb{N}$ and $m<n$ write*

$$\alpha_{m,n}=\gamma_m\gamma_{m+1}\cdots\gamma_{n-1}$$

and for all $n\in\mathbb{N}$ let $\alpha_{n,n}$ denote the identity automorphism of $G_{n(\bmod d)}$.

Let S be the set of the ordered triples (m,a_i,n), where $m,n\in\mathbb{N}$, $0\leqq i\leqq d-1$ and $a_i\in G_i$. Define a multiplication in S by the rule that

$$(m,a_i,n)(p,b_j,q)=(m-n+t,(a_i\alpha_{u,w})(b_j\alpha_{v,w}),q-p+t)$$

where $t=\max\{n,p\}$, $u=nd+i$, $v=pd+j$ and $w=\max\{u,v\}$. (The multiplication in S can be defined equivalently by the rule $(m,a_i,n)(p,b_j,q)=(m+p-r,(a_i\alpha_{u,w})(b_j\alpha_{v,w}),\ n+q-r)$ where $r=\min\{n,p\}$, $u=nd+i$, $v=pd+j$ and $w=\max\{u,v\}$). Denote the so formed groupoid by $S(d,G_i,\gamma_i)$. Then $S(d,G_i,\gamma_i)$ is a simple regular ω-semigroup with exactly d D-classes and any simple regular ω-semigroup is isomorphic to a semigroup $S(d,G_i,\gamma_i)$. For $n\in\mathbb{N}$ and $i=0,\dots,d-1$ write $e_i^n=(n,e_i,n)$, where e_i is the identity of the group G_i. The elements e_i^n are the idempotents of $S(d,G_i,\gamma_i)$ and we have

$$e_0^0>e_1^0>\cdots>e_{d-1}^0>e_0^1>\cdots>e_{d-1}^1>e_0^2>\cdots$$

In the remainder of the paper the endomorphism $\alpha_{i,i+d}$ of G_i will be indicated by α_i. Moreover, in view of the above theorem, we will denote a simple regular ω-semigroup by $S(d,G_i,\gamma_i)$.

Definition 1.1. (see [2, 2 and 3], [1, 2]). Let $S=S(d,G_i,\gamma_i)$. A *congruence* μ on the set E of idempotents is called *uniform* if $(e_i^n,e_j^m)\in\mu$ implies that

$$(e_i^{n+p},e_j^{m+p})\in\mu \text{ for all integers } p\geqq-\min\{m,n\}.$$

Put $G=G_0\times G_1\times\cdots\times G_{d-1}$, the cartesian product of G_i, $i=0,1,\dots,d-1$. A *subset* A of G will be called *γ-admissible* if

(i) $A=A_0\times\cdots\times A_{d-1}$, for some $A_i\subseteq G_i$, $i=0,1,\dots,d-1$.

(ii) $A_i\trianglelefteq G_i$, for $i=0,1,\dots,d-1$.

(iii) $A_{d-1}\gamma_{d-1}\subseteq A_0$ and $A_i\gamma_i\subseteq A_{i+1}$, for $i=0,1,\dots,d-2$.

A subset A of G will be called *normal* if it satisfies (i) and (ii) above. If $A = A_0 \times \cdots \times A_{d-1}$ and $B = B_0 \times \cdots \times B_{d-1}$ are normal subsets of G we define $A \cdot B = A_0B_0 \times \cdots \times A_{d-1}B_{d-1}$. $A \cdot B$ is a normal subset and, if A and B are γ-admissible, $A \cdot B$ is γ-admissible.

Let μ be a uniform congruence on $E(S)$ and A a γ-admissible subset of G. Put

$$\mu\text{-rad } A_i = \{a_i \in G_i \mid a_i\alpha_{nd+i,md+j} \in A_j \text{ for some } n,m,j \text{ such that} (e_i^n, e_j^m) \in \mu \text{ and } e_j^m \leqq e_i^n\}$$

and

$$\mu\text{-rad } A = \mu\text{-rad } A_0 \times \cdots \times \mu\text{-rad } A_{d-1}.$$

If μ-rad $A = A$, μ and A are called *linked.*

Remark 1.2. (see [2], Lemma 3.1 and Lemma 3.2]) Let μ and μ' be uniform congruences of $E(S)$ and let A and A' be γ-admissible subsets of G. Then

(i) μ-rad A is a γ-admissible subset of G and $A \subseteq \mu$-rad A,

(ii) $A \subseteq A'$ implies μ-rad $A \subseteq \mu$-rad A',

(iii) $\mu \leqq \mu'$ implies μ-rad $A \subseteq \mu'$-rad A,

(iv) μ-rad $A = \mu$-rad$(\mu$-rad $A)$.

Hence, if we denote by 1 the γ-admissible subset $\{e_0\} \times \cdots \times \{e_{d-1}\}$ of G, it follows that

(v) μ-rad $1 \subseteq \mu$-rad A,

(vi) $A \cdot \mu$-rad $1 \subseteq \mu$-rad A.

Theorem B. (See [2], Theorems 4.2, 5.1, 5.2, 5.3). *Let $S = S(d, G_i, \gamma_i)$, let μ be a uniform congruence on E, let A be a γ-admissible subset of G, and suppose that μ and A are linked. Then*

$$\tau = \{((m, a_i, n), (p, b_j, q)) \in S \times S \mid (a_i\alpha_{u,w})(b_j^{-1}\alpha_{v,w}) \in A_w (\text{mod } d),$$
$$\text{where } u = nd + i,\ v = qd + j,\ w = \max\{u, v\};\ m - n = p - q;\ (e_i^m, e_j^p) \in \mu\}$$

is a congruence on S contained in $[\iota, \sigma \vee H]$ such that $\operatorname{tr}\tau = \mu$ and $A^\tau = A$, where $A = A_0^\tau \times \cdots \times A_{d-1}^\tau$ and $A_i^\tau = \{a_i \in G_i \mid (0, a_i, 0)\tau e_i^0\}$, $i = 0, \ldots, d-1$.

Conversely, let τ be a congruence on $S = S(d, G_i, \gamma_i)$ contained in $[\iota, \sigma \vee H]$, then τ is of the above form with $\mu = \operatorname{tr}\tau$ and $A = A^\tau$.

Moreover, let ρ, λ be congruences on $S = S(d, G_i, \gamma_i)$ contained in $[\iota, \sigma \vee H]$. Then

(i) $\rho \leqq \lambda$ *if and only if* $\operatorname{tr}\rho \leqq \operatorname{tr}\lambda$ *and* $A^\rho \subseteq A^\lambda$,

(ii) $\operatorname{tr}(\rho \vee \lambda) = \operatorname{tr}\rho \vee \operatorname{tr}\lambda$, $\operatorname{tr}(\rho \wedge \lambda) = \operatorname{tr}\rho \wedge \operatorname{tr}\lambda$,

(iii) $A^{\rho \vee \lambda} = (\operatorname{tr}\rho \vee \operatorname{tr}\lambda)$-rad $A^\rho \cdot A^\lambda$, $A^{\rho \wedge \lambda} = A^\rho \cap A^\lambda$.

Lemma 1.3. *Let $S = S(d, G_i, \gamma_i)$. Let μ and v be uniform congruences on $E(S)$ and let A*

be a γ-admissible subset of G. If A is linked with both μ and ν, then it is also linked with $\mu \vee \nu$.

Proof. We must prove that $(\mu \vee \nu)$-rad $A = A$. Since $A = A_0 \times \cdots \times A_{d-1}$, first we prove that $(\mu \vee \nu)$-rad $A_i = A_i$ for every $i = 0, \ldots, d-1$. Let $g_i \in (\mu \vee \nu)$-rad A_i, then $g_i \in G_i$, $g_i \alpha_{nd+i, md+j} \in A_j$ with $n, m \in \mathbb{N}$, $j = 0, \ldots, d-1$ such that $e_j^m \leqq e_i^n$, $e_j^m \ (\mu \vee \nu)\ e_i^n$. Hence there exist $e_i^n = e_{i_1}^{n_1}, \ldots, e_{i_r}^{n_r}, \ldots, e_{i_k}^{n_k} = e_j^m \in E$ $(0 \leqq i_r \leqq d-1,\ n \leqq n_r \leqq m)$ such that $e_{i_1}^{n_i} \geqq \cdots \geqq e_{i_r}^{n_r} \geqq \cdots \geqq e_{i_k}^{n_k}$; $e_{i_1}^{n_1}\ \mu\ e_{i_2}^{n_2}$, $e_{i_2}^{n_2}\ \nu\ e_{i_3}^{n_3}, \ldots, e_{i_{k-1}}^{n_{k-1}}\ \nu\ e_{i_k}^{n_k}$. Hence we have $g_i \alpha_{nd+i, md+j} = (g_i \alpha_{nd+i, n_{k-1}d+i_{k-1}}) \alpha_{n_{k-1}d+i_{k-1}, md+j} \in A_j$. Since we have $g_i \alpha_{nd+i, n_{k-1}d+i_{k-1}} \in G_{i_{k-1}}$, $e_{i_{k-1}}^{n_{k-1}}\ \nu\ e_j^m$, $e_{i_{k-1}}^{n_{k-1}} \geqq e_j^m$, it follows that $g_i \alpha_{nd+i, n_{k-1}d+i_{k-1}} \in \nu$-rad $A_{i_{k-1}}$. Hence, since A is linked with ν, we have ν-rad $A = A$ whence ν-rad $A_{i_{k-1}} = A_{i_{k-1}}$, thus $g_i \alpha_{nd+i, n_{k-1}d+i_{k-1}} \in A_{i_{k-1}}$. Repeating the same argument, since we have also $A = \mu$-rad A, we obtain $g_i \in A_i$. It follows that $(\mu \vee \nu)$-rad $A_i = A_i$, hence $(\mu \vee \nu)$-rad $A = A$.

Lemma 1.4. *Let $S = S(d, G_i, \gamma_i)$. If $L(S)$ is modular, then the following condition holds*

$$A \cdot \mu\text{-rad}\, 1 = \mu\text{-rad}\, A \tag{$*$}$$

for every γ-admissible subset A of G and for every uniform congruence μ on $E(S)$.

Proof. Let A be a γ-admissible subset of G and μ a uniform congruence on $E(S)$. We remark that every γ-admissible subset is linked with the identity on $E(S)$ which is obviously uniform, moreover μ-rad 1 is a γ-admissible subset and it is linked with the uniform congruence μ.

Hence, by Theorem B, there exist in $L(S)$ three congruence ρ, λ, τ such that ρ, λ are idempotent separating with $A^\rho = A \cdot \mu$-rad 1, $A^\lambda = \mu$-rad A, and τ has trace μ with $A^\tau = \mu$-rad 1. Thus, we obtain, by Theorem B (i) and by Remark 1.2 (vi)

$$\rho \leqq \lambda \tag{1}$$

and by Theorem B (ii)

$$\mathrm{tr}(\rho \wedge \tau) = \mathrm{tr}(\lambda \wedge \tau) = \iota, \quad \mathrm{tr}(\rho \vee \tau) = \mathrm{tr}(\lambda \vee \tau). \tag{2}$$

Moreover by Theorem B (iii), and by Remark 1.2 (ii), (iv), (v), (vi) $A^{\lambda \vee \tau} = \mu$-rad$(\mu$-rad $A \cdot \mu$-rad 1$) = \mu$-rad$(\mu$-rad $A) = \mu$-rad A, $A^{\rho \vee \tau} = \mu$-rad$(A \cdot \mu$-rad 1 $\cdot\, \mu$-rad 1$) = \mu$-rad$(A \cdot \mu$-rad 1$) = \mu$-rad A, thus

$$A^{\rho \vee \tau} = A^{\lambda \vee \tau} = \mu\text{-rad}\, A, \quad A^{\rho \wedge \tau} = A^{\lambda \wedge \tau} = \mu\text{-rad}\, 1. \tag{3}$$

Since $L(S)$ is modular, (1), (2), (3) imply $\rho = \lambda$, whence, by Theorem 2, condition $(*)$ holds.

Lemma 1.5. *Let $S = S(d, G_i, \gamma_i)$. If condition $(*)$ holds, then*

$$\mu\text{-rad}\,1\cdot\nu\text{-rad}\,1=(\mu\vee\nu)\text{-rad}\,1$$

for every μ, ν uniform congruences on $E(S)$.

Proof. Putting $A=\nu$-rad 1, from condition $(*)$ it follows that ν-rad $1\cdot\mu$-rad $1=\mu$-rad$(\nu$-rad $1)$. Hence, by Remark 1.2 (iv), μ-rad$(\nu$-rad $1\cdot\mu$-rad $1)=\mu$-rad$(\mu$-rad$(\nu$-rad $1))=\mu$-rad$(\nu$-rad $1)=\nu$-rad $1\cdot\mu$-rad 1, hence the γ-admissible subset ν-rad $1\cdot\mu$-rad 1 is linked with μ. Analogously we can prove that such a subset is linked with ν, hence by Lemma 1.3 it is linked with $\nu\vee\mu$. From Remark 1.2 (v) it follows that ν-rad$\cdot\mu$-rad $1\supseteq(\nu\vee\mu)$-rad 1. Moreover condition (iii) of Remark 1.2 implies ν-rad $1\cdot\mu$-rad $1\subseteq(\nu\vee\mu)$-rad 1 hence ν-rad $1\cdot\mu$-rad $1=(\nu\vee\mu)$-rad 1.

Lemma 1.6. *Let $S=S(d,G_i,\gamma_i)$. If condition $(*)$ holds, then $[\iota,\sigma\vee H]$ is a modular sublattice of $L(S)$.*

Proof. Let ρ, λ, τ be congruences on S contained in $[\iota,\sigma\vee H]$ and such that $\rho\leqq\lambda$, $\rho\vee\tau=\lambda\vee\tau$, $\rho\wedge\tau=\lambda\wedge\tau$. We must prove that $\rho=\lambda$, that is tr $\rho=$ tr λ, $A^\rho=A^\lambda$ (see Theorem B). By Theorem B, in $L(E)$ we have tr $\rho\leqq$ tr λ, tr $\rho\vee$ tr $\tau=$ tr $\lambda\vee$ tr τ, tr $\rho\wedge$ tr $\tau=$ tr $\lambda\wedge$ tr τ; moreover, since E is an ω-chain, $L(E)$ is a modular lattice (see, e.g., [**4**, Theorem 2]) and tr $\rho=$ tr λ. If we put $\nu=$ tr $\rho=$ tr λ, $\mu=$ tr τ, then $\nu\vee\mu$ is a uniform congruence on $E(S)$ (being the trace of the congruence $\lambda\vee\tau$ of S) and $A^\lambda\cdot A^\tau$ is a γ-admissible subset of G (see [**2**, p. 164)]. By condition $(*)$, we have

$$(\nu\vee\mu)\text{-rad}(A^\lambda\cdot A^\tau)=A^\lambda\cdot A^\tau\cdot(\nu\vee\mu)\text{-rad}\,1. \tag{4}$$

Moreover $A^\tau=\mu$-rad $A^\tau\supseteq\mu$-rad 1 and $A^\lambda=\nu$-rad $A^\lambda\supseteq\nu$-rad 1 so, by Lemma 1.5, $A^\lambda\cdot A^\tau\supseteq\mu$-rad $1\cdot\nu$-rad $1=(\mu\vee\nu)$-rad 1. Hence (4) implies $(\nu\vee\mu)$-rad$(A^\lambda\cdot A^\tau)=A^\lambda\cdot A^\tau$, whence, by Theorem B (iii), $A^{\lambda\vee\tau}=A^\lambda\cdot A^\tau$.

Analogously we prove that $A^{\rho\vee\tau}=A^\rho\cdot A^\tau$. Since $\lambda\vee\tau=\rho\vee\tau$, we have $A^\rho\cdot A^\tau=A^\lambda\cdot A^\tau$ and also $A^\rho\wedge A^\tau=A^{\rho\wedge\tau}=A^{\lambda\wedge\tau}=A^\lambda\wedge A^\tau$. From the modularity of the lattice of normal subgroups of a group (see also [**1**, p. 464]), we have $A^\rho=A^\lambda$ and the lemma is proved.

Lemma 1.7. *Let $S=S(d,G_i,\gamma_i)$. If $[\iota,\sigma\vee H]$ is a modular sublattice of $L(S)$, then $L(S)$ is modular.*

Proof. Let λ, ρ, τ be congruences on S such that $\rho\leqq\lambda$, $\rho\vee\tau=\lambda\vee\tau$, $\rho\wedge\tau=\lambda\wedge\tau$. We must prove that $\lambda=\rho$. By an argument similar to the one used in the proof of the preceding lemma, we have tr $\rho=$ tr λ. Let tr $\rho=$ tr $\lambda=\nu$, tr $\tau=\mu$. If ν, μ are both different from the universal relation $\omega_{E(S)}$ of $E(S)$, the congruences ρ, λ, τ belong to $[\iota,\sigma\vee H]$ (see [**2**, Remark p. 164]) and by the modularity of such sublattice we get $\lambda=\rho$. If, ν, μ are both equal to $\omega_{E(S)}$, we have $\lambda=\rho$ because the lattice of group congruences on an inverse semigroup is modular (see [**10**, Corollary III.2.7]). Hence it remains to examine the following cases

(i) $\nu=\omega_{E(S)}$, $\mu\neq\omega_{E(S)}$

(ii) $\mu=\omega_{E(S)}$, $\nu\neq\omega_{E(S)}$

We will use an argument analogous to the one used by Munn in the proof of Lemma 10 of [**8**].

Case (i). Since τ is not a group congruence, τ belongs to $[\iota,\sigma\vee H]$ hence $\tau=\tau\wedge(\sigma\vee H)$. So, since $\lambda\wedge(\sigma\vee H)$, τ, $\sigma\in[\iota,\sigma\vee H]$ and $\sigma\leqq\lambda\wedge(\sigma\vee H)$ (λ being a group congruence), it follows from the modularity of $[\iota,\sigma\vee H]$ that $(\lambda\wedge\tau)\vee\sigma=(\lambda\wedge(\sigma\vee H)\wedge\tau)\vee\sigma=(\lambda\wedge(\sigma\vee H))\wedge(\sigma\vee\tau)=\lambda\wedge((\sigma\vee H)\wedge(\sigma\vee\tau))=\lambda\wedge(\sigma\vee\tau)$. Analogously we can prove that $(\rho\wedge\tau)\vee\sigma=\rho\wedge(\sigma\vee\tau)$. From $\lambda\wedge\tau=\rho\wedge\tau$ and $\lambda\vee\tau=\rho\vee\tau$, we deduce

$$\lambda\wedge(\sigma\vee\tau)=\rho\wedge(\sigma\vee\tau) \tag{5}$$

$$\lambda\vee(\sigma\vee\tau)=\rho\vee(\sigma\vee\tau) \tag{6}$$

The congruences λ, ρ, $\tau\vee\sigma$ are group congruences satisfying (5) and (6). Moreover $\rho\leqq\lambda$ and thus, from the modularity of the lattice of the group congruences of S, we have $\lambda=\rho$.

Case (ii). Since τ is a group congruence, $\tau\geqq\sigma$ and so $\tau\vee\sigma=\tau$, moreover τ, $\lambda\vee\sigma$, $\sigma\vee H$ are group congruences, and λ is not a group congruence, hence $\lambda\leqq\sigma\vee H$, so $\lambda\vee\sigma\leqq\sigma\vee H$. Since the lattice of the group congruences is modular, we have $(\lambda\vee\tau)\wedge(\sigma\vee H)=((\lambda\vee\sigma)\vee\tau)\wedge(\sigma\vee H)=(\tau\wedge(\sigma\vee H))\vee(\lambda\vee\sigma)=(\tau\wedge(\sigma\vee H))\vee\lambda$. Analogously we can deduce that $(\rho\vee\tau)\wedge(\sigma\vee H)=(\tau\wedge(\sigma\vee H))\vee\rho$. Thus, from $\lambda\vee\tau=\rho\vee\tau$ it follows

$$(\tau\wedge(\sigma\vee H))\vee\lambda=(\tau\wedge(\sigma\vee H))\vee\rho. \tag{7}$$

Moreover from $\lambda\wedge\tau=\rho\wedge\tau$ it follows

$$(\tau\wedge(\sigma\vee H))\wedge\lambda=(\tau\wedge(\sigma\vee H))\wedge\rho. \tag{8}$$

The congruences λ, ρ, $\tau\wedge(\sigma\vee H)$ belong to the modular sublattice $[\iota,\sigma\vee H]$ of $L(S)$ and satisfy (7) and (8), moreover, $\rho\leqq\lambda$, hence we have again $\rho=\lambda$ and the lemma is proved.

Now we are able to state the following characterization of simple regular ω-semigroups whose lattice of congruences is modular.

Theorem 1.8. *Let $S=S(d,G_i,\gamma_i)$. $L(S)$ is modular is and only if, for every γ-admissible subset $A=A_0\times\cdots\times A_{d-1}$ of G, one of the following conditions holds*

(i) $A\cdot\mu$-rad $1=\mu$-rad A *for every uniform congruence* μ *on* $E(S)$,

(ii) $A_0\gamma_{d-1}^{-1}=A_{d-1}\cdot\ker\gamma_{d-1}$ *and* $A_{k+1}\gamma_k^{-1}=A_k\cdot\ker\gamma_k$ *for every* k *with* $0\leqq k\leqq d-2$.

Proof. From previous lemmas we can deduce that the congruences lattice of S is modular if and only if condition (i) holds.

Now we prove that (i) implies (ii). For $0\leqq k\leqq d-1$, let μ_k the congruence on E defined by

$$\mu_k=\{(e_i^n,e_j^m)\mid i=j,\ n=m \text{ if } j\neq k;\text{ or } i=j-1, n=m \text{ if } j=k\neq 0;$$

$$\text{or } i=d-1,\ n=m-1 \text{ if } j=k=0\},$$

and let $A=A_0\times\cdots\times A_{d-1}$ be a γ-admissible subset of G, it immediately follows that μ_k-rad $A=A_0\times\cdots\times A_{k-2}\times A_k\gamma_{k-1}^{-1}\times A_k\times\cdots\times A_{d-1}$ for every $0<k\leqq d-1$ and μ_0-rad $A=A_0\times\cdots\times A_{d-2}\times A_0\gamma_{d-1}^{-1}$. So from (i) we deduce condition (ii).

Now we will prove that (ii) implies (i). From [2, 2], and from definition of μ-rad A it follows that it is enough to prove that

$$A_{j(\text{mod } d)}\alpha_{i,j}^{-1}=A_{i(\text{mod } d)}\cdot\ker\alpha_{i,j} \tag{9}$$

for every positive integers i, j with $i<j$. Denote by $\bar{j}$ and $\bar{i}$ the integers j (mod d) and i (mod d) respectively, and distinguish two cases:

Case 1. Let $j-i<d$. Suppose $\bar{i}<\bar{j}$. The subset

$$A_0\times\cdots\times A_{\bar{i}}\times A_{\bar{j}}\alpha_{\bar{i}+1,\bar{j}}^{-1}\times A_{\bar{j}}\alpha_{\bar{i}+2,\bar{j}}^{-1}\times\cdots\times A_{\bar{j}}\alpha_{\bar{j}-1,\bar{j}}^{-1}\times A_{\bar{j}}\times\cdots\times A_{d-1}$$

is a γ-admissible subset of G. hence, by (ii), we have $A_{\bar{j}}\alpha_{\bar{i}+1,\bar{j}}^{-1}\gamma_{\bar{i}}^{-1}=A_{\bar{i}}\cdot\ker\gamma_{\bar{i}}\subseteq A_{\bar{i}}\ker\alpha_{\bar{i},\bar{j}}$; since obviously $A_{\bar{j}}\alpha_{\bar{i}+1,\bar{j}}^{-1}\gamma_{\bar{i}}^{-1}\supseteq A_{\bar{i}}\ker\alpha_{\bar{i},\bar{j}}$ the statement follows. If $\bar{j}<\bar{i}$, we consider the γ-admissible subset

$$A_0\alpha_{0,\bar{j}}^{-1}\times\cdots\times A_{\bar{j}}\alpha_{\bar{j}-1,\bar{j}}^{-1}\times A_{\bar{j}}\times\cdots\times A_{\bar{i}}\times A_{\bar{j}}\alpha_{\bar{i}+1,\bar{j}}^{-1}\times\cdots\times A_{\bar{j}}\alpha_{d-1,\bar{j}}^{-1}$$

and analogously we deduce (9).

Case 2. Let $j-i\geqq d$, suppose $\bar{i}<\bar{j}$ then $\alpha_{i,j}=\alpha_{\bar{i},\bar{j}}\alpha_{\bar{j}}^n$ for some positive integer n. We establish (9) by induction on n, the case $n=0$ being proved by Case 1. We consider the γ-admissible subset

$$A_{\bar{j}}(\alpha_{0,\bar{j}}\alpha_{\bar{j}}^{n-1})^{-1}\times\cdots\times A_{\bar{j}}(\alpha_{\bar{i},\bar{j}}\alpha_{\bar{j}}^{n-1})^{-1}\times A_{\bar{j}}(\alpha_{\bar{i}+1,\bar{j}}\alpha_{\bar{j}}^{n})^{-1}\times\cdots$$

$$\times A_{\bar{j}}(\alpha_{\bar{j}}^{n})^{-1}\times A_{\bar{j}}(\alpha_{\bar{j}+1,\bar{j}+d}\alpha_{\bar{j}}^{n-1})^{-1}\times\cdots\times A_{\bar{j}}(\alpha_{d-1,\bar{j}+d}\alpha_{\bar{j}}^{n-1})^{-1},$$

by (ii) and by the induction hypothesis we have

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$$A_{\bar{j}}(\alpha_{\bar{i}+1,\bar{j}}\alpha_{\bar{j}}^{n})^{-1}\gamma_{i}^{-1} = A_{\bar{j}}(\alpha_{\bar{i},\bar{j}}\alpha_{\bar{j}}^{n-1})^{-1}\cdot\ker\gamma_{\bar{i}} = A_{\bar{j}}\cdot\ker\alpha_{\bar{i},\bar{j}}\alpha_{\bar{j}}^{n-1}\cdot\ker\gamma_{\bar{i}}$$

$$= A_{\bar{j}} = A_{\bar{j}}\cdot\ker\alpha_{\bar{i},\bar{j}}\alpha_{\bar{j}}^{n-1} \subseteq A_{\bar{j}}\cdot\ker\alpha_{\bar{i},\bar{j}}\alpha_{\bar{j}}^{n}.$$

Since $A_{\bar{j}}\cdot\ker\alpha_{\bar{i},\bar{j}}\alpha_{\bar{j}}^{n} \subseteq A_{\bar{j}}(\alpha_{\bar{i}+1,\bar{j}}\alpha_{\bar{j}}^{n})^{-1}$, we have condition (9). Analogously we proceed when $\bar{i} \geqq \bar{j}$,

Remark 1.9. If S is a bisimple ω-semigroup then S is a simple regular ω-semigroup with $d=1$, hence from Theorem 1.8 we reobtain the Munn's result (see [**8**, 2]). In fact, the congruences on a bisimple ω-semigroup are either idempotent separating or group congruences (see [**8**, 1]), the γ-admissible subsets coincide with the α-admissible normal subgroups of Munn and ω_E-rad $A = \text{rad}\,A$.

From Theorem 1.8, we can also deduce Theorem 4.1 of [**1**].

Remark 1.10. For every positive integer d there exists a simple regular ω-semigroup $S=S(d, G_i, \gamma_i)$ whose lattice of congruences is modular. In fact if γ_i is trivial for every $i=0,1,\ldots,d-1$, then condition (ii) of Theorem 1.8 holds. Furthermore we prove that if a simple regular ω-semigroup S satisfies (ii) of Theorem 1.8 and there exists a k such that γ_k is trivial, then γ_i is trivial for every i. Let us consider the γ-admissible subset

$$A_k = \{e_0\} \times \cdots \times \{e_{k-1}\} \times G_k \times \{e_{k+1}\} \times \cdots \times \{e_{d-1}\},$$

from condition (ii) of Theorem 1.8 we deduce $\ker\gamma_{k-1} = G_{k-1}$, so, γ_{k-1} is trivial. Repeating this argument, we can prove that γ_i is trivial for every $i=0,1,\ldots,d-1$ (we remark that triviality of γ_0 implies triviality of γ_{d-1}).

From the previous remark we deduce that, differently from what happens in the bisimple ω-semigroup (see [**8**, Corollaries 2 and 5]), the finiteness or the simplicity of G_i do not imply, in general, the modularity of $L(S)$. We also remark that it is possible to construct $S=S(d, G_i, \gamma_i)$ with a modular lattice of congruences and γ_i non trivial. We can for example take $d=2$, $G_0=S_3$, $G_1=\{e_1, g\}$, γ_0 such that $x\gamma_0=e_1$ if x belongs to the alternating subgroup of G_0, otherwise $x\gamma_0=g$, γ_1 such that $e_1\gamma_1=i$, $g\gamma_1=(12)$, where i denotes the identity of S_3.

Finally we remark that the modularity of $L(S)$ does not imply that the congruences of $L(S)$ are permutable (see [**3**]).

2. The general case

First we recall that, for a regular ω-semigroup, Munn in [**7**] proved the following result:

Theorem C. *Let S be a regular ω-semigroup.*

(*i*) *If S has no kernel, then it is the union of an ω-chain of groups*

(ii) *If the kernel of S coincides with S, then S is a simple regular ω-semigroup*

(iii) *If S has a proper kernel, then S is a (retract) ideal extension of a simple regular ω-semigroup K by a semigroup H^0, where H is a finite chain of groups and H^0 is obtained from H by adjoining a zero. Moreover this extension is determined by means of a homomorphism of H into the group of units of K.*

The characterization of semigroups of type (i) whose lattice of congruence is modular can be easily deduced from the following:

Theorem D. (See [4, Theorem 3]). *Let S be a semilattice of groups and E its semilattice of idempotents. Then $L(S)$ is modular if and only if E is a tree and S has trivial multiplication.*

In fact, from Theorem D immediately follows:

Theorem 2.1. *Let $S=[\Omega, H_j, \phi_j]$ be a regular ω-semigroup without kernel. Then $L(S)$ is modular if and only if ϕ_j are trivial for every $j\in\mathbb{N}$. ($S=[\Omega, H_j, \phi_j]$ indicates the Clifford representation of the ω-chain Ω of the groups H_j ($j\in\mathbb{N}$) with connecting homomorphisms $\phi_j=\phi_{j,j+1}$.)*

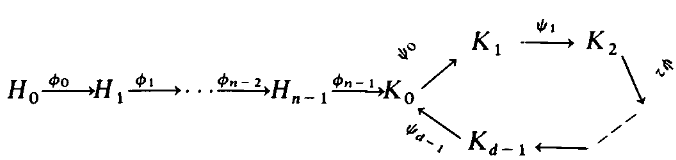
Theorem 1.8 gives the characterization of semigroups of type (ii) whose lattice of congruences is modular. To achieve the aim of the paper, it remains to examine the semigroups of type (iii). Such a semigroup S is the disjoint union of a finite chain of groups H and of a simple regular ω-semigroup K which is an ideal of S.

For H we use the short Clifford representation $[n, H_j, \phi_j]$, where n is the length of the chain, $j=0,\dots,n-1$, $\phi_j=\phi_{j,j+1}$ for $0\leqq j\leqq n-2$, and use for K the notation $K=K(d, K_i, \psi_i)$. Moreover, since the group of units of K is $\{(0,k_0,0)\,|\,k_0\in K_0\}$, the homomorphism ϕ which induces the retract extension of K by H^0 can be constructed by means of a homomorphism $\phi_{n-1}\colon H_{n-1}\to K_0$ in the following way

$\phi\colon h_{n-1}\to(0, h_{n-1}\phi_{n-1}, 0)$ for every $h_{n-1}\in H_{n-1}$

$\phi\colon h_i\to(0, (((h_i\phi_i)\phi_{i+1})\cdots)\phi_{n-1}, 0)$ for every $h_i\in H_i$: $0\leqq i\leqq n-2$.

So S can be represented by the following diagram of homomorphisms



We will denote S by the notation $S=S([n, H_j, \phi_j]; K(d, K_i, \psi_i); \phi_{n-1})$, or shortly,

$S=S(H;K;\phi_{n-1})$. Moreover we will indicate the idempotents of S by: f_j is the identity of H_j $(j=0,\ldots,n-1)$, $e_i^m=(m,e_i,m)$ $(m\in\mathbb{N};\ i=0,\ldots,d-1;\ e_i$ the identity of $K_i)$ are the idempotents of K. Therefore

$$f_0>f_1>\cdots>f_{n-1}>e_0^0>e_1^0>\cdots>e_{d-1}^0>e_0^1>\cdots$$

Lemma 2.2. *Let $S=S(H;K;\phi_{n-1})$ and let $L(S)$ be modular. Then*

(a) *$L(H)$ is modular*

(b) *$L(K)$ is modular*

(c) *ϕ_{n-1} is trivial.*

Proof. First we prove (a). Let ρ_K be the Rees congruence of S modulo K. Since $L(S)$ is modular, $L(S/\rho_K)$, that is $L(H^0)$, is modular whence also $L(H)$ is modular (see, e.g., [**4**, Corollary of Theorem 1 and proof of Theorem 4]).

Now we prove (b). For every congruence ρ' on K, denote by ρ the congruence defined by $a\rho b$ if and only if either $a,b\in K$ and $a\rho' b$ or $a=b$. It is immediate that ρ is a congruence on S and that the mapping $\rho'\to\rho$ is an injective lattice homomorphism from $L(K)$ into $L(S)$. Consequently, since $L(S)$ is modular, so is $L(K)$.

Finally we prove (c). Let ρ, λ, τ, be relations on S respectively defined by $a\ \rho\ b$ if and only if either $a=b$ or $a,b\in K$, or $a,b\in H_{n-1}$ and $a\phi_{n-1}=b\phi_{n-1}$, $a\ \lambda\ b$ if and only if either $a=b$, or a, $b\in K$, or $a,b\in H_{n-1}$, $a\ \tau\ b$ if and only if either $a=b$, or $a,b\in H_{n-1}$ and $a\phi_{n-1}=b\phi_{n-1}$, or $a\in K$, $b\in H_{n-1}$ and $b\phi_{n-1}=a$, or $b\in K$, $a\in H_{n-1}$ and $a\phi_{n-1}=b$.

It is straightforward to verify that ρ,λ,τ are congruences on S such that $\rho\leqq\lambda$, $\rho\vee\tau=\lambda\vee\tau$, $\rho\wedge\tau=\lambda\wedge\tau$. Since $L(S)$ is modular, we have $\rho=\lambda$, hence $a\phi_{n-1}=b\phi_{n-1}$ for every $a,b\in H_{n-1}$, thus ϕ_{n-1} is trivial.

Lemma 2.3. *Let $S=(H;K;\phi_{n-1})$ with ϕ_{n-1} trivial. Then for all congruences ρ, τ on S we have*

(a) $(\rho\vee\tau)_{|K}=\rho_{|K}\vee\tau_{|K}$

(b) $(\rho\vee\tau)_{|H}=\rho_{|H}\vee\tau_{|H}$

(c) $(\rho\wedge\tau)_{|K}=\rho_{|K}\wedge\tau_{|K}$

(d) $(\rho\wedge\tau)_{|H}=\rho_{|H}\wedge\tau_{|H}$

(e) $\ker\rho=\ker\rho_{|H}\cup\ker\rho_{|K}$

Proof. First we prove condition (a). We immediately have that $(\rho\vee\tau)_{|K}\geqq\rho_{|K}\vee\tau_{|K}$, hence we verify that

$$(\rho\vee\tau)_{|K}\leqq\rho_{|K}\vee\tau_{|K}.$$

Let $a, b \in K$ such that $a\ (\rho \vee \tau)_{|K}\ b$. Then there exist $x_1, x_2, \ldots, x_r \in S$ such that $a\ \rho\ x_1$, $x_1\ \tau\ x_2, \ldots, x_r\ \tau\ b$. Hence $a = ae_0^0\ \rho\ x_1e_0^0$, $x_1e_0^0\ \tau\ x_2e_0^0, \ldots, x_re_0^0\ \tau\ be_0^0 = b$. All the elements $x_ie_0^0$ belong to K, thus $a\ (\rho_{|K} \vee \tau_{|K})\ b$.

Now we prove condition (b). It is enough to prove that $(\rho \vee \tau)_{|H} \leqq \rho_{|H} \vee \tau_{|H}$. Let $a, b \in H$ with $a\ (\rho \vee \tau)_{|H}\ b$. Then there exist $x_1, x_2, \ldots, x_r \in S$ such that

$$a\ \rho\ x_1, x_1\ \tau\ x_2, \ldots, x_r\ \tau\ b. \tag{10}$$

If all the elements $x_1, x_2, \ldots, x_r \in S$ belong to H, then $(\rho_{|H} \vee \tau_{|H})\ b$. Suppose, on the contrary, that there are in (10) some elements of K, i.e. that (10) contains sequences of the type

$$h\ \sigma_1\ k_1, k_1\ \sigma_2\ k_2, \ldots, k_{s-1}\ \sigma_s\ h', \tag{11}$$

where $h, h' \in H$, $k_1, k_2, \ldots, k_{s-1} \in K$ and, for every $i = 1, \ldots, s$, either $\sigma_i = \rho$ or $\sigma_i = \tau$.

Suppose, without loss of generality, that $h \in H_i$ and $h' \in H_j$, $i, j \in \{0, 1, \ldots, n-1\}$, $i \geqq j$. Thus, since ϕ_{n-1} is trivial, we have, using the first pair of (11), $h^{-1}h\ \sigma_1\ h^{-1}k_1$ hence $f_i\ \sigma_1\ k_1$ and $f_i\ \sigma_1\ h$, using the last, $k_{s-1}(h')^{-1}\ \sigma_s\ h'(h')^{-1}$ hence $k_{s-1}\ \sigma_s\ f_j$, $k_{s-1}f_i\ \sigma_s\ f_jf_i$, $k_{s-1}\ \sigma_s\ f_i$ and $f_i\ \sigma_s\ h'$, hence we can replace sequence (11) with the sequence $h\ \sigma_1\ f_i$, $f_i\ \sigma_s\ h'$ whose elements are in H, whence $a\ (\rho_{|H} \vee \tau_{|H})\ b$.

The proofs of (c) and (d) are trivial and obviously hold if H and K are replaced by an arbitrary subsemigroup T and S.

Finally we must prove (e). It is obvious that $\ker \rho \supseteq \ker \rho_{|H} \cup \ker \rho_{|K}$. Suppose that $\ker \rho \supset \ker \rho_{|H} \cup \ker \rho_{|K}$. Two cases are possible.

Case 1: there exists $x \in K$ such that $x\ \rho\ f_j$ for some $j = 0, 1, \ldots, n-1$ and $x \notin \ker \rho_{|K}$.

Case 2: there exists $y \in H$ such that $y\ \rho\ e_i^m$ for some $i = 0, 1, \ldots, d-1$, $m \in \mathbb{N}$, $y \notin \ker \rho_{|H}$.

In the first case from $x\ \rho\ f_j$ it follows $x^{-1}x\ \rho\ x^{-1}f_j$ and, since f_j is the identify of H_j and $x^{-1}x$ is an idempotent e_i^m of K, we get $e_i^m\ \rho\ x^{-1}e_0^0 = x^{-1}$; hence $x^{-1} \in \ker \rho_{|K}$ and $x \in \ker \rho_{|K}$, a contradiction.

In the second case from $y\ \rho\ e_i^m$ it follows $y^{-1}y\ \rho\ y^{-1}e_i^m$ and, since $y^{-1}y$ is the identity of the group H_j and ϕ_{n-1} is trivial, $f_j\ \rho\ e_0^0\ e_i^m$, $f_j\ \rho\ e_i^m$, $f_j\ \rho\ y$, hence we obtain the contradiction $y \in \ker\ \rho_{|H}$. Thus $\ker \rho = \ker \rho_{|H} \cup \ker \rho_{|K}$.

Lemma 2.4. *Let $S = S(H; K; \phi_{n-1})$ and suppose that the following conditions hold*

(a) *$L(H)$ is modular*

(b) *$L(K)$ is modular*

(c) *ϕ_{n-1} is trivial*

Then $L(S)$ is modular.

Proof. Let ρ, λ, τ be congruences on S such that $\rho \leqq \lambda$, $\rho \vee \tau = \lambda \vee \tau$, $\rho \wedge \tau = \lambda \wedge \tau$.

Since the mapping $\rho \to \mathrm{tr}\ \rho$ is a lattice homomorphism of $L(S)$ into $L(E)$ (see e.g. [**10**, Theorem III.2.5]) and $L(E)$ is modular, E being an ω-chain (see e.g. [**4**, Theorem 2]), we have $\mathrm{tr}\ \rho = \mathrm{tr}\ \lambda$. On the other hand, by Lemma 2.3, we have

$$(\lambda \vee \tau)_{|K} = \lambda_{|K} \vee \tau_{|K}$$

$$(\rho \vee \tau)_{|K} = \rho_{|K} \vee \tau_{|K}$$

$$(\lambda \wedge \tau)_{|K} = \lambda_{|K} \wedge \tau_{|K}$$

$$(\rho \wedge \tau)_{|K} = \rho_{|K} \wedge \tau_{|K}$$

Since $\rho_{|K} \leqq \lambda_{|K}$ it follows from the modularity of $L(K)$ that $\rho_{|K} = \lambda_{|K}$. Analogously we can prove $\rho_{|H} = \lambda_{|H}$. From condition (e) of Lemma 2.3, we deduce also $\ker \lambda = \ker \lambda_{|K} \cup \ker \lambda_{|H} = \ker \rho_{|K} \cup \ker \rho_{|H} = \ker \rho$. Thus, since $\mathrm{tr}\ \rho = \mathrm{tr}\ \lambda$ and $\ker \lambda = \ker \rho$, we have $\rho = \lambda$ (see, for instance [**10**, Theorem III.1.5]), hence $L(S)$ is modular.

The above lemmas, Theorem D and Theorem 1.8, give us the following theorem which completely characterizes the regular ω-semigroups whose lattice of congruences is modular.

Theorem 2.5. *Let $S = S([n, H_j, \phi_j]; K(d, K_i, \psi_i); \phi_{n-1})$ be a regular ω-semigroup with proper kernel. Then $L(S)$ is modular if and only if the following hold.*

(a) $\phi_0, \phi_1, \ldots, \phi_{n-1}$ *are trivial,*

(b) $A_0\psi_{d-1}^{-1} = A_{d-1} \cdot \ker \psi_{d-1}$ *and* $A_{k+1}\psi_k^{-1} = A_k \cdot \ker \psi_k$ $(0 < k \leqq d-1)$ *for every ψ-admissible subset $A = A_0 \times \cdots \times A_{d-1}$ of $K_0 \times \cdots \times K_{d-1}$.*

Remark 2.6. We remark that Theorems 2.1 and 2.5 could be deduced either from Theorem 1.21 or from Theorem 5.2 of [**6**]. Since the proofs of these theorems have not been published, we have given here a direct proof of our result.

After the authors sent the paper to this journal, they knew that M. Petrick proved in "Congruences on strong semilattices of regular simple semigroups", *Semigroup Forum* **37** (1988), 167–199, a theorem from which our Theorem 2.5 can be immediately deduced.

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