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SELF-SMALL ABELIAN GROUPS

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Abstract

This paper investigates self-small abelian groups of finite torsion-free rank. We obtain a new characterization of infinite self-small groups. In addition, self-small groups of torsion-free rank 1 and their finite direct sums are discussed.

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1. Introduction

It is the goal of this paper to investigate *self-small abelian groups*. Introduced by Arnold and Murley [6], these are the abelian groups G such that, for every index set I and every $\alpha \in \mathrm{Hom}(G, G^{(I)})$, there is a finite subset $J \subseteq I$ with $\alpha(G) \subseteq \bigoplus_J G$. Since every torsion-free abelian group of finite rank is self-small, and all self-small torsion groups are finite by [6, Proposition 3.1], our investigation focuses on mixed groups. Section 2 introduces several important classes of self-small mixed groups, and discusses their basic properties. In Section 3 we give a new characterization of infinite self-small groups of finite torsion-free rank (Theorem 3.1), while Section 4 extends the notion of completely decomposable groups from the torsion-free case to the case of mixed groups.

Although we use the standard notation from [12], some terminology shall be mentioned for the benefit of the reader. A mixed abelian group G is *honest* if tG is not a direct summand of G, where $tG = \bigoplus_p G_p$ denotes the torsion subgroup of G, and G_p is its *p-torsion subgroup*. The set $S(G) = \{p | G_p \neq 0\}$ is the *support of* G. The *torsion-free rank of* G, denoted by $r_0(G)$, is the rank of $\overline{G} = G/tG$. Furthermore, a subgroup U of G is *full* if G/U is torsion.

Given a class $\mathcal{C}$ of abelian groups, the objects of the category $\mathbb{Q}\mathcal{C}$ are the groups in $\mathcal{C}$, while its morphisms are the *quasi-homomorphisms*

$$\mathrm{Hom}_{\mathbb{Q}\mathcal{C}}(A, B) = \mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{Hom}(A, B).$$

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This makes $\mathbb{Q}\mathcal{C}$ a full subcategory of $\mathbb{Q}Ab$ where Ab is the category of all groups [14]. Groups A and B, which are isomorphic in $\mathbb{Q}Ab$, are called *quasi-isomorphic*; and we write $A \sim B$ in this case. The discussion of quasi-properties often gives rise to properties of a group which hold for all but finitely many primes in a subset S of the set P of all primes. We say that such a property holds for *almost all primes* in S.

For sets S_1 and S_2, we write $S_1 \subseteq^{\cdot} S_2$ if there is a finite subset T of S_1 such that $S_1 \setminus T \subseteq S_2$. The sets S_1 and S_2 are *quasi-equal* ($S_1 \doteq S_2$) if $S_1 \subseteq^{\cdot} S_2$ and $S_2 \subseteq^{\cdot} S_1$.

2. Classes of self-small mixed groups

We use the symbol $\mathcal{S}$ to denote the class of infinite self-small groups which have finite torsion-free rank. For $G \in \mathcal{S}$, choose a full free subgroup F of G, and let $D(G)$ be the quasi-equality class of the set $\{p | (G/F)_p \text{ is divisible}\}$. To see that $D(G)$ is well defined, consider full free subgroups F_1 and F_2 of a group $G \in \mathcal{S}$. Then $F_1 \cap F_2$ is a full free subgroup of G; and $F_i / F_1 \cap F_2$ is finite for $i = 1, 2$. Hence, for almost all primes p, $(G/F_1)_p$ is divisible if and only if $(G/F_2)_p$ is divisible because $G/F_i \cong [G/F_1 \cap F_2]/[F_i/F_1 \cap F_2]$.

THEOREM 2.1 ([6, Proposition 3.6] and [3, Theorem 2.2]). *The following are equivalent for a reduced group G with $r_0(G) < \infty$.*

(a) *G is self-small.*
(b) *Each G_p is finite, and $S(G) \subseteq^{\cdot} D(G)$.*
(c) *Each G_p is finite and* $\mathrm{Hom}(G, tG)$ *is torsion.* □

Moreover, groups in $\mathcal{S}$ satisfy a weak projection condition.

COROLLARY 2.2 [8, Lemma 2.1]. *Let G be a self-small group of finite torsion-free rank, and F be a full free subgroup of G. For every prime p, consider a decomposition $G = G_p \oplus G(p)$ with corresponding canonical projection $\pi_p : G \to G_p$. Then $G_p = \pi_p(F)$ for almost all primes p.* □

The objects of the category *WALK* are the abelian groups. The *WALK-maps* are $\mathrm{Hom}_W(G, H) = \mathrm{Hom}(G, H)/\mathrm{Hom}(G, tH)$, and $E_W(G) = \mathrm{Hom}_W(G, G)$ is the *WALK*-endomorphism ring of G. The *WALK*-endomorphism ring of a self-small group G of finite torsion-free rank is $\overline{E}(G) = E(G)/tE(G)$ since $\mathrm{Hom}(G, tG) = tE(G)$ by Theorem 2.1. Finally, the symbol $G \cong_W H$ indicates that G and H are isomorphic in *WALK*.

THEOREM 2.3. *The following are equivalent for self-small groups G and H.*

(a) *$G \cong_W H$.*
(b) *There exist decompositions $G = G' \oplus U$ and $H = H' \oplus V$ with U and V finite such that $G' \cong H'$.*

PROOF. It suffices to show (a) implies (b). If G and H are isomorphic in *WALK*, then there exist homomorphisms $\alpha : G \to H$ and $\beta : H \to G$ such that $[1_G - \beta\alpha](G)$ and $[1_H - \alpha\beta](H)$ are torsion. By the remarks preceding the theorem, there exists a finite

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set I of primes such that $(1_G - \beta\alpha)(G) \subseteq \bigoplus_I G_p$ and $(1_H - \alpha\beta)(H) \subseteq \bigoplus_I H_p$. Since G_p and H_p are finite, $G = [\bigoplus_I G_p] \oplus G'$ and $H = [\bigoplus_I H_p] \oplus H'$. It is now easy to see that α and β induce isomorphisms between G' and H'. □

In particular, the last result shows that a self-small group G is honest if and only if it is not *WALK*-isomorphic to a torsion-free group. Moreover, arguing as in the proof of Theorem 2.3, we obtain the following corollary.

COROLLARY 2.4. *Let $G \in \mathcal{S}$. If $\overline{e}_1, \ldots, \overline{e}_n$ are pairwise orthogonal idempotents in $E_W(G)$ with $\overline{e}_1 + \cdots + \overline{e}_n = 1$, then there exist pairwise orthogonal idempotents $e_1, \ldots, e_n, e$ of $E(G)$ such that $e_i \in \overline{e}_i$ for all $i = 1, \ldots, n$, $e(G)$ is finite, and $e_1 + \cdots + e_n + e = 1_G$.* □

There are two subclasses of $\mathcal{S}$ which have been studied extensively over the last decade by several authors. The first of these is the class $\mathcal{G}$, which consists of the self-small mixed groups G of finite torsion-free rank such that G/tG is divisible (for example, see [1–3, 10, 13, 15]). A reduced group G of finite torsion-free rank belongs to $\mathcal{G}$ if and only if G_p is finite for all primes p, and G can be embedded as a pure subgroup into $\prod_p G_p$ such that it satisfies the *projection condition*: for some (equivalently any) full free subgroup $F \subseteq G$, one has $\pi_p(F) = G_p$ for almost all p. Here, π_p is the natural projection of the direct product onto G_p.

Unfortunately, the direct sum of a group in $\mathcal{G}$ and of a subgroup of $\mathbb{Q}$ need not be in $\mathcal{S}$.

PROPOSITION 2.5.

(a) *There exist a group $G \in \mathcal{G}$ and a subgroup H of $\mathbb{Q}$ such that $G \oplus H \notin \mathcal{S}$.*

(b) *If $\{G_1, \ldots, G_k\}$ are groups of finite torsion-free rank, then $\bigoplus_{i=1}^k G_i \in \mathcal{S}$ if and only if each $(G_i)_p$ is finite for all primes p, and $\bigcup_{i=1}^k S(G_i) \subseteq^{\cdot} \bigcap_{i=1}^k D(G_i)$.*

PROOF. (a) For each prime p, let a_p be a generator of the group $\mathbb{Z}/p\mathbb{Z}$. Define G to be the pure subgroup of $\prod_p \langle a_p \rangle$ generated by $\oplus_p \langle a_p \rangle$ and the element $a = (a_p)_p$. Clearly, $G \in \mathcal{G}$. We consider the subgroup H of $\mathbb{Q}$ generated by $\{1/p \mid p \text{ a prime}\}$. Since $S(G \oplus H)$ is the set of all primes, while $D(G \oplus H)$ is empty, $G \oplus H$ is not self-small.

(b) In [6], it was shown that $\mathcal{S}$ is closed under direct summands. Moreover, it is not hard to check that $S(\bigoplus_{i=1}^k G_i) = \bigcup_{i=1}^k S(G_i)$ and $D(\bigoplus_{i=1}^k G_i) = \bigcap_{i=1}^k D(G_i)$. □

The second class of mixed groups is $\mathcal{D}$, the class of *quotient divisible (qd-) groups* [11]. It consists of those abelian groups G for which tG is reduced, and which contain a full free subgroup F of finite rank such that G/F is the direct sum of a divisible and a finite group (see also [4]). The torsion-free qd-groups are precisely the classical quotient divisible groups introduced by Beaumont and Pierce [7]. It is easy to see that $\mathcal{G} \subseteq \mathcal{D}$.

Furthermore, qd-groups are also characterized by a projection condition. Let W be a set of primes. For $p \in W$, select a finitely generated $\widehat{\mathbb{Z}}_p$-module M_p. A subgroup $G \leq \prod_p M_p$ satisfies the *p-adic projection condition* if it contains a full

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free subgroup F of finite rank such that $\pi_p(F)$ generates M_p as a $\widehat{\mathbb{Z}}_p$-module for every $p \in W$. Here, $\pi_p : \prod_q M_q \to M_p$ again denotes the canonical projection.

THEOREM 2.6 [11]. *An infinite, reduced group G of finite torsion-free rank is quotient divisible if and only if there exist a set W of primes and finitely generated p-adic modules $\{M_p | p \in W\}$ such that G can be embedded as a pure subgroup into $\prod_{p\in W} M_p$ satisfying the p-adic projection condition.*

In particular, $\mathcal{D} \subseteq \mathcal{S}$ since G_p is finite whenever $G \in \mathcal{D}$ by Theorem 2.6.

3. A new characterization

Let W be a nonempty set of primes. A group G is *essentially W-reduced* if G does not contain an infinite subgroup which is divisible by all primes $p \in W$. The symbol $\mathcal{DR}(W)$ denotes the class of essentially W-reduced groups G of finite torsion-free rank which have a full free subgroup F such that $W \subseteq D(G/F)$. To see that every group G in $\mathcal{DR}(W)$ is self-small, observe that $S(G) \subseteq^{\cdot} W$ since G is essentially W-reduced. In view of the fact that $W \subseteq D(G/F)$, we obtain that G_p is finite for every $p \in W$ because G has finite torsion-free rank. If $p \in S(G) \setminus W$, then G_p is divisible by all primes in W. Thus, G_p is finite in this case too. Now apply Theorem 2.1.

THEOREM 3.1.

(a) *Let G be an honest self-small mixed group of finite torsion-free rank. Then $G \cong_W H$ such that H is an extension of a finite-rank torsion-free $S(G)$-divisible group X by a group Y in $\mathcal{DR}(S(G))$.*

(b) *Let W be a nonempty set of primes. Every extension H of a W-divisible finite-rank torsion-free group by a group from $\mathcal{DR}(W)$ is self-small.*

PROOF. (a) Let G be an honest mixed self-small group of finite torsion-free rank. As we have already noted, G_p is finite for each $p \in S(G)$. Since G is honest, $S(G)$ must be infinite. For each $p \in S(G)$, fix a decomposition $G = G_p \oplus G(p)$. Then $K = \bigcap_{S(G)} G(p)$ is torsion-free and pure in G. Let $X = \bigcap_{S(G)} p^\omega G$. For $p \in S(G)$, there exists a positive integer k_p such that $p^{k_p} G = p^{k_p} G(p)$. Thus, $p^\omega G = p^\omega G(p) \subseteq G(p)$ for all $p \in S(G)$. Consequently, $X \subseteq K$, and X is a torsion-free group which is p-divisible for all $p \in S(G)$. Furthermore, since tG cannot be a summand of G, $r_0(X) < r_0(G)$.

Let $\{x_1, \ldots, x_n\} \subseteq G$ be a maximal independent subset consisting of elements of infinite order such that $\{x_1, \ldots, x_m\}$, for some $m < n$, is maximal independent in X. Write

$$S(G) \setminus D(G/\langle x_1, \ldots, x_n\rangle) = \{p_1, \ldots, p_l\}.$$

The modularity law yields

$$G = \left(\bigoplus_{i=1}^{l} G_{p_i}\right) \oplus \left(\bigcap_{i=1}^{l} G(p_i)\right).$$

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If $\rho : G \to \bigcap_{i=1}^{l} G(p_i)$ is the canonical projection, then the full free subgroup

$$F = \langle x_1, \ldots, x_m, \rho(x_{m+1}), \ldots, \rho(x_n)\rangle \subseteq \bigcap_{i=1}^{l} G(p_i)$$

has the property that $\bigcap_{i=1}^{l} G(p_i)/F$ is p-divisible for all primes

$$p \in S\Big(\bigcap_{i=1}^{l} G(p_i)\Big) = S(G) \setminus \{p_1, \ldots, p_l\}.$$

Moreover, for every $i = 1, \ldots, l$, we have

$$\Big(\bigcap_{j=1}^{l} G(p_j)/F\Big)_{p_i} = R_i \oplus D_i$$

with R_i finite and D_i divisible.

If $F \subseteq F' \subseteq (\bigcap_{i=1}^{l} G(p_i))$ satisfies $F'/F = \bigoplus_{i=1}^{l} R_i$, then F' is a free group such that $\bigcap_{i=1}^{l} G(p_i)/F'$ is p-divisible for all $p \in S(G)$. Moreover, F'/F is a direct summand of $\bigcap_{i=1}^{l} G(p_i)/F$. Hence, its nonzero elements have finite p_i-heights for all $i \in \{1, \ldots, l\}$. Then, every element $y \in F' \setminus F$ has finite p-height for all $p \in \{p_1, \ldots, p_l\}$. We claim that F' has a basis which contains the maximal independent system $\{x_1, \ldots, x_m\}$ of X.

To see this, it is enough to prove that $\langle x_1, \ldots, x_m\rangle$ is a pure subgroup of F'. Suppose that $y \in F'$ such that $py \in \langle x_1, \ldots, x_m\rangle$. If $y \in F$, then obviously $y \in \langle x_1, \ldots, x_m\rangle$. If $y \notin F$, then $p \in \{p_1, \ldots, p_l\}$. Hence, the p-height of y in $(\bigcap_{i=1}^{l} G(p_i))$ is finite. But, since X is p-divisible (it is q-divisible for all $q \in S(G)$), the p-height of py is infinite, and this is not possible since

$$G_p \cap \Big(\bigcap_{i=1}^{l} G(p_i)\Big) = 0.$$

If

$$Y = \Big(\bigcap_{i=1}^{l} G(p_i)\Big)\Big/ X,$$

then

$$\overline{F} = (F' + X)/X \cong F'/(F' \cap X)$$

is a free subgroup of Y, and $Y/\overline{F}$ is p-divisible for all $p \in S(G)$. Since Y is also essentially $S(G)$-reduced, $Y \in \mathcal{DR}(S(G))$.

(b) If $0 \to X \to H \to Y \to 0$ is an exact sequence such that X is a torsion-free group which is p-divisible for all $p \in W$ and $Y \in \mathcal{DR}(W)$, then $S(H) \subseteq S(Y) \subseteq W$. Let $\{x_1, \ldots, x_m\}$ be a maximal independent subset of X; and choose

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$y_{m+1}, \ldots, y_n \in H$ such that $\{y_{m+1} + X, \ldots, y_n + X\}$ is a maximal independent subset of Y such that $Y/\langle y_{m+1} + X, \ldots, y_n + X\rangle$ is p-divisible for all $p \in W$. Then $F = \langle x_1, \ldots, x_m, y_{m+1}, \ldots, y_n\rangle$ is a full free subgroup of H; and it is easy to see that H/F is p-divisible for all $p \in W$. By Theorem 2.1, H is self-small. □

The next result shows that the groups in $\mathcal{DR}(W)$ satisfy a projection condition similar to the one for qd-groups (see Theorem 2.6). We remind the reader that, for a set W of primes, a subgroup U of an abelian group G is W-pure if $p^nU = U \cap p^nG$ for all $p \in W$ and $n < \omega$.

THEOREM 3.2. *Let G be a group of finite torsion-free rank. If W is a nonempty set of primes, then $G \in \mathcal{DR}(W)$ if and only if G is isomorphic to a W-pure subgroup of a direct product of (finitely generated) p-adic modules $\prod_W M_p$ satisfying the p-adic projection condition.*

PROOF. Suppose that $G \in \mathcal{DR}(W)$. For each $p \in W$, write $G = G_p \oplus G(p)$ with G_p finite, and let $X = \bigcap_W p^\omega G$. As in the proof of Theorem 3.1, $X = 0$ since X is torsion-free and W-divisible, and G is essentially W-reduced. As in Theorem 2.6, there is a W-pure embedding $G \subseteq M = \prod_W M_p$, where each M_p is the p-adic completion of $G/p^\omega G$.

For each $p \in W$, let π_p be the natural projection of M onto M_p. Since G/F is p-divisible, $\widehat{\mathbb{Z}}_p\pi_p(F) = M_p$ for each p. Thus, we have shown that G satisfies the p-adic projection condition and that each M_p is finitely generated of rank no greater than $r_0(G)$.

Conversely, suppose that G is a W-pure subgroup of $M = \prod_W M_p$ which satisfies the p-adic projection condition. Then, $(G/F)_p$ is divisible for some full free $F \le G$ and all $p \in W$. In addition, any subgroup G of M is essentially W-reduced since each M_p is a reduced p-adic module. □

We remind the reader that $\overline{G}$ denotes the group G/tG where tG is the torsion subgroup of G.

COROLLARY 3.3. *Let G be a self-small group of finite torsion-free rank n. If G_p is of rank n for almost all $p \in S(G)$, then $\overline{G}$ is p-divisible for almost all $p \in S(G)$, and $X = \bigcap_{p\in S(G)} p^\omega G = 0$.*

PROOF. As a consequence of Theorem 3.1, we obtain an exact sequence $0 \to Y \to G \to H \to 0$ in which $H \in \mathcal{DR}(W)$, where W is quasi-equal to $S(G)$, and $Y = B \oplus X$ for a finite direct summand B of G such that $X = \bigcap_{p\in S(G)} p^\omega G$ is torsion-free and p-divisible for almost all $p \in S(G)$. We view H as a W-pure subgroup of $\prod_{p\in W} M_p$ as in Theorem 3.2. Since almost all p-components G_p are direct summands in the corresponding p-adic modules M_p, we observe that almost all of the p-adic modules M_p are generated by at least n elements. Therefore, the torsion-free rank of H is n, hence $X = 0$. Moreover, if $M_p = G_p \oplus N_p$, then the minimal number of generators for M_p is $n + m$, where m is the minimal number of generators of the free p-adic module N_p. From all these observations, we obtain $N_p = 0$ for almost

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all p, and modulo a finite direct summand, G is a W-pure subgroup of $\prod_{p\in W} G_p$. Hence, $\overline{G}$ is p-divisible for all $p \in W$. □

COROLLARY 3.4. *The following are equivalent for an essentially $S(G)$-reduced group G.*

(a) *G is self-small.*
(b) *$G \in \mathcal{DR}(S(G))$.*
(c) *G can be embedded as an $S(G)$-pure subgroup into a direct product of (finitely generated) p-adic modules $\prod_{p\in S(G)} M_p$ such that G satisfies the p-adic projection condition.* □

Observe that W-purity and the projection condition are independent.

EXAMPLE 3.5. Let S be an infinite set of primes. For each prime $p \in S$, let a_p be a generator of $\mathbb{Z}/p\mathbb{Z}$ and b_p be a generator of $\mathbb{Z}/p^2\mathbb{Z}$.

(a) Consider the subgroup G of $\prod_S \langle a_p\rangle$ which is generated by $\oplus_S \langle a_p\rangle$ and the additional element $a = (a_p)_S$. The group G satisfies both the ordinary and the p-adic projection condition and $r_0(G) = 1$. But $G/\mathbb{Z}a \cong \oplus_S \mathbb{Z}/p\mathbb{Z}$. Thus, $D(G)$ is empty, while $S(G) = S$. By Theorem 2.1, G is not self-small.

(b) Consider the pure subgroup H of $\prod_{p\in S}\langle b_p\rangle$ generated by $\oplus_{p\in S}\langle b_p\rangle$ and the additional element $b = (pb_p)_S$. Then H is not self-small since $D(H)$ is empty. Observe that H does not satisfy the (p-adic) projection condition.

4. Direct sums of self-small groups of torsion-free rank 1

A self-small group of finite torsion-free rank is *completely decomposable* if it is isomorphic to a direct sum of groups whose torsion-free rank is 1. By Theorem 2.3, *WALK*-isomorphic self-small groups are quasi-isomorphic. On the other hand, two quasi-isomorphic self-small groups, whose torsion-free rank is 1, are *WALK*-isomorphic by [9, Lemma 3.1]. Hence, we obtain the following result.

THEOREM 4.1. *The following are equivalent for completely decomposable, self-small groups $A = \bigoplus_{i=1}^m A_i$ and $B = \bigoplus_{j=1}^k B_j$ where the A_i and B_j have torsion-free rank* 1.

(a) *$A \sim B$.*
(b) *$m = k$; and, after some re-indexing, $A_i \cong_W B_i$ for all $i = 1, \ldots, m$.*
(c) *$A \cong_W B$.*

If G is a self-small group of torsion-free rank 1, then G_p is cyclic for almost all primes p by Corollary 2.2. Conversely, let S be an infinite set of primes, and consider the torsion group $T = \bigoplus_S \langle a_p\rangle$, where a_p is a generator of $\mathbb{Z}/p^{k_p}\mathbb{Z}$ with $0 < k_p < \infty$ for each $p \in S$. Define $\tau = \mathrm{type}(T)$ to be the type with characteristic (χ_p) defined by $\chi_p = k_p$ for $p \in S$ and $\chi_p = 0$ otherwise.

Note that τ is *locally free* in the sense that all of its entries are finite. The set S is a representative of the *support* of τ where the support of a locally free type $\tau = [(\chi_p)]$

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is the set of primes $S(\tau) = \{p \mid \chi_p > 0\}$. The set $S(\tau)$ is defined only up to quasi-equality.

For T as above, regard T as a pure subgroup of $\widehat{T} = \prod_S \langle a_p \rangle$. Consider $a = (a_p) \in \widehat{T}$, and choose a subgroup X of $\mathbb{Q}$ containing $\mathbb{Z}_{S^{-1}}$ where $\mathbb{Z}_{S^{-1}}$ is the subring of $\mathbb{Q}$ generated by $\mathbb{Z}$ and $\{1/p \mid p \in S\}$. Define $A(T, X)$ to be the inverse image of $X(a + T) \subseteq \widehat{T}/T$ under the natural factor map $\widehat{T} \to \widehat{T}/T$. Because $A/\mathbb{Z}a$ is p-divisible for all $p \in S$, Theorem 2.1 yields that $A(T, X)$ is a self-small group such that $tA = T$ and $A/T(A) \cong X$.

THEOREM 4.2. *Let A be a self-small group of torsion-free rank 1.*

(a) *$A \cong_W A'$ where A' is a rank-one torsion-free group, or $A' \cong A(T, X)$ for some appropriately chosen T and X.*
(b) *A/tA is divisible for almost all primes in $S(A)$.*

PROOF. (a) Since A is self-small, each A_p must be finite. If $S(A)$ is also finite, then $A \cong_W A'$ where A' is a torsion-free group of rank 1. Therefore, we may assume that $S = S(A)$ is infinite. Choose an element $a \in A$ of infinite order, and let a_p be the projection $\pi_p(a)$ of a into A_p associated with any direct sum decomposition $A = A_p \oplus B(p)$. By Theorem 2.1, we can modify A by a *WALK*-isomorphism in such a way that $A/\mathbb{Z}a$ is divisible for all $p \in S$. Following Corollary 2.2, each A_p is cyclic of order p^{k_p} with generator a_p. Hence, $T = tA = \bigoplus_{p \in S} \langle a_p \rangle$.

There is a natural map $\lambda : A \to \widehat{T} = \prod_S A_p$ extending the inclusion $T \to \widehat{T}$. Note that $\lambda(a) = (a_p) \in \widehat{T}$. It is now easy to check that λ is an embedding of A into $\widehat{T}$ and that $\lambda(A)$ satisfies the projection condition.

Let X be the rank-one torsion-free group defined by $\lambda(A)/T = X(a + T)$. Since $\lambda(A)$ satisfies the projection condition, $\lambda(A)/T$ is p-divisible for all $p \in S$. Equivalently, $pX = X$ for $p \in S$. Since the modified group A is isomorphic to $A(T, X)$, our original A is *WALK*-isomorphic to $A(T, X)$.

(b) Observe $S(A) \subseteq^{\cdot} D(A)$ by Theorem 2.1. Since A has torsion-free rank 1, we obtain $D(A) = D[A/tA]$ using (a). □

In particular, $\overline{A} = A/tA$ is divisible by almost all primes in $S(A)$ whenever A is a completely decomposable self-small group.

Let $\mathcal{S}_1$ denote the set of $\cong_W$ equivalence classes of torsion-free rank-one self-small groups. The symbol $\mathcal{T}$ indicates the set of ordered pairs (τ, σ) where τ is a locally free type and σ is a type such that $\sigma(p) = \infty$ for almost all $p \in S(\tau)$.

For $[A] \in \mathcal{S}_1$, write $\tau_A = \text{type}(tA)$ and $\sigma_A = \text{type}(A/tA)$. The proof of Corollary 4.2 shows that A is determined up to *WALK*-isomorphism by the pair of types (τ_A, σ_A), which we call *the pair of types of A*. In particular, we obtain that the map $[A] \to (\tau_A, \sigma_A)$ is a bijection between the set $\mathcal{S}_1$ and the set $\mathcal{T}$.

For a self-small group A of torsion-free rank 1, we may assume that $S(A) \subset D(A)$ since there exists a group which is *WALK*-isomorphic to A and has this property. Furthermore, we may assume that there is an element $a \in A$ with $\langle \pi_p(a) \rangle = A_p$ for all $p \in S$. Recall that the sequence $u_p(a) = (h_p(a), h_p(pa), \ldots)$ is the p-indicator

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of $a \in A$. Suppose that $\sigma_A = [(m_p)]$. The *height matrix* of A is the $\omega \times \omega$ matrix whose rows are indexed by the primes such that the pth row is $u_p(a)$.

It is easy to see the following.

(Ia) If $p \notin S$ and $0 \leq m_p < \infty$, then $u_p(a)$ has no gaps; in this case $u_p(a) = (m_p, m_p + 1, \ldots)$.

(Ib) If $p \notin S$ and $m_p = \infty$, then A is p-divisible, that is, $u_p(a) = (\infty, \infty, \ldots)$.

(II) If $p \in S$ and $T_p(A) \cong Z(p^{k_p})$ with $0 < k_p < \infty$, then $u_p(a) = (0, 1, \ldots, k_p - 1, \infty, \ldots)$.

Consequently, every self-small group of torsion-free rank 1 is determined up to *WALK*-isomorphism by its height matrix.

THEOREM 4.3. *Let A and C be rank-one self-small groups with corresponding pairs of types (τ_A, σ_A) and (τ_C, σ_C), respectively. Choose characteristics $(k_p^A) \in \tau_A$, $(k_p^C) \in \tau_C$ and $(m_p^A) \in \sigma_A$. The following are equivalent.*

(a) $\mathrm{Hom}_W(A, C) \neq 0$.

(b) (i) $\sigma_A \leq \sigma_C$;

(ii) $m_p^A = 0$ *for almost all* $p \in S(C) \setminus S(A)$; *and*

(iii) $k_p^A \geq k_p^C$ *for almost all* $p \in S(A) \cap S(C)$.

PROOF. To prove that (a) implies (b), let $f : A \to C$ be a homomorphism and $a \in A$ an element of infinite order such that $c = f(a)$ also has infinite order. Then the induced homomorphism $\overline{f} : \overline{A} \to \overline{C}$ is nonzero, hence $\sigma_A \leq \sigma_C$. Moreover, $u_p(a) \leq u_p(c)$ for all primes p. Since these p-indicators satisfy conditions (Ia), (Ib), and (II) for almost all p, (ii) and (iii) also hold.

For the reverse implication, we may assume without loss the generality that $A = A(S, X)$ with $S = \bigoplus_p \mathbb{Z}/p^{k_p^A}\mathbb{Z}$ and that X is a subgroup of $\mathbb{Q}$ of type σ_A. Similarly, we need to consider only the case $C = A(T, Y)$ where $T = \oplus_p \mathbb{Z}/p^{k_p^C}\mathbb{Z}$ and $Y \subseteq \mathbb{Q}$ of type σ_C. Hence,

$$A \subseteq \widehat{S} = \prod_p \mathbb{Z}/p^{k_p^A}\mathbb{Z}a_p$$

where $a = (a_p) \in A$ satisfies $A/S = X(a + S)$. Similarly,

$$C \leq \widehat{T} = \prod_p \mathbb{Z}/p^{k_p^C}\mathbb{Z}c_p$$

where $c = (c_p) \in C$ with $C/T = Y(c + T)$. By (i), no generality is lost if we only consider the case $1 \in X \subseteq Y$. Moreover, as a consequence of Theorem 2.3, we may assume that (ii) and (iii) are valid for all $p \in S(C) \setminus S(A)$ and for all $p \in S(A) \cap S(C)$, respectively.

Using either condition (ii) for $p \in S(C) \setminus S(A)$, or the remarks preceding the theorem for $p \in S(A) \cap S(C)$, we obtain $h_p(a) = 0$ for all $p \in S(C)$. By [5, Section 1] for $p \in S(C) \setminus S(A)$, and by condition (iii) for $p \in S(A) \cap S(C)$, we have $A/p^{k_p^C}(A) \cong \mathbb{Z}/p^{k_p^C}\mathbb{Z}$ for all $p \in S(C)$. There exists a unique homomorphism $\varphi_p : A \to C_p$ with $\varphi_p(a) = c_p$ for all $p \in S(C)$. Let $\varphi : A \to \prod_{S(C)} C_p$ be the homomorphism induced by the maps $\{\varphi_p\}_{p \in S(C)}$. Then $\varphi(a) = c$ and $\varphi(tA) \subseteq tC$.

If x is an element of infinite order in A, then there exists a nonzero integer k such that $1/k \in X$ and $kx \in \langle a \rangle + tA$. Hence, $k\varphi(x) \in \langle c \rangle + tC$. Since $1/k \in X \subseteq Y$, we have $\varphi(x) \in C$. Thus, $\varphi(A) \subseteq C$; and φ induces a homomorphism from A into C whose image is not a torsion group. □

A group $A \in \mathcal{S}$ is *WALK-homogeneous completely decomposable* provided $A = (\bigoplus_{i=1}^n C_i) \oplus B$ such that B is finite, $r_0(C_i) = 1$, and $C_i \cong C_j$ for all $i, j = 1, \ldots, n$. By [5, Theorem 2.3], a homogeneous almost completely decomposable group is completely decomposable. We now show that this also holds for self-small mixed groups.

THEOREM 4.4. *A self-small group A of finite torsion-free rank such that $A \doteq \bigoplus_{i=1}^n C_i$ with $r_0(C_i) = 1$ and $C_i \cong C_j$ for all $i, j = 1, \ldots, n$ is WALK-homogeneous completely decomposable.*

PROOF. Let $m > 0$ be an integer such that $mA \subseteq C = \bigoplus_{i=1}^n C_i \subseteq A$. Since C is self-small by [9, Lemma 2.5], the projection condition implies that the p-components of the groups C_i are cyclic for almost all primes p. Hence, we can assume that, modulo a *WALK*-isomorphism, the following conditions hold:

(i) for every prime p, the p-component of C_i is a cyclic group;
(ii) for every prime divisor $p|m$, the p-component of A is zero;
(iii) $\overline{A}$ is p-divisible for all $p \in S(A)$.

These conditions guarantee that, for every prime p, either $A_p = 0$ or $A_p \cong (\mathbb{Z}/p^{k_p}\mathbb{Z})^n$ for some integer $k_p > 0$. Note that A has a unique decomposition $A = A_p \oplus A(p)$ for every $p \in S(A)$ since $\overline{A}$ is p-divisible. Let $\pi_p : A \to A_p$ be the canonical projection.

Since $m\overline{A} \subseteq \bigoplus_{i=1}^n \overline{C}_i \subseteq \overline{A}$, [5, Theorem 2.3] yields that $\overline{A}$ is homogeneous completely decomposable. Let R be a subgroup of $\mathbb{Q}$ which has the same type as $\overline{A}$. Write $\overline{A} = \bigoplus_{i=1}^n R\overline{a}_i$. We can assume that, for every prime p, $A_p = \langle \pi_p(a_1), \ldots, \pi_p(a_n) \rangle$. Because of $A_p \cong (\mathbb{Z}/p^{k_p}\mathbb{Z})^n$, we obtain $A_p = \bigoplus_{i=1}^n \langle \pi_p(a_i) \rangle$. Observe that the subgroup $\langle \pi_p(a_1), \ldots, \pi_p(a_n) \rangle$ does not have p^{nk_p} elements if the elements $\pi_p(a_1), \ldots, \pi_p(a_n)$ are not independent. For every prime p, let $\pi_{pi} : A \to \langle \pi_p(a_i) \rangle$ be the canonical projection induced by the decomposition $A = (\bigoplus_{i=1}^n \langle \pi_p(a_i) \rangle) \oplus A(p)$.

For every index i, consider the subgroups

$$\begin{aligned}\langle a_i \rangle_\star &= \{a \in A \mid sa \in \langle a_i \rangle, 0 \neq s \in \mathbb{Z}, \text{ and } \forall p \in \mathbb{P}, \pi_p(a) \in \langle \pi_p(a_i) \rangle\} \\ &= \{a \in A \mid sa \in \langle a_i \rangle, 0 \neq s \in \mathbb{Z}, \text{ and } \forall p \in \mathbb{P}, \ j \neq i \Rightarrow \pi_{pj}(a) = 0\}\end{aligned}$$

where $\mathbb{P}$ denotes the set of all primes. It is easy to see that $\sum_{i=1}^n \langle a_i \rangle_\star = \bigoplus_{i=1}^n \langle a_i \rangle_\star$ and that it contains tA. If $\overline{x} = x + tA \in R\overline{a}_i$, then there exist coprime integers u, s, with $u \neq 0$ and $s/u \in R$, and $t \in tA$ such that $ux = sa_i + t$. Let S be the set of primes

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dividing u, s, and the order of t. It is not hard to see that $\pi_p(x) \in \pi_p(a_i)$ for all $p \in \mathbb{P} \setminus S$. Therefore, $x' = x - \sum_{p \in S} \pi_p(x)$ satisfies $\pi_p(x') \in \pi_p(a_i)$ for all $p \in P$. Observe that $p \in S$ yields $\pi_p(x') = 0$, from which we obtain $\overline{x} = \overline{x'} \in (\langle a_i \rangle_\star + T(A))$. Thus,

$$R\overline{a}_i \subseteq (\langle a_i \rangle_\star + tA)/tA.$$

Since the converse inclusion is obvious,

$$(\langle a_i \rangle_\star + T(A))/T(A) = R\overline{a}_i$$

for every index i; and

$$A = \sum_{i=1}^{n} \langle a_i \rangle_\star = \bigoplus_{i=1}^{n} \langle a_i \rangle_\star$$

where the groups $\langle a_i \rangle_\star$ are isomorphic mixed groups of torsion-free rank 1. □

Arguing as in the last proof, we obtain the following result.

COROLLARY 4.5. *Let A be a self-small group of torsion-free rank* 1. *If C is a self-small group of torsion-free rank n such that $tC \cong tA^n$ and $\overline{C} \cong \overline{A}^n$, then $C \cong A^n$.* □

References

[1] U. Albrecht, 'Mixed Abelian groups with Artinian quasi-endomorphism ring', *Comm. Algebra* **25**(11) (1997), 3497–3511.

[2] ———, '*A*-projective resolutions and an Azumaya theorem for a class of mixed abelian groups', *Czechoslovak Math. J.* **51(126)**(1) (2001), 73–93.

[3] U. Albrecht, P. Goeters and W. Wickless, 'The flat dimension of Abelian groups as *E*-modules', *Rocky Mountain J. Math.* **25**(2) (1995), 569–590.

[4] U. Albrecht and W. Wickless, *Finitely Generated and Cogenerated QD-groups, Rings, Modules, Algebras, and Abelian Groups*, Lecture Notes in Pure and Appl. Math, 236 (Dekker, New York, 2004), pp. 13–26.

[5] D. M. Arnold, *Finite Rank Torsion Free Abelian Groups and Rings*, Lecture Notes in Mathematics, 931 (Springer-Verlag, Berlin, 1982).

[6] D. M. Arnold and C. E. Murley, 'Abelian groups, A, such that $H\mathrm{om}(A, —)$ preserves direct sums of copies of A', *Pacific J. Math.* **56** (1975), 7–21.

[7] R. A. Beaumont and R. S. Pierce, 'Torsion-free rings', *Illinois J. Math.* **5** (1961), 61–98.

[8] S. Breaz, 'Self-small Abelian groups as modules over their endomorphism rings', *Comm. Algebra* **31** (2003), 4911–4924.

[9] ———, 'Quasi-decompositions for self-small abelian groups', *Comm. Algebra* **32** (2004), 1373–1384.

[10] A. Fomin and W. Wickless, 'Self-small mixed abelian groups G with $G/t(G)$ finite rank divisible', *Comm. Algebra* **26** (1998), 3563–3580.

[11] ———, 'Quotient divisible abelian groups', *Proc. Amer. Math. Soc.* **126** (1998), 45–52.

[12] L. Fuchs, *Infinite Abelian Groups* Vols. I and II (Academic Press, New York, 1970/1973).

[13] S. Glaz and W. Wickless, 'Regular and principal projective endomorphism rings of mixed abelian groups', *Comm. Algebra* **22** (1994), 1161–1176.

[14] E. Walker, 'Quotient categories and quasi-isomorphisms of abelian groups', in: *Proc. Colloq. Abelian Groups* (Akademiai Kiado, Budapest, 1963), pp. 147–162.

[15] W. Wickless, 'A funtor from mixed groups to torsion free groups', *Contemp. Math.* **171** (1995), 407–419.

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