LOCALLY SOLUBLE GROUPS WITH MIN-*n*.

Dedicated to the memory of Hanna Neumann

H. HEINEKEN and J. S. WILSON*

(Received 29 June 1972)

Communicated by M. F. Newman

It was shown by Baer in [1] that every soluble group satisfying Min-*n*, the minimal condition for normal subgroups, is a torsion group. Examples of non-soluble locally soluble groups satisfying Min-*n* have been known for some time (see McLain [2]), and these examples too are periodic. This raises the question whether all locally soluble groups with Min-*n* are torsion groups. We prove here that this is not the case, by establishing the existence of non-trivial locally soluble torsion-free groups satisfying Min-*n*. Rather than exhibiting one such group G, we give a general method for constructing examples; the reader will then be able to see that a variety of additional conditions may be imposed on G. It will follow, for instance, that G may be a Hopf group whose normal subgroups are linearly ordered by inclusion and are all complemented in G; further, that the countable groups G with these properties fall into exactly $2^{\aleph_0}$ isomorphism classes. Again, there are exactly $2^{\aleph_0}$ isomorphism classes of countable groups G which have hypercentral non-nilpotent Hirsch-Plotkin radical, and which at the same time are isomorphic to all their non-trivial homomorphic images.

As a by-product, we shall also show the existence of locally soluble torsion-free groups which are characteristically simple and whose proper non-trivial normal subgroups are linearly ordered by inclusion, with the order type $\boldsymbol{Z}$ of the integers.

1. Treble products

1.1 Our results depend on some properties of the *treble product* of three groups. This is a particular case of the twisted wreath product introduced by Neumann in [3]; however, in order to make our arguments clearer, we use here a

* This work was done while the second author held a Royal Society European Programme Fellowship at the University of Erlangen-Nürnberg.

different notation from that adopted by Neumann. Suppose we are given three groups A, B and C and homomorphisms

$$s : B \to \mathrm{Aut}\, A \qquad \text{and} \qquad t : C \to \mathrm{Aut}\, B,$$

where Aut X denotes the automorphism group of X. Let $W = A \wr C$ be the (standard, restricted) wreath product of A and C. We form the free product F of W and B, and write T for the quotient group of F by the normal subgroup K generated by all $b^{t(c)}c^{-1}b^{-1}c$ and all $a^{s(b)}b^{-1}a^{-1}b$, where a, b and c run through the elements of A, B and C respectively. Then we call T the treble product of A, B and C, and we denote it by $\mathrm{Tr}(A, B, C; s, t)$ or, suppressing reference to s and t, by $\mathrm{Tr}(A, B, C)$.

Obviously $W \cap K = B \cap K = 1$. We may therefore identify A, B and C with their images in T. It is not hard to see that the subgroups $\langle A, B\rangle$ and $\langle B, C\rangle$ of T are split extensions of A and B, and that T itself is a split extension of the normal closure D of A in T by BC; further D is the direct product of the subgroups $c^{-1}Ac$ $(c \in C)$ of T.

In the terminology of Neumann [3], T is a twisted wreath product of A by BC.

1.2 We collect here some results about normal subgroups, and in particular *minimal normal subgroups*, of the treble product T of three given groups A, B and C. In the special case $B = 1$, Lemma 1 and Lemma 3 reduce to well known properties of wreath products.

LEMMA 1. *Suppose that $N \lhd A$ is a minimal normal subgroup of AB, and that N is not contained in the centre Z of A. Then the normal closure N^C of N in T is a minimal normal subgroup of T.*

PROOF. Let M be a normal subgroup of T such that $1 \neq M \subseteq N^C$. We must show that $N^c \subseteq M$ for some $c \in C$; it then follows immediately that $M = N^C$, and that N^C is a minimal normal subgroup of T. If $L_c = [A^c, M] \neq 1$ for some $c \in C$, then, because L_c is normal in A^cB and contained in N^c, we have $N^c = L_c \subseteq M$, as required. If $L_c = 1$ for each $c \in C$, then M is contained in the centre of A^C, and its projection in each subgroup A^c is contained both in N^c and Z^c, and therefore is trivial; thus $M = 1$, a contradiction.

It is easy to see that the conclusion of Lemma 1 need no longer hold if the condition $N \nsubseteq Z$ is deleted. However, we require for later use a criterion valid also in the case $N \subseteq Z$. One such is provided by

LEMMA 2. *Suppose that $N \lhd A$ is a minimal normal subgroup of AB. If, for every element $c \neq 1$ of C, there is a two-variable word $p_c(a, b)$ and there is an element x_c of B, such that*

(i) $p_c(1, b) = 1$ *for all* b *in* B,
(ii) $p_c(a, x_c) \neq 1$ *for all* $a \neq 1$ *in* N, *and*
(iii) $p_c(a, cx_cc^{-1}) = 1$ *for all* a *in* N,

then N^C *is a minimal normal subgroup of* T.

Proof. Let M be a normal subgroup of T such that $1 \neq M \subseteq N^C$. Again it will be enough to show that some conjugate of N is contained in M. Suppose that this is not the case, and let

$$N_1, \cdots, N_k$$

be a minimal collection of conjugates of N such that

$$Q = M \cap N_1 N_2 \cdots N_k \neq 1.$$

Then $k \geqq 2$. We may clearly assume $N_1 = N$. Let $N_k = N^c$, so that $c \neq 1$. We choose a non-trivial element q of Q, and write $q' = p_c(q, x_c)$. Then, by condition (i), q' lies in the normal closure of $\langle q \rangle$ in $\langle q, x_c \rangle$, and therefore in Q. Further, if q_i denotes the projection of q in N_i, then the projection of q' in N_i is $p_c(q_i, x_c)$, for each i. By (ii), $p_c(q_1, x_c) \neq 1$, so that $q' \neq 1$. On the other hand,

$$p_c(q_k, x_c) = c^{-1} p_c(cq_kc^{-1}, cx_cc^{-1})c = 1,$$

from condition (iii). Thus

$$1 \neq q' \in M \cap N_1 N_2 \cdots N_{k-1}.$$

But this is in contradiction to the choice of $N_1, \cdots, N_k$, and the Lemma follows.
For later convenience we include the

Remark. *Let* N *be a minimal normal subgroup of* AB, *contained in* A, *and suppose that there is an element* y *of* B *such that* $y^{-1}ny = n^2$ *for all* $n \in N$. *If* $[[y, c], n] \neq 1$ *whenever* $1 \neq c \in C$ *and* $1 \neq n \in N$, *then the conditions of Lemma 2 are satisfied with*

$$p_c(a, b) = a^{-2}b^{-1}ab, \text{ and } x_c = c^{-1}yc,$$

for all $c \neq 1$ *in* C.

Lemma 3. *Let* N *be a normal subgroup of* AB, *contained in* A, *such that* $N \cap X \neq 1$ *for every normal subgroup* $X \neq 1$ *of* AB. *Then* $N^C \cap M \neq 1$ *for every normal subgroup* $M \neq 1$ *of* T.

Proof. Let $1 \neq M \lhd T$. If $A^C B \cap M = 1$, then M centralizes A^C and so normalizes A; but since the normalizer of A is evidently $A^C B$, we have a contradiction. Therefore $M_1 = A^C B \cap M$ is a non-trivial normal subgroup of T.

We choose an element $m \neq 1$ of M_1, with, say, $m = db$, where $d \in A^C$ and $b \in B$. We may assume $d \neq 1$, for otherwise we would have $1 \neq M_1 \cap AB$ and

https://doi.org/10.1017/S1446788700017079 Published online by Cambridge University Press

$N \cap M_1 = 1$. Replacing m by a conjugate if necessary, we may further assume that the projection a of d in A is non-trivial. We write R for the direct product of all conjugates of A except A itself. Then

$$1 \neq ab \in RM_1 \cap AB \lhd AB,$$

so that $(RM_1 \cap AB) \cap N \neq 1$, and, in particular, $RM_1 \cap A \neq 1$. Let $1 \neq a' = r'm'$ with $a' \in A$, $r' \in R$ and $m' \in M_1$. We cannot have $a' = r'$, since the group generated by A and R is their direct product; therefore $m' = r'a'^{-1}$ is a non-trivial element of $A^C \cap M_1$. It follows that $M_2 = A^C \cap M_1$ is a non-trivial normal subgroup of T.

Let

$$A_1, \cdots, A_k$$

be a minimal collection of conjugates of A such that $A_1 \cdots A_k \cap M_2 \neq 1$. We denote by N_j the subgroup of A_j conjugate to N, and write

$$T_j = N_1 \cdots N_j A_{j+1} \cdots A_k \cap M_2$$

for $0 \leqq j \leqq k$ (with the obvious conventions for $j = 0$ and $j = k$). Then $T_0 \neq 1$. Suppose $j < k$ and $T_j \neq 1$. The projection of T_j in A_{j+1} is a non-trivial normal subgroup of $A_{j+1}B$, and so has non-trivial intersection with N_{j+1}; it therefore follows that $T_{j+1} \neq 1$. Thus we have

$$N^C \cap M = N^C \cap M_2 \neq 1,$$

as required.

Combining Lemmas 1, 2 and 3 we now have

LEMMA 4. *Suppose that the normal subgroup N of AB is contained in every non-trivial normal subgroup of AB and satisfies either the conditions of Lemma* 1 *or those of Lemma* 2. *Then N^C is contained in every non-trivial normal subgroup of T.*

1.3 We now show how the treble product construction may be iterated, to produce a *treble product tower*. Let ρ be an ordinal number. Suppose we are given a family $\{A_\sigma; 0 \leqq \sigma < \rho\}$ of non-trivial groups and a family $\{\theta_{\sigma+1}; 1 \leqq \sigma+1 < \rho\}$ of homomorphisms, with $\theta_{\sigma+1}$ a homomorphism from $A_{\sigma+1}$ into Aut A_σ for each σ. We define an ascending sequence $K_\sigma (\sigma \leqq \rho)$ of groups, and an auxiliary sequence $L_\sigma (\sigma < \rho)$ as follows:
$K_1 = L_1 = A_0$, and K_2 is the split extension of A_0 by A_1. If $K_{\sigma+1}$ is defined and is the split extension of a subgroup L_σ by A_σ, then

$$K_{\sigma+2} = \mathrm{Tr}(L_\sigma, A_\sigma, A_{\sigma+1}).$$

So $K_{\sigma+2}$ has $K_{\sigma+1} = L_\sigma A_\sigma$ as a subgroup, and $K_{\sigma+2}$ is a split extension of $L_{\sigma+1} = (K_{\sigma+1})^{K_{\sigma+2}}$ by $A_{\sigma+2}$. If σ is a limit ordinal and K_τ is defined for all $\tau < \sigma$, with $K_\tau \subset K_{\tau+1}$ for all τ, then we define $K_\sigma = \cup \{K_\tau; \tau < \sigma\}$, and we define $K_{\sigma+1} = K_\sigma \wr A_\sigma$, the standard wreath product of K_σ by A_σ. Then $K_\sigma \subset K_{\sigma+1}$, and $K_{\sigma+1}$ is a split extension of the base group L_σ of $K_{\sigma+1}$ by A_σ.

Thus the groups K_σ are defined for all ordinals $\sigma \leqq \rho$. The group K_ρ will be called the treble product tower of the groups A_σ $(0 \leqq \sigma < \rho)$, and it will be denoted by $\mathrm{Trt}(A_\sigma; 0 \leqq \sigma < \rho)$.

LEMMA 5. *Suppose that $\rho > 2$ and that N is a minimal normal subgroup of $K_2 = \langle A_0, A_1 \rangle$, contained in A_0. Suppose further that the hypotheses of Lemma 1 or of Lemma 2 are satisfied for N, with $A = A_0$, $B = A_1$ and $C = A_2$. Then N^{K_ρ} is a minimal normal subgroup of K_ρ.*

PROOF. For $\rho = 3$ the statement is true by Lemma 1 or Lemma 2; we therefore assume $\rho > 3$ and argue by induction on ρ. If $\rho - 2$ exists, we have

$$K_\rho = \mathrm{Tr}(L_{\rho-2}, A_{\rho-2}, A_{\rho-1}),$$

and $N^{K_{\rho-1}}$ is a non-central minimal normal subgroup of $K_{\rho-1}$. We may therefore apply Lemma 1 (with $A = L_{\rho-2}$, $B = A_{\rho-2}$ and $C = A_{\rho-1}$) to deduce that N^{K_ρ} is a minimal normal subgroup of K_ρ. If $\rho - 1$ exists and is a limit ordinal, then K_ρ is the standard wreath product of $K_{\rho-1}$ and $A_{\rho-1}$, and the result again follows from Lemma 1, with $B = 1$. Finally, if ρ is a limit ordinal, and if M is a non-trivial normal subgroup of K_ρ contained in N^{K_ρ}, then $M \cap N^{K_\sigma} \neq 1$ for some $\sigma < \rho$; and since $M \cap N^{K_\sigma}$ is a normal subgroup of K_σ contained in N^{K_σ}, we have $N^{K_\sigma} = M \cap N^{K_\sigma}$ and $N^{K_\rho} = M$. Thus N^{K_ρ} is a minimal normal subgroup of K_ρ, as required.

LEMMA 6. *Suppose that $\rho > 2$ and that N is a normal subgroup of $K_2 = \langle A_0, A_1 \rangle$, contained in A_0, such that $N \cap X \neq 1$ for all non-trivial normal subgroups X of K_2. Then $N^{K_\rho} \cap Y \neq 1$ for all non-trivial normal subgroups Y of K_ρ.*

The proof by induction on ρ using Lemma 3 is similar to the proof of Lemma 5, and we omit it.

Combining Lemma 5 and Lemma 6, we have

LEMMA 7. *Suppose that $\rho > 2$, and suppose that the minimal normal subgroup N of $A_0 A_1$ is contained in every non-trivial normal subgroup of $A_0 A_1$. If either the conditions of Lemma 1 or those of Lemma 2 are satisfied for N, with $A = A_0$, $B = A_1$ and $C = A_2$, then N^{K_ρ} is a minimal normal subgroup of K_ρ, and is contained in every non-trivial normal subgroup of K_ρ.*

2. Groups satisfying Min-*n*

2.1 L_ρ-*groups*. Every locally soluble group satisfying Min-*n* has an ascending invariant series with Abelian factors; this is a straightforward consequence of the result of McLain [4] that each minimal normal subgroup of a locally soluble group is Abelian. The first groups which we construct are locally soluble torsion-free groups G which have unique ascending invariant series with Abelian factors, of length any given limit ordinal ρ. The normal subgroups of such groups are linearly ordered by inclusion, of order type $\rho + 1$, so that, *a fortiori*, the groups satisfy Min-*n*. For brevity, we call a locally soluble group whose non-trivial normal subgroups are linearly ordered of order type $\rho + 1$ a L_ρ-group.

We begin with a general lemma concerning treble product towers.

LEMMA 8. *Let* $G = \mathrm{Trt}(A_\sigma; 0 \leqq \sigma < \rho)$, *and suppose that*

(a) A_σ *has no* $A_{\sigma+1}$*-invariant subgroups other than* A_σ *and* 1, *for each* σ *satisfying* $1 \leqq \sigma + 1 < \rho$, *and*

(b) *for each* σ *satisfying* $2 \leqq \sigma + 2 < \rho$, *the conditions of Lemma* 4 *are satisfied with* $A = N = A_\sigma$, $B = A_{\sigma+1}$ *and* $C = A_{\sigma+2}$.

Then any proper non-trivial normal subgroup of G *is either of the form* K_σ^G *for some* $\sigma < \rho$, *or, in the case when* ρ *is not a limit ordinal, contains* $L_{\rho-1}$.

PROOF. Let $1 \neq M \lhd G$. We may assume $M \subset K_\rho = G$. Let σ be the smallest ordinal with $K_\sigma \nsubseteq M$; then σ is not a limit ordinal, and, from Lemmas 5 and 6, $\sigma > 1$. Thus $\sigma = \tau + 1$ for an ordinal τ, and $K_\tau^G \subseteq M$. But G is a split extension of K_τ^G by $H = \langle A_\phi; \tau < \phi < \rho \rangle$, which is itself a treble product tower. If $\tau + 2 \leqq \rho$, then, again by Lemmas 5 and 6, every non-trivial normal subgroup of H contains $A_{\tau+1} = A_\sigma$; thus by choice of σ, we must have $K_\tau^G = M$. Otherwise, $\sigma = \tau + 1 = \rho$, and M contains both $K_{\rho-1}$ and its normal closure $L_{\rho-1}$. The proof of Lemma 8 is complete.

We now suppose in Lemma 8 that ρ is a limit ordinal and that each subgroup A_σ is Abelian. Then each subgroup generated by finitely many subgroups A_σ is an iterated extension of Abelian groups, and so is soluble; and it follows that G is locally soluble. Thus, from the Lemma, G is an L_ρ-group. If each subgroup A_σ is torsion-free, so also is G. Further since A_σ is a faithful module for $A_{\sigma+1}$ for every σ, each minimal normal subgroup of a non-Abelian quotient group of G will coincide with its centralizer. Thus, in order to exhibit the existence of torsion-free L_ρ-groups with unique ascending series with Abelian factors, it will be enough to show how we may choose sequences $(A_\sigma; 0 \leqq \sigma < \rho)$ of Abelian torsion-free groups, with A_σ a faithful irreducible $A_{\sigma+1}$-module for each σ, and with each triple $(A_\sigma, A_{\sigma+1}, A_{\sigma+2})$ satisfying the conditions of Lemma 2.

Let F be a field of real numbers which is closed under forming (real) *n*th roots, for all natural numbers *n*. Examples of countable such fields are (a) the field of real algebraic numbers and (b) the field of all real algebraic numbers with soluble Galois groups; the real field itself is an uncountable example. The group Q_F of all matrices

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \qquad (a, b \in F, a > 0)$$

is a split extension of

$$S_F = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}; b \in F \right\} \text{ by } P_F = \left\{ \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}; a \in F, a > 0 \right\}.$$

The groups S_F and P_F are isomorphic divisible Abelian groups, and P_F operates faithfully and irreducibly on S_F. The ring of endomorphisms induced by Q_F in S_F is isomorphic to F itself, so $Q_{F_1} \cong Q_{F_2}$ if and only if $F_1 \cong F_2$.

Now we consider $\mathrm{Trt}(A_\sigma; 0 \leqq \sigma < \rho)$, where all A_σ are Abelian torsion-free divisible groups of countable rank and the split extensions $A_\sigma A_{\sigma+1}$ are isomorphic to Q_{F_σ}, where the F_σ are countable fields of real numbers which are closed under forming (real) nth roots. By the Remark following Lemma 2, each triple $(A_\sigma, A_{\sigma+1}, A_{\sigma+2})$ satisfies the conditions of Lemma 2. Thus, from Lemma 8, G is an L_ρ-group.

Every normal subgroup of G is complemented. For each σ there is a maximal subgroup T_σ such that $K_\sigma \subset T_\sigma$ and $K_{\sigma+1} T_\sigma = G$.

If $\rho = \omega$ and, for some natural number k, we have a set of isomorphisms s_σ such that

$$(A_\sigma A_{\sigma+1})^{s_\sigma} = A_{\sigma+k} A_{\sigma+k+1}, \; A_\sigma^{s_\sigma} = A_{\sigma+k}, \; A_{\sigma+1}^{s_\sigma} = A_{\sigma+k+1},$$

and such that $s_\sigma s_{\sigma+1}^{-1}$ is the identity mapping on $A_{\sigma+1}$, then there is an isomorphism s defined by

$$(a_1 a_2 \cdots a_r)^s = a_1^{s_{\sigma(1)}} a_2^{s_{\sigma(2)}} \cdots a_r^{s_{\sigma(r)}}$$

for $a_i \in A_{\sigma(i)}$ mapping $\mathrm{Trt}(A_\sigma; 0 \leqq \sigma < \omega) = G$ onto $\mathrm{Trt}(A_\sigma; k \leqq \sigma < \omega) \cong G/K_k^G$, and G is non-Hopfian. Indeed, by suitable choices for the groups Q_{F_σ}, we can ensure that G has precisely k isomorphism classes of non-trivial quotient groups.

On the other hand, if F_1 and F_2 are two non-isomorphic countable fields of real numbers closed under forming (real) nth roots, and if f is a function defined on the positive integers taking the values 1 and 2, we may define $G_f = \mathrm{Trt}(A_\sigma; 0 \leqq \sigma < \omega)$ with all A_σ Abelian torsion-free divisible of countable rank and

$A_0 A_1 \cong Q_{F_1}$,
$A_\sigma A_{\sigma+1} \cong Q_{F_2}$ if σ is not a square, and
$A_\sigma A_{\sigma+1} \cong Q_{F_{f(n)}}$ for $\sigma = n^2$.

If $f_1 \neq f_2$, then $G_{f_1} \not\cong G_{f_2}$; and G_f is not isomorphic to any of its proper quotients, since the distribution of the fields is not periodic. So there are at least $2^{\aleph_0}$ isomorphism classes of Hopfian groups G_f, and if $f(n) = 1$ for infinitely many n, the quotient groups of G_f are mutually non-isomorphic. However there are only $2^{\aleph_0}$ isomorphism classes of countable groups, and we conclude that there are exactly $2^{\aleph_0}$ isomorphism classes of countable Hopfian torsion-free L_ω-groups.

2.2 *Normal subgroup lattices which are not linearly ordered.* Of course, not every torsion-free locally soluble group satisfying Min-n is an L_ρ-group for some ordinal ρ, because any finite direct product of groups of the sort constructed above in section 2.1 satisfies Min-n. We next suggest how two of the many possible modifications of the construction of L_ρ-groups can be used to establish the existence of locally soluble torsion-free groups satisfying Min-n, none of whose non-trivial quotients are directly decomposable or have their normal subgroups linearly ordered by inclusion.

Let $G_1 = \mathrm{Trt}(A_\sigma; 0 \leqq \sigma < \rho)$, where ρ is a limit ordinal, and where the operation of each $A_{\sigma+1}$ on A_σ is defined in the following way: A_σ is considered as the additive group of a vector space of finite dimension n_σ over, for example, the field F of real algebraic numbers, and $A_{\sigma+1}$ operates on A_σ as the multiplicative group of positive elements of F. Using Lemmas 5 and 6 for the group $P_\sigma = \mathrm{Trt}(A_\tau; \sigma \leqq \tau < \rho)$ and for quotient groups of P_σ by normal subgroups contained in $A_\sigma^{P_\sigma}$, it follows that every normal subgroup of P_σ is contained in or contains A_σ^{P} . The lattice $\mathscr{L}$ of normal subgroups of G_1 has a sublattice

$$\mathscr{S} = \{1, K_\sigma^{G_1}; 0 \leqq \sigma < \rho\},$$

of order type $\rho + 1$, consisting of the elements comparable with all elements of $\mathscr{L}$; and the interval of the lattice between $K_\sigma^{G_1}$ and $K_{\sigma+1}^{G_1}$ is isomorphic to the lattice of subspaces of an F-space of dimension n_σ. G_1 is locally soluble, torsion-free and satisfies Min-n; however if $n_\sigma > 1$ for at least one $\sigma < \rho$, the lattice $\mathscr{L}$ of normal subgroups of G_1 is not even distributive.

Let $G_2 = \mathrm{Trt}(A_\sigma; 0 \leqq \sigma < \rho)$, where ρ is a limit ordinal. We express each group A_σ as a direct product of two isomorphic factors, which we regard as the additive groups of two non-isomorphic fields of real numbers closed under formation of (real) nth roots for all n. We let $A_{\sigma+1}$ operate on the direct factors as the multiplicative groups of positive elements of the two corresponding fields. This time, the lattice $\mathscr{L}$ of normal subgroups has a sublattice $\mathscr{S}$ of elements comparable with all elements of $\mathscr{L}$, and the intervals of $\mathscr{L}$ between any two neighbouring elements of $\mathscr{S}$ are isomorphic to the non-linear lattice of order 4.

2.3 Once we have constructed locally soluble groups satisfying Min-n which do not possess central factors, it is possible to construct new ones by using *wreath products*. This is a consequence of the following Lemma which is probably well known (cf. Hall [5]; p. 425).

https://doi.org/10.1017/S1446788700017079 Published online by Cambridge University Press

LEMMA 9. *Let X and Y be transitive (faithful) permutation groups, and let W be the (permutational) wreath product of X and Y, with base group D. If X has no non-trivial central factors, then any normal subgroup H of W is either of the form N^Y with $N \lhd X$ or DM with $M \lhd Y$.*

It follows in particular from Lemma 9 that the wreath product of two groups satisfying Min-n and having no central factors has the same properties. Indeed, it can be deduced from Lemma 9 by induction on the ordinal ρ that the wreath product $W = \mathrm{Wr}(G_\sigma; \sigma < \rho)$ (in the sense of Hall [6], p. 175) satisfies Min-n whenever the groups $G_\sigma (\sigma < \rho)$ satisfy Min-n and have no non-trivial central factors. If the groups G_σ are locally soluble, so is W, and if the G_σ have their normal subgroups linearly ordered by inclusion, so does W. Thus we have another source of locally soluble groups satisfying Min-n; and in particular, using the periodic L_ω-group defined by McLain in [2], we see that there are $\mathrm{L}_{\omega 2}$-groups with elements both of finite and infinite order.

2.4 *Properly hypercentral Hirsch-Plotkin radicals.* In this section we construct torsion-free locally soluble groups G satisfying Min-n, whose normal subgroups are linearly ordered by inclusion, and all of whose non-trivial quotient groups have non-nilpotent hypercentral Hirsch-Plotkin radicals. We begin by constructing a treble product $T = \mathrm{Tr}(A, B, C)$, with A, B and C Abelian torsion-free divisible groups, whose normal subgroups contained in A^C are linearly ordered by inclusion and satisfy the minimal condition, and whose Hirsch-Plotkin radical is non-Abelian.

We take for A, B and C Abelian torsion-free divisible groups of countable rank, and consider A as a vector space of countable dimension over a countable field $\mathfrak{k}$ of real numbers closed under the formation of the (real) nth roots of positive elements for all natural numbers n. Let $e_0, e_1, \cdots$ be a $\mathfrak{k}$-basis of A, and let E_i be the $\mathfrak{k}$-subspace generated by $e_0, \cdots, e_i$. We write B as a direct product $F_0 \times F_+$ of two divisible groups of countable rank, and let one of them, F_+, operate on A as the group of scalar multiplications by positive elements of $\mathfrak{k}$. Thus the subgroups of A invariant under F_+ are just the $\mathfrak{k}$-subspaces of A.

Let us denote by η the $\mathfrak{k}$-linear mapping of A which maps e_i onto e_{i-1} for $i > 0$ and e_0 onto the zero vector. We choose a subfield $\mathfrak{h}$ of $\mathfrak{k}$ (not necessarily closed under taking roots) whose additive group is isomorphic to that of $\mathfrak{k}$. For each $w \in \mathfrak{h}$ we define $\xi(w)$ by the formal power series for $(1+\eta)^w$:

$$\xi(w) = \sum_{n=0}^{\infty} \frac{w(w-1)\cdots(w-n+1)}{1 \cdot 2 \cdot \cdots n} \eta^n.$$

Because η is a locally nilpotent endomorphism of A, the $\xi(w)$ are all well defined $\mathfrak{k}$-linear mappings of A. Furthermore we have $\xi(kw) = (\xi(w))^k$ for all integers k, and $\xi(w_1)\xi(w_2) = \xi(w_1 + w_2)$ for all $w_1, w_2 \in \mathfrak{h}$. Therefore the set L of all linear mappings $\xi(w)$ is an Abelian torsion-free divisible group of automorphisms of A

and has countable rank. We choose an isomorphism of F_0 and L, and use it to define the operation of F_0 on A. The automorphisms of A induced by F_+ and F_0 centralize each other; we may therefore consider $B = F_0 \times F_+$ as operating on A, and form the split extension of A by B. It is then easy to verify that

(a) the non-trivial normal subgroups of AB are just the subgroups E_i and the subgroups containing A,

(b) AF_0 is hypercentral, and E_{i-1} is the ith term of its upper central series; further AF_0 is the Hirsch-Plotkin radical of AB,

(c) AB/E_i is isomorphic to AB for each i, and

(d) if $a \in A$ and $a \notin E_0$, then the centralizer of a in AB is A; if $a \in A$ and $a \notin E_1$, then $[b, a] \notin E_0$ for all $b \neq 1$ of B.

We take an element $y \in E_1, y \notin E_0$. Then $[y, F_0]$ is a subgroup of E_0 isomorphic to the additive group of $\mathfrak{h}$, and the normalizer P of $[y, F_0]$ in F_+ is such that $[y, F_0]P$ is isomorphic to $Q_{\mathfrak{h}}$ as defined in section 2.1. Since there are $2^{\aleph_0}$ non-isomorphic subfields $\mathfrak{h}$ of $\mathfrak{k}$ with additive group isomorphic to that of $\mathfrak{k}$, there are $2^{\aleph_0}$ isomorphism classes for the extensions AB.

It remains to define the action of C on B. This we may do by requiring that BC be isomorphic to AB under an isomorphism $f : AB \to BC$ such that

$$A^f = B, \quad E_0^f = F_0 \quad \text{and} \quad B^f = C,$$

and such that E_1^f does not contain the element $x \in B$ which satisfies $a^{-2}x^{-1}ax = 1$ for all $a \in A$. Then we may use Lemma 4, together with the Remark after Lemma 2, to deduce that E_0^C is contained in every non-trivial normal subgroup of $T = \mathrm{Tr}(A, B, C)$. Because $\mathrm{Tr}(A, B, C)$ and $\mathrm{Tr}(A/E_i, B, C)$ are isomorphic under the map which acts as the identity on BC and maps e_j onto the coset $E_i e_{i+j+1}$ for each j, it follows furthermore that every non-trivial normal subgroup of T either is one of the subgroups E_i^C or contains A^C.

We now use the group T as the starting point for the construction of a treble product tower

$$G = \mathrm{Trt}(A_k; 0 \leqq k < \omega).$$

We take for all of the A_k Abelian torsion-free divisible groups, and for each k, two subgroups $A_{k,0}$ and $A_{k,1}$ satisfying

$$A_{k,0} \subset A_{k,1} \text{ and } A_k/A_{k,1} \cong A_{k,1}/A_{k,0} \cong A_{k,0}.$$

We set

$$A_0 = A, \quad A_{0,i} = E_i \quad \text{for } i = 0, 1,$$
$$A_1 = B, \quad A_{1,0} = F_0 \text{ and } A_2 = C.$$

We let the operation of B on A, the operation of C on B, and the isomorphism $f = f_0$ be as already defined with $A_{1,1} = A_{0,1}^f$. For $k \geqq 0$, the split extension of A_{k+1} by A_{k+2} is taken isomorphic to the split extension of A_k by A_{k+1} under an isomorphism f_k such that

$$A_k^{f_k} = A_{k+1},\ A_{k,0}^{f_k} = A_{k+1\ 0},$$

$$A_{k,1}^{f_k} = A_{k+1.1},\ A_{k+1}^{f_k} = A_{k+2},$$

and such that the element x_{k+1} of A_{k+1} which squares each element of A_k is not contained in $(A_{k,1})^{f_k}$.

By Lemma 7, $(A_{0,0})^G$ is contained in every non-trivial normal subgroup of G. For each i, G and $\mathrm{Trt}(A_0/A_{0,i}, A_k;\ 1 \leqq k < \omega) = G/(A_{0,i})^G$ are isomorphic under a map which fixes $\mathrm{Trt}(A_k;\ 1 \leqq k < \omega)$ elementwise and maps $A_{0,j}$ onto $A_{0,i+j+1}/A_{0,i}$ for all j, and it follows that every non-trivial normal subgroup of T either is a subgroup $A_{0,i}^G$ or contains A_0^G. The same argument applied to the groups G/K_n^G or their isomorphic images $\mathrm{Trt}(A_k;\ n \leqq k < \omega)$ shows that the normal subgroups of G are linearly ordered, of order type $\omega^2 + 1$. The Hirsch-Plotkin radical of G is properly contained in $(A_0A_{1,1})^G$, and so coincides with $(A_0A_{1,0})^G$, which is hypercentral with upper central height $\omega + 1$. If N is a proper normal subgroup, satisfying $K_n^G \subseteq N \subset K_{n+1}^G$, then $G/N \cong \mathrm{Trt}(A_k;\ n \leqq k < \omega)$. This shows that the Hirsch-Plotkin radical of every non-trivial quotient group of G is hypercentral with upper central height $\omega + 1$. Further, since G is a union of soluble normal subgroups, G is locally soluble.

If we now assume that the isomorphisms f_k are so chosen that $f_k(f_{k+1})^{-1}$ is the identity map on A_{k+1} for each $k \geqq 0$, then the maps $f_k : A_k \to A_{k+1}$ extend to an isomorphism f from G to $\mathrm{Trt}(A_k;\ 1 \leqq k < \omega)$; and f^n is an isomorphism of G and $\mathrm{Trt}(A_k;\ n \leqq k < \omega)$, which is isomorphic to G/K_n^G. Since we have already remarked that all of the non-trivial quotient groups $G/A_{0,i}^G$ are isomorphic to G, it follows that G is isomorphic to all of its non-trivial quotients.

The group G thus constructed is of course countable. We have already remarked that there are $2^{\aleph_0}$ possible isomorphism classes for the split extension A_0A_1, and since non-isomorphic groups A_0A_1 give rise to non-isomorphic treble product towers, it follows that there are (exactly) $2^{\aleph_0}$ mutually non-isomorphic choices for G.

We mention one further property of G. It is clear that all of the subgroups K_n^G are complemented in G; however none of the other subgroups $(K_nA_{n+1.i})^G$ is complemented, since otherwise $A_{n+1.i}$ would be complemented in $A_{n+1}A_{n+2}$, which is not the case. Thus in each quotient the minimal normal subgroup is not complemented, and G has no maximal subgroups.

3. Characteristically simple groups

Our remarks about characteristically simple groups stem from the following

LEMMA 10. *Let G be a group whose normal subgroups are linearly ordered by inclusion. If there is an isomorphism s from G onto a subgroup H, and if H is* *complemented in G by a subgroup $N \lhd G$ such that $G = \langle N, N^s, N^{s^2}, \cdots, \rangle$, then G may be embedded in a characteristically simple group $\bar{G}$ whose normal subgroups are linearly ordered by inclusion, in such a way that G is complemented in $\bar{G}$ by a subgroup $C \lhd \bar{G}$.*

If Ω is the linearly ordered set of all $K \lhd G$ with $K \subset N$, then the order type of the set of proper non-trivial normal subgroups of $\bar{G}$ is $\mathbf{Z} \times \Omega$, where $\mathbf{Z}$ denotes the set of integers with its natural order and where the Cartesian product $\mathbf{Z} \times \Omega$ is lexicographically ordered.

PROOF. We write $N_0 = N$, and for each integer $k > 0$ we take a group N_k isomorphic to N and an isomorphism t_k of N_k onto N_{k-1}. Let $H = G_{-1}$. Beginning with $G = G_0$ and $s = s_0$, we construct inductively an ascending sequence G_k ($k \geqq 0$) of groups and a sequence s_k ($k \geqq 0$) of isomorphisms of G_k onto G_{k-1} such that G_k is a split extension of N_k by G_{k-1} and such that the restriction of s_k to G_{k-1} is s_{k-1} for each $k > 0$. Suppose the sequences defined as far as G_k and s_k; we define G_{k+1} to be the split extension of N_{k+1} by G_k, where the operation of G_k on N_{k+1} is defined by

$$g^{-1}ng = ((g^{-1})^{s_k} n^{t_{k+1}} g^{s_k})^{t(k+1)^{-1}}$$

for all g in G_k and n in N_{k+1}. It is then easy to see that the map s_{k+1} defined by

$$(ng)^{s_{k+1}} = n^{t_{k+1}} g^{s_k} \quad (n \in N_{k+1}, g \in G_k)$$

is an isomorphism of G_{k+1} onto G_k whose restriction to G_k is s_k. Thus the G_k's and s_k's may be defined for all k. We write $\bar{G}$ for the union of the groups G_k and t for the automorphism of $\bar{G}$ whose restriction to G_k is s_k for each k.

Suppose now that U and V are distinct normal subgroups of $\bar{G}$; then $U \cap G_k \neq V \cap G_k$ for some k. We assume $U \cap G_k \nsubseteq V \cap G_k$. Then $U \cap G_m \nsubseteq V \cap G_m$ for all $m \geqq k$; and since these intersections are normal subgroups of G_m, which is isomorphic to G, we have $V \cap G_m \subset U \cap G_m$. So

$$V = \bigcup \{V \cap G_m;\ m \geqq k\} \subseteq \bigcup \{U \cap G_m;\ m \geqq k\} = U.$$

It follows that the normal subgroups of $\bar{G}$ are linearly ordered.

We define $C = \langle N_1, N_2, \cdots \rangle$. Then C is a normal subgroup of $\bar{G}$ which complements G in $\bar{G}$. For each integer k, we write $C_k = C^{(t^k)}$. Then

$$\bar{G} = \bigcup \{C_k;\ k \geqq 0\} \text{ and } 1 = \bigcap \{C_k;\ k < 0\}.$$

Thus if R is a non-trivial proper normal subgroup of $\bar{G}$, we cannot have $R \subset C_k$ for every integer k or $C_k \subset R$ for every k. Hence there is a least m satisfying $R \subset C_m$, and we have

$$C_{m-1} \subseteq R \subset C_m.$$

Since

$$R \subset C_m = C_{m-1}^t \subseteq R^t,$$

R cannot be a characteristic subgroup of $\bar{G}$, and it follows that $\bar{G}$ is characteristically simple. Because $C_m = C_{m-1}N^{t^m}$, we have $R = C_{m-1}(N^{t^m} \cap R)$, and $N^{t^m} \cap R$ is a normal subgroup of G^{t^n}. The map

$$R \to (m, (N^{t^m} \cap R)^{t^{-m}})$$

is easily verified to be a 1-1 order preserving correspondence of the set of proper non-trivial normal subgroups of $\bar{G}$ and $\mathbf{Z} \times \Omega$. This concludes the proof of Lemma 10.

In a natural extension of the notation of section 2.1, we call a group an L_Ω-group if it is locally soluble and if Ω is the order type of its set of proper non-trivial normal subgroups. Applying Lemma 10 with G one of the L_ω-groups isomorphic to every non-trivial quotient group constructed in section 2.1, we obtain a torsion-free characteristically simple $L_{\mathbf{Z}}$-group $\bar{G}$. Since every normal subgroup of $\bar{G}$ has a complement, $\bar{G}$ has many maximal subgroups, the subgroups $G^{(t^i)}C^{(t^{i+1})}$ in the notation of the Lemma for instance. We may also apply Lemma 10 with G one of the torsion-free L_{ω^2}-groups isomorphic to every non-trivial quotient constructed in section 2.4. We deduce that there are torsion-free $L_{\mathbf{Z}\times\omega}$-groups, all of whose proper non-trivial quotient groups are isomorphic and have non-nilpotent hypercentral Hirsch-Plotkin radicals.

An example of a periodic characteristically simple $L_{\mathbf{Z}}$-group may be constructed by Lemma 10, using the periodic L_ω-group M described by McLain in [2].

A different approach to characteristically simple groups is provided by the wreath powers of Hall [6]. A linearly ordered set Ω is called *irreducible* if, for all x and y in Ω with $x < y$, there is an order automorphism θ of Ω such that $y < x\theta$. This implies in particular that Ω has neither a greatest nor a least element. In Theorem D of [6], Hall proved that the derived group W' of the wreath power

$$W = \mathrm{Wr}S^\Omega$$

of a group S over a linearly ordered set Ω is characteristically simple, provided only that Ω is irreducible. If S is an L_Σ-group with no non-trivial central factors, it follows readily using Lemma 9 that W is an $L_{\Omega\times(1+\Sigma)}$-group with no non-trivial central factors, and since S must be perfect, we have further that $W = W'$. Thus, once we have constructed locally soluble groups with no non-trivial central factors, we may construct characteristically simple such groups. Taking for Ω the set of integers and for S the wreath product $G \wr M$ of a torsion-free L_ω-group G and the periodic L_ω-group M mentioned above, we obtain a characteristically simple $L_{\mathbf{Z}\times\omega}$-group with elements of both finite and infinite order.

References

[1] R. Baer, 'Irreducible groups of automorphisms of abelian groups', *Pacific J. Math.* 14 (1964), 385–406.
[2] D. H. McLain, 'Finiteness conditions in locally soluble groups', *J. London Math. Soc.* 34 (1959), 101–107.
[3] B. H. Neumann, 'Twisted wreath products of groups', *Arch. Math.* 14 (1963), 1–6.
[4] D. H. McLain, 'On locally nilpotent groups', *Proc. Cambridge Philos. Soc.* 52 (1956), 5–11.
[5] P. Hall, 'Finiteness conditions for soluble groups', *Proc. London Math. Soc.* 4 (1954), 419–436.
[6] P. Hall, 'Wreath powers and characteristically simple groups', *Proc. Cambridge Philos. Soc.* 58 (1962), 170–184.

Mathematisches Institut
Universität Erlangen-Nürnberg
852 Erlangen
Bismarckstr. 1 1/2.
Fed. Rep. of Germany

Christ's College
Cambridge
Great Britain