ON SOME ASYMPTOTIC PROPERTIES CONCERNING HOMOGENEOUS DIFFERENTIAL PROCESSES

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1. Introduction. About the behaviour of brownian motion at time point ∞ there are many results by P. Lévy and A. Khintchine etc. The method of W. Feller[1] is applicable to a similar discussion about a homogeneous differential process. In this paper we shall study, applying his method, the properties of a homogeneous differential process.

Let $\{X(t, \omega);\ 0 \leqq t < \infty,\ \omega \in \Omega\}$[2] be a homogeneous differential process such that $E(X(t)) = mt$ and $V(X(t)) = \sigma^2 t$.[3] After P. Lévy we shall define the concept of upper class and lower class with respect to a homogeneous differential process as follows: if the set of t such that

$$X(t, \omega) > \sigma\sqrt{t}\,\phi(t)$$

is bounded (unbounded) for almost all ω, then we say that $\phi(t)$ belongs to the upper (lower) class with respect to $\{X(t);\ 0 \leqq t < \infty\}$. Then we may prove the following three theorems. In these theorems, the distribution function of $X(t)$ is denoted by $V_t(x)$.

THEOREM 1. *Let $\{X(t, \omega);\ 0 \leqq t < \infty,\ \omega \in \Omega\}$ be a right continuous*[4] *homogeneous differential process satisfying the following conditions:*

$$(1) \qquad \int_{|x-m|<z} |x-m|^3 dV_1(x) = O(z(\log\log z)^{-1/2}) \quad \textit{as} \quad z \to \infty.$$

For any $\varepsilon > 0$,

$$(2) \qquad \int_{|x-m|>z} (x-m)^2 dV_1(x) = O((\log\log z)^{-(2+\varepsilon)})$$

or

$$(2)' \qquad \int_{|x-m|>z} (x-m)^2 dV_1(x) = o((\log\log z)^{-2}) \quad \textit{as} \quad z \to \infty.$$

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[1] W. Feller: "The law of the iterated logarithm for identically distributed random variables." Ann. of Math. vol. **47** (1946).

[2] ω is the probability parameter.

[3] The symbols E and V denote the expectation and the variance respectively.

[4] This is not an essential restriction.

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There exist two positive numbers α *and* N *such that, for* $0 \leqq t \leqq \alpha$[5] *and* $0 < N \leqq b - a$,

$$\int_{a \leqq |x - mt| < b} dV_t(x) = O\Big(t \int_{a \leqq |x-m| < b} dV_1(x)\Big) \quad \text{as} \quad a \to \infty, \tag{3}$$

uniformly in t.

Then a monotone non-decreasing right continuous function $\phi(t)$ *belongs to the upper (lower) class if, and only if,*

$$\int^{\infty} \frac{1}{t} \phi(t) e^{-\frac{1}{2}\phi^2(t)} dt \in \mathfrak{C}(\mathfrak{D}).^{6)} \tag{4}$$

Example 1. For a Poisson process, the conditions (1), (2) and (3) are well satisfied.

Example 2. For a process of Pearson type, that is, a differential process $\{X(t, \omega);\ 0 \leqq t < \infty\}$ such that

$$P_r\{X(t, \omega) \leqq x\} = \begin{cases} \int_0^x \frac{e^{-y} y^{t-1}}{\Gamma(t)} dy & \text{if} \quad x \geqq 0 \\ 0 & \text{otherwise,} \end{cases}$$

the conditions (1), (2) and (3) are well satisfied.

THEOREM 2. *Let* $\{X(t, \omega);\ 0 \leqq t < \infty,\ \omega \in \Omega\}$ *be a right continuous homogeneous differential process with symmetric distribution function* $V_t(x)$. *Then, in Theorem* 1, *we may remove the assumption* (1) *and palliate* (2) *as follows:*

$$\int_{|x-m| > z} (x-m)^2 dV_1(x) = O((\log \log z)^{-1}) \tag{2''}$$

Example 3. For a Gaussian process, the conditions (2)″ and (3) are well satisfied.

THEOREM 3. *Let* $\{X(t, \omega);\ 0 \leqq t < \infty,\ \omega \in \Omega\}$ *be a right continuous homogeneous differential process. If* $E((X(t) - mt)^4)$ *is finite, the criterion* (4) *is valid.*

2. Proofs. Without loss of generality we may assume that $m = 0$ and $\sigma = 1$.

LEMMA 1. *Let* $\phi(t)$ *be a monotone non-decreasing right continuous function. If* $\phi(t)$ *does not belong to the upper class, then there exists a monotone increasing sequence* $\{t_k\}$ *such that* $\{\phi_k = \phi(t_k)\}$ *does not belong to the upper class with respect to* $\{X_k;\ X_k = X(t_k) - X(t_{k-1})\}$.

Proof. Let $\phi(t)$ be a function which does not belong to the upper class. Then there exists a set $\Omega^* \subseteqq \Omega$ with positive probability such that, for any $T > 0$,

[5] We may assume $\alpha \leqq 1$ without losing generality.

[6] $\in \mathfrak{C}(\mathfrak{D})$ denotes the convergence (divergence) of the integrals.

there exists $t(\omega) > T$ such as

(5) $$X(t, \omega) > \sqrt{t}\,\phi(t) \quad \text{when} \quad \omega \in \Omega^*.$$

Since $X(t, \omega)$ is right continuous in t, we have

(6) $$X(r, \omega) > \sqrt{r}\,\phi(r) \quad \text{when} \quad \omega \in \Omega^*,$$

with a rational number $\gamma(\omega)(> T)$. Let us put

$$P_r(\Omega^*) = c > 0.$$

Let $\{r_i\}$ be the set of all rational numbers. We shall define $\Omega_n^{(1)}$ as follows:

$$\Omega_n^{(1)} = \{\omega \in \Omega^* \ ; \ {}^\exists r_i \leqq n,\ X(r_i, \omega) > \sqrt{r_i}\,\phi(r_i)\} \quad (n = 1, 2, \ldots),$$

where "${}^\exists r_i \leqq n$" means that there exists at least one r_i which does not exceed n. Then we have, by (6) (with exception of the set of zero measure),

(7) $$\bigcup_n \Omega_n^{(1)} = \Omega^*, \quad \Omega_1^{(1)} \subseteqq \Omega_2^{(1)} \subseteqq \ldots \subseteqq \Omega_n^{(1)} \subseteqq \ldots .$$

Hence, for any $\varepsilon > 0$, we may take n_1 such as

(8) $$P_r(\Omega_{n_1}^{(1)}) \geqq c - \varepsilon/2.$$

Let us put

(9) $$\Omega_{r_i} = \{\omega \in \Omega^* \ ; \ X(r_i, \omega) > \sqrt{r_i}\,\phi(r_i)\}.$$

Then we have

$$\bigcup_{r_i \leqq n} \Omega_{r_i} = \Omega_{n_1}^{(1)},$$

so that, if i_1 is sufficiently large, we obtain

(10) $$P_r(\bigcup_{i \leqq i_1} \Omega_{r_i}) \geqq P_r(\Omega_{n_1}^{(1)}) - \varepsilon/2 \geqq c - \varepsilon.$$

Rearranging $\{r_i \ ; \ i \leqq i_1\}$ according to the order of magnitude, we obtain the set $\{t_1, \ldots, t_{i_1}\}$. Again we shall adopt the following definition:

(11) $$\Omega_n^{(2)} = \begin{cases} \{\omega \in \Omega^*;\ {}^\exists r_i,\ \max(i_1, t_{i_1}) < r_i \leqq n \\ \qquad X(r_i, \omega) > \sqrt{r_i}\,\phi(r_i)\} & \text{and if } n > \max(i_1, t_{i_1}), \\ \text{empty set} & \text{otherwise.} \end{cases}$$

Then by (6)

$$\bigcup_n \Omega_n^{(2)} = \Omega^*, \quad \Omega_1^{(2)} \subseteqq \Omega_2^{(2)} \subseteqq \ldots \subseteqq \Omega_n^{(2)} \subseteqq \ldots .$$

Accordingly there exists n_2 such that

$$P_r(\Omega_{n_2}^{(2)}) \geqq c - \varepsilon^2/2$$

and

$$\bigcup_{\max(t_1, t_{i_1}) < r_i \leqq n_2} \Omega_{r_i} = \Omega_{n_2}^{(2)}.$$

Therefore, if i_2 is sufficiently large, we have

$$(12) \qquad P_r\Big(\bigcup_{i_1<i\leqq i_2} \Omega_{r_i}\Big) \geqq P_r(\Omega_{n_2}^{(2)}) - \varepsilon^2/2 \geqq c - \varepsilon^2.$$

By the same method as in the previous discussion we have a monotone sequence $\{t_{i_1+1}, \ldots, t_{i_2}\}$. Repeating this process, we have a monotone sequence such that

$$(13) \qquad t_1 < t_2 < \ldots < t_{i_1} < \ldots < t_{i_j} < t_{i_j+1} \ldots < t_{i_{j+1}} < \ldots, \qquad (t_i \to \infty \quad \text{as} \quad i \to \infty)$$

and

$$P_r\Big(\bigcup_{i_{j-1}<i\leqq i_j} \Omega_{t_i}\Big) \geqq c - \varepsilon^j.$$

Hence, if $\varepsilon < c/2$, we obtain

$$(14) \qquad P_r\Big(\bigcap_j \bigcup_{i_{j-1}<i\leqq i_j} \Omega_{t_i}\Big) \geqq c - (\varepsilon + \varepsilon^2 + \ldots + \varepsilon^n + \ldots) = c - \frac{\varepsilon}{1-\varepsilon} > 0.$$

(9) and (14) show that $\{\phi_i\}$ does not belong to the upper class with respect to $\{X_i\}$.

According to the following lemma which will be proved after the method of W. Feller, we can exchange in Lemma 1 the condition "$\{\phi_k\}$ does not belong to the upper class" by the condition "$\{\phi_k\}$ belongs to the lower class."

LEMMA 2. *Let the conditions in Theorem* 1 *be satisfied. Let* $\{t_k\}$ *be a monotone increasing sequence such that* $t_k \to \infty$ (*as* $k \to \infty$) *and* $t_k - t_{k-1} \leqq \alpha \leqq 1$. *Then the monotone increasing sequence* $\{\phi_k = \phi(t_k)\}$ *belongs to the upper* (*lower*) *class with respect to* $\{X_k\,;\ X_k = X(t_k) - X(t_{k-1})\}$ *if, and only if,*

$$(15) \qquad \sum_k \frac{t_k - t_{k-1}}{t_k} \phi_k e^{-\frac{1}{2}\phi_k^2} \in \mathfrak{C}(\mathfrak{D}).$$

Theorem 1 is a simple corollary to Lemma 1 and Lemma 2.

Proof of Theorem 1.

a) The case of convergence. Let us suppose that $\phi(t)$ does not belong to the upper class. Then, according to Lemma 1 and Lemma 2, there exists a monotone increasing sequence $\{t_k\}$ such that $t_k - t_{k-1} \leqq \alpha$ and $\{\phi_k = \phi(t_k)\}$ belongs to the lower class with respect to $\{X_k\,;\ X_k = X(t_k) - X(t_{k-1})\}$. Hence by Lemma 2

$$\sum_k \frac{t_k - t_{k-1}}{t_k} \phi_k e^{-\frac{1}{2}\phi_k^2} \in \mathfrak{D}.$$

On the other hand, by the monotony of $\phi(t)$ and the assumption of convergence,

$$\sum_k \frac{t_k - t_{k-1}}{t_k} \phi_k e^{-\frac{1}{2}\phi_k^2} \leqq \sum_k \int_{t_{k-1}}^{t_k} \frac{1}{t} \phi(t) e^{-\frac{1}{2}\phi^2(t)}\, dt$$

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$$= \int^{\infty} \frac{1}{t} \phi(t) e^{-\frac{1}{2}\phi^2(t)} dt \in \mathfrak{C}.$$

This is a contradicition. So $\phi(t)$ must belong to the upper class.

b) The case of divergence. Let us consider the monotone increasing sequences $\{t_k = k\alpha\}$ and $\{\phi_k = \phi(t_k)\}$. Then we have

$$\int^{\infty} \frac{1}{t} \phi(t) e^{-\frac{1}{2}\phi^2(t)} dt = \sum_k \int_{t_{k-1}}^{t_k} \frac{1}{t} \phi(t) e^{-\frac{1}{2}\phi^2(t)} dt$$
$$\leqq \sum_k \frac{t_k - t_{k-1}}{t_{k-1}} \phi_{k-1} e^{-\frac{1}{2}\phi^2_{k-1}}$$
$$= \sum_k \frac{t_k - t_{k-1}}{t_k} \phi_k e^{-\frac{1}{2}\phi^2_k}.$$

Thus the divergence of the integrals yields that of the series (15). Therefore, by Lemma 2, $\{\phi_k\}$ must belong to the lower class and accordingly $\phi(t)$ belongs to the lower class with respect to the process $\{X(t, \omega)\}$.

Now our purpose is to prove Lemma 2. We put

$$\eta_k^2 = \begin{cases} t_k(\log\log t_k)^{-3} & \text{for } t_k > 80, \\ \text{arbitrary in such a way that } \{\eta_k\} \text{ becomes a} & \\ \text{monotone increasing sequence} & \text{for } t_k \leqq 80. \end{cases} \tag{16}$$

Furthermore we put

$$P_r\{X_k \leqq x\} = F_k(x), \tag{17}$$

$$b_k = \int_{|x| < \eta_k} x^2 dF_k(x), \tag{18}$$

$$B_n = \sum_{k=1}^n b_k, \tag{19}$$

$$\mu_k' = -\int_{|x| < \eta_k} x dF(x), \quad \mu_k'' = -\int_{\eta_k \leqq |x| < t_k^{1/2}} x dF_k(x), \tag{20}$$

$$\mu_k''' = -\int_{t_k^{1/2} \leqq |x|} x dF_k(x)$$

and

$$\sigma_k^2 = b_k - \mu_k'^2, \quad s_n^2 = \sum_{k=1}^n \sigma_k^2. \tag{21}$$

We shall introduce three new sequences of random variables as follows;

$$X_k' = \begin{cases} X_k + \mu_k' & \text{if } |X_k| < \eta_k \\ \mu_k' & \text{otherwise,} \end{cases} \tag{22}$$

$$X_k'' = \begin{cases} X_k + \mu_k'' & \text{if } \eta_k \leqq |X_k| < t_k^{1/2} \\ \mu_k'' & \text{otherwise,} \end{cases} \tag{23}$$

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$$X_k''' = \begin{cases} X_k + \mu_k''' & \text{if} \quad |X_k| \geqq t_k^{1/2} \\ \mu_k''' & \text{otherwise.} \end{cases} \tag{24}$$

Then we have

$$X_k = X_k' + X_k'' + X_k''', \tag{25}$$

and the variables of each of the three sequences are mutually independent. Moreover

$$E(X_k) = E(X_k') = E(X_k'') = E(X_k''') = 0 \tag{26}$$

and

$$V(X_k') = \sigma_k^2. \tag{27}$$

If we define S_n' as follows

$$S_n' = X_1' + X_2' + \ldots + X_n', \tag{28}$$

(the sums S_n'' and S_n''' are defined similarly), then we have

$$V(S_n') = s_n^2. \tag{29}$$

LEMMA 3. *With probability one*

$$S_n''' = O(t_n^{1/2}(\log\log t_n)^{-1/2}). \tag{30}$$

Proof. From the assumption (3) we have

$$\begin{aligned}\sum_k P_r\{X_k''' \neq \mu_k'''\} &= \sum_k \int_{|x| \geqq t_k^{1/2}} dF_k(x) = O(1)\sum_k (t_k - t_{k-1}) \int_{|x| \geqq t_k^{1/2}} dV_1(x) \\ &= O(1)\sum_{k=1}^{\infty}\sum_{j=k}^{\infty}(t_k - t_{k-1})\int_{t_j^{1/2} \leqq |x| < t_{j+1}^{1/2}} dV_1(x) \\ &= O(1)\sum_{j=1}^{\infty}\sum_{k=1}^{j}(t_k - t_{k-1})\int_{t_j^{1/2} \leqq |x| < t_{j+1}^{1/2}} dV_1(x) \\ &= O(1)\sum_{j=1}^{\infty} t_j \int_{t_j^{1/2} \leqq |x| < t_{j+1}^{1/2}} dV_1(x) = O(1)\int_{-\infty}^{\infty} x^2 dV_1(x) < \infty.\end{aligned}$$

Thus, by Borel-Cantelli's lemma, it follows that with probability one there will be only finitely many k such that $X_k''' \neq \mu_k'''$. So, by the assumptions (1) and (2), we have

$$\begin{aligned}|S_n'''| &= O(1)\{1 + |\sum_{k=1}^{n}\int_{|x| \geqq t_k^{1/2}} x dF_k(x)|\} = O(1)\{1 + \sum_{k=1}^{n}(t_k - t_{k-1})\int_{|x| \geqq t_k^{1/2}} |x| dV_1(x)\} \\ &= O(1)\{1 + \sum_{k=1}^{n}\sum_{j=k}^{\infty}(t_k - t_{k-1})\int_{t_j^{1/2} \leqq |x| < t_{j+1}^{1/2}} |x| dV_1(x)\} \\ &= O(1)\{1 + \sum_{j=1}^{n-1}\sum_{k=1}^{j}(t_k - t_{k-1})\int_{t_j^{1/2} \leqq |x| < t_{j+1}^{1/2}} |x| dV_1(x) \\ &\qquad + \sum_{j=n}^{\infty}\sum_{k=1}^{n}(t_k - t_{k-1})\int_{t_j^{1/2} \leqq |x| < t_{j+1}^{1/2}} |x| dV_1(x)\}\end{aligned}$$

$$= O(1)\{1+\sum_{j=1}^{n-1} t_j \int_{t_j^{1/2} \leqq |x| < t_{j+1}^{1/2}} |x| dV_1(x) + \sum_{j=n}^{\infty} t_n \int_{t_j^{1/2} \leqq |x| < t_{j+1}^{1/2}} |x| dV_1(x)\}$$

$$= O(1)\{1+\int_{|x|<t_n^{1/2}} |x|^3 dV_1(x) + t_n \int_{t_n^{1/2} \leqq |x|} |x| dV_1(x)\}$$

$$= O(1)\left(\frac{t_n}{\log\log t_n}\right)^{1/2}$$

This proves the lemma.

LEMMA 4. *With probability one*

$$S_n'' = O((t_n \log\log\log t_n)^{1/2}). \tag{31}$$

Proof. According to a theorem of L. Kronecker,[7] it will be sufficient to prove that the series

$$\sum_n \frac{1}{(t_n \log\log\log t_n)^{1/2}} X_n'' \tag{32}$$

converges with probability one. By a theorem of Khintchine and Kolmogoroff,[8] it is sufficient to show that

$$\sum_n \frac{1}{t_n \log\log\log t_n} E(X_n''^2) \in \mathfrak{C}. \tag{33}$$

To prove (33) we shall consider the following function

$$S(n) = \min_{k \in T_n} k, \quad T_n = \{k \,;\, \frac{t_k^{1/2}}{(\log\log t_k)^{3/2}} > t_n^{1/2}\}. \tag{34}$$

Obviously $S(n)$ is monotone non-decreasing. Hence we can define the inverse function of $S(n)$ as follows

$$S^{-1}(n) = \min_{S(l) \geqq n} l. \tag{35}$$

By the definition (23), we obtain

$$\sum_n \frac{1}{t_n \log_{(3)} t_n} E(X_n''^2) \leqq \sum_n \frac{1}{t_n \log_{(3)} t_n} \int_{\eta_n \leqq |x| < t_n^{1/2}} x^2 dF_n(x)$$

$$= O(1) \sum_n \frac{t_n - t_{n-1}}{t_n \log_{(3)} t_n} \int_{\eta_n \leqq |x| < t_n^{1/2}} x^2 dV_1(x)$$

$$= O(1) \sum_n \frac{t_n - t_{n-1}}{t_n \log_{(3)} t_n} \sum_{k=n}^{S(n)} \int_{\eta_k \leqq |x| < \eta_{k+1}} x^2 dV_1(x)$$

$$= O(1) \sum_k \int_{\eta_k \leqq |x| < \eta_{k+1}} x^2 dV_1(x) \sum_{n=S^{-1}(k)}^{k} \frac{t_n - t_{n-1}}{t_n \log_{(3)} t_n}$$

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7) K. Knopp: Theorie und Anwendung der Unendlichen Reihen, 2 ed., Berlin, 1924, p. 127.

8) A. Kolmogoroff: Grundbegriffe der Wahrscheinlichkeitsrechung, Berlin, 1933, p. 59.

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$$= O(1)\sum_k \int_{\eta_k \leqq |x| < \eta_{k+1}} x^2 dV_1(x) < \infty,$$

where $\log_{(k)}$ denotes the k-times iterated logarithm. This proves the lemma.

LEMMA 5. *For any $\delta > 0$ the probability is zero that there exist infinite many n for which the inequalities*

$$S_n'' > \delta t_n^{1/2} / (\log\log t_n)^{1/2} \tag{36}$$

and

$$S_n' > \delta(t_n \log\log t_n)^{1/2} \tag{37}$$

hold simultaneously.

Proof. Let us denote by A_n the event that there exists at least one t_k such that

$$10\, t_n \log_{(2)}^{-3} t_n < t_k \leqq t_n \quad \text{and} \quad X_k'' \neq \mu_k'', \tag{38}$$

and by $\bar{A}_n$ its complementary event. Choosing m for which $[t_m] = [10\, t_n \log_{(2)}^{-3} t_n]$[9] holds, we have

$$\begin{aligned}
\left|\sum_{k=m}^{n} \mu_k''\right| &= O(1)\sum_{k=m}^{n} (t_k - t_{k-1}) \int_{\eta_k \leqq |x| < t_k^{1/2}} |x|\, dV_1(x) \\
&= O(1)\sum_{k=m}^{n} (t_k - t_{k-1}) \frac{1}{\eta_k (\log_{(2)} \eta_k)^{2+\varepsilon}} \\
&= O(1)\sum_{k=m}^{n} \frac{t_k - t_{k-1}}{t_k^{1/2} (\log_{(2)} t_k)^{1/2+\varepsilon}} \\
&= O(1) \frac{t_n^{1/2}}{(\log\log t_n)^{1/2+\varepsilon}}.
\end{aligned}$$

Accordingly, if $\bar{A}_n$ occurs, then we have by Lemma 4

$$\begin{aligned}
S_n'' &= S_m'' + (S_n'' - S_m'') = O((t_m \log_{(3)} t_m)^{1/2}) + O\left(\frac{t_n^{1/2}}{(\log\log t_n)^{1/2+\varepsilon}}\right) \\
&= o(t_n^{1/2} / (\log\log t_n)^{1/2}).
\end{aligned}$$

This excludes (36). Therefore, for sufficiently large n, the event (36) will occur only in conjunction with the event A_n with probability one. Let B_n denote the event of a simultaneous realization of (37) and A_n. It suffices to prove that the probability that B_n occurs for infinitely many n is zero. To this purpose, we consider the event

$$C_\nu = \sum_{e^{\nu-1} < t_m \leqq e^\nu} B_n \tag{39}$$

which implies the realization of at least one B_n with

[9] $[x]$ denotes the largest integer which does not exceed x.

(40) $$e^{\nu-1} < t_n \leqq e^{\nu}.$$

Our lemma will be proved if we show that

(41) $$\sum_{\nu} P_r(C_\nu) < \infty.$$

Put

(42) $$P_\nu = \sum_{e^\nu \log^{-3}\nu < t_k \leqq e^\nu} P_r\{X_k'' \neq \mu_k''\}.$$

Then we obtain

(43) $$\sum_{\nu} \frac{P_\nu}{(\log \nu)^{100}} < \infty$$ [10]

and

(44) $$P_r(C_\nu) = O(1) P_\nu / (\log \nu)^{100}$$ [11]

Accordingly the series (41) converges.

LEMMA 6. *For any monotone increasing sequence* $\{\phi_n\}$ *the divergence (convergence) of the series* (15) *is a necessary and sufficient condition that with probability one the inequality*

(45) $$\sum_{k=1}^{n} X_k > B_n^{1/2} \phi_n$$

be satisfied for infinitely (only finitely) many n.

Proof. Without loss of generality, we may assume that

(46) $$\log\log t_n \leqq \phi_n^2 \leqq 4 \log\log t_n.$$ [12]

If a and b are sufficiently large and $b - a \geqq N$, then we have by the assumption (3)

$$\int_a^b x^2 dF_n(x) = O(1)(t_n - t_{n-1}) \int_a^b x^2 dV_1(x).$$

So we have

$$\begin{aligned} \sigma_n^2 = b_n - \mu_n'^2 &= \int_{|x|<\eta_n} x^2 dF_n(x) - \left(\int_{|x|<\eta_n} x dF_n(x) \right)^2 \\ &= t_n - t_{n-1} - \int_{|x|\geqq\eta_n} x^2 dF_n(x) - \left(\int_{|x|<\eta_n} x dF_n(x) \right)^2 \\ &= t_n - t_{n-1} - O(1)(t_n - t_{n-1})(\log\log t_n)^{-(2+\varepsilon)}. \end{aligned}$$

Thus $t_n - t_{n-1}/\sigma_n^2 \to 1$ and therefore $t_n/s_n^2 \to 1$ as $n \to \infty$. So the divergence (convergence) of (15) is equivalent to

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10) 11) 12) loc. cit. 1).

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(47) $$\sum_n \frac{\sigma_n^2}{s_n^2}\phi_n e^{-\frac{1}{2}\phi_n^2} \in \mathfrak{D}(\mathfrak{L}).$$

According to a theorem of W. Feller,[13] (47) implies that with probability one there are infinitely (only finitely) many n such that

(48) $$S_n' > s_n(\phi_n + c/\phi_n),$$

where c is an arbitrary constant. From the definition (19) and (21), we have

$$B_n - s_n^2 = \sum_{k=1}^{n} \mu_k'^2$$

and

$$|\mu_k'| = \left|\int_{|x|\geqq\eta_k} x dF_k(x)\right| \leqq \frac{1}{\eta_k}\int_{|x|\geqq\eta_k} x^2 dF_k(x) \leqq \frac{t_k - t_{k-1}}{\eta_k}.$$

Hence $B_n - s_n^2 = O((\log t_n)^2)$ and we may take $B_n^{1/2}$ for s_n in (48), so we have

$$S_n' > B_n^{1/2}(\phi_n + c/\phi_n).$$

Hence, using Lemma 3 and Lemma 5, the divergence of (15) yields that with probability one there exist infinitely many n for which the ineqalities

(49) $$S_n' > B_n^{1/2}(\phi_n + c/\phi_n)$$

and

(50) $$|S_n'' + S_n'''| < M(t_n/\log\log t_n)^{1/2},$$

where c is an arbitrary constant and M is a sufficiently large number, hold simultaneously. Let us put $c = 2M$ in (49). Then we see that with probability one there exist infinitely many n such that

(51) $$\sum_{k=1}^{n} X_k = S_n' + S_n'' + S_n''' > B_n^{1/2}\phi_n.$$

Conversely if (51) holds for infinitely many n with probability one, it follows, by (30) and (31), that with probability one

$$S_n' > \frac{1}{2}B_n^{1/2}\phi_n$$

for infinitely many n appearing in (51). From Lemma 5, it follows that with probability one there exist infinitely many n for which (50) and (51) hold simultaneously, so that we have

$$S_n' > B_n^{1/2}(\phi_n - 2M/\phi_n) > s_n(\phi_n - 2M/\phi_n).$$

This means that $\{\phi_n - 2M/\phi_n\}$ belongs to the lower class with respect to $\{X_n'\}$.

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13) W. Feller: "The general form of the so-called law of the iterated logarithm." Trans. Amer. Math. Soc. vol. **54** (1943), pp. 373-402.

Then, by a theorem of W. Feller,[14] we have

$$\frac{\sigma_n^2}{s_n^2}(\phi_n - 2M/\phi_n)e^{-\frac{1}{2}(\phi_n - 2M/\phi_n)^2} \in \mathfrak{D},$$

and accordingly

$$\sum_n \frac{\sigma_n^2}{s_n^2}\phi_n e^{-\frac{1}{2}\phi_n^2} \in \mathfrak{D}.$$

This is equivalent to the divergence of (15).

Now Lemma 2 will be proved easily.

Proof of Lemma 2. If the series (15) diverges, then it is clear, from Lemma 6, that for any constant c

$$\sum_{k=1}^{n} X_k > B_n^{1/2}(\phi_n + c/\phi_n)$$

will be satisfied for infinitely many n with probability one. Therefore it is sufficient to show that

(52) $$t_n^{1/2} - B_n^{1/2} = O(t_n^{1/2}/\phi_n^2)$$

or, by (46),

(53) $$t_n - B_n = O(t_n/\log\log t_n).$$

But we have

$$t_n - B_n = \sum_{k=1}^{n}\int_{|x|\geqq \eta_k} x^2 dF_k(x) = O(1)\sum_{k=1}^{n}(t_k - t_{k-1})\int_{|x|\geqq \eta_k} x^2 dV_1(x)$$
$$= O(1)(t_n/\log\log t_n + t_n\int_{|x|\geqq t_n^{1/2}/\log_{(2)}^2 t_n} x^2 dV_1(x))$$

(the first term on the right is the contribution of the terms in the sum with $t_k < t_n/\log\log t_n$, and the integral is an upper bound for the contribution of the remaining terms). Hence, by the assumption (2), we have

$$t_n - B_n = O(t_n/\log\log t_n).$$

The converse is trivial.

Proof of Theorem 2. In the proof of Theorem 1, the condition (1) was used to evaluate

$$\sum_{k=1}^{n}\int_{|x|\geqq t_k^{1/2}} x dF_k(x) = O(t_n/\log\log t_n)^{1/2}.$$

But this is equal to zero in our case. Also the condition (2) was used to evaluate

[14] loc. cit. 13).

$$\left|\sum_{k=1}^{n}\mu_k''\right| = O(t_n/(\log\log t_n)^{1+2\varepsilon})^{1/2}$$

and

$$t_n - B_n = O(t_n/\log\log t_n).$$

In our case the former is equal to zero and for the latter the condition $(2)''$ is sufficient. These prove Theorem 2.

Proof of Theorem 3. Let $t = q/p$ be a rational number. Then we have

$$\begin{aligned} E((X(t))^4) &= \int_{-\infty}^{\infty} x^4 dV_t(x) = \int (X(q/p))^4 P(d\omega) \\ &= \int\{(X(1/p) - X(0)) + (X(2/p) - X(1/p)) \\ &\quad + \ldots + (X(q/p) - X(q-1/p))\}^4 P(d\omega) \\ &= q\int_{-\infty}^{\infty} x^4 dV_{1/p}(x) + 3q(q-1)/p^2. \end{aligned}$$

Put $p = q$ and $E((X(1))^4) = a$. Then we obtain

$$a = E((X(1))^4) = p\int_{-\infty}^{\infty} x^4 dV_{1/p}(x) + 3(p-1)/p,$$

and accordingly

$$\int_{-\infty}^{\infty} x^4 dV_{1/p}(x) = (a - 3(p-1)/p)/p.$$

Hence

$$\begin{aligned} E((X(t))^4) &= E((X(q/p))^4) = a\,q/p - 3(1-q/p)\,q/p \\ &= at - 3t(1-t). \end{aligned} \tag{54}$$

Since $E((X(t))^4)$ is a monotone increasing function of t, (54) holds for any real number t.

Let us consider the sequence $\{X_k\}$ in Lemma 2. Using the notations in the previous proofs we have

$$\begin{aligned} P_r\{|X_k| \geqq \eta_k\} &= \int_{|x|\geqq\eta_k} dF_k(x) \\ &\leqq \frac{1}{\eta_k^4}\int_{-\infty}^{\infty} x^4 dF_k(x) < a(t_k - t_{k-1})/\eta_k^4, \end{aligned}$$

and so

$$\sum_k P_r\{|X_k| \geqq \eta_k\} \leqq a\sum_k (t_k - t_{k-1})(\log\log t_k)^6/t_k^2 \in \mathfrak{C}.$$

According to Borel-Cantelli's lemma, it follows that with probability one there will be only finitely many k such that $|X_k| \geqq \eta_k$.

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Put

$$X_k' = \begin{cases} X_k + \int_{|x| \geqq \eta_k} x dF_k(x) & \text{if } |X_k| < \eta_k \\ \int_{|x| \geqq \eta_k} x dF_k(x) & \text{otherwise.} \end{cases}$$

Then we have

$$E(X_k') = 0, \quad V(X_k') = t_k - t_{k-1} + O(1)(t_k - t_{k-1})/\eta_k^2$$

and so

$$t_n - \sum_{k=1}^{n} V(X_k') = O((\log t_n)^2).$$

On the other hand, we heve

$$\sum_k \left| \int_{|x| \geqq \eta_k} x dF_k(x) \right| \leqq a \sum_k (t_k - t_{k-1})/\eta_k^3 \in \mathfrak{C}.$$

Hence

$$\sum_{k=1}^{n} X_k = \sum_{k=1}^{n} X_k' + O(1).$$

By a theorem of W. Feller[15] the criterion (4) is valid for $\{X_k'\}$ and so for $\{X_k\}$. Thus we may apply Lemma 2, and Theorem 3 will be proved similarly as in the previous proof of Theorem 1.

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15) loc. cit. 13).

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