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Resultants of Chebyshev Polynomials: A Short Proof

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Abstract. We give a simple proof of the value of the resultant of two Chebyshev polynomials (of the first or the second kind), values lately obtained by D. P. Jacobs, M. O. Rayes, and V. Trevisan.

In [Lou], we gave a simple proof of the value of the resultant of cyclotomic polynomials, a result obtained in [Apo, Die]. Here, we show that our method applies readily to the computation of resultants of Chebyshev polynomials. We explain in six steps how one can simply deduce such resultants. Let us mention that whereas the authors in [JRT] have to know beforehand the result, for they prove it by induction, we will deduce it from the definitions of the resultants (see (1) and (2) below) and from a simple tool (see Lemma 2).

Step 1. The Chebyshev polynomials of the first kind $T_n(x)$, $n \geq 1$, are defined by

$$T_n(x) = \cos(n \cdot \arccos x) = 2^{n-1} \prod_{k=1}^{n} \Big(x - \cos\Big(\frac{2k-1}{n} \frac{\pi}{2} \Big) \Big) \in \mathbf{Z}[x].$$

The resultant of two such polynomials is given by

$$\operatorname{res}(T_m, T_n) = 2^{mn-m-n} \prod_{k=1}^{m} \prod_{l=1}^{n} \Big(2\cos\Big(\frac{2k-1}{m} \frac{\pi}{2} \Big) - 2\cos\Big(\frac{2l-1}{n} \frac{\pi}{2} \Big) \Big) \tag{1}$$

(*e.g.*, see [JRT, (3.3)]).
To begin with, $\operatorname{res}(T_m, T_n) = 0$ if and only if there exist k and l such that

$$\frac{2k-1}{m} \frac{\pi}{2} \equiv \pm \frac{2l-1}{n} \frac{\pi}{2} \pmod{2\pi},$$

i.e., such that $m_1(2l-1) \equiv \pm n_1(2k-1) \pmod{4gm_1n_1}$, where $m = gm_1$, $n = gn_1$ and $g = \gcd(m,n)$. If m_1 and n_1 are odd, then by taking $k = (m_1+1)/2$ and $l = (n_1+1)/2$ we obtain $\operatorname{res}(T_m, T_n) = 0$. If m_1 or n_1 is even, then one of them is odd (since $\gcd(m_1, n_1) = 1$), hence $m_1(2l-1) \equiv \pm n_1(2k-1) \pmod{4}$ has no solution and $\operatorname{res}(T_m, T_n) \neq 0$. Hence, we have the following proposition.

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Proposition 1 (See [JRT, Corollary 4.2]) *For $m, n \geq 1$, we have* $\operatorname{res}(T_m, T_n) = 0$ *if and only if $m_1 := m/\gcd(m, n)$ and $n_1 := n/\gcd(m, n)$ are odd.*

Step 2. So, from now on we may assume that m_1 or n_1 is even, the other one being odd, since $\gcd(m_1, n_1) = 1$. We write $m_1 = 2^\alpha M$, $n_1 = 2^\beta N$, and $g = 2^\gamma G$, where M, N, and G are odd, α or β is equal to zero and the other one is not, and where $\gcd(M, N) = 1$. As in [Lou], two algebraic integers α and β are called *associated* if there exists an algebraic unit ϵ such that $\beta = \epsilon\alpha$. To prove that the rational integer $\operatorname{res}(T_m, T_n)$ is equal to some $R \in \mathbf{Z}$, it suffices to prove that $\operatorname{res}(T_m, T_n)$ is associated with R and that $\operatorname{res}(T_m, T_n)$ and R are of the same sign. We set $\zeta_x = \exp(2i\pi/x)$. Now, if a and b are rational multiples of π, then

$$2\cos a - 2\cos b = 4\sin\Big(\frac{a+b}{2}\Big)\sin\Big(\frac{a-b}{2}\Big) = -e^{ib}(1 - e^{i(a+b)})(1 - e^{i(a-b)})$$

is an algebraic integer.

In particular, by (1), it follows that 2^{mn-m-n} divides $\operatorname{res}(T_m, T_n) \in \mathbf{Z}$.

Step 3. Now we will use a standard result from cyclotomic fields (see *e.g.*, [Lou, Lemma 1] for a proof).

Lemma 2 *Let $x \geq 1$ and y be coprime integers. Then $1 - \zeta_x^y$ is associated with $1 - \zeta_x$. Moreover, $1 - \zeta_x$ is associated with 1, except if x is a power of some prime p, in which case $(1 - \zeta_x)^{\phi(x)}$ is associated with p.*

Now, for $a = \frac{2k-1}{m}\frac{\pi}{2}$ and $b = \frac{2l-1}{n}\frac{\pi}{2}$, we have $1 - e^{i(a\pm b)} = 1 - \zeta_x^y$ with $x = 4gm_1n_1/\delta \geq 1$ and $y = ((2k-1)n_1 \pm (2l-1)m_1)/\delta$ coprime, where

$$\delta := \gcd\big(4gm_1n_1, (2k-1)n_1 \pm (2l-1)m_1\big) = \gcd\big(gm_1n_1, (2k-1)n_1 \pm (2l-1)m_1\big)$$

(for $(2k-1)n_1 \pm (2l-1)m_1$ is odd). Hence, x is divisible by 4. Therefore, if x is a power of a prime, then x is a power of 2.

Consequently, $\operatorname{res}(T_m, T_n)$ is always a perfect power of 2, up to the sign, by (1) and Lemma 2.

Step 4. Moreover, we have $\delta = \gcd(GMN, 2^\beta(2k-1)N \pm 2^\alpha(2l-1)M)$ (for $(2k-1)n_1 \pm (2l-1)m_1$ is odd), and

$$x = 2^{\alpha+\beta+\gamma+2}\frac{GMN}{\gcd(GMN, 2^\beta(2k-1)N \pm 2^\alpha(2l-1)M)}$$

is a power of 2 if and only if GMN divides $2^\beta(2k-1)N \pm 2^\alpha(2l-1)M$, in which case $x = 2^{\alpha+\beta+\gamma+2}$. It follows that if t denotes the number of $(k, l) \in \{1, \dots, m\} \times \{1, \dots, n\}$ such that GMN divides $2^\beta(2k-1)N \pm 2^\alpha(2l-1)M$, then $\operatorname{res}(T_m, T_n)$ is associated with, hence up to its sign equal to, $R := 2^{mn-m-n} \times 2^{2\cdot t/\phi(2^{\alpha+\beta+\gamma+2})}$, by (1) and Lemma 2.

Step 5. Since $\operatorname{res}(T_m, T_n) = \operatorname{res}(T_n, T_m)$, we may assume that $\alpha = 0$ and $\beta \geq 1$. Then GMN divides $2^\beta(2k-1)N \pm 2^\alpha(2l-1)M$ if and only if M divides $2k-1$ and $2^\beta\frac{2k-1}{M} \pm (2l-1) \equiv 0 \pmod{GM}$. Hence, k is unique mod M and there are

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$m/M = m/m_1 = g$ possible choices for k. Now, for a given k we have $2l - 1 \equiv \mp 2^{\beta}\frac{2k-1}{M}N \pmod{GN}$, where GN is odd. Hence, l is unique mod GN, so that there are $n/GN = 2^{\beta+\gamma}n/gn_1 = 2^{\beta+\gamma} = 2^{\alpha} + \beta + \gamma$ choices for n.

Hence, $t = g \cdot 2^{\alpha+\beta+\gamma}$ and $\operatorname{res}(T_m, T_n)$ is associated with $R = 2^{mn-m-n+q}$.

Step 6. Finally the sign of $\operatorname{res}(T_m, T_n)$ being equal to $(-1)^{mn/2}$ (count the number of negative terms in (1), *i.e.*, the number of indices (k, l) for which $(2k-1)/m > (2l-1)/n$), as in [JRT] we obtain the following theorem.

Theorem 3 (See [JRT, Theorem 4.5]) *For $m, n \geq 1$ we have*

$$\operatorname{res}(T_m, T_n) = \begin{cases} 0 & m/\gcd(m,n) \text{ and } n/\gcd(m,n) \text{ are odd},\\ 2^{mn-m-n+\gcd(m,n)} & \text{otherwise.}\end{cases}$$

The Chebyshev polynomials of the second kind, $U_n(x)$, $n \geq 1$, are defined by

$$U_n(x) = \frac{T'_{n+1}(x)}{n+1} = \frac{\sin((n+1)\arccos x)}{\sin(\arccos x)} = 2^n \prod_{k=1}^{n}\Big(x - \cos\Big(\frac{k\pi}{n+1}\Big)\Big) \in \mathbf{Z}[x].$$

The resultant of two such polynomials is given by

$$(2) \qquad \operatorname{res}(U_m, U_n) = 2^{mn}\prod_{k=1}^{m}\prod_{l=1}^{n}\Big(2\cos\Big(\frac{k\pi}{m+1}\Big) - 2\cos\Big(\frac{l\pi}{n+1}\Big)\Big).$$

To begin with, $\operatorname{res}(U_m, U_n) = 0$ if and only if there exist k with $1 \leq k \leq m$ and l with $1 \leq l \leq n$ such that

$$\frac{k\pi}{m+1} \equiv \pm\frac{l\pi}{n+1} \pmod{2\pi},$$

i.e., such that $m_1 l \equiv \pm n_1 k \pmod{2gm_1n_1}$, where $m+1 = gm_1$, $n+1 = gn_1$ and $g = \gcd(m+1, n+1)$. If $g > 1$, then by taking $k = m_1 \leq m$ and $l = n_1 \leq n$ we obtain $\operatorname{res}(T_m, T_n) = 0$. If $g = 1$, then $m_1 = m+1$ must divide $k < m$ and $n_1 = n+1$ must divide $l < n$, which cannot occur. Hence $\operatorname{res}(T_m, T_n) \neq 0$ and $\operatorname{res}(U_m, U_n) = 0$ if and only if $\gcd(m+1, n+1) > 1$, as in [JRT, Corollary 5.2]. So we now assume that $m_1 := m+1$ and $n_1 := n+1$ are coprime. With $a = k\pi/m_1$ and $b = l\pi/n_1$, we have $1 - e^{i(a\pm b)} = 1 - \zeta_x^y$ with $x = 2m_1n_1/\delta$, $y = (kn_1 \pm lm_1)/\delta$ and $\delta = \gcd(2m_1n_1, kn_1 \pm lm_1)$. But

$$x' := \frac{m_1n_1}{\gcd(m_1n_1, kn_1 \pm lm_1)} = \frac{m_1}{\gcd(m_1, k)}\frac{n_1}{\gcd(n_1, l)}$$

divides x, and x' is clearly not a prime power (for $k < m_1$, $l < n_1$ and $\gcd(m_1, n_1) = 1$ yields that x' is the product of two coprime integers greater than 1). Hence, each factor in (2) is associated with 1, by Lemma 2. Therefore, $\operatorname{res}(U_m, U_n)$ is associated with 2^{mn}, and we obtain the following theorem.

Theorem 4 (See [JRT, Theorem 5.6]) *For $m, n \geq 1$ we have*

$$\operatorname{res}(U_m, U_n) = \begin{cases} 0 & \gcd(m+1, n+1) > 1,\\ (-1)^{mn/2}2^{mn} & \text{otherwise.}\end{cases}$$

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