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Perturbations of graphs for Newton maps I: bounded hyperbolic components

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Abstract. We consider graphs consisting of finitely many internal rays for degenerating Newton maps and state a convergence result. As an application, we prove that a hyperbolic component in the moduli space of quartic Newton maps is bounded if and only if every element has degree 2 on the immediate basin of each root. This provides the first complete description of bounded hyperbolic components in a complex two-dimensional moduli space.

Key words: Newton maps, internal rays, hyperbolic components, Newton graphs
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1. *Introduction*

For $d \geq 2$, denote by Rat_d the space of rational maps of degree d in one complex variable. Via parameterizing coefficients, the space Rat_d is an open dense subset of the $2d+1$-dimensional complex projective space $\mathbb{P}^{2d+1}$. The boundary $\partial\mathrm{Rat}_d := \mathbb{P}^{2d+1} \setminus \mathrm{Rat}_d$ consists of so-called degenerate rational maps. A sequence in Rat_d is *degenerate* if its limit is a degenerate rational map. It is of interest to understand the interplay of dynamics for a degenerate sequence and its limit. The goal of this paper is to explore this interplay in a significant slice of Rat_d, namely Newton family. We show that under natural assumptions, the dynamics preserves stably when Newton maps approach to $\partial\mathrm{Rat}_d$. Once this result is at our disposal, we can describe completely the boundedness of hyperbolic components in the moduli space of quartic Newton maps.

1.1. *Statements of main results.*

For a degree $d \geq 2$ complex polynomial $P(z)$ with simple roots, its Newton map is defined by

$$f_P(z) = z - \frac{P(z)}{P'(z)}.$$

Denote by NM_d the space of degree d Newton maps. Then NM_d is a d-dimensional subspace in Rat_d and hence in $\mathbb{P}^{2d+1}$. Let $\overline{\mathrm{NM}}_d$ be the closure of NM_d in $\mathbb{P}^{2d+1}$. For $f \in \overline{\mathrm{NM}}_d$, denote by $\hat{f}$ the reduction of f, see §2.1. We are interested in the case where $\hat{f}$ has degree at least 2, see Lemma 5.6. Then $\hat{f}$ is a Newton map for a polynomial with possible multiple roots. For more details, we refer the reader to [**21**].

Now consider the basin of roots of $\hat{f}$. Let $\mathcal{U}$ be a set consisting of finitely many components of such basins. The boundary of each $U \in \mathcal{U}$ is locally connected [**5**, **29**]. Provided that $\hat{f}$ is forward invariant and post-critically finite on $\bigcup_{U\in\mathcal{U}} U$, each $U \in \mathcal{U}$ carries landed internal rays $I_{(U,u)}(t)$ of $\hat{f}$ for $t \in \mathbb{R}/\mathbb{Z}$, where $u \in U$ is the center of U. Let Γ be a (not necessarily connected) graph consisting of finitely many (pre)periodic internal rays in $\bigcup_{U\in\mathcal{U}} U$, that is,

$$\Gamma := \bigcup_{U\in\mathcal{U}, t\in T_U} I_{(U,u)}(t),$$

with $T_U \subseteq \mathbb{Q}$ for every $U \in \mathcal{U}$. Here we allow $T_U = \emptyset$ for some $U \in \mathcal{U}$. The canonical paradigms of such graphs are the Newton graphs (see §4.1) formulated recently by Drach *et al* [**4**] and the alternative graphs for cubic Newton maps (see §4.2) based on Roesch's work in [**25**].

Let $\{f_n\}_{n\geq 1} \subset \mathrm{NM}_d$ be a sequence such that f_n converges to f. As we will see in §3.1, each $(U, u) \in \mathcal{U}$ has a deformation (U_n, u_n) at f_n and the map $f_n : (U_n, u_n) \to \mathbb{C}$ converges to $\hat{f} : (U, u) \to \mathbb{C}$ under the Carathéodory topology in the sense of McMullen [**16**, §5.1]. If, in addition, the local degrees $\deg_{u_n} f_n = \deg_u \hat{f}$ for every $(U, u) \in \mathcal{U}$, we denote $f_n \xrightarrow{\deg} f$ on $\mathcal{U}$. In this case, the deformation (U_n, u_n) is unique at f_n. Moreover, a Böttcher coordinate of $\hat{f}$ on $U \in \mathcal{U}$ naturally deduces a Böttcher coordinate of f_n on the deformation (U_n, u_n) of (U, u), see §3.1. Then we can define the corresponding internal rays in U_n, which either land on ∂U_n or terminate at f_n-iterated preimages of critical points in U_n, see §3.2. For examples satisfying the above conditions, see Lemma 3.5.

Our first result concerns the perturbations of Γ.

THEOREM 1.1. *Let $f \in \overline{\mathrm{NM}}_d$ with $\mathcal{U}, \Gamma$ defined as above. Let $\{f_n\}_{n\geq 1} \subset \mathrm{NM}_d$ be a sequence such that $f_n \xrightarrow{\deg} f$ on $\mathcal{U}$. Suppose that for each $U \in \mathcal{U}$ and $t \in T_U$:*

(i) *the orbit of the landing point of $I_{(U,u)}(t)$ is eventually repelling periodic and avoids the critical points of $\hat{f}$; and*

(ii) *the corresponding internal ray $I_{(U_n,u_n)}(t)$ lands on ∂U_n for all large n.*

Then for all large n, the graph

$$\Gamma_n := \bigcup_{U\in\mathcal{U}, t\in T_U} I_{(U_n,u_n)}(t)$$

is homeomorphic to Γ, and Γ_n converges to Γ, as $n \to \infty$, in the Hausdorff metric topology.

Furthermore, if Γ is $\hat{f}$-invariant, then there exist homeomorphisms $\varphi_n : \Gamma \to \Gamma_n$ such that $\varphi_n \circ f = f_n \circ \varphi_n$ on Γ for all large n.

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Remark 1.2

(1) In Theorem 1.1, the assumption (i) guarantees that the perturbations of the orbit of the landing point of $I_{(U,u)}(t)$ are well controlled, see Proposition 2.4. In Proposition 3.9, we provide sufficient condition for the assumption (ii).
(2) In the present paper, we mainly use the first part of the conclusion in Theorem 1.1 to classify the bounded hyperbolic components. In the sequel [**11**], we will apply the both parts of the conclusion in Theorem 1.1 to characterize the unbounded hyperbolic components.

The technique of perturbations of internal rays already appear in complex dynamics for the non-degenerate maps, see e.g. [**10**, **12**, **24**]. Theorem 1.1 generalizes it to the degenerate case within the Newton family. The key point of the proof, differing from the non-degenerate case, is an elaborate argument to the internal rays landing at holes of f, where the locally uniform convergence fails.

In principle, our above theorem provides a combinational method to study degenerate sequences of Newton maps in the parameter space and hence that in moduli space. In a certain sense it asserts that, under the assumptions, part of the dynamics of the degenerate map $\hat{f}$ embeds into the dynamics of non-degenerate maps f_n. Thus it allows us to control the dynamics of f_n by that of $\hat{f}$.

Now we apply Theorem 1.1 to study the boundedness of hyperbolic components in the moduli space of quartic Newton maps. Since the point ∞ is the unique repelling fixed point for Newton maps, the moduli space of degree d Newton maps is defined by

$$\mathrm{nm}_d := \mathrm{NM}_d/\mathrm{Aut}(\mathbb{C}),$$

modulo the action by conjugation of the group of affine maps. We mention here that the space nm_d has complex dimension $d-2$. Recall that a rational map is *hyperbolic* if each critical point converges under iteration to a (super)attracting cycle, equivalently, it is uniformly expanding in a neighborhood of its Julia set, see [**16**, §3.4]. The space of hyperbolic Newton maps descends an open subset in nm_d, and each component of this subset is a *hyperbolic component* in nm_d. Endowing nm_d the quotient topology, we say a hyperbolic component in nm_d is *bounded* if it has compact closure in nm_d, and *unbounded* otherwise.

A hyperbolic component $\mathcal{H} \subset \mathrm{nm}_d$ is of *immediate escaping type* if each element in $\mathcal{H}$ is the conjugacy class of a Newton map having degree at least 3 in the immediate basin of some root.

THEOREM 1.3. *Let $\mathcal{H} \subset \mathrm{nm}_4$ be a hyperbolic component. Then $\mathcal{H}$ is unbounded if and only if $\mathcal{H}$ is of immediate escaping type (see Figure 1).*

For the boundedness of hyperbolic components, motivated by a result of Kleinian groups [**28**, Theorem 1.2], McMullen [**15**] conjectures that every hyperbolic component with Sierpínski Julia set is bounded in the moduli space of degree d rational maps. In his celebrated work [**17**, Remark 7.2], Milnor proposed the study of this topic in quadratic case. If the moduli space has complex dimension at least 2, there are only few already known results: for a hyperbolic component in the moduli space of bicritical rational maps,

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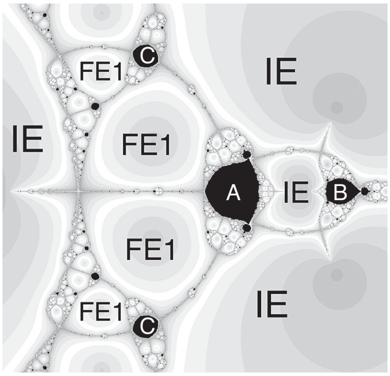

FIGURE 1. The c-plane for the family of Newton maps f_{P_c} for the polynomials $P_c(z) = z^4/12 - cz^3/6 + (4c-3)z/12 + (3-4c)/12$, see [22, Figure 1]. The critical points of f_{P_c} are the four roots of $P_c(z)$, 0 and c. The map f_{P_c} has a superattracting 2-cycle $0 \to 1 \to 0$. The letters indicate the types of hyperbolic components, see §5.1. Our result asserts that the hyperbolic components indicated by A, B, C, or FE1 are bounded in nm_4.

if each element possesses two distinct (super)attracting cycles of period at least 2, then it is bounded, see [6, Theorem 1] and [23, Theorem 1.1]; for quartic Newton maps, the second author and Pilgrim proved that a hyperbolic component in nm_4 is bounded if each element has two distinct (super)attracting cycles of period at least 2 [22, Main Theorem].

All the previous known bounded hyperbolic components are of so-called type D, that is, each element has maximal number of (super)attracting cycles. We point out here that the type-D components are semi-algebraic, but the components of other types are possible transcendental objects, see [20, Theorem 1 and Conjecture 2]. Our boundedness result gives the first non-semi-algebraic bounded hyperbolic components in a complex two-dimensional moduli space. Moreover, it strengthens the result [22, Theorem 1.3].

1.2. *Strategy of the proof of Theorem 1.3.* One direction of Theorem 1.3 is the result [22, Theorem 1.4]: if $\mathcal{H}$ is of immediate escaping type, then $\mathcal{H}$ is unbounded. Now we give an overview of the proof of the reverse implication. Differing from the analytic argument in [6] and the arithmetic argument in [22, 23], our argument relies on the combinatorial properties of Newton maps and applies Theorem 1.1. The proof goes by contradiction as follows. Suppose $\mathcal{H}$ is unbounded and not of immediate escaping type. Then we obtain an unbounded sequence $[f_n] \in \mathcal{H}$. Passing to a subsequence, we can assume that $[f_n]$ has a lift $f_n \in \mathrm{NM}_4$ such that f_n converges to $f \in \partial\mathrm{NM}_4$ with reduction $\hat{f}$ having degree 2 or 3 and no roots of f_n collide in $\mathbb{C}$ as $n \to \infty$, see Lemma 5.6. It follows that at least one non-fixed critical point c_n of f_n diverges to ∞. We derive a contradiction case by case.

Case 1: $\deg \hat{f} = 2$. In this case, we consider rational internal rays in the immediate basins of the roots of $\hat{f}$ and the corresponding perturbations for f_n. Theorem 1.1 implies that $\deg f_n = 2$ and hence leads to a contradiction.

Case 2: $\deg \hat{f} = 3$ and $\mathcal{H}$ is of type A, B, C, or D. It turns out that the Newton graphs of $\hat{f}$ are disjoint with the unique non-fixed critical point c of $\hat{f}$. Applying Theorem 1.1 to the Newton graphs of $\hat{f}$, we bound the immediate basins of the (super)attracting cycles of periods at least 2 for f_n. We obtain a contradiction by arguing the location of forward orbit of the critical point c_n.

Case 3: deg $\hat{f} = 3$ and $\mathcal{H}$ is of type FE1 or FE2. In this case, the critical point c could be an iterated preimage of ∞. Then we can not apply Theorem 1.1 directly to the Newton graphs as in the previous case. Alternatively, using Rosech's results in [**25**] on cut angles, we construct a natural Jordan curve $\mathcal{C}$ consisting of (pre)periodic internal rays of $\hat{f}$ such that the orbit of $\mathcal{C}$ is away from the critical point c. Then Theorem 1.1 works for the curve $\mathcal{C}$. Thus, we can continue to analyze the location of the related critical points and the corresponding Fatou components of f_n, and obtain a contradiction.

We remark that our proof of Theorem 1.3 highly relies on the behavior of the critical point c_n for f_n, see Lemma 5.3. We do not expect an analogy of such behavior holding for Newton maps of higher degrees. However, it would be interesting to apply Theorem 1.1 to investigate the boundedness of hyperbolic components in nm_d for $d \geq 5$.

1.3. *Structure of the paper.* This paper is organized as follows. In §2, we introduce the relevant preliminaries about degenerate rational maps and Newton maps. Section 3 contains the proof of Theorem 1.1. In §4, we investigate some dynamical graphs for Newton maps, and in §5, we prove Theorem 1.3.

2. *Preliminaries*

In this section, we give background materials. In §2.1, we provide basic definitions and properties of degenerate rational maps. Section 2.2 concerns the (degenerate) Newton maps.

2.1. *Degenerate rational maps.* As mentioned in §1, the space Rat_d is naturally identified as an open and dense subset of $\mathbb{P}^{2d+1}$. We say each element $f \in \mathbb{P}^{2d+1} \setminus \mathrm{Rat}_d$ is a *degenerate rational map* of degree d. For such f, there exist two homogeneous polynomials $F(X, Y)$ and $G(X, Y)$ of degree d in $\mathbb{C}[X, Y]$ such that $f = [F : G]$ is in homogeneous coordinates and $H_f := \gcd[F, G]$ is a polynomial in $\mathbb{C}[X, Y]$ of degree at least 1. We can rewrite

$$f = H_f \hat{f},$$

where $\hat{f}$ is a rational map of degree less than d. We say each zero of H_f is a *hole* of f and denote $\mathrm{Hole}(f)$ the set of holes of f. Moreover, we call $\hat{f}$ the *reduction* of f. For convenience, if f is a rational map of degree d, we define $H_f = 1$ and then $\hat{f} = f$.

Let $\{f_n\}_{n\geq 1}$ be a sequence of rational maps of degree $d \geq 1$. We say f_n converges *semi-algebraically* to a (degenerate) rational map f if the coefficients of f_n converge to the coefficients of f in $\mathbb{P}^{2d+1}$. Compare to the algebraically convergence in [**1**]. Without special emphasis, we mean semi-algebraically convergence when we consider convergence for a sequence in $\mathbb{P}^{2d+1}$. The semi-algebraical convergence implies locally uniform convergence away from holes.

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LEMMA 2.1. [2, Lemma 4.1] *Let $\{f_n\}_{n\geq 1}$ be a sequence of degree $d \geq 1$ rational maps. If f_n converges to $f = H_f \hat{f} \in \mathbb{P}^{2d+1}$, then f_n converges locally uniformly to $\hat{f}$ outside* $\mathrm{Hole}(f)$.

Suppose that each f_n possesses a cycle of fixed period. If the limit of these cycles is away from the holes of *f*, Lemma 2.1 immediately implies that this limit is also a cycle for $\hat{f}$. We state as follows and omit the proof.

LEMMA 2.2. *Let $\{f_n\}_{n\geq 1}$ be a sequence of degree $d \geq 2$ rational maps. Suppose that f_n converges to $f = H_f \hat{f} \in \mathbb{P}^{2d+1}$ with $\deg \hat{f} \geq 1$. Assume $\mathcal{O}_n$ is a cycle of f_n of period $m \geq 1$ and suppose that $\mathcal{O}_n$ converges to $\mathcal{O}$ in $\mathbb{P}^1$. If $\mathcal{O} \cap \mathrm{Hole}(f) = \emptyset$, then $\mathcal{O}$ is a cycle of $\hat{f}$ of period q with $q \mid m$. Furthermore: (1) if $\mathcal{O}_n$ is attracting, then $\mathcal{O}$ is non-repelling; (2) if $q < m$, then $\mathcal{O}$ is parabolic.*

If the limit intersects the holes of *f*, we have the following basins shrinking result.

LEMMA 2.3. [22, Proposition 2.8] *Let $\{f_n\}_{n\geq 1}$ be a sequence of degree $d \geq 2$ rational maps. Assume that f_n converges to $f = H_f \hat{f} \in \mathbb{P}^{2d+1}$. Assume $\deg \hat{f} \geq 2$ and $\infty \in \mathrm{Hole}(f)$ is a fixed point of $\hat{f}$. Let $\{z_n^{(0)}, \ldots, z_n^{(m-1)}\}$ be a (super)attracting cycle of f_n of period $m \geq 2$, and let $U_n^{(k)}$ be the Fatou component containing $z_n^{(k)}$. Suppose $z_n^{(k)} \to z^{(k)}$ for $k = 0, \ldots, m-1$ with $z^{(0)} = \infty$ and $z^{(i)} \neq \infty$ for some $1 \leq i \leq m-1$. Then:*

(1) *$U_n^{(0)}$ converges to ∞ in the sense that, for any $\epsilon > 0$, the component $U_n^{(0)}$ is contained in the disk $\{z : \rho(z, \infty) < \epsilon\}$ for all large n, where ρ is the sphere metric; and*

(2) *there exists a neighborhood V of ∞ such that $U_n^{(i)} \cap V = \emptyset$ for all large n.*

Now we state a straightforward result about the perturbations of periodic points.

LEMMA 2.4. *Let $f = H_f \hat{f} \in \mathbb{P}^{2d+1}$ with $\deg \hat{f} \geq 1$. Then the following holds.*

(1) *For $z_0 \in \widehat{\mathbb{C}}$ and $j \geq 1$, denote $z_i := \hat{f}^i(z_0)$ for $0 \leq i \leq j$. Suppose z_i avoids the critical point of $\hat{f}$ for all $0 \leq i \leq j-1$. Let $z_j(g)$ be a holomorphic map defined in a neighborhood of $f \in \mathbb{P}^{2d+1}$ with $z_j(f) = z_j$. Then for each $0 \leq i \leq j-1$, there exists a holomorphic map $z_i(g)$ defined in a neighborhood of f such that $z_i(f) = z_i$ and $\hat{g}^{j-i}(z_i(g)) = z_j(g)$. Moreover, if z_i avoids the holes of f for all $0 \leq i \leq j-1$, then $z_i(g)$ is the unique point near z_i such that $\hat{g}^{j-i}(z_i(g)) = z_j(g)$, which implies $z_i(g) = \hat{g}^i(z_0(g))$ for all $0 \leq i \leq j-1$.*

(2) *Let $\mathcal{O} = \{\xi_0, \ldots, \xi_{k-1}\}$ be an attracting (respectively repelling) cycle of $\hat{f}$. If $\mathcal{O} \cap \mathrm{Hole}(f) = \emptyset$, then for each g close to f, there exists a unique attracting (respectively repelling) cycle $\mathcal{O}(g) := \{\xi_0(g), \ldots, \xi_{k-1}(g)\}$ of g such that each $\xi_i(g)$ is a holomorphic map near f with $\xi_i(f) = \xi_i$.*

Proof. By pre and post composition of Möbius transformations, we can assume $z_0, \ldots, z_j \in \mathbb{C}$. For $g = H_g \hat{g} \in \mathbb{P}^{2d+1}$ close to *f*, we have $\deg \hat{g} \geq 1$. Then for $0 \leq i \leq j-1$, the iteration g^{j-i} is well defined, see [2, Lemma 2.2]. Consider the holomorphic function

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$$F_i(g,z) := g^{j-i}(z) - z_j(g)$$

on $\Lambda_f \times D(z_j)$, where $\Lambda_f \subseteq \mathbb{P}^{2d+1}$ is a neighborhood of f and $D(z_j) \subseteq \mathbb{C}$ is a neighborhood of z_j. By the assumptions, we have that $F_i(f, z_i) = 0$ and

$$\frac{\partial F_i}{\partial z}|_{(f,z_i)} = (\hat{f}^{j-i})'(z_i) \neq 0.$$

Then the implicit function theorem implies there exists a holomorphic function $z_i(g)$ near f such that $\hat{g}^{j-i}(z_i(g)) = z_j(g)$. If $\{z_0, \ldots, z_{j-1}\} \cap \mathrm{Hole}(f) = \emptyset$, the function $\hat{g}^{j-i}(z)$ is holomorphic in z in a fixed neighborhood of z_i for each g close to f. It follows from Hurwitz's theorem (see [7]) that $g^{j-i}(z) - z_j(g)$ has a unique root near z_i for g close to f. Thus statement (1) follows.

For statement (2), note that the cycle $\mathcal{O} \cap \mathrm{Hole}(f) = \emptyset$. Applying the implicit function theorem on $G(g,z) := g^k(z) - z$, we obtain the expected cycle $\mathcal{O}(g)$ of g for g close to f. □

For $f = H_f \hat{f} \in \mathbb{P}^{2d+1}$, assume $\hat{f}$ has an attracting cycle $\mathcal{O}$ and denote Ω the immediate basin of $\mathcal{O}$. If $\Omega \cap \mathrm{Hole}(f) = \emptyset$, Lemma 2.4 implies that for g close to f, the map $\hat{g}$ has an attracting cycle $\mathcal{O}(g)$. Denote by $\Omega(g)$ the immediate basin of $\mathcal{O}(g)$. Then we have the following lemma.

LEMMA 2.5. *Assume that $\Omega \cap \mathrm{Hole}(f) = \emptyset$ and let $E \subset \Omega$ be any compact set. Then $E \subseteq \Omega(g)$ for any g sufficiently close to f.*

This above result is well known in the case where f is a rational map of degree d, see [3, Lemma 6.3]. Our assumption $\Omega \cap \mathrm{Hole}(f) = \emptyset$ guarantees that the argument in the non-degenerate case also works in our case. Here we omit the proof.

2.2. *Newton maps.*

For a degree $d \geq 2$ complex polynomial $P(z)$ with simple roots, its Newton map

$$f_P(z) := z - \frac{P(z)}{P'(z)}$$

is a degree d rational map having d superattracting fixed points at the roots of P. The only other fixed point is at ∞. The holomorphic index formula (see [18, Theorem 12.4]) asserts that the point ∞ is the unique repelling fixed point of f_P. The critical points of f_P are the roots of P and the zeros of P''. Moreover, the poles of f_p are the zeros of P'.

Recall that NM_d is the space of degree d Newton maps and $\overline{\mathrm{NM}}_d$ is the closure of NM_d in $\mathbb{P}^{2d+1}$. Then for each $f = H_f \hat{f} \in \overline{\mathrm{NM}}_d$, there exists a polynomial Q degree at most d with possible multiple roots such that $\hat{f}$ is the Newton map of Q. Each root r of Q is a (super)attracting fixed point of $\hat{f}$ with multiplier $1 - 1/n_r$, where n_r is the multiplicity of r as a zero of Q. Moreover, again $\hat{f}$ has only one more fixed point at ∞, which is repelling. It follows that each hole of f is either a multiple root of Q or ∞. Furthermore, $\infty \in \mathrm{Hole}(f)$ if and only if $\deg Q < d$. For more details about degenerate Newton maps, we refer the reader to [21].

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For $f = H_f \hat{f} \in \overline{\mathrm{NM}}_d$ with $\deg \hat{f} \geq 2$, the Fatou components of $\hat{f}$ have well-studied topological structure. By a result of Shishikura [27], all Fatou components of $\hat{f}$ are simply connected, and hence the Julia set of $\hat{f}$ is connected. Moreover, the boundary of each component of the basins of roots is locally connected, see [5, 29].

3. *Proof of Theorem 1.1*

The goal of this section is to prove Theorem 1.1. We define Böttcher coordinates on the deformations in §3.1 and prove the convergence of Böttcher coordinates (Proposition 3.7). To do that, we introduce the convergence preserving the degrees at centers (Definition 3.3). In §3.2, we use the Böttcher coordinates on the deformations to define the corresponding internal rays, and then show a convergence result on these rays (Proposition 3.8). Finally, we prove Theorem 1.1 in §3.3.

3.1. *Perturbation of Böttcher coordinates.* Let $f = H_f \hat{f} \in \overline{\mathrm{NM}}_d$ with $\deg \hat{f} \geq 2$ and denote by $\Omega_{\hat{f}}$ the union of basins of the roots of $\hat{f}$. Let $\mathcal{U}$ be a finite subset of components of $\Omega_{\hat{f}}$ such that if $U \in \mathcal{U}$, then $\hat{f}(U) \in \mathcal{U}$. Recall that $\hat{f}$ is post-critically finite on $\bigcup_{U \in \mathcal{U}} U$ if the critical points in any $U \in \mathcal{U}$ have finite orbits. For such $\hat{f}$ and $\mathcal{U}$, one can choose a system of Böttcher coordinates $\{\phi_U : U \to \mathbb{D}\}_{U \in \mathcal{U}}$ satisfying

$$\phi_{\hat{f}(U)} \circ \hat{f} \circ \phi_U^{-1}(z) = z^{d_U}, \quad z \in \mathbb{D}, \text{ where } d_U := \deg(\hat{f}|_U),$$

and $u := \phi_U^{-1}(0)$ is called the *center* of U. Moreover, we have the following proposition.

PROPOSITION 3.1. *The set $\bigcup_{U \in \mathcal{U}} U$ is disjoint with the holes of f.*

Proof. By §2.2, a possible hole of f is either ∞ or a geometrically attracting fixed point of $\hat{f}$. Since $\hat{f}$ is post-critically finite on $\bigcup_{U \in \mathcal{U}} U$, the conclusion holds. □

To abuse notation, we denote the set of pointed sets (U, u) with $U \in \mathcal{U}$ also by $\mathcal{U}$. Let $\{f_n\}_{n \geq 1}$ be a sequence in NM_d such that f_n converges to f. By Proposition 3.1 and Lemma 2.5, for $(U, u) \in \mathcal{U}$, the point u belongs to the unique component U_n of Ω_{f_n} for all large n; and combining Proposition 2.4, the component U_n contains a preperiodic point u_n with the same preperiod and period as that of u such that $u_n \to u$ as $n \to \infty$. We call such (U_n, u_n) a *deformation* of (U, u) at f_n. Note that such u_n is not necessarily unique. For example, if u is a preperiodic critical point, it possibly splits into two preperiodic points of f_n contained in U_n.

To the end of this subsection, under natural assumptions, we define a Böttcher coordinate ϕ_{U_n} on the deformations U_n of U and show a convergence result of ϕ_{U_n}. First, we consider the convergence of deformations.

Recall the definition of Carathéodory topology on a set of pointed sets and a set of holomorphic functions respectively, following [16, §5.1]. Let $\mathcal{V}$ be a set of open simply connected pointed sets (V, v) in $\mathbb{C}$. The *Carathéodory topology* on $\mathcal{V}$ is defined by the following convergence: (V_n, v_n) converges to (V, v) if and only if:

(i) v_n converges to v;
(ii) for any compact $K \subset V$, we have $K \subset V_n$ for all large n; and

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(iii) for any domain W containing v, if $W \subset V_n$ for infinitely many n, then $W \subset V$.

Denote $\mathcal{G}$ the set of holomorphic functions defined on $(V, v) \in \mathcal{V}$. Then the *Carathéodory topology* on $\mathcal{G}$ is defined as follows. Let $g : (V, v) \to \mathbb{C}$ and $g_n : (V_n, v_n) \to \mathbb{C}$ be functions in $\mathcal{G}$. We say g_n converges to g if: (1) (V_n, v_n) converges to (V, v) in $\mathcal{V}$; and (2) g_n converges to g uniformly on any compact subset of V as $n \to \infty$.

PROPOSITION 3.2. *Let $\{f_n\}_{n\geq 1}$, $f = H_f \hat{f}$ and $\mathcal{U}$ be as above. For any $(U, u) \in \mathcal{U}$, the holomorphic maps $f_n : (U_n, u_n) \to \mathbb{C}$ converge to $\hat{f} : (U, u) \to \mathbb{C}$, where (U_n, u_n) is a deformation of (U, u) at f_n.*

Proof. We first show that (U_n, u_n) converge to (U, u). By the definition of (U_n, u_n), we have $u_n \to u$ as $n \to \infty$. Proposition 3.1 and Lemma 2.5 imply that any compacted set contained in U is contained in U_n for sufficiently large n. Now pick an open and connected set $W \subset U_n$ for infinitely many n with $u \in W$. We assume on the contrary that $W \not\subset U$. Then W contains a eventually repelling preperiodic point $z \in \partial U$ such that its orbit avoids the critical points and holes of $\hat{f}$, since the repelling periodic points are dense in the boundary of any immediate basin of a root of $\hat{f}$, and the critical points and holes of $\hat{f}$ are finite. By Proposition 2.4, the point z is perturbed to an eventually repelling preperiodic point of f_n, which also belongs to W for sufficiently large n. It contradicts that $W \subset U_n$. Thus (U_n, u_n) converges to (U, u). The locally uniform convergence of f_n follows immediately from Lemma 2.1. This completes the proof. □

For our propose, we use the following definition.

Definition 3.3. Let f_n and f be as above. We say f_n converges to f preserving the degrees at the centers, denoted by $f_n \xrightarrow{\text{deg}} f$, on $\mathcal{U}$ if for each $(U, u) \in \mathcal{U}$, a deformation (U_n, u_n) of (U, u) satisfies the local degrees property $\deg_u \hat{f} = \deg_{u_n} f_n$.

If $f_n \xrightarrow{\text{deg}} f$ on $\mathcal{U}$, it follows immediately that any $(U, u) \in \mathcal{U}$ has a unique deformation (U_n, u_n). We call such u_n a *center* of U_n. In this case, set

$$\mathcal{U}_n := \{(U_n, u_n) : (U_n, u_n) \text{ as the deformation of } (U, u) \in \mathcal{U}\}.$$

We mention here that the set U_n may contain several distinct centers.

Remark 3.4. If a critical point c of $\hat{f}$ is contained in the boundaries of distinct (U, u) and (U', u') in $\mathcal{U}$, it is possible that U_n coincides with U'_n and it contains the critical point of f_n perturbed from c (see Figure 2). In this case, both u_n and u'_n are centers of $U_n = U'_n$, and hence f_n is not post-critically finite on the union of U_n with $(U_n, u_n) \in \mathcal{U}_n$.

The following result states a natural sufficient condition for the convergence $f_n \xrightarrow{\text{deg}} f$, which we will use repeatedly in §5. The proof is straightforward, so we omit it.

LEMMA 3.5. *Let f_n, f, and $\mathcal{U}$ be as above. Assume that $\hat{f}$ has degree 2 on every immediate basin of roots in $\mathcal{U}$ and degree 1 on all other elements in $\mathcal{U}$. Then, $f_n \xrightarrow{\text{deg}} f$ on $\mathcal{U}$.*

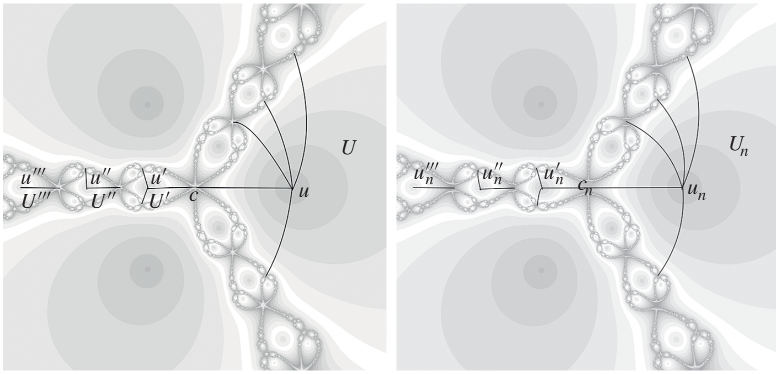

FIGURE 2. (a) The dynamical plane of the Newton map f for the polynomial $z^3 - 1$. The letters indicate Fatou components U, U', U'', and U''' with centers u, u', u'', and u''' respectively. The arcs indicate internal rays. The critical point $c = 0$ is contained in $\partial U \cap \partial U'$. (b) The dynamical plane of the Newton map f_n for the polynomial $z^3 + z/n - 1$ with indicated Fatou component U_n. The critical point $c_n \in U_n$. The points u_n, u'_n, u''_n, and u'''_n are all in U_n and centers of U_n. The set (U_n, u_n) is the deformation of (U, u); the set (U_n, u'_n) is the deformation of (U', u'); the set (U_n, u''_n) is the deformation of (U'', u''); and the set (U_n, u'''_n) is the deformation of (U''', u'''). The corresponding rays in U_n either land on ∂U_n or terminate at the iterated preimages of c_n.

From now on, we assume that $f_n \xrightarrow{\text{deg}} f$. Since $\hat{f}$ is post-critically finite on $\bigcup_{U\in\mathcal{U}} U$, by Proposition 3.1 and Lemma 2.1, we have the following straightforward result and again omit the proof.

LEMMA 3.6. *If (U_n, u_n) is the deformation of $(U, u) \in \mathcal{U}$ at f_n, then $(f_n(U_n), f_n(u_n))$ is the deformation of $(\hat{f}(U), \hat{f}(u))$.*

Lemma 3.6 suggests that for each $(U_n, u_n) \in \mathcal{U}_n$, we have a Böttcher coordinate $\phi_{(U_n,u_n)}$ near u_n such that

$$\phi_{(U_n,u_n)}(z)^{d_U} = \phi_{(f_n(U_n), f_n(u_n))} \circ f_n(z) \tag{3.1}$$

for z near u_n, and that

$$\phi'_{(U_n,u_n)}(u_n) \to \phi'_{(U,u)}(u) \text{ as } n \to \infty. \tag{3.2}$$

The map $\phi_{(U_n,u_n)}$ extends conformally until meeting an iterated preimage of critical points of f_n. Then there exists a maximum $r_n \leq 1$ such that $\psi_{(U_n,u_n)} := \phi^{-1}_{(U_n,u_n)} : \mathbb{D}_{r_n} \to U_n$ is well defined.

Denote by $\psi_{(U,u)}$ the inverse of $\phi_{(U,u)}$. The following result asserts that $\psi_{(U_n,u_n)}$ converges to $\psi_{(U,u)}$ locally uniformly on $\mathbb{D}$, which is well known in Carathéodory topology, see e.g. [16, Theorem 5.1].

PROPOSITION 3.7. *For $(U, u) \in \mathcal{U}$, let $(U_n, u_n) \in \mathcal{U}_n$ be the deformation of (U, u). Then $\psi_{(U_n,u_n)}$ converges to $\psi_{(U,u)}$ locally uniformly on $\mathbb{D}$.*

3.2. *Perturbation of internal rays.* In §3.1, we perturb a Böttcher coordinate in $(U, u) \in \mathcal{U}$ to obtain a Böttcher coordinate $\phi_{(U_n,u_n)}$ in $(U_n, u_n) \in \mathcal{U}_n$. In this subsection, we use the

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inverse map $\psi_{(U_n,u_n)}$ to define the internal rays in (U_n, u_n) and prove a convergence result on internal rays.

Now we define internal rays of f_n in (U_n, u_n) as follows. For each $\theta \in \mathbb{R}/\mathbb{Z}$, let r_θ be the maximal radius such that $\psi_{(U_n,u_n)}$ extends along $(0, r_\theta)e^{2\pi i\theta}$. If $r_\theta < 1$, then arc $\psi_{(U_n,u_n)}((0, r_\theta)e^{2\pi i\theta})$ terminates at an iterated preimage of critical points of f_n, and if $r_\theta = 1$, the arc $\psi_{(U_n,u_n)}((0, 1)e^{2\pi i\theta})$ accumulates and factually lands on ∂U_n. In the latter case, we call

$$I_{(U_n,u_n)}(\theta) := \psi_{(U_n,u_n)}([0, 1]e^{2\pi i\theta})$$

the *landed internal ray in* (U_n, u_n) *of angle* θ. Note that f_n sends a landed internal ray of (U_n, u_n) to a landed internal ray of $(f(U_n), f(u_n))$. Also, since U_n may contain more than one center, it may possess several groups of landed interval rays. In this case, each such ray starts from a center of U_n and rays from distinct groups are disjoint (see Figure 2).

The following result asserts that the internal rays of eventually periodic angles converge.

PROPOSITION 3.8. *For $(U, u) \in \mathcal{U}$, assume that the internal ray $I_{(U,u)}(\theta)$ of angle θ lands at an eventually repelling periodic point. For all large n, suppose that $I_{(U_n,u_n)}(\theta)$ is a landed internal ray in $(U_n, u_n) \in \mathcal{U}_n$. Then $I_{(U_n,u_n)}(\theta) \to I_{(U,u)}(\theta)$ as $n \to \infty$.*

Proof. To ease notation, we write $I(\theta)$, $I_n(\theta)$, ψ, and ψ_n for $I_{(U,u)}(\theta)$, $I_{(U_n,u_n)}(\theta)$, $\psi_{(U,u)}$, and $\psi_{(U_n,u_n)}$, respectively. Set $\delta := \deg(\hat{f}|_U)$ and let z_0 be the landing point of $I(\theta)$. It is sufficient to show that, given any $\eta > 0$, for all large n, we have $d_H(I(\theta), I_n(\theta)) < \eta$, where d_H is the Hausdorff metric.

First assume that $I(\theta)$ is periodic of period $p \geq 1$. Then u is a super-attracting fixed point of $\hat{f}$. Define

$$D_\epsilon := \{z \in \widehat{\mathbb{C}} : \rho(z, z_0) < \epsilon\},$$

where ρ is the spherical metric. Shrinking ϵ if necessary, we may assume $\hat{f}|_{D_\epsilon}$ is injective and $\overline{D}_\epsilon \subseteq \hat{f}^p(D_\epsilon)$. We claim that for any sufficiently large n and any component D'_n of $f_n^{-p}(D_\epsilon)$, either $\overline{D'_n} \subseteq D_\epsilon$ or $\overline{D'_n} \subseteq \widehat{\mathbb{C}} \setminus \overline{D}_\epsilon$. Indeed, if $p > 1$, the landing point z_0 of $I(\theta)$ is not a hole of f^p, see §2.1 and [2, Lemma 2.2]. It follows from Lemma 2.1 that f_n^p converges uniformly to $\hat{f}^p$ near z_0, and hence $f_n^p|_{D_\epsilon}$ is injective and $\overline{D}_\epsilon \subseteq f_n^p(D_\epsilon)$ for all large n. Then in this case, the claim follows. Now we consider the case where $p = 1$. Then $z_0 = \infty$. If $z_0 = \infty$ is not a hole of f, the claim follows by previous argument. If $z_0 = \infty$ is a hole of f, then f_n fails to converge uniformly to $\hat{f}$ near ∞. In this case, we prove the claim by contradiction. Suppose that the claim fails. Then there exists a subsequence, denoted also by $\{f_n\}$, such that for each f_n, there exists a component D'_n of $f_n^{-1}(D_\epsilon)$ with $\overline{D}'_n \cap \partial D_\epsilon \neq \emptyset$. Choose a point $w_n \in \overline{D}'_n \cap \partial D_\epsilon$. Passing to subsequence if necessary, we may assume $w_n \to w$. Then $w \in \partial D_\epsilon$. By Lemma 2.1, the sequence f_n converges uniformly to $\hat{f}$ on ∂D_ϵ. It follows that as $n \to \infty$,

$$f_n(w_n) \to \hat{f}(w).$$

Note that $\overline{D}_\epsilon \subseteq f(D_\epsilon)$. Then

$$\hat{f}(\partial D_\epsilon) \cap \overline{D}_\epsilon = \emptyset.$$

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We have that $f(w) \notin \overline{D}_\epsilon$. However,

$$f_n(w_n) \in f_n(\overline{D}'_n) = \overline{D}_\epsilon,$$

which implies $f(w) \in \overline{D}_\epsilon$. It is a contradiction. Therefore, the claim holds.

Since $I(\theta)$ lands at z_0, there exists $0 < r < 1$ such that

$$\psi((r, 1)e^{2\pi i\theta}) \subseteq U \cap D_\epsilon.$$

Pick $0 < s < 1$ such that $s^{\delta^p} > r$. Then the segment $\psi([s^{\delta^p}, s]e^{2\pi i\theta}) \subseteq I(\theta)$ belongs to $U \cap D_\epsilon$. It follows from Proposition 3.7 that for all large n,

$$d_H(\psi_n([0, s]e^{2\pi i\theta}), \psi([0, s]e^{2\pi i\theta})) < \epsilon. \tag{3.3}$$

Define

$$\gamma_{n,0} : [0, 1] \to \psi_n([s^{\delta^p}, s]e^{2\pi i\theta})$$

to be an arc such that $\gamma_{n,0}(0) = \psi_n(s^{\delta^p} e^{2\pi i\theta})$ and $\gamma_{n,0}(1) = \psi_n(se^{2\pi i\theta})$. Then,

$$\gamma_{n,0}([0, 1]) \subseteq D_\epsilon \cap U_n.$$

Note that $f_n^p(\gamma_{n,0}(1)) = \gamma_{n,0}(0)$. Lift $\gamma_{n,0}$ to an arc $\gamma_{n,1}$ based at $\gamma_{n,0}(1)$. Since $I_n(\theta)$ is landed, inductively we obtain a sequence of arcs $\gamma_{n,k}$ such that $\gamma_{n,k+1}$ is a lift by f_n of $\gamma_{n,k}$ based at the endpoint of $\gamma_{n,k}$ which is not in $\gamma_{n,k-1}$.

Now we claim that for sufficiently large n, the arc $\gamma_{n,k} \subset D_\epsilon$. We prove the claim by induction on k. The claim holds for $k = 0$ by the definition of $\gamma_{n,0}$. Suppose that for $k \geq 0$, the arc $\gamma_{n,k} \subseteq D$. Since $\gamma_{n,k+1}$ is a preimage of $\gamma_{n,k}$ under f_n, there exists a component D' of $f_n^{-1}(D_\epsilon)$ containing $\gamma_{n,k+1}$. Since the intersection point of $\gamma_{n,k+1} \subseteq D'$ and $\gamma_{n,k} \subseteq D_\epsilon$ belongs to D_ϵ, it follows that $D' \cap D_\epsilon \neq \emptyset$. By the previous claim, we have $D' \subseteq D_\epsilon$, and hence $\gamma_{n,k+1} \subseteq D_\epsilon$, which completes the induction.

Note that for all large n,

$$I_n(\theta) = \psi_n([0, s]e^{2\pi i\theta}) \bigcup (\cup_{k\geq 0}\gamma_{n,k}) \cup \{z_n\},$$

where z_n is the landing point of $I_n(\theta)$. According to the estimate in equation (3.3) and the fact that $\gamma_{n,k} \subseteq D_\epsilon$, we have

$$d_H(I(\theta), I_n(\theta)) < \epsilon.$$

By choosing $\epsilon < \eta$, we prove the proposition under the periodicity assumption.

In the strictly preperiodic case, we set $(V, v) := \hat{f}(U, u)$ and $I_V(\theta') = \hat{f}(I(\theta))$. Let (V_n, v_n) be the deformation of (V, v) with $f_n(U_n, u_n) = (V_n, v_n)$. Inductively, it is sufficient to prove $d_H(I(\theta), I_n(\theta)) < \epsilon$ under the assumption that $\lim_{n\to\infty} d_H(I_V(\theta'), I_{V_n}(\theta')) = 0$.

Define D_ϵ as above. By Proposition 3.7, there exists $0 < s < 1$ such that for all large n,

$$d_H(\psi_n([0, s]e^{2\pi it}), \psi([0, s]e^{2\pi it})) < \epsilon \text{ and } \psi_n(se^{2\pi it}) \in D_\epsilon.$$

Set $L'_n := \psi_{(V_n,v_n)}([s^\delta, 1]e^{2\pi\theta'})$ and $L' := \psi_{(V,v)}([s^\delta, 1]e^{2\pi\theta'})$. Since $I_{V_n}(\theta') \to I_V(\theta')$, L'_n and L' are contained in $\hat{f}(D_\epsilon)$ for large n. Since $I_n(\theta)$ is a landed internal ray for all large n, there is a lift L_n of L'_n based at the point $\psi_n(se^{2\pi i\theta})$. Denote by L the lift of

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L' based at the point $\psi(se^{2\pi i\theta})$. Note that in this case, we have $z_0 \notin \mathrm{Hole}(f)$. Then f_n converges uniformly to $\hat{f}$ on D_ϵ. Thus for sufficiently large n,

$$f(D_\epsilon) \subset f_n(D_{2\epsilon}).$$

Hence we have $L_n \subset D_{2\epsilon}$ and $L \subseteq D_{2\epsilon}$. Note $I(\theta) = \psi([0, s]e^{2\pi it}) \cup L$ and $I_n(\theta) = \psi_n([0, s]e^{2\pi it}) \cup L_n$. It follows that

$$d_H(I(\theta), I_n(\theta)) < 2\epsilon.$$

Choose $\epsilon < \eta/2$. This completes the proof. □

3.3. *Proof of Theorem 1.1.* Now we begin to prove Theorem 1.1 and state a sufficient condition for the assumption (ii) in Theorem 1.1.

Proof of Theorem 1.1. Under the assumptions (i) and (ii), by Proposition 3.8, we have that for each $(U, u) \in \mathcal{U}$ and $t \times T_U$, the internal rays $I_{(U_n,u_n)}(t)$ converge to $I_{(U,u)}(t)$ as $n \to \infty$. It follows immediately that Γ_n converges to Γ as $n \to \infty$.

We are going to check that Γ_n is homeomorphic to Γ for large n. It is sufficient to show that for any (U, u), $(U'u') \in \mathcal{U}$ and $t \in T_U$, $t' \in T_{U'}$, the rays $I_{(U,u)}(t)$ and $I_{(U',u')}(t')$ land at a common point only if $I_{(U_n,u_n)}(t)$ and $I_{(U_n',u_n')}(t')$ land at a common point for all large n. Assume first that this common landing point, denoted by z, is periodic. If $z = \infty$, the unique possible hole of f is in the Julia set of $\hat{f}$, where we have $t = t' = 0$. It follows that $I_{(U_n,u_n)}(t)$ and $I_{(U_n',u_n')}(t')$ land at the fixed points of f_n in the Julia set, which can be only ∞. If $z \neq \infty$, the conclusion follows immediately from Lemma 2.4(2) and Proposition 3.8. In the case where z is preperiodic, we have this result by combining the periodic case, Lemma 2.4(1), and Proposition 3.8 since the orbit of z avoids the critical points of $\hat{f}$.

Assume now that Γ is $\hat{f}$-invariant. Let $I := I_{(U,u)}(t)$ be a periodic internal ray in Γ of period p. Since $I_n := I_{(U_n,u_n)}(t)$ is landed by the assumption, the inverse $\psi_{(U_n,u_n)}$ of the Böttcher coordinate on (U_n, u_n) (defined after Lemma 3.6) can be extended to $[0, 1]e^{2\pi it}$. So we are able to define a homeomorphism $\varphi_{n,I} : I_{(U,u)}(t) \to I_{(U_n,u_n)}(t)$ by $\varphi_{n,I} := \psi_{(U_n,u_n)} \circ \phi_{(U,u)}|_I$. It follows from equation (3.1) that $\varphi_{n,I} \circ \hat{f}^p = f_n^p \circ \varphi_{n,I}$ on I. If $I' := I_{(U,u)}(t')$ is mapped to I by $\hat{f}^k$, we get a homeomorphism $\varphi_{n,I'} : I' \to I_n' := I_{(U_n',u_n')}(t')$ by lifting $\varphi_{n,I}$ along the maps $\hat{f}^k : I' \to I$ and $f_n^k : I_n' \to I_n$. Finally, define a map φ_n on Γ such that $\varphi_n|_I = \varphi_{n,I}$ on each internal ray I in Γ. Then φ_n is the desired homeomorphism from Γ to Γ_n. □

To end this section, we state a sufficient condition to guarantee that assumption (ii) in Theorem 1.1 holds. Note that for any internal ray $I \subset \Gamma$, there exists an smallest integer $k_I \geq 0$ such that $\hat{f}^{k_I}(I)$ is contained in the immediate basin of a root of $\hat{f}$.

PROPOSITION 3.9. *For any internal ray $I \subset \Gamma$, let $k_I \geq 0$ be as above. Under assumption (i) in Theorem 1.1, if the perturbation of $\hat{f}^{k_I}(I)$ at f_n is landed for all large n, then assumption (ii) in Theorem 1.1 holds. In particular, under assumption (i) in Theorem 1.1, if $\deg f_n|_{U_n'} = \deg \hat{f}|_{U'}$ for all immediate basins $U' \in \mathcal{U}$, then assumption (ii) in Theorem 1.1 holds.*

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Proof. Consider an internal ray $I \subset \Gamma$ and it's perturbation $I_n \subset \Gamma_n$. Let $U \in \mathcal{U}$ be the component such that $I \subset U$, and let U_n be the deformation of U at f_n. We show I_n lands on ∂U_n. If U is the immediate basin of a root of $\hat{f}$, the conclusion follows immediately since in this case, $k_I = 0$.

Now we consider the case where U is not the immediate basin of a root of $\hat{f}$. We set $V = \hat{f}(U)$ and $I' = \hat{f}(I)$. By an induction argument, it suffices to show I_n lands on ∂U_n under the assumption that I'_n lands on ∂V_n, where I'_n is the perturbation of I' and V_n is the deformation of V at f_n. Fix notation as in the proof of Proposition 3.8. Since the orbit of the landing point of the internal ray I avoids the critical points of $\hat{f}$, we can apply a similar argument in Proposition 3.8 and obtain that

$$I_n = I_{(U_n,u_n)}(t) = \psi_{(U_n,u_n)}([0,s]e^{2\pi it}) \cup L_n,$$

where L_n is a lift of $\psi_{(V_n,v_n)}([s^\delta,1]e^{2\pi it'})$ based at $\psi_{(U_n,u_n)}(se^{2\pi it})$. Note that

$$\psi_{(V_n,v_n)}(e^{2\pi it'}) \in \partial V_n.$$

It follows that I_n land on ∂U_n.

If $\deg f_n|_{U'_n} = \deg \hat{f}|_{U'}$ for all immediate basins $U' \in \mathcal{U}$, since $f_n \xrightarrow{\deg} f$ and $\hat{f}$ is post-critically finite on U', it follows that for the center u'_n of U'_n, the degrees $\deg_{u'_n} f_n = \deg f_n|_{U'_n}$. Hence u'_n is the unique critical point of f_n. It follows that all internal rays in U'_n are landed. Thus the conclusion follows. □

4. *Invariant graphs for Newton maps*

In this section, we introduce suitable dynamical graphs of Newton maps for later use to prove Theorem 1.3. In §4.1, we recall the *Newton graphs* given by Drach *et al* [**4**]. In §4.2, we first state Roesch's result on cut angles and then construct invariant graphs differing from the Newton graphs for cubic Newton maps. In §4.3, we generalize Roesch's cut angles result to quartic Newton maps.

4.1. *Newton graphs.*

Let $f \in \mathrm{NM}_d$ with $d \geq 2$. Recall that Ω_f is the union of basins of its roots. Assume that f is post-critically finite on Ω_f. The dynamics of f can be characterized by an invariant graph that is the so-called Newton graph. Such a graph was first constructed in [**4**] and then applied to study the dynamics of corresponding maps, see [**5**, **8**, **9**, **14**, **13**, **29**]. In this subsection, we state briefly the construction of Newton graphs and list some properties.

Let r be a root of f and denote by $\Omega_f(r)$ its immediate attracting basin. The fixed internal rays in $\Omega_f(r)$ land at fixed points in $\partial\Omega_f(r)$. Since the only Julia fixed point of f is at ∞, all fixed internal rays in Ω_f have a common landing point at ∞. We denote by Δ_0 the union of all fixed internal rays in Ω_f together with ∞. Then $f(\Delta_0) = \Delta_0$. For any $m \geq 0$, denote by Δ_m the connected component of $f^{-m}(\Delta_0)$ that contains ∞. Following [**4**], we call Δ_m the *Newton graph* of f at level m. The vertex set of Δ_n consists of iterated preimages of fixed points of f contained in Δ_n.

A crucial property for Newton graphs is the following.

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LEMMA 4.1. [4, Theorem 3.4] *There exists $M \geq 0$ such that the Newton graph Δ_M contains all poles of f. Hence $\Delta_{m+1} = f^{-1}(\Delta_m)$ and $\Delta_m \subseteq \Delta_{m+1}$ for any $m \geq M$.*

The Newton graphs induce naturally a puzzle structure for f on $\widehat{\mathbb{C}}$. Let Δ_f denote the Newton graph of f with the least level such that Δ_f contains all poles and all critical points that map to fixed points under iteration. Set X_0 the complement of the union of the disks $\{z \in U : \phi_U(z) < 1/2\}$ for all connected components U of Ω_f with $U \cap \Delta_f \neq \emptyset$, where ϕ_U is the Böttcher coordinate on U. Define $G_0 := (\Delta_f \cap X_0) \cup \partial X_0$. Then G_0 is a finite graph consisting of segments of internal rays and equipotential lines in Ω_f. For each $m \geq 0$, we define $X_m := f^{-m}(X_0)$ and $G_m := f^{-m}(G_0)$. Then each X_m is connected and the interior $\mathrm{int}(X_m)$ contains the Julia set J_f of f. For each $m \geq 0$, the closures of the components of $X_m \setminus G_m$ are called *puzzle pieces of level m*. It follows that the puzzle pieces of different levels have a nested structure. For each $z \in J_f$, denote $E_m(z)$ the union of puzzle pieces of level m which contains z. Then $z \in \mathrm{int}(E_m(z))$. Moreover, $E_m(z)$ are puzzle pieces for all m if and only if z is not an iterated preimage of ∞.

PROPOSITION 4.2. [5, Corollary 1.2] and [29, Theorem 1.1] *If z is on the boundary of a component of Ω_f, then*

$$\bigcap_{m \geq 0} E_m(z) = \{z\}.$$

In particular, the boundary of any component of basins of the roots is locally connected.

4.2. *An alternative graph for cubic Newton maps.* In this subsection, we focus on the case where $f \in \mathrm{NM}_3$. Except for some special cases, we construct an invariant graph away from the unique non-fixed critical point. Our graph is based on Roesch's work in [25, §3] and differs from the Newton graphs introduced above.

Let r_1, r_2, and r_3 be the roots of f and let Ω_1, Ω_2, and Ω_3 be the corresponding immediate basins respectively. Note that f has another critical point denoted by c. In this subsection, we always assume $c \notin \Omega_1 \cup \Omega_2 \cup \Omega_3$ and c is not a pole, that is, $f(c) \neq \infty$.

Under the assumptions, we have that f has two distinct poles, denoted by ξ_1 and ξ_2. An orientation argument implies that $\partial\Omega_1$, $\partial\Omega_2$, and $\partial\Omega_3$ cannot intersect at a common pole. By counting the preimages of Ω_i basins, we have that there is a unique pole at which exactly two $\partial\Omega_i$ basins intersect. Up to reindexing, we can assume $\xi_1 \in \partial\Omega_1 \cap \partial\Omega_2$. It follows that $\xi_2 \in \partial\Omega_3$ and $\xi_2 \notin \partial\Omega_1 \cup \partial\Omega_2$.

For $i = 1, 2$, and 3, denote by $I_i(\theta)$ the internal ray in Ω_i of angle $\theta \in \mathbb{R}/\mathbb{Z}$. Following Roesch [25], we say an angle θ is a *cut angle* in Ω_1 if there exists $\theta' \in \mathbb{R}/\mathbb{Z}$ such that $I_1(\theta)$ and $I_2(\theta')$ land at a common point. It turns out that θ is a cut angle in Ω_1 if and only if $1 - \theta$ is a cut angle in Ω_2. For the basin Ω_3, the only cut angle is 0. Let Θ be the set of cut angles in Ω_1. It follows immediately that $0, 1/2 \in \Theta$. Label Ω_1 such that Ω_3 and $I_{\Omega_1}(\theta)$ are in the same complementary component of the curve

$$\gamma(0, 1/2) := I_1(0) \cup I_1(1/2) \cup I_2(0) \cup I_2(1/2) \tag{4.1}$$

for any $\theta \in (0, 1/2)$, and define

$$\alpha := \inf\{\theta : \theta \in \Theta\},$$

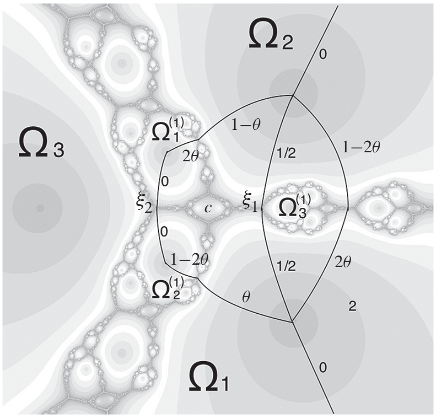

FIGURE 3. The dynamical plane of the Newton map for the polynomial $z^3/3 - z^2/2 + 1$. The curve $\gamma(0, 1/2)$ consists of the internal rays $I_1(0)$, $I_1(\frac{1}{2})$, $I_2(0)$, and $I_2(\frac{1}{2})$. The angle $\theta \notin \Theta$ but $2\theta \in \Theta$. A curve in Lemma 4.3(5) consists of indicated internal rays except those in $\gamma(0, 1/2)$. In this section, we continue to use this example in the subsequent figures.

where inf is obtained under the order by identifying $\mathbb{R}/\mathbb{Z}$ with $(0, 1]$. In fact, the local connectivity of $\partial\Omega_1$ and $\partial\Omega_2$ implies that $\alpha \in \Theta$.

Now we summarize the properties of the cut angles for later use. We use the following notation. Let $\Omega_i^{(1)}$ be the preimage of Ω_i disjoint from Ω_i. Then $c \notin \Omega_i^{(1)}$. For $j \geq 1$, if $\Omega_i^{(j)}$ is a domain such that $f^j : \Omega_i^{(j)} \to \Omega_i$ is a homeomorphism, then an internal ray $I_i(\theta)$ in Ω_i deduces an internal ray $I_i^{(j)}(\theta)$ in $\Omega_i^{(j)}$ satisfying $I_i^{(j)}(\theta) = f^{-j}(I_i(\theta))$.

LEMMA 4.3. [25, §3] *Fix the notation as above. The following statements hold.*

(1) *If the orbit of a rational angle θ is contained in $[\alpha, 1]$, then $\theta \in \Theta$.*
(2) *The angle $0 < \alpha < 1/2$. Furthermore, the periodic angles $1 - 1/(2^n - 1)$ belong to Θ for all large n.*
(3) *Assume $0 < \theta < 1/2$ with $2\theta \in \Theta$. Then $\theta + 1/2 \in \Theta$. Furthermore, if $\theta \in \Theta$, then $I_1^{(1)}(2\theta)$ and $I_2^{(1)}(1 - 2\theta)$ land at a common point; if $\theta \notin \Theta$, then $I_1(\theta)$ and $I_2^{(1)}(1 - 2\theta)$ land at a common point, as well as $I_2(1 - \theta)$ and $I_1^{(1)}(2\theta)$. The two landing points are distinct.*
(4) *The curve $\gamma(0, 1/2)$ defined in equation (4.1) separates Ω_3 and $\Omega_3^{(1)}$.*
(5) *Let $0 < \theta < 1/2$ with $2\theta \in \Theta$. If $\theta \notin \Theta$, then the curve*

$$I_1(1/2) \cup I_1(\theta) \cup I_2^{(1)}(1-2\theta) \cup I_2^{(1)}(0) \cup I_1^{(1)}(0) \cup I_1^{(1)}(2\theta) \cup I_2(1-\theta) \cup I_2(1/2)$$

separates c and ∞.

Figure 3 provides an example to illuminate the curves in the above lemma.

Let $\gamma(0, 1/2)$ be as in equation (4.1). Then the complement of $\gamma(0, 1/2)$ in $\widehat{\mathbb{C}}$ contains two components. Denote by D the one that is disjoint with Ω_3. It follows from Lemma 4.3(4) that $\Omega_3^{(1)} \subset D$.

By Lemma 4.3(2), we can choose a rational angle $\theta \in (0, 1/2)$ satisfying:

(i) $\theta \notin \Theta$, but $2\theta \in \Theta$;

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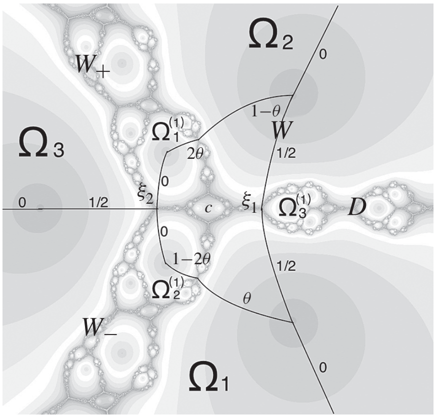

FIGURE 4. The curve $\mathcal{L}$ consists of the indicated internal rays except $I_1(1/2)$ and $I_2(1/2)$. The boundary of D consists of $I_1(0)$, $I_1(1/2)$, $I_2(1/2)$, and $I_2(0)$.

(ii) there exists $k \geq 1$ such that $\eta := 2^k\theta \in (1/2, 1)$; and

(iii) the orbit of the landing point of $I_1(\theta)$ avoids c and ∞.

Define

$$\mathcal{L} := I_3(0) \cup I_3(1/2) \cup I_1(0) \cup I_1(\theta) \cup I_2(0) \cup I_2(1-\theta) \\ \cup I_2^{(1)}(1-2\theta) \cup I_2^{(1)}(0) \cup I_1^{(1)}(0) \cup I_1^{(1)}(2\theta).$$

Then Lemma 4.3(3) implies that $\mathcal{L}$ is a connected graph. Moreover, $\widehat{\mathbb{C}} \setminus \mathcal{L}$ has three components. We label W the one disjoint with Ω_3. In the remaining two components, we label W_- the one intersecting with Ω_1 and label W_+ the one intersecting with Ω_2 (see Figure 4). By Lemma 4.3(5), it immediately follows that $D \cup \overline{\Omega}_3^{(1)} \subseteq W$ and $c \in W \setminus \overline{D}$. In particular, $\xi_1 \in W$. Moreover, we have $I_3(3/4) \subseteq W_-$ and $I_3(1/4) \subseteq W_+$.

Now consider the components of $f^{-1}(\Omega_1^{(1)})$ and $f^{-1}(\Omega_2^{(1)})$. Note that $f^{-1}(\Omega_2^{(1)})$ has a component whose boundary contains the landing point of $I_1((1+\theta)/2)$. Since $I_1((1+\theta)/2) \subset D$, this component is also contained in D. Hence it does not contain c since $c \in W \setminus \overline{D}$. Note that the landing points of $I_3(1/4)$ and $I_3(3/4)$ are contained in the boundaries of the two remaining components of $f^{-1}(\Omega_2^{(1)})$ respectively. We denote by $\Omega_2^{(2)}$ the component whose boundary contains the landing point of $I_3(3/4)$. Then $I_2^{(2)}(0)$ and $I_3(3/4)$ land at a common point. Moreover, $\Omega_2^{(2)} \subset W_-$ since $I_3(3/4) \subseteq W_-$. It follows that $c \notin \Omega_2^{(2)}$. By Lemma 4.3(3), we have $I_1(\theta)$ and $I_2^{(1)}(1-2\theta)$ land at a common point. It follows that $I_1(\theta/2)$ and $I_2^{(2)}(1-2\theta)$ land at a common point since $I_1(\theta/2) \subseteq W_-$. Similarly, denote by $\Omega_1^{(2)}$ the component of $f^{-1}(\Omega_1^{(1)})$ contained in W_+. Then $c \notin \Omega_1^{(2)}$. Moreover, $I_1^{(2)}(0)$ and $I_3(1/4)$ land at a common point, as well as $I_1^{(2)}(2\theta)$ and $I_2(1-\theta/2)$. Define the Jordan curve

$$\mathcal{C} := I_3(1/4) \cup I_3(3/4) \cup I_2^{(2)}(0) \cup I_2^{(2)}(1-2\theta) \cup I_1(\theta/2) \cup I_1(\eta) \\ \cup I_2(1-\eta) \cup I_2(1-\theta/2) \cup I_1^{(2)}(2\theta) \cup I_1^{(2)}(0). \tag{4.2}$$

See Figure 5 for an illustration for the curve $\mathcal{C}$.

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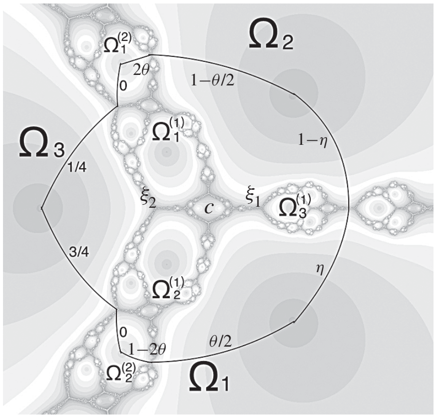

FIGURE 5. The curve $\mathcal{C}$ consists of the indicated internal rays. For this θ, we have $\eta = 2\theta$.

We show that the critical point c is not in the iterations of $\mathcal{C}$ and separated by $\mathcal{C}$ from ∞. More precisely, we have the following lemma.

LEMMA 4.4. *Let $\mathcal{C}$ be as above. Then the following statements hold.*

(1) *The orbit of any Julia point in $\mathcal{C}$ is disjoint with the critical points of f.*

(2) *Denote V the bounded component of $\widehat{\mathbb{C}} \setminus \mathcal{C}$. Then*

$$\overline{\Omega}_1^{(1)} \cup \overline{\Omega}_2^{(1)} \cup \overline{\Omega}_3^{(1)} \cup \{\xi_1, \xi_2, c\} \subset V.$$

Proof. The Julia points in $\mathcal{C}$ are the landing points of $I_3(1/4)$, $I_3(3/4)$, $I_1(\theta/2)$, $I_1(\eta)$, and $I_2(1-\theta/2)$. By the choice of θ, the orbits of the landing points of $I_1(\theta/2)$, $I_1(\eta)$, and $I_2(1-\theta/2)$ are away from c. Since

$$c \in W \setminus \{\infty, \xi_2\} \subseteq \widehat{\mathbb{C}} \setminus \overline{\Omega}_3,$$

it follows that $c \notin \partial\Omega_3$, and hence the orbits of the landing points of $I_3(1/4)$ and $I_3(3/4)$ are disjoint with c. Then statement (1) holds.

Statement (2) follows immediately from the construction of $\mathcal{C}$ and Lemma 4.3 (4), (5). □

Since θ is rational, there is a positive integer $k > 1$ such that the graph

$$G := \bigcup_{j=0}^{k} f^j(\mathcal{C})$$

is invariant. Lemma 4.4 immediately implies that $c \notin G$. Moreover, obviously our graph G is distinct from the Newton graphs of f.

4.3. *Cut angles for quartic Newton maps.* In this subsection, we generalize part of results in [25, §3] from the cubic case to a quartic case. Throughout this subsection, we assume that $f \in \mathrm{NM}_4$ has degree 2 in the immediate basin of each root.

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Let r_1, r_2, r_3, and r_4 be the roots of f and denote by $\Omega_1, \Omega_2, \Omega_3$, and Ω_4 the corresponding immediate basins. Then there exist $1 \le i < j \le 4$ such that $\partial\Omega_i \cap \partial\Omega_j$ contains a pole. Hence the internal rays $I_i(1/2)$ and $I_j(1/2)$ land at a common point. We say that f is of *separable type* if there exist $1 \le i < j \le 4$ such that $I_i(1/2)$ and $I_j(1/2)$ land at a common pole and each component of $\widehat{\mathbb{C}} \setminus \gamma(0, 1/2)$ contains a pole of f, where

$$\gamma(0, 1/2) := I_i(0) \cup I_i(1/2) \cup I_j(0) \cup I_j(1/2).$$

If f is not of separable type, we can choose $1 \le i < j \le 4$ such that $I_i(1/2)$ and $I_j(1/2)$ land at a common pole, but a component D of $\widehat{\mathbb{C}} \setminus \gamma(0, 1/2)$ does not contain a pole of f. Relabeling the roots of f, we set $i = 1$ and $j = 2$. Furthermore, we can set $I_1(\theta) \in D$ if and only if $\theta \in (1/2, 1)$. Hence $I_2(\theta') \in D$ if and only if $\theta' \in (0, 1/2)$. We now consider the cut angles in Ω_1. An angle $\theta \in \mathbb{R}/\mathbb{Z}$ is a *cut angle* in Ω_1 if there exists $\theta' \in \mathbb{R}/\mathbb{Z}$ such that $I_1(\theta)$ and $I_2(\theta')$ land at a common point. If θ is a cut angle in Ω_1, then the corresponding $\theta' = 1 - \theta$. Denote Θ the set of all cut angles in Ω_1 and set

$$\alpha := \inf\{\theta : \theta \in \Theta\},$$

where inf is obtained under the order by identifying $\mathbb{R}/\mathbb{Z}$ with $(0, 1]$. Since $\widehat{\mathbb{C}} \setminus \overline{D}$ contains $\Omega_3 \cup \Omega_4$, it follows that $\alpha > 0$. By the local connectivity of $\partial\Omega_1$ and $\partial\Omega_2$, we have $\alpha \in \Theta$ and Θ is a closed set in $\mathbb{R}/\mathbb{Z}$.

Now we state some properties of the cut angles. Since we are interested in hyperbolic maps, see §5, we further assume that f is hyperbolic in the following result.

PROPOSITION 4.5. *Let f be hyperbolic and not of separable type. With the above notation, the following statements hold.*

(1) *For any $\theta \in \Theta$, $(\theta + 1)/2 \in \Theta$.*
(2) *Let θ be a periodic angle. If the orbit of θ belongs to $(\alpha, 1)$, then $\theta \in \Theta$.*
(3) *The angles $\alpha \in (0, 1/2)$ and there exist periodic angles in $(\alpha, 1/2) \cap \Theta$.*

Proof. For statement (1), since $(\theta + 1)/2 > 1/2$, the internal rays $I_1((\theta + 1)/2) \subseteq D$. Suppose $(\theta + 1)/2 \notin \Theta$. Since $f(I_1((\theta + 1)/2)) = I_1(\theta)$ and $\theta \in \Theta$, there exists a component $\Omega_2^{(1)}$ of $f^{-1}(\Omega_2)$ disjoint with Ω_2 such that $\Omega_2^{(1)}$ contains the landing point of $I_1((\theta + 1)/2)$. Note that f is hyperbolic and hence the landing point of $I_2(1/2)$ is not a critical point. It follows that $\overline{\Omega_2^{(1)}} \subseteq D$. Hence D contains a pole of f. It contradicts the choice of D.

To prove statement (2), let p be the period of the angle θ. Under the assumptions of f, the unique fixed angle is 0. It follows that $p > 1$. Define

$$\gamma(0, \alpha) := I_1(0) \cup I_1(\alpha) \cup I_2(0) \cup I_2(1 - \alpha).$$

Since $\alpha \le 1/2$, there exists a component of $\widehat{\mathbb{C}} \setminus \gamma(0, \alpha)$ containing D. Denote this component by W. It follows that the only possible pole of f contained in W is the common landing point of $I_1(1/2)$ and $I_2(1/2)$. Hence the only component of $f^{-1}(\Omega_1)$ (respectively $f^{-1}(\Omega_2)$) intersecting with W is Ω_1 (respectively Ω_2) itself.

For each $0 \le i \le p$, denote z_i the landing point of $I_1(2^i\theta)$ and by w_i the landing point of $I_2(2^i(1 - \theta)) = I_2(1 - 2^i\theta)$. Since θ is p-periodic, the points $z_0, \ldots, z_{p-1}$ (respectively

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$w_0, \ldots, w_{p-1}$) are pairwise disjoint and $z_0 = z_p$ (respectively $w_0 = w_p$). Moreover, the assumption of θ implies that $z_0, \ldots, z_{p-1}, w_0, \ldots, w_{p-1} \in W$. Suppose $\theta \notin \Theta$. Then $z_0 \neq w_0$. As Θ is closed, we can choose an arc ℓ_0 in $W \setminus \{I_1(t) \cup I_2(1-t) : t \in \Theta\}$ joining the points $z_0 = z_p$ and $w_0 = w_p$ such that ℓ_0 is disjoint with $\Omega_1 \cup \Omega_2$. Let ℓ_1 be the lift of ℓ_0 based at z_{p-1}. By the choice of ℓ_0, we have

$$\ell_1 \subset W \setminus \{I_1(t) \cup I_2(1-t) : t \in \Theta\}$$

and

$$\ell_1 \cap (\Omega_1 \cup \Omega_2) = \emptyset.$$

Note that the endpoint of ℓ_1 is on the boundary of a preimage of Ω_2. By the previous paragraph, this preimage is Ω_2 itself. Note also that w_{p-1} is the unique preimage of w_p on $\partial\Omega_2$ such that w_{p-1} and z_{p-1} are in the same component of $W \setminus (I_1(1/2) \cup I_2(1/2))$. Hence the endpoint of ℓ_1 is w_{p-1}.

Inductively, for each $m \geq 1$, we get an arc $\ell_{mp} \subseteq W$ joining z_0 and w_0 which is a lift of ℓ_0 by f^{pm}. Choose ℓ_0 such that it does not intersect the closure of the forward orbits of the critical points of f. Since f is hyperbolic, it is uniformly expanding near its Julia set. It follows that the length of ℓ_{mp} converges to 0 as $m \to \infty$. Then $z_0 = w_0$, a contradiction. Hence $\theta \in \Theta$ and statement (2) follows.

Now we prove statement (3). Note that $\alpha \in (0, 1/2]$. Suppose, on the contrary, that $\alpha = 1/2$. According to statement (1), the angles $1 - 1/2^n \in \Theta$ for all $n \geq 1$. Choose an angle $\eta \in \Theta$ close to 1 and define

$$\gamma(0, \eta) := I_1(0) \cup I_1(\eta) \cup I_2(1-\eta) \cup I_2(0).$$

Let D_η be a component of $\widehat{\mathbb{C}} \setminus \gamma(0, \eta)$ contained in D. We can choose η sufficiently close to 1 such that D_η contains no critical values of f. Since $\alpha = 1/2$, then $I_1(\eta/2)$ and $I_2(1 - \eta/2)$ land at distinct points. Denote by $\Omega_1^{(1)}$ the component of $f^{-1}(\Omega_1)$ such that $I_2(1 - \eta/2)$ and $I_1^{(1)}(\eta)$ land at a common point and denote $\Omega_2^{(1)}$ the component of $f^{-1}(\Omega_2)$ such that $I_1(\eta/2)$ and $I_2^{(1)}(1-\eta)$ land at a common point. Since f is hyperbolic, its Julia set contains no critical points. It follows that there exists a component D'_η of $f^{-1}(D_\eta)$ whose boundary contains the arc

$$I_1(1/2) \cup I_2(1/2) \cup I_1(\eta/2) \cup I_2(1-\eta/2) \cup I_2^{(1)}(1-\eta) \cup I_1^{(1)}(\eta).$$

Note that the two arcs $I_1(\eta/2) \cup I_2^{(1)}(1-\eta)$ and $I_2(1-\eta/2) \cup I_1^{(1)}(\eta)$ are disjoint and mapped to the same arc $I_1(\eta) \cup I_2(1-\eta)$ under f. Then the proper map $f : D'_\eta \to D_\eta$ has degree at least 2. It implies that D'_η contains at least one critical point. Hence D_η contains a critical value. It contradicts the choice of D_η.

For the second part of statement (3), let $\theta_n := 1 - 1/(2^n - 1)$. Then θ_n is periodic with period n. If $0 \leq i < n - 1$, we have

$$2^i \theta_n = 1 - 2^i/(2^n - 1) \in (1/2, 1).$$

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For $i = n - 1$, we have

$$2^{n-1}\theta_n = \frac{1}{2}\left(1 - \frac{1}{2^n - 1}\right) \in (0, 1/2).$$

Since $\alpha < 1/2$, it follows that $2^{n-1}\theta_n \in (\alpha, 1)$ for sufficiently large n. Then $\theta_n \in \Theta$ by statement (2), and hence $2^{n-1}\theta_n$ is as required. □

5. *The boundedness of hyperbolic components*

In this section, we aim to prove Theorem 1.3. In §5.1, we classify the hyperbolic components into several types and state known boundedness results. Section 5.2 contains two key lemmas for the proof of Theorem 1.3: one concerns the orbit of a critical point and the limit of an attracting cycle; the other one concerns the combinatorial property of the limit function. Then we prove Theorem 1.3 in §5.3.

5.1. *Classification of hyperbolic components and known results.* Let $f \in \mathrm{NM}_4$ be the Newton map of the quartic polynomial P. Then the finite fixed points of f are the zeros of P, and the critical points of f are the zeros of P and zeros of P''. Hence zeros of P are the superattracting fixed points of f. We call any other (super)attracting cycles of f a *free (super)attracting cycle*. Then any free (super)attracting cycle has period at least 2. Moreover, we say a critical point c of f is *additional* if $P''(c) = 0$. Hence f has two additional critical points, counted with multiplicity. According to the orbits of the additional critical points, the hyperbolic components in the moduli space $\mathrm{nm}_4 := \mathrm{NM}_4/\mathrm{Aut}(\mathbb{C})$ belong to the following seven types, see [22]. The same classification is also for hyperbolic components in NM_4.

Type A. Adjacent critical points. The two additional critical points belong to the same component of the immediate basin of a free (super)attracting cycle.

Type B. Bitransitive. Each of the two additional critical points belongs to the immediate basin of a free (super)attracting period cycle, with two distinct components.

Type C. Capture. Only one additional critical point belongs to the immediate basin of a free (super)attracting cycle, but the orbit of the other additional critical point eventually lies in this immediate basin.

Type D. Disjoint (super)attracting orbits. The two additional critical points belong to the immediate basins of two distinct free (super)attracting cycles.

Type IE. Immediate escape. Some additional critical point in the immediate basin of a root.

Type FE1. One future escape. Only one additional critical point in the basin (but not immediate basin) of a root, while the other additional critical point is in the immediate basin of a free (super)attracting cycle.

Type FE2. Two future escapes. The two additional critical points belong to the basins (but not immediate basins) of one or two roots.

The above classification is an analogy of that for quadratic rational maps [17] and for cubic polynomials [19].

Recall that a hyperbolic component in nm_4 is bounded if it has a compact closure in nm_4. Since the type-D hyperbolic components have semi-algebraic boundaries, an arithmetic argument shows that such components are bounded.

PROPOSITION 5.1. [22, Main Theorem] *The hyperbolic components of type D in nm_4 are bounded.*

In contrast, all hyperbolic components of type IE are unbounded.

PROPOSITION 5.2. [22, Theorem 1.4] *Let $\mathcal{H} \subset \mathrm{nm}_4$ be a hyperbolic component. If $\mathcal{H}$ is of type IE, then $\mathcal{H}$ is unbounded in* nm_4.

In the remainder of this section, we give more bounded hyperbolic components in nm_4. In fact, we show the condition in Proposition 5.2 is also necessary.

5.2. *Key lemmas.* To prove Theorem 1.3, we need two key lemmas.

Let $\{f_n\} \subset \mathrm{NM}_4$ be a sequence converging to $f = H_f \hat{f} \in \overline{\mathrm{NM}}_4$ such that $\mathrm{Hole}(f) = \{\infty\}$ and $\deg \hat{f} = 3$. Then f_n has a unique additional critical point c_n converging to ∞ as $n \to \infty$. We suppose that all maps f_n are in a same hyperbolic component in NM_4 and assume that f_n has an attracting cycle $\mathcal{O}_n = \{w_n^{(0)}, \ldots, w_n^{(m-1)}\}$ of period $m \geq 2$. Our first lemma states the orbit of c_n and the limit of $\mathcal{O}_n$.

LEMMA 5.3. *Let f_n, f, c_n, and $\mathcal{O}_n$ be as above. Then the following statements hold.*

(1) *Given any $k \geq 0$ and small $\epsilon > 0$, the points $c_n, f_n(c_n), \ldots, f_n^k(c_n)$ are in the ϵ-neighborhood of ∞ for all large n.*

(2) *Suppose $\mathcal{O}_n$ converges to $\mathcal{O}$ as $n \to \infty$. Then $\mathcal{O} \neq \{\infty\}$.*

(3) *If $\infty \in \mathcal{O}$, then c_n is not in the immediate basin of $\mathcal{O}_n$.*

Proof. Denote by $r_{1,n}, r_{2,n}, r_{3,n}$, and $r_{4,n}$ the roots of f_n. Since $\mathrm{Hole}(f) = \{\infty\}$ and $\deg \hat{f} = 3$, we may assume $r_{4,n} \to \infty$, as $n \to \infty$, and for $1 \leq i \leq 3$, the point $r_{i,n}$ is outside the ϵ-neighborhood of ∞ for all large n. Define $M_n(z) := r_{4,n}z$ and let $g_n := M_n^{-1} \circ f_n \circ M_n$. Then $g_n \in \mathrm{NM}_4$ with roots at $r_{1,n}/r_{4,n}, r_{2,n}/r_{4,n}, r_{3,n}/r_{4,n}$, and 1. Let $g = H_g \hat{g}$ be the degenerate Newton map of the polynomial $z^3(z-1)$. Then g_n converges locally uniformly to $\hat{g}$ away from $\mathrm{Hole}(g) = \{0\}$. Note that $\hat{g}$ has a critical point at $\tilde{c} = 1/2$ and $\tilde{c}$ is attracted to the attracting fixed point 0. Given any $k \geq 0$, the point $\tilde{c}$ is not in $\mathrm{Hole}(g^k) = \bigcup_{i=0}^{k-1} \hat{g}^{-i}(0)$. It follows that there exists $\epsilon_0 = \epsilon_0(k) > 0$ such that $|\hat{g}^j(\tilde{c})| > \epsilon_0$ for all $0 \leq j \leq k$. By Lemma 2.1, we have $|g_n^j(\tilde{c}_n)| > \epsilon_0$ for all large n. Note that for the maps f_n, we have $f_n^j(c_n) = M_n(g_n^j(\tilde{c}_n))$. It follows that $|f_n^j(c_n)| > r_{4,n}\epsilon_0$ for all $0 \leq j \leq k$. Thus, statement (1) follows.

For statement (2), suppose in contrast that $\mathcal{O} = \{\infty\}$. Then all $w_i^{(0)}$s converge to ∞. In the following argument, we may pass to subsequences if necessary to obtain limits. Relabeling the indices, we may assume $w_n^{(i)}/w_n^{(0)}$ does not converge to 0 for all $0 \leq i \leq m-1$. Write $L_n(z) = w_n^{(0)} z$. Then

$$\mathcal{O}_n' := \{1, w_n^{(1)}/w_n^{(0)}, \ldots, w_n^{(m-1)}/w_n^{(0)}\}$$

is an attracting cycle of $h_n := L_n^{-1} \circ f_n \circ L_n \in \mathrm{NM}_4$. Denote by $\mathcal{O}'$ the limit of $\mathcal{O}'_n$. Then $0 \notin \mathcal{O}'$. Assume that $h_n \to h = H_h \hat{h} \in \overline{\mathrm{NM}}_4$. Note that $\mathrm{Hole}(h) \subset \{0, \infty\}$ and $1 \leq \deg \hat{h} \leq 2$.

If $\deg \hat{h} = 1$, then at least three roots of h_n collide to 0 as $n \to \infty$ and the remaining root either collides to 0 or diverges to ∞. For otherwise, $\hat{h}$ would have degree 2. It follows that $\hat{h}(z) = 3z/4$ or $h(z) = 2z/3$. Thus, $\hat{h}$ has an attracting fixed point at 0 and a repelling fixed point at ∞. Moreover, $\mathrm{Hole}(h^j) = \mathrm{Hole}(h) \subset \{0, \infty\}$ for all $j \geq 1$. It follows that $\mathcal{O}' \cap \mathrm{Hole}(h) = \emptyset$. Then by Lemma 2.2, the set $\mathcal{O}'$ is a non-repelling cycle of $\hat{h}$. Note $1 \in \mathcal{O}'$ is not a fixed point of $\hat{h}$. It is a contradiction since all the periodic points of $\hat{h}$ are fixed points.

If $\deg \hat{h} = 2$, then $\mathrm{Hole}(h) = \{0\}$. Moreover, $\hat{h}$ has an attracting fixed point at 0, a superattracting fixed point at the limit r of $r_4^{(n)}/w_n^{(0)}$, and a repelling fixed point at ∞. Since $0 \notin \mathcal{O}'$, then $\mathcal{O}' \cap \mathrm{Hole}(h) = \emptyset$. By Lemma 2.2, the set $\mathcal{O}'$ is a non-repelling cycle of $\hat{h}$ of period at least 2. It follows that $\hat{h}$ has at least three non-repelling cycles: two (super)attracting fixed points 0 and r, and one non-repelling cycle $\mathcal{O}'$. It contradicts the Fatou–Shishikura inequality (see [**26**]) which asserts that $\hat{f}$ has at most two non-repelling cycles. Therefore, we have $\mathcal{O} \neq \{\infty\}$ and the conclusion follows.

Now we prove statement (3). For $0 \leq j \leq m-1$, denote by $U(w_n^{(j)})$ the Fatou component of f_n containing $w_n^{(j)}$. Suppose in contrast that $c_n \in \bigcup_{j=0}^{m-1} U(w_n^{(j)})$. Then $c_n \in U(w_n^{(j_0)})$ for some $0 \leq j_0 \leq m-1$. By relabeling the index, we can assume that $j_0 = 0$. If $w_n^{(0)} \to \infty$, by statement (2), there exists $1 \leq j \leq m-1$ such that $w_n^{(j)} \not\to \infty$. It follows from Lemma 2.3 that the basin $U(w_n^{(j)})$ stays outside a neighborhood of ∞ for all large n. Since $f_n^j(c_n) \in U(w_n^{(j)})$, statement (1) implies that $c_n \not\to \infty$. If $w_n^{(0)} \not\to \infty$, some $w_n^{(\ell)}$ with $1 \leq \ell \leq m-1$ must converge to ∞ since $\infty \in \mathcal{O}$. Again by Lemma 2.3, the basin $U(w_n^{(0)})$ stays outside a neighborhood of ∞ for all large n. Hence $c_n \not\to \infty$. It contradicts the assumption that $c_n \to \infty$. Hence $c_n \notin \bigcup_{j=0}^{m-1} U(w_n^{(j)})$. □

COROLLARY 5.4. *Let f_n, f, c_n, $\mathcal{O}_n$, and $\mathcal{O}$ be as in Lemma 5.3 and let $\widetilde{\mathcal{H}} \subset \mathrm{NM}_4$ be the hyperbolic component containing f_ns. Assume $\widetilde{\mathcal{H}}$ is of type A, B, C, or D. If c_n is in the basin of $\mathcal{O}_n$, then $\infty \notin \mathcal{O}$.*

Proof. If $\widetilde{\mathcal{H}}$ is of type A, B, or D, then c_n is in the immediate basin of $\mathcal{O}_n$. By Lemma 5.3(3), it follows that $\mathcal{O} \subseteq \mathbb{C}$. If $\mathcal{H}$ is of type C, suppose $\infty \in \mathcal{O}$. By Lemma 5.3(2), there exist periodic points $w_n^{(i)}$ and $w_n^{(j)}$ in $\mathcal{O}_n$ such that $w_n^{(i)} \to \infty$ but $w_n^{(j)} \not\to \infty$. It follows from Lemma 2.3 that the basin $U(w_n^{(j)})$ stays outside a neighborhood of ∞ for all large n. Moreover, by Lemma 5.3(3), the critical point c_n is not in the immediate basin of $\mathcal{O}_n$. Then there exists k, independent of n, such that $f_n^k(c_n) \in U(w_n^{(j)})$. It contradicts Lemma 5.3(1). Hence $\infty \notin \mathcal{O}$. □

Recall from §4.3 that a quartic Newton map $f \in \mathrm{NM}_4$ is of separable type if f has two distinct immediate basins Ω_i and Ω_j of roots such that the corresponding internal rays $I_i(1/2) \in \Omega_i$ and $I_j(1/2) \in \Omega_j$ land at a pole and the curve $I_i(0) \cup I_i(1/2) \cup I_j(1/2) \cup I_j(0)$ separates the remaining poles of f. We say a hyperbolic component $\mathcal{H}$ of nm_4 is of *separable type* if each element in $\mathcal{H}$ is of separable type; equivalently, there is an element of separable type in $\mathcal{H}$. Otherwise, we say $\mathcal{H}$ is of *inseparable type*.

Our next key lemma asserts that a non-type-IE hyperbolic component is of inseparable type under an extra assumption on its lift.

LEMMA 5.5. *Let* $\mathcal{H} \subset \mathrm{nm}_4$ *be a non-type-IE hyperbolic component and let* $\widetilde{\mathcal{H}} \subset \mathrm{NM}_4$ *be a lift of* $\mathcal{H}$. *Suppose there exists a sequence* $\{f_n\} \subset \widetilde{\mathcal{H}}$ *such that* f_n *converges to* $f = H_f \hat{f} \in \overline{\mathrm{NM}_4}$ *with* $\mathrm{Hole}(f) = \{\infty\}$ *and* $\deg(\hat{f}) = 3$. *Then* $\mathcal{H}$ *is of inseparable type. Moreover, all poles of* $\hat{f}$ *are simple.*

Proof. By the assumptions, $\hat{f}$ has three roots, denoted by r_1, r_2, and r_3 respectively. Let Ω_1, Ω_2, and Ω_3 be the corresponding immediate basins. Moreover, since $\mathcal{H}$ is of non-type IE, the map $\hat{f}$ has a unique critical point c with $c \notin \bigcup_{i=1}^3 \Omega_i$. By Lemma 3.5, we have $f_n \xrightarrow{\deg} \hat{f}$ on $\{\Omega_1, \Omega_2, \Omega_3\}$. Relabeling r_1, r_2, and r_3, we may assume that there exists a pole of $\hat{f}$ in the intersection $\partial\Omega_1 \cap \partial\Omega_2$. For $1 \le i \le 3$, denote by $(\Omega_{i,n}, r_{i,n})$ the deformation of (Ω_i, r_i) at f_n. Then each $r_{i,n}$ is a root of f_n and $\partial\Omega_{1,n} \cap \partial\Omega_{2,n}$ contains a pole of f_n. Let $r_{4,n}$ be the remaining root of f_n. Then $r_{4,n} \to \infty$, as $n \to \infty$. Denote $\Omega_{4,n}$ its immediate basin.

In contrast, we assume $\mathcal{H}$ is of separable type. Consider the internal rays in $\Omega_{1,n}$ and $\Omega_{2,n}$. Set

$$\gamma_n(0, 1/2) := I_{1,n}(0) \cup I_{1,n}(1/2) \cup I_{2,n}(0) \cup I_{2,n}(1/2).$$

Then each component of $\widehat{\mathbb{C}} \setminus \gamma_n(0, 1/2)$ contains a pole of f_n, and hence contains $\Omega_{3,n}$ or $\Omega_{4,n}$. We denote D_n the one containing $\Omega_{4,n}$, and assume that $I_{1,n}(\theta) \subseteq D_n$ if and only if $\theta \in (1/2, 1)$.

Since $\Omega_{4,n} \subset D_n$, there exists a minimal $k \ge 2$ such that the landing point z_n of $I_{2,n}(1/2^k)$ is not in $\partial\Omega_{1,n}$. Let $\Omega_{1,n}^{(1)}$ be the component of $f_n^{-1}(\Omega_{1,n})$ such that $z_n \in \partial\Omega_{1,n}^{(1)}$. Then $\Omega_{1,n}^{(1)} \neq \Omega_{1,n}$ and $\Omega_{1,n}^{(1)} \subseteq D_n$. Note that $\Omega_{1,n}^{(1)}$ contains no critical points. For otherwise, $\partial\Omega_{1,n}^{(1)}$ and hence D_n would contain two poles of f_n, which is impossible. Then $\partial\Omega_{1,n}^{(1)}$ contains a unique pole of f_n, which coincides with the one on $\partial\Omega_{4,n}$. Set $I_{1,n}^{(1)}(t)$ the internal ray in $\Omega_{1,n}^{(1)}$ landing at z_n. By Proposition 3.8, the landing point z_n of $I_{2,n}(1/2^k)$ converges to the landing point z of $I_2(1/2^k)$. Note that the pole of f_n in $\partial\Omega_{1,n} \cap \partial\Omega_{2,n}$ (respectively $\partial\Omega_{3,n}$) converges to the pole of $\hat{f}$ in $\partial\Omega_1 \cap \partial\Omega_2$ (respectively $\partial\Omega_3$). Thus, the pole of f_n in $\partial\Omega_{4,n} \cap \partial\Omega_{1,n}^{(1)}$ converges to ∞ as $n \to \infty$. For otherwise, these poles converge to poles of $\hat{f}$, contradicting $\deg \hat{f} = 3$. Similarly, the center of $\Omega_{1,n}^{(1)}$ converges to ∞. Then, passing to subsequences if necessary, we have that the arcs $I_{1,n}^{(1)}(t)$ converge to a continuum ℓ containing ∞ and z.

Recall that $\psi_{1,n}^{(1)} : \mathbb{D} \to \Omega_{1,n}^{(1)}$ and $\psi_{1,n} : \mathbb{D} \to \Omega_{1,n}$ are the inverses of the Böttcher coordinates on $\Omega_{1,n}^{(1)}$ and $\Omega_{1,n}$, respectively. Let q be any point in $\ell \setminus \{\infty\}$. There exists $q_n \in I_{1,n}^{(1)}(t)$ with $q_n \to q$. We write $q_n = \psi_{1,n}^{(1)}(s_n e^{2\pi i t})$. Since $q \neq \infty$, we have $f_n(q_n) \to \hat{f}(q)$. Note that

$$f_n(q_n) = f_n \circ \psi_{1,n}^{(1)}(s_n e^{2\pi i t}) = \psi_{1,n}(s_n e^{2\pi i t}) \in I_{1,n}(t).$$

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Since $I_{1,n}(t) \to I_1(t)$, the point $\hat{f}(q)$ belongs to $I_1(t)$. We claim in fact that $\hat{f}(q) \in \partial\Omega_1$. Otherwise, q belongs to either Ω_1 or the other component $\Omega_1^{(1)}$ of $\hat{f}^{-1}(\Omega_1)$. Note that $\Omega_1^{(1)} \cap D_n = \emptyset$ for large n. By Lemma 2.5, we have $q_n \notin \Omega_{1,n}^{(1)}$. It is a contradiction. By this claim, any point in $\ell \setminus \{\infty\}$ maps under $\hat{f}$ to the landing point of $I_1(t)$. It is impossible. Thus, $\mathcal{H}$ is of inseparable type.

Now we show all poles of $\hat{f}$ are simple. Let Θ be the set of angles θ such that $I_{1,n}(\theta)$ and $I_{2,n}(1-\theta)$ land at a common point. Since $\mathcal{H}$ is of inseparable type, by Proposition 4.5(3), there exists a periodic angle $\theta \in \Theta \cap (0, 1/2)$. According to Proposition 3.8, the internal rays $I_1(\theta)$ and $I_2(1-\theta)$ land at a common point. This implies c cannot be a pole of $\hat{f}$, since otherwise c is a common point of $\partial\Omega_i$, $i = 1, 2, 3$, impossible. □

5.3. *Proof of Theorem 1.3.* To prove Theorem 1.3, we first state the following lift result.

LEMMA 5.6. *For $d \geq 3$, let $[g_n] \in \mathrm{nm}_d$ be a sequence such that $[g_n] \to \infty$. Then there exists a sequence $f_{n_i} \in \mathrm{NM}_d$ such that $[f_{n_i}] = [g_{n_i}]$ and f_{n_i} converges to $f = H_f \hat{f} \in \partial\mathrm{NM}_d$ with* $\mathrm{Hole}(f) = \{\infty\}$ *and* $\deg \hat{f} \geq 2$. *Moreover, if all $[g_n]$ terms are contained in a same hyperbolic component in nm_d, then f_{n_i} terms are contained in the same hyperbolic component in NM_d.*

Proof. Since $[g_n] \to \infty$, there exists a subsequence g_{n_i} such that g_{n_i} converges to an element in $\partial\mathrm{NM}_d$. We first normalize the roots of g_{n_i} by affine maps to obtain a sequence $\tilde{g}_{n_i} \in \mathrm{NM}_d$ such that 0 and 1 are two roots of $\tilde{g}_{n_i}$. Note $[\tilde{g}_{n_i}] = [g_{n_i}]$. It follows that $[\tilde{g}_{n_i}] \to \infty$ and hence $\{\tilde{g}_{n_i}\}$ contains a subsequence converging to an element in $\partial\mathrm{NM}_d$. We also denote this subsequence by $\{\tilde{g}_{n_i}\}$. We can further assume all roots of $\tilde{g}_{n_i}$ converge in $\widehat{\mathbb{C}}$. Denote $r_{1,n_i}, \ldots, r_{d,n_i}$ the roots of $\tilde{g}_{n_i}$. Choose $1 \leq m_0 < m_1 \leq d$ such that

$$|r_{m_0,n_i} - r_{m_1,n_i}| = O(|r_{\ell,n_i} - r_{k,n_i}|)$$

for all $1 \leq \ell < k \leq d$ with $r_{\ell,n_i} \not\to \infty$ and $r_{k,n_i} \not\to \infty$, as $n_i \to \infty$. Define

$$M_{n_i}(z) := \frac{z - r_{m_1,n_i}}{r_{m_0,n_i} - r_{m_1,n_i}}$$

and set $f_{n_i} := M_{n_i} \circ \tilde{g}_{n_i} \circ M_{n_i}^{-1}$. Then f_{n_i} has roots at 0, 1 and no roots colliding in $\mathbb{C}$. Then the sequence f_{n_i} is the desired sequence.

The remaining part of the lemma follows from the connectedness of the quotient group $\mathrm{Aut}(\mathbb{C})$. □

Proof of Theorem 1.3. By Proposition 5.2, it suffices to show that if $\mathcal{H} \subset \mathrm{nm}_4$ is not of type IE, then $\mathcal{H}$ is bounded in nm_4. The proof goes by contradiction.

Suppose $\mathcal{H}$ is unbounded. Let $\{[f_n]\}_{n\geq 0}$ be a degenerated sequence in $\mathcal{H}$. Passing to a subsequence, by Lemma 5.6, we can assume that all f_n belong to a hyperbolic component in NM_4, and f_n converges to $f = H_f \hat{f} \in \overline{\mathrm{NM}}_4$ with $\mathrm{Hole}(f) = \{\infty\}$ and $\deg \hat{f} = 2$ or 3. We deduce the contradiction case by case.

Case 1: deg $\hat{f} = 2$. Let (Ω_1, r_1) and (Ω_2, r_2) be the immediate basins of roots of $\hat{f}$. By Lemma 3.5, we have that $f_n \xrightarrow{\text{deg}} f$ on $\{\Omega_1, \Omega_2\}$. Denote $(\Omega_{1,n}, r_{1,n})$ and $(\Omega_{2,n}, r_{1,n})$ the deformations of (Ω_1, r_1) and (Ω_2, r_2) at f_n respectively. In this case, the Julia set of $\hat{f}$ is

$$J_{\hat{f}} = \partial\Omega_1 = \partial\Omega_2,$$

which is a Jordan curve and contains no critical points. Given any rational angle θ, the internal rays $I_1(\theta)$ and $I_2(1-\theta)$ land at a common point. By Theorem 1.1, for all large n, the internal rays $I_{1,n}(\theta)$ and $I_{2,n}(1-\theta)$ land at a common point. Since all f_n belong to the same hyperbolic component, we get that the internal rays $I_{1,0}(t)$ and $I_{2,0}(1-t)$ of f_0 land together for all $t \in \mathbb{Q}$. Then the boundaries $\partial\Omega_{1,0}$ and $\partial\Omega_{2,0}$ coincide. It follows that f_0 is conjugate to $z \mapsto z^2$, which is a contradiction.

Case 2: deg $\hat{f} = 3$. In this case, $\hat{f} \in \mathrm{NM}_3$. Moreover, the unique additional critical point c of $\hat{f}$ is not in the immediate basins of the roots of $\hat{f}$. For otherwise, f_n would possess an additional critical point in the immediate basin of some root, which is a contradiction.

Let c_n be the additional critical point of f_n such that c_n converges to ∞. Now we proceed our argument according to the type of $\mathcal{H}$.

Case 2(i): *$\mathcal{H}$ is of type A,B,C, or D.* Let $\mathcal{O}_n$ be the free (super)attracting cycle of f_n such that c_n is in the basin of $\mathcal{O}_n$. Denote by $\mathcal{O}$ the limit of $\mathcal{O}_n$. By Corollary 5.4, we have that $\mathcal{O} \subseteq \mathbb{C}$. Then by Lemma 2.2, the set $\mathcal{O}$ is a non-repelling cycle of $\hat{f}$ of period at least 2. It follows that the critical point c is not an iterated preimage of ∞ under $\hat{f}$. Moreover, $\hat{f}$ is post-critically finite on $\Omega_{\hat{f}}$.

Consider the Newton graph $\Delta_m(\hat{f})$ of $\hat{f}$ at level m. Applying Proposition 4.2 to $z = \infty$, for a sufficiently large m, we obtain a Jordan curve $\gamma \subseteq \Delta_m(\hat{f})$ such that the orbit $\mathcal{O}$ is contained in the bounded component of $\widehat{\mathbb{C}} \setminus \gamma$. Let $\mathcal{U}$ be the collection of components of $\Omega_{\hat{f}}$ intersecting $\Delta_m(\hat{f})$. Then $\hat{f}(U) \in \mathcal{U}$ for $U \in \mathcal{U}$. By Lemma 3.5, we have that $f_n \xrightarrow{\text{deg}} f$ on $\mathcal{U}$.

Set $\delta := d_H(\infty, \gamma)$. By Theorem 1.1, the curve γ is perturbed to a Jordan curve $\gamma_n \subseteq \Delta_m(f_n)$ such that $\mathcal{O}_n$ is contained in the bounded component of $\widehat{\mathbb{C}} \setminus \gamma_n$ and $d_H(\gamma_n, \gamma) < \delta/3$ for all large n. Since the immediate basin of $\mathcal{O}_n$ is disjoint with $\Delta_m(f_n)$ for all n, it is contained in the bounded component of $\widehat{\mathbb{C}} \setminus \gamma_n$.

If $\mathcal{H}$ is of type A, B, or D, then the critical point c_n is in the immediate basin of $\mathcal{O}_n$. The above argument immediately implies that the distance between c_n and ∞ is at least $\delta/3$, a contradiction to $c_n \to \infty$.

If $\mathcal{H}$ is of type C, since the critical point c_n converges to ∞, the above argument implies that c_n is not in the immediate basins of $\mathcal{O}_n$. In this case, there exists $k > 0$ such that $f_n^k(c_n)$ belongs to the immediate basin of $\mathcal{O}_n$ for all n, which stays outside the $\delta/3$ neighborhood of ∞. It contradicts Lemma 5.3(1).

Case 2(ii): *$\mathcal{H}$ is of type FE1 or FE2.* First, differing from Case 2(i), the additional critical point of $\hat{f}$ may be an iterated preimage of ∞. So the assumptions of Theorem 1.1 may fail for the Newton graphs of $\hat{f}$. Alternatively, we apply Theorem 1.1 to the Jordan curve $\mathcal{C}$ constructed in §4.2 in the following argument.

By Lemma 5.5, the additional critical point c of $\hat{f}$ is not a pole. We can thus use the results in §4.2. Inheriting the notation in §4.2, by Lemma 4.4, we obtain a Jordan curve $\mathcal{C}$ consisting of some internal rays in Ω_1, Ω_2, Ω_3, $\Omega_1^{(2)}$, and $\Omega_2^{(2)}$ such that the orbits of the landing points of these rays are disjoint with the critical points of $\hat{f}$ and the bounded component of $\widehat{\mathbb{C}} \setminus \mathcal{C}$ contains $\overline{\Omega}_1^{(1)}$, $\overline{\Omega}_2^{(1)}$, $\overline{\Omega}_3^{(1)}$, c, and the poles of $\hat{f}$.

Set

$$\mathcal{U} := \{\Omega_1, \Omega_2, \Omega_3, \Omega_1^{(1)}, \Omega_2^{(1)}, \Omega_1^{(2)}, \Omega_2^{(2)}\}.$$

Then $\hat{f}(U) \in \mathcal{U}$ for $U \in \mathcal{U}$. Moreover, by Lemma 3.5, we have that $f_n \xrightarrow{\deg} f$ on $\mathcal{U}$. By applying Theorem 1.1 to $\mathcal{C}$, for all large n, we obtain a Jordan curve $\mathcal{C}_n$ consisting of internal rays of f_n in $\Omega_{1,n} \cup \Omega_{2,n} \cup \Omega_{3,n} \cup \Omega_{1,n}^{(2)} \cup \Omega_{2,n}^{(2)}$ with the same angles as those of $\hat{f}$ in $\Omega_1 \cup \Omega_2 \cup \Omega_3 \cup \Omega_1^{(2)} \cup \Omega_2^{(2)}$. Then the bounded component of $\widehat{\mathbb{C}} \setminus \mathcal{C}_n$ contains $\overline{\Omega}_{1,n}^{(1)}$, $\overline{\Omega}_{2,n}^{(1)}$, $\overline{\Omega}_{3,n}^{(1)}$, two poles of f_n, and the closures of the two preimages of $\Omega_{4,n}$ disjoint with $\Omega_{4,n}$. Moreover, the unbounded component of $\widehat{\mathbb{C}} \setminus \mathcal{C}_n$ contains $\overline{\Omega}_{4,n}$.

For the additional critical point c_n of f_n with $c_n \to \infty$, we claim that there exists a minimal integer $k \geq 1$ such that

$$f_n^k(c_n) \in \Omega_{1,n} \cup \Omega_{2,n} \cup \Omega_{3,n} \cup \Omega_{4,n}.$$

To prove this claim, it suffices to consider the case where f_n has a free (super)attracting cycle $\mathcal{O}_n$. Suppose $\mathcal{O}_n$ converges to $\mathcal{O}$. If $\infty \in \mathcal{O}$, the claim follows from Lemma 5.3(3). If $\mathcal{O} \subseteq \mathbb{C}$, by Lemma 2.2, the set $\mathcal{O}$ is the non-repelling cycle of $\hat{f}$ of period at least 2. It follows that $\hat{f}^j(c) \neq \infty$ for all $j \geq 0$. Moreover, $\hat{f}$ is post-critically finite in the basins of the roots. With the same argument in Case 2(i), we obtain that the immediate basin of $\mathcal{O}_n$ is disjoint with a fixed neighborhood of ∞. Hence the claim follows since $c_n \to \infty$.

We also claim that $\partial U(f_n^i(c_n)) \cap \partial\Omega_{4,n} \neq \emptyset$ for all $0 \leq i \leq k-1$. By Lemma 5.3(1), for each $0 \leq i \leq k-1$ and all large n, the Fatou component $U(f_n^i(c_n))$ containing $f_n^i(c_n)$ is not contained in the bounded domain of $\widehat{\mathbb{C}} \setminus \mathcal{C}_n$. Furthermore, none of these Fatou components intersect $\mathcal{C}_n$. Indeed, if $U(f_n^i(c_n))$ intersects $\mathcal{C}_n$ for some $0 \leq i \leq k-1$, then $U(f_n^i(c_n))$ coincides with either $\Omega_{1,n}^{(2)}$ or $\Omega_{2,n}^{(2)}$. It then follows that $U(f_n^{i+1}(c_n))$ coincides with either $\Omega_{1,n}^{(1)}$ or $\Omega_{2,n}^{(1)}$. Note $\Omega_{1,n}^{(1)}$ and $\Omega_{2,n}^{(1)}$ are both in the bounded component of $\widehat{\mathbb{C}} \setminus \mathcal{C}_n$. It contradicts Lemma 5.3(1). Therefore, for $0 \leq i \leq k-1$, the component $U(f_n^i(c_n))$ is contained in the unbounded component of $\widehat{\mathbb{C}} \setminus \mathcal{C}_n$.

By previous argument, the closure of any non-fixed preimage of $\Omega_{1,n}$, $\Omega_{2,n}$, $\Omega_{3,n}$, or $\Omega_{4,n}$ either belongs to the bounded component of $\widehat{\mathbb{C}} \setminus \mathcal{C}_n$ or intersects with $\partial\Omega_{4,n}$ at a pole. Then

$$\partial U(f_n^{k-1}(c_n)) \cap \partial\Omega_{4,n} \neq \emptyset.$$

Note that $\Omega_{4,n}$ is the unique component of $f_n^{-1}(\Omega_{4,n})$ contained in the unbounded component of $\widehat{\mathbb{C}} \setminus \mathcal{C}_n$. Since each $U(f_n^i(c_n))$ is in the unbounded component of $\widehat{\mathbb{C}} \setminus \mathcal{C}_n$, then for all $0 \leq i \leq k-1$,

$$\partial U(f_n^i(c_n)) \cap \partial\Omega_{4,n} \neq \emptyset.$$

The claim is proved.

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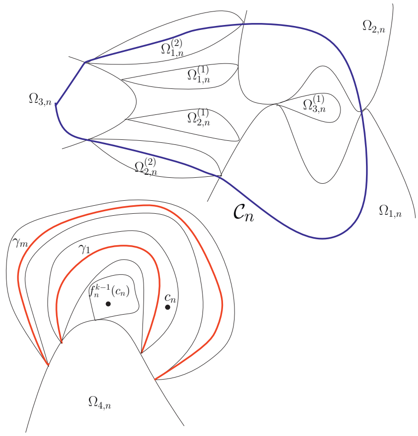

FIGURE 6. Sketch of the proof in Case 2(ii).

Moreover, we claim in fact that $k \geq 2$. Indeed, if $k = 1$, then the Fatou component $U(c_n)$ contains two poles of f_n. Note that a bounded component of $\widehat{\mathbb{C}} \setminus \mathcal{C}_n$ contains two poles of f_n and its complement contains the other pole. We then get a contradiction since $U(c_n)$ is contained in the unbounded component of $\widehat{\mathbb{C}} \setminus \mathcal{C}_n$.

Note that all f_n are in the same hyperbolic component, then all quantities defined for f_n and properties satisfied by f_n for n large also hold for f_0. We deduce the contradiction by f_0. Suppose $\partial U(f_0(c_0))$ intersects $\partial\Omega_{4,0}$ at the landing point of $I_{4,0}(\theta)$. Since $U(c_0)$ contains a critical point and is contained in the unbounded component of $\widehat{\mathbb{C}} \setminus \mathcal{C}_0$, the intersection $\partial U(c_0) \cap \partial\Omega_{4,0}$ contains the landing points of $I_{4,0}(\theta/2)$ and $I_{4,0}((1+\theta)/2)$. We consider an arc $\gamma_1 \subset \overline{U(c_0)}$ joining these two landing points and avoiding the orbits of critical points of f_0. Let γ_2 be the lift of γ_1 based at the landing point of $I_{4,0}(\theta/2^2)$. Since γ_1 does not intersect with $\mathcal{C}_0$, the endpoint of γ_2 belongs to $\partial\Omega_{4,0}$. Note also that the preimages of $\gamma_1(1)$ on $\partial\Omega_{4,0}$ are the landing points of the internal rays in $\Omega_{4,0}$ of angles $(1+\theta)/4$ or $(3+\theta)/4$. Since $(1+\theta)/4 \in (\theta/2, (1+\theta)/2)$, it follows that the endpoint of γ_2 is the landing point of $I_{4,0}((3+\theta)/4)$.

Inductively, for every $m \geq 1$, define γ_{m+1} to be the lift of γ_m based at the landing point of $I_{4,0}(\theta/2^{m+1})$. Then the endpoint of γ_{m+1} is the landing point of $I_{4,0}(1-(1-\theta)/2^{m+1})$. Note that for large m, each γ_m is an arc joining two points of $\partial\Omega_{4,0}$ in different components of $\partial\Omega_{4,0} \setminus (I_{4,0}(0) \cup I_{4,0}(1/2))$ near ∞. Moreover, the intersection of γ_m and $\overline{\Omega}_{1,0} \cup \overline{\Omega}_{2,0} \cup \overline{\Omega}_{3,0} \cup \overline{\Omega}_{4,0}$ is the endpoint of γ_m. It follows that the length of γ_m has a positive infinitum as $m \to \infty$. However, since f_0 is uniformly expanding near the Julia set, the length of γ_m decreases to 0 as $m \to \infty$. It is a contradiction (see Figure 6). □

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