ONTARIO MATHEMATICAL MEETINGS 1968

Four meetings were held during the year, 1968. For the first time, centres outside Toronto were used. The details of each meeting (time, place and invited address) and the abstracts of papers presented are as follows:

Sixth meeting: January 20, University of Waterloo (Abstracts 68.1 to 68.8) P.M. Cohn (University of London, and Rutgers, the State University), Dependence in rings.

Seventh meeting: March 30, University of Toronto (Abstracts 68.9 to 68.15) Abraham Robinson (Yale University), Germs and monads.

Eighth meeting: November 2, McMaster University, Hamilton (Abstracts 68.16 to 68.26) A. Rosenberg (Cornell University), Some recent results on the Brauer group of rings.

Ninth meeting: December 14, University of Toronto (Abstracts 68.27 to 68.40) Marc Kac (Rockefeller University), Some mathematical problems in statistical mechanics.

68.1 G. Alexits (Hungarian Academy of Sci. Math. Research Inst. and University of Waterloo)
On the Characterization of Classes of Functions by Best Linear Approximation

For a real Banach space B containing a sequence $\{y_\nu\}$, define the nth best $\{y_\nu\}$ - approximation of $x \in B$:

$$E_n^{(B)}(x, \{y_\nu\}) = \inf\{ \| x - (a_1 y_1 + \ldots + a_n y_n) \|_B : a_i \in R\} .$$

For a subset C of B, $E_n^{(B)}(C, \{y_\nu\}) = \sup\{E_n^{(B)}(x, \{y_\nu\}) : x \in C\}$. If $\{E_n\}$ is a positive non-increasing null sequence, $C(\{E_n\}, \{y_\nu\})$ is the set of all those elements $x \in B$ with $E_n^{(B)}(x, \{y_\nu\}) \leq E_n (n=1, 2, \ldots)$. A set $C \subseteq B$ is $\{E_n\}$ - characterizable if there exists a basis of approximation $\{y_\nu\}$ such that $C(\{KE_n\}, \{y_\nu\}) \subseteq C \subseteq C(\{E_n\}, \{y_\nu\})$ for each n where $K \leq 1$ is an absolute constant.

Assume that B is a Banach space contained in a Hilbert space such that the B-norm dominates the Hilbert space norm. Write $\{a_n\} \approx \{b_n\}$ for two positive sequences if $\{{}^{a_n}/b_n\}$ and $\{{}^{b_n}/a_n\}$ are both

https://doi.org/10.1017/S0008439500029866 Published online by Cambridge University Press

bounded, and $E_n^{(B)}(C) = \inf \{E_n^{(B)}(C, \{y_\nu\}) : \{y_\nu\} \subseteq B\}$.

THEOREM. Let $\{\xi_\nu\}$ be a bounded orthonormal system and C a closed set $\{E_n\}$ - characterizable by $\{\xi_\nu\}$ - approximation. If $E_{2n} \geq c E_n$ for some constant c and every n, then

$$\{E_n\} \approx \{E_n^{(B)}(C)\} .$$

This theorem can be applied to approximation by ν-times differentiable functions whose rth derivatives satisfy a Lipschitz condition and by rational functions.

68.2 E. Zakon (University of Windsor)
Non-Standard Models of the Real Axis

68.3 G.F. Duff (University of Toronto)
On Rearrangement Identities

For a real-valued function $f \in C^1[0,b]$, the equi-measurable decreasing rearrangement f^* of f is defined as m^{-1} where $m(y)$ is the measure of the set $\{x | f(x)>y\}$. If $n(y)$ denotes the number of roots x_k of $f(x) = y$, then

$$\frac{1}{|f^{*\prime}(x)|} = \sum_{k=1}^{n} \frac{1}{|f'(x_k)|} .$$

From this relation we can deduce integral inequalities such as

$$\int |f^{*\prime}(x)|^p \, dx \leq \int \frac{|f'(x)|^p}{n(f(x))^p} \, dx , \qquad p > 1 .$$

Also, if we define an "equivariational transform" F of f by $dF = n(f)df^*$, we can establish an arc length inequality of the form

$$\int \sqrt{1 + (F')^2} \, dx \leq \int \sqrt{1 + (f')^2} \, dx .$$

Various generalisations involving convex functions and higher dimensions are also possible.

68.4 R.G. Stanton (York University) and J.G. Kalbfleisch (University of Waterloo)
Covering Problems for Dichotomized Matchings

Olga Taussky and John Todd have posed the following problem concerning an abelian group G with n base elements g_i $(i = 1, \ldots, n)$ each of order p, where p need not be a prime. Let S be the set of $1 + n(p-1)$ distinct powers of the base elements. Then one is required to determine the minimal integer $\sigma(n, p)$ for which there exists a subset H of G, with H containing $\sigma(n, p)$ elements, such that each element of G may be written as a product of an element of H and an element of S.

In this paper, attention is restricted to the case of $\sigma(n, 2)$. If the set H contains y_i elements whose entries are made up of i ones and $n-i$ zeros, then

$$(n-i+1)y_{i-1} + y_i + (i+1)y_{i+1} \geq \binom{n}{i}$$

for $i = 0, 1, \ldots, n$. Consideration of these inequalities allows one to determine $\sigma(n, p)$ for $p = 2$, $n = 2, \ldots, 7$. The values are $2, 2, 4, 7, 12, 16$. Of these, the value $\sigma(6, 2) = 12$ is new, and some new uniqueness properties are found. The method shows that $\sigma(8, 2)$ is either 31 or 32, and it is announced that different considerations, to be published later, show that $\sigma(8, 2) = 32$.

68.5 H.P. Heinig (McMaster University)
An Extension of Plancherel's Theorem

Let M denote the real line and m the Lebesgue measure on M. Define f^* to be the equimeasurable decreasing rearrangement of $|f|$ by $f^*(t) = \inf\{y > 0 : \lambda_{|f|}(y) \leq t\}$, where $\lambda_{|f|}(y) = m\{x \in M : |f(x)| > y, \forall\ y \geq 0\}$, and f^{**} by $f^{**}(s) = 1/s \int_0^s f^*(t)\,dt$, $s > 0$. If $L_{q,p}(M)$, $1 \leq p < \infty$, $1 < q < \infty$ is the class of measurable functions f on M such that

$$\|f\|_{q,p} = \left[(q-1)/q^2 \int_0^\infty [f^{**}(t)]^p\, t^{p/q-1}\, dt \right]^{1/p} < \infty, \quad \text{where}$$

$1 < p < \infty$, $1 < q < \infty$, $\|f\|_{q,\infty} = \sup_t t^{1/q} f^{**}(t) < \infty$,

$L_{\infty,\infty}(M) \equiv L_\infty(M)$, and $L_{1,1}(M) = L_1(M)$, then the following extension of Plancherel's theorem holds:

THEOREM: If $f \in L_{q,p}(M)$, $1 < q < 2$, $1 \leq p \leq \infty$ then as $\lambda \to \infty$,

$\hat{f}_\lambda(x) = 1/\sqrt{2\pi} \int_{-\lambda}^{\lambda} e^{ixt} f(t)dt$, converges in the $L_{q',p}$-norm to $\hat{f}$, called Fourier transform of f and $||\hat{f}||_{q',p} \leq A\, ||f||_{q,p}$, where $q' + q = q'q$.

COROLLARY. If $1 < q < 2$, $q \leq p \leq q'$, $\int_{-\infty}^{\infty} |x|^{p/q-1} |f(x)|^p dx < \infty$, then $\hat{f}_\lambda$ converges in mean to $\hat{f}$ and for $p \leq s \leq q'$,

$$\left\{ \int_{-\infty}^{\infty} |x|^{s/q'-1} |\hat{f}(x)|^s dx \right\}^{1/s} \leq A \left\{ \int_{-\infty}^{\infty} |x|^{p/q-1} |f(x)|^p dx \right\}^{1/p} .$$

68.6 H.H. Crapo (University of Waterloo)
Simplicial Geometries

A geometric theory of combinatorial topology may be founded upon the following simple observations concerning Betti numbers.

For a fixed n-element set T, and any non-negative integer $k \leq n$, let T_k be the set of all k-element subsets of T. With each subset $A \subseteq T_k$, associate the simplicial complex $S(A)$, with simplices

$$S(A) = T_0 \cup T_1 \cup \ldots \cup T_{k-1} \cup A .$$

Let $\alpha_i(A)$ and $\beta_i(A)$ be the total number of simplices, and the Betti number, respectively, calculated for simplices of cardinality i in the complex $S(A)$.

The integer-valued function r, defined on subsets $A \subseteq T_k$ by

$$r(A) = \binom{n-1}{k-1} - \beta_{k-1}(A) = |A| - \beta_k(A)$$

increases by at most one when an element is added to a set A, has value $r(\emptyset) = 0$, and satisfies the semimodular inequality

$$r(A \cup B) + r(A \cap B) \leq r(A) + r(B) .$$

https://doi.org/10.1017/S0008439500029866 Published online by Cambridge University Press

THEOREM. $r(A)$ measures the geometric rank of the subset A in some geometry $G(T_k)$ of rank $\binom{n-1}{k-1}$ on the set T_k of $\binom{n}{k}$ points.

The lattice of flats of a geometry $G(T_k)$ for $k = 2$ is the lattice of all partitions of the set T, ordered by refinement. For higher values of k, the simplicial geometries have not previously been studied. Two general results are available.

THEOREM. In a simplicial geometry $G(T_k)$, the set B of points (k-element simplices) containing any fixed element $b \in T$ form a basis for the geometry.

THEOREM. On an n-element set T, the geometries $G(T_k)$ and $G(T_{n-k})$ are orthogonal, for $k = 0, \ldots, n$. (Alexander duality)

(G. C. Rota and the author are including an exposition of these results in the book Trends in Lattice Theory, soon to appear in the Van Nostrand series.)

68.7 F.P. Cass and D. Borwein (University of Western Ontario)
Multiplication Theorems for Strong Summability

Some theorems concerning the strong Nörlund summability of the Cauchy product of two given series are established which generalise known theorems about strong Cesaro summability.

68.8 Tae Ho Chae (McMaster University)
On Compact Topological Lattices of Finite Dimensions

In 1947, I. Kaplansky proved that a compact semi-simple topological ring is isomorphic and homeomorphic with a cartesian direct sum of finite simple rings; this implies that any compact Boolean topological lattice is always totally disconnected.

However, in the proof of his theorem, Kaplansky utilized duality theorem in the sense of a topological group. Professor A.D. Wallace had suggested the possibility of a proof of the latter theorem (Boolean ring case) which is independent of the duality theorem. In this connection, we show, without using the duality theorem, that any compact Boolean topological lattice of finite dimension is always totally disconnected.

With the use of duality theorem, this theorem can be generalised as follows: a locally compact and locally convex Boolean topological lattice is totally disconnected. And we show: If L is a compact complemented modular topological lattice and if $a \wedge L = F(a \wedge L)$ (the boundary of $a \wedge L$) for all non-zero elements a of L and dually, then L is totally disconnected. A.D. Wallace conjectured that the center of a compact connected topological lattice L of codimension n contains at most 2^n elements. L.W. Anderson showed that if L is distributive, then Wallace's conjecture is true. In the second section, we prove that the conjecture is always true. It is also shown that if L is a topological lattice with 0 and 1 with codimension n, then L is iseomorphic with the n-cell if and only if L satisfies (i) L is distributive and contains n-independent elements $x_1, \ldots, x_n$ over 0 whose union is 1. (ii) each $x_i \wedge L$ is separable, connected and locally compact.

68.9 A. Tsutsumi (University of Toronto)
<u>On a Generalised Goursat Problem</u>

We consider the equation

$$\left(\frac{\partial}{\partial t_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial t_m}\right)^{\alpha_m} u(t_1, \ldots, t_m, x)$$

$$= \left[\sum_{\beta,\gamma} a_{\beta\gamma}(t_1, \ldots, t_m, x) \left(\frac{\partial}{\partial t_1}\right)^{\beta_1} \cdots \left(\frac{\partial}{\partial t_m}\right)^{\beta_m} \left(\frac{\partial}{\partial x}\right)^{\gamma} u(t_1, \ldots, t_m, x)\right]$$

$$+ f(t_1, \ldots, t_m, x),$$

with data

$$\left(\frac{\partial}{\partial x}\right)^{k} u(t_1, \ldots, t_m, x) \Big|_{t_i=0} = \phi_{ik} \Big| \; 0 \le k \le \alpha_i - 1, \quad 1 \le i \le m,$$

under the compatibility conditions on ϕ_{ik};

$$\left(\frac{\partial}{\partial t_i}\right)^{\ell} \phi_{jk} \Big|_{t_i=0} = \left(\frac{\partial}{\partial t_j}\right)^{k} \phi_{i\ell} \Big|_{t_j=0}$$

where the summation $\sum_{\beta,\gamma}$ is done for β, γ such that

$$|\gamma| \ge |\beta| + \gamma, \quad \alpha_i > \beta_i, \quad i=1,\ldots,m.$$

We have an unique existence theorem of the solution of the above equation in the function class of δ-geverey with respect to x for

$$\delta : 1 \leq \delta \leq \min_{\beta, \gamma} \frac{|\alpha| - |\beta|}{\gamma}$$ and continuously differentiable up to

the order α with respect to t, which generalises the results of A. Friedman (Trans. Amer. Math. Soc. 98) and L. Hörmander (Theorem 5.1.1 ; Springer 1963).

68.10 R.G. Lintz (McMaster University)
Cauchy's Problem for Generalised Differential Equations

The idea of derivatives in general topological spaces has been introduced in (Notices Amer. Math. Soc. - 648-76; August, 1967). As a consequence, we can consider also differential equations in general topological spaces. To do this, we have to consider in the spaces X and Y a structure of Gauss space and then if we are given an open set M in X and an open set M' in Y it is possible to prove the existence of solution of the equation Df = g, where g is a special g-function, satisfying initial conditions relatively to the pair (M, M'). For arbitrary g-function g this problem is not yet solved.

68.11 D.Ž. Djoković (University of Waterloo)
A Representation Theorem for $(X_1-1)(X_2-1)\ldots(X_n-1)$ and its Applications

Let R be a commutative ring with unity. We prove that the polynomial

$$(1) \qquad (n!)^{2^s} \prod_{i=1}^{n} (X_i - 1)$$

for some integer $s > 0$ is contained in the ideal of $R[X_1, \ldots, X_n]$ which is generated by all polynomials of the form

$$(2) \qquad (X_{i_1} X_{i_2} \ldots X_{i_k} - 1)^n$$

where $1 \leq i_1 < i_2 < \ldots < i_k \leq n$.

The proof of this is based on the identity

$$\sum_{k=1}^{n} (-1)^k \sum_{1 \le i_1 < i_2 < \ldots < i_k \le n} (X_{i_1} X_{i_2} \ldots X_{i_k} - 1)^n$$

$$= \sum_{m=1}^{n} (-1)^m \binom{n}{m} \prod_{i=1}^{n} (X_i^m - 1) .$$

This representation of the polynomial (1) as a combination of polynomials (2) (with polynomial coefficients) leads to a similar representation of the iterated difference operator

$$\Delta_{u_1} \Delta_{u_2} \cdots \Delta_{u_n} .$$

By using this representation we prove a generalisation of a recent result of M. A. McKiernan: If $f : R \to R$ satisfies

$$\Delta_u^{n+1} f(x) = 0$$

for all $u, x \in R$ and a fixed positive integer n then

$$f(x) = \sum_{k=0}^{n} g_k^*(x)$$

where $g_k : R^k \to R$ is multiadditive and $g_k^*(x) = g_k(x, x, \ldots, x)$.

We show that the same theorem is valid (essentially) if we take f to be a function which maps an abelian semigroup into an abelian group.

68.12 E. Zakon (University of Windsor)
On Uniform Spaces with a Nested Base

A uniform space X with a nested base ("nested space") always has a base $\mathbf{V}$ which is either countable or consists of clopen entourages which are equivalence relations, so that each $V \in \mathbf{V}$ induces a partition of X into disjoint clopen neighborhoods $V[x]$. Such a base is called standard if it is well ordered by inverse inclusion and is of the least possible order type J under that well ordering. Notation: $(X, \mathbf{V}, J)$. $(X, \mathbf{V}, J)$ is said to be pseudocomplete if $\cap_{\nu < \eta} V_\nu [x_\nu] \neq \emptyset$ for every decreasing sequence of neighborhoods $V_\nu [x_\nu]$ $(V_\nu \in \mathbf{V})$ of order type $\eta < J$. We say that X has few isolated points if some neighborhood is free of such points.

SOME THEOREMS. 2.4 (a). $(X, \mathbf{V}, J)$ is metrizable if it is hereditarily Lindelöf, or separable, or totally bounded, or has a non-discrete subspace with one of these properties.

3.2. A pseudocomplete nested space $(X, \mathbf{V}, J)$ with few isolated points is metrizable if: (a') For each $V \in \mathbf{V}, X$ can be covered by less than $2^{\aleph_0}$ neighborhoods $V[x]$; or (b') every open covering of X has a subcovering of power $< 2^{\aleph_0}$; or (c') every set of power $\geq 2^{\aleph_0}$ in X has a limit point; or (d') some neighborhood $V[x]$ without isolated points has one of these three properties, as a subspace of $(X, \mathbf{V}, J)$.

68.13 W.A. O'N. Waugh (University of Toronto)

Conditional Probabilities in a Birth and Death Process

A Markov process may, in general, possess one or more sets of absorbing states. The author has described a method for deriving probabilities conditional on absorption in a given set, from unconditional probabilities for the same process. The purpose of the present work is to extend this result to the non-Markovian age-dependent branching process. There is a single absorbing state: zero, or extinction of the population, and all other states (the positive integers) are transient. The process is conveniently described in terms of a family tree, and we shall make use of probability measures on a space of possible family trees. The process is well defined when one has given the distribution $G(t)$ of life-lengths of individuals, and the respective probabilities $q_0(t)$ and $q_2(t)$ that life ends in death without issue or in binary fission, given that it ends at age t. Let ρ be the probability of death, and σ the probability of binary fission (unconditionally). Also, let $B^*(t|b)$ be the life-length distribution conditional on binary fission, and $D^*(t|d)$, be that for death without issue.

Let A be any event and let E be the event "extinction". We write for the probability in the conditioned process

$$P(A|E) = \tilde{P}(A)$$

and adopt the same convention of a tilde for all probabilities in the conditioned process. Our result is that

(a) $\tilde{\rho} = \sigma$ and $\tilde{\sigma} = \rho$

(b) $\tilde{D}^*(t|d) = D^*(t|d)$ and $\tilde{B}^*(t|b) = B^*(t|b)$

(a) is just what would be obtained by applying the author's earlier result for Markov processes to the imbedded discrete-time Galton-Watson process, while (b) implies that, given his reproductive history an individual's life is independent of the ultimate fate of the population.

From this result, the conditional probabilities $\tilde{G}(t)$ $\tilde{q}_0(t)$ and $\tilde{q}_2(t)$ can be obtained, and mean life-lengths and other properties of the process obtained.

(Reference: W.A. O'N. Waugh, Age-dependent Branching Processes Under a Condition of Ultimate Extinction. Biometrika 55 (1968) 291-296.)

68.14 M.P. Heble (University of Toronto)
On the Homotopy Groups of the General Linear Group of an Infinite-dimensional Banach Space

The basic hypothesis made is:

> X is an infinite-dimensional complex Banach space with a countable basis, and with the further property that every closed linear subspace in X has a complementary closed linear subspace in X. The basis elements are assumed to be normalised.

We consider $L(X, X)$ the linear space of continuous linear operators $A: X \to X$, with the topology defined by the operator norm:

$$\|A\| = \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|} .$$

Denote by $GL = GL(X)$ the group of elements $A \in L(X, X)$ such that A, A^{-1} both belong to $L(X, X)$. GL becomes a topological group with the above topology. The theorem proved is:

THEOREM. All homotopy groups of GL vanish:

$$\pi_k(GL) = 0, \; k = 0, 1, 2, \ldots .$$

68.15 J.A. Baker (University of Waterloo)
Measurability Implies Continuity for Solutions to a General System of Functional Equations

68.16 C.E. Haff and G. Berman (University of Waterloo)
The Construction of Λ-Kernels for Coloring a Graph

Let G be a finite, undirected graph, without loops. Define the integer-valued function $\Lambda(G)$ by

$$(1) \qquad \Lambda(G) = \max_{G' \subset G} \ \min_{x \in V(G)} \ \mathrm{val}(G', x)$$

where $\mathrm{val}(G', x)$ is the number of edges of G' incident with x and $V(G)$ is the vertex set of G. Szekeres and Wilf* have shown that

$$(2) \qquad \kappa(G) \leq \Lambda(G) + 1,$$

where $\kappa(G)$ is the chromatic number of G.

A Λ-kernel of G is a maximal independent set $C \subseteq V(G)$ such that $\Lambda(G - G[C] \leq \Lambda(G) - 1$. A method is given for partitioning $V(G)$ by Λ-kernels. This yields a sharpening of the inequality (2),

$$(3) \qquad \kappa(G) \leq \Lambda^*(G) + 1 \leq \Lambda(G) + 1,$$

where $\Lambda^*(G)$ is the cardinality of the partition. A corresponding coloration of G in $\Lambda^*(G) + 1$ - colors is determined.

(*G. Szekeres and H.S. Wilf, An Inequality for the Chromatic Number of a Graph, Journal of Combinatorial Theory 4, (1968) 1-3.)

68.17 C.E. Billigheimer (McMaster University)
Symmetric Difference Operators in a Hilbert Space

We have discussed properties of the formally self-adjoint fourth order difference operator P acting on sequences of complex numbers

$y = \{y_n\}_{-2}^{\infty}$ defined by

$$(Py)_n = d_n y_{n+2} + c_n y_{n+1} + b_n y_n + c_{n-1} y_{n-1} + d_{n-2} y_{n-2} \quad (n \geq 0)$$

where b_n, c_n, d_n are real numbers, $d_n > 0$.

Regarding the operator P, which is analogous to a fourth order self-adjoint differential operator, as an unbounded operator in the Hilbert space ℓ^2 of sequences $\{y_n\}_{-2}^{\infty}$ which are of summable square such that $\sum_0^\infty |y_n|^2 < \infty$, we consider the classification of P according to the deficiency indices (m, m) $(0 \leq m \leq 4)$ of the closed symmetric operator A with minimal domain, or equivalently the number m of linearly independent solutions of the recurrence relations

$$y'_n = \lambda y_n \qquad (n \geq 0) \qquad (*)$$

which are of summable square.

By considering symmetric extensions of A we show that for real λ, which are not eigenvalues of A, the number of summable square solutions $m(\lambda)$ satisfies $m(\lambda) \leq m$.

In the case of self-adjoint extensions, we obtain the classical eigenfunction expansion theorem for a sequence in ℓ^2 in terms of a unique spectral function, which corresponds to the resolution of the identity for the self-adjoint operator. We also obtain the unique Green's function for $(*)$ for $\operatorname{Im} \lambda \neq 0$ and the corresponding resolvent operator. In the quasi-regular case, $m = 4$, the resolvent operator is completely continuous and the spectrum is discrete.

We have obtained by a direct method the theorem that there always exist at least two linearly independent solutions of summable square of the unrestricted recurrence relations $y'_n = \lambda y_n$ $(n \geq 0)$. Also, if for one value of λ these recurrence relations have four linearly independent solutions of summable square, then this is true for all values of λ. These two theorems are also derivable by Hilbert space methods.

The above results are parallel to those for the analogous continuous case of an even-order differential operator, obtained by Hilbert space methods by Kodaira (1950) and Glazman (1951) and directly by Everitt (1957), which generalize the second-order differential operator case with its fundamental limit-point, limit-circle distinction discussed by Weyl (1910), Stone (1932), Titchmarch (1946), Levitan (1950), Yosida (1950), Levinson (1951), and others.

68.18 R.A. Day (McMaster University)
<u>The Characterization of Lattice Equations in Universal Algebra</u>

In this paper we give characterizations of certain properties that hold in the congruence lattice of every algebra of an equational class (variety) by equations of the equational class. Mal'cev (Mat. Sb (N.S.) 35 (77) (1954) 3-20) has characterized permutability (i.e. every pair of congruence relations θ, ψ in every congruence lattice satisfy $\theta \circ \psi = \psi \circ \theta$) by the existence of a ternary term $p(x, y, z)$ that satisfies the equations $a = p(a, b, b)$ and $b = p(a, a, b)$. Pixley (Proc. AMS 14 (1963) 105-109) gave a similar result for permutability and distributivity while Jonsson (Math. Scand. (to appear)) has characterized distributivity alone.

The author (Can. Math. Bull. (to appear)) has shown that every congruence lattice of an equational class is modular if and only if there exists a finite sequence $m_o, \ldots, m_n$ of quaternary terms satisfying the equations:

(M1) $a = m_o(a, b, c, d)$ and $d = m_n(a, b, c, d)$

(M2) $m_i(a, b, b, a) = a$ $(i = 0, 1, \ldots, n)$

(M3) $m_i(a, a, b, b) = m_{i+1}(a, a, b, b)$ (i even)

(M4) $m_i(a, b, b, d) = m_{i+1}(a, b, b, d)$ (i odd).

Direct proofs were also given (in terms of the characterizing equations) that both distributivity and permutability imply modularity.

The above results raise the following general problem: can every property that possibly holds in congruence lattices be characterized by equations if it holds for every congruence lattice in an equational class; or more precisely, what distinguishes those properties that are characterized by equations?

68.19 Y.L. Park (Laurentian University)
<u>On the Projective Cover of the Stone-Čech Compactification of a Completely Regular Hausdorff Space</u>

Let C be the category of compact Hausdorff spaces and continuous maps. For $E \in C$, let $O(E)$ be its topology, and $\Lambda(E)$ be the space of maximal filters $M \subseteq O(E)$ whose topology is generated by the set $\Lambda_W(E) = \{M \mid W \in M, M \in \Lambda(E)\}$ for each $W \in O(E)$. The following are proved: 1) Let X and Y be the topological spaces such that X is a dense subspace of Y; then $\Lambda(Y) \cong \Lambda(X)$ under the mapping $M' \to M' | X$, $M' \in \Lambda(Y)$; 2) For any $M \in \Lambda(X)$, if $U \in M$, then $M \in \Gamma_{\Lambda(X)} \lim{}^{-1}_{\beta X}(U)$; 3) Let $\phi : K \to \beta X$ be a projective cover of βX in C; then for each dense subset D of X, $\phi^{-1}(D)$ is dense in K and $K = \beta\phi^{-1}(D)$. Let $\mathfrak{D}$ be a filter base of dense subsets of X and $Q^*_{\mathfrak{D}}(X)$ be the direct limit of the direct system $(C^*(D))_{D \in \mathfrak{D}}$ with $(\phi_D)_{D \in \mathfrak{D}}$ as a family of the limit homomorphisms. A function $f \in C^*(D)$ defines $f \circ \phi$ on $\phi^{-1}(D)$, and $f \circ \phi$ has a unique continuous extension $\tilde{f}$ to K for $D \in \mathcal{D}$. Let $u_f \in Q^*_{\mathcal{D}}(X)$ with $u_f = \phi_D(f)$ and $f \in C^*(D)$ for some $D \in \mathcal{D}$. The mapping $u_f \to \tilde{f}$ is a norm preserving monomorphism. If $\mathcal{D}$ contains all disconnected dense open subsets of X, then the maximal ideal space of $Q^*_{\mathcal{D}}(X)$ endowed with the Stone-topology is homeomorphic to K. Hence K is homeomorphic to the maximal ideal space of the maximal ring of quotients of $C(X)$.

68.20 L. J. Mordell (University of Toronto)

The Integer Solutions of the Equation $ax^2 + by^2 + c = 0$ in Quadratic Fields

The following result is proved. Let a, b, c, be rational integers such that $(b, c) = (c, a) = (a, b) = 1$ and a and b are square free. Then integer solutions of the equation $ax^2 + by^2 + c = 0$ exist in a quadratic field $Q(t)$ if and only if there exist rational integers p, q, d, d_1 such that

$$ap^2 + bq^2 = d; \quad (ap, bq) = d_1,$$

and either d is some divisor of a, b, c and

$$t^2 + abk^2/d_1 + c/d = 0,$$

or d is some even divisor of $2abc$ and

$$t^2 + t + \frac{1}{4}(1 + abk/d_1{}^2) + c/d = 0.$$

Also k is an integer such that the equations have integer coefficients.

The values of x and y are expressed simply in terms of t.

68.21 B. Banaschewski (McMaster University)

Another Algebraic Characterization of $C^\infty(\mathbf{R}^n)$

An algebra A (with unit e) over the real number field $\mathbf{R}$ is called real semi-simple if and only if the intersection of its real maximal ideals is zero, and convex if and only if, for every $f \in A$, $e + f^2$ is invertible.

PROPOSITION. An $\mathbf{R}$ - algebra A with unit e is isomorphic to the $\mathbf{R}$ - algebra $C^\infty(\mathbf{R}^n)$ if and only if it is real semi-simple and convex, and there exist elements $u_1, \ldots, u_n \in A$ with the following properties:

(K) The ideals $\Sigma A(u_i - a_i e)$, $(a_1, \ldots, a_n) \in R^n$, are exactly the real maximal ideals of A.

(S) For each $(a_1, \ldots, a_n) \in R^n$ and each invertible $f \in A$, there exist $\alpha \neq 0$ in $\mathbf{R}$ and $g \in A$ such that

$$\Sigma (u_i - a_i e)^2 + f^2 = \alpha^2 e + g^2$$

(D) There exist derivations $\partial_1, \ldots, \partial_n : A \to A$ such that $\partial_i u_i = e$ and $\partial_i u_k = 0$ for $i \neq k$.

(M) A has no non-trivial unitary, real semi-simple, and convex algebra extension in which (K), (S), and (D) still hold for the elements $u_1, \ldots, u_n \in A$.

The present characterization of $C^\infty(\mathbf{R}^n)$ is similar to that given in An Algebraic Characterization of $C^\infty(\mathbf{R}^n)$, by B. Banaschewski, Bull. Acad. Polon. Sci. 16 (1968) 169-174 , but a good deal simpler in some aspects; nonetheless, further simplifications would be welcome. It should be mentioned that the maximality condition (M) is not implied by the conditions preceeding it: the algebra of all real-analytic functions on $\mathbf{R}^n$ is a unitary, real semi-simple, and convex proper subalgebra of $C^\infty(\mathbf{R}^n)$ in which (K) - (D) hold for the Cartesian coordinate functions. Incidentally, this algebra is clearly not isomorphic to $C^\infty(\mathbf{R}^n)$; more generally, $C^\infty(\mathbf{R}^n)$ cannot be isomorphic to any proper unitary subalgebra containing the Cartesian coordinate functions, a consequence of its maximality property given by the Proposition.

68.22 K. L. Duggal (University of Windsor)
Singular Riemannian Structures Compatible with π -Structures

Riemannian structure (briefly R_π-structure) on π-structure is defined by the knowledge of a complex metric $G = (g_{ij})$, of rank n_1, satisfying the relation $JG = \lambda G$. By setting $G = JA + \lambda A$, where $A = (a_{ij})$ is a field of symmetric tensors on Vm, of rank m, one can always obtain from (a_{ij}) an R_π-structure. We introduce R_π-adapted bases $(e_{1'}) = (e_{\alpha'}, e_{\alpha^*'})$ such that the vectors $(e_{\alpha'})$ are orthonormal. It is easy to show that the set $O(n_1, n_2)$ of the transformation matrices of two R_π-adapted bases is a Lie-subgroup of $G(n_1, n_2)$.

The set $E_R(Vm)$ of the R_π-adapted bases relative to different points of Vm admits a natural structure of principal fibre bundle and consequently one is able to define R_π-connection (infinitesimal connection) on $E_R(Vm)$. As $E_R(Vm)$ is subbundle of the fibre bundle $E_c(Vm)$ of all the bases so any R_π-connection defines cannonically a linear connection with which it can be identified and conversely. One can prove that $\nabla g_{ij} = 0$ in an R_π-connection. Knowing that $\nabla F^i_j = 0$, we conclude that a complex linear connection can be identified with an R_π-connection if and only if $\nabla F^i_j = \nabla g_{ij} = 0$.

Further it can be proved that V_m has an R_π-structure if and only if there exists a complex linear connection whose holonomy group is a subgroup of $O(n_1, n_2)$. Finally we show that the first characteristic form ψ_1 is zero for any R_π-connection.

68.23 F.H. Northover (Carleton University)
<u>Linear Integral Equations</u>

Apart from the establishment of a few scattered results the bulk of the theory of the homogeneous linear integral equation is restricted to the case - admittedly an important one - in which the kernel is symmetric. An extensive and detailed theory - the well-known Hilbert-Schmidt theory - has been built up for this case.

In the present work, an extensive theory covering the general kernel has been built up. It is based upon the idea of expressing the solutions of a linear integral equation at an eigenvalue $\lambda = \lambda_0$, say, as the limit function of the solution of the corresponding non-homogeneous linear equation for $\lambda \neq \lambda_0$, when $\lambda \to \lambda_0$. Under certain circumstances, this kind of representation works for solutions of the homogeneous equation at $\lambda = \lambda_0$, and also for solutions of the non-homogeneous equation at $\lambda = \lambda_0$ (when such exist).

Various <u>explicit</u> expressions for the general solution of the linear equation at $\lambda = \lambda_0$, are obtained, and, in the case of the non-homogeneous equation, necessary and sufficient conditions are obtained for the existence of a solution at $\lambda = \lambda_0$.

As a by-product, a stronger version of the well-known theorem that the maximum number of linearly independent solutions obtainable from a homogeneous equation at an eigenvalue is finite, is obtained, with an expression for the upper bound of such a number.

Also, necessary and sufficient conditions are obtained for the ability of the simple form

$$\int_a^b t(y)\, D^{(\ell)}(x, y; \lambda_0)\, dy$$

to comprise all solutions. Here, $D(x, y; \lambda_0)$ is the "first Fredholm minor", ℓ the least number such that the derivative indicated (taken with respect to λ) is not identically zero at λ_0, and $t(y)$ is is any continuous function.

68.24 E. Hotzel (McMaster University)
<u>On Semigroups whose Non-trivial Left Congruence Classes are Left Ideals</u>

A left congruence λ of a semigroup S is called a Rees left congruence if there exists a left ideal L such that $(a, b) \in \lambda$ if and only if $a, b \in L$ or $a = b$. A Rees left congruence is a special case of a left congruence whose non-trivial classes (i.e. classes containing at least two elements) are left ideals. In the following a description is given of the semigroups S without zero which have the property that

(A) all left congruences of S are Rees left congruences or more generally that

(B) all non-trivial left congruence classes in S are left ideals.

Semigroups with property (A) have been completely characterized under the supposition of commutativity in [1] and under the supposition of the existence of the zero element in [2] ((A) is equivalent to (B) under these suppositions).

In every semigroup S the set $R(S)$ of right zeros is an ideal. It contains at most three elements if S has property (A).

THEOREM 1. <u>Let</u> S <u>be a semigroup such that</u> $R(S)$ <u>is not empty. Then</u> S <u>has property</u> (B) <u>if and only if</u> $S/R(S)$ <u>has property</u> (A).

THEOREM 2. <u>Let</u> S <u>be a semigroup which has exactly two right zeros</u> q <u>and</u> r. <u>Then</u> S <u>has property</u> (A) <u>if and only if</u> $S\setminus\{q\}$ <u>or</u> $S\setminus\{r\}$ <u>is a subsemigroup of</u> S <u>which has property</u> (A).

THEOREM 3. <u>Let</u> S <u>be a semigroup which has exactly three right zeros. Then</u> S <u>has property</u> (A) <u>if and only if it is isomorphic to one of the following 10 semigroups</u>:

	p	q	r	1
p	p	q	r	p
q	p	q	r	q
r	p	q	r	r
1	p	q	r	1

	p	q	r	a	1
p	p	q	r	p	p
q	p	q	r	p	q
r	p	q	r	q	r
a	p	q	r	p	a
1	p	q	r	a	1

	p	q	r	a	b
p	p	q	r	p	p
q	p	q	r	p	r
r	p	q	r	r	p
a	p	q	r	a	p
b	p	q	r	b	p

	p	q	r	a	b
p	p	q	r	p	r
q	p	q	r	p	p
r	p	q	r	r	r
a	p	q	r	a	r
b	p	q	r	b	r

	p	q	r	a	b
p	p	q	r	p	p
q	p	q	r	p	q
r	p	q	r	r	p
a	p	q	r	a	p
b	p	q	r	p	b

	p	q	r	a	b	c
p	p	q	r	p	p	p
q	p	q	r	p	r	r
r	p	q	r	r	r	p
a	p	q	r	a	a	p
b	p	q	r	b	b	p
c	p	q	r	c	c	p

THEOREM 4. Let S be a non-empty semigroup without right zeros. Then S has property (B) if and only if it is a two element left zero semigroup or a cyclic group of prime order.

The non-trivial part of the proof of Theorem 4 may be given in the following steps:

1) S is right cancellative. 2) If e is an idempotent in S then it is the identity element of S, or S is a two element left zero semigroup. 3) S is left simple.

[1] E.S. Ljapin, Semisimple commutative associative systems (Russian), Izv. Akad. Nauk SSSR 14, (1950) 367-380.

[2] E. Hotzel; Halbgruppen mit ausschließlich reesschen Linkskongruenzen, submitted to Math. Zeitschrift.

68.25 T.D. Howroyd (Universities of Melbourne and Waterloo)
On Functional Equations in Many Variables and Simultaneous Functional Equations in a Single Variable

Let $\underline{S}$ be a set and $\underline{H}: \underline{S}^n \times [0,1]^n \to \underline{S}$. If $\phi : [0,1] \to \underline{S}$ and

$$(1) \quad \phi\left(\sum_1^n x_i/n\right) = H\left(\phi(x_1), \ldots, \phi(x_n); x_1, \ldots, x_n\right)$$

then

$$(2) \quad \phi((x+p)/n) = H(\phi(x), \phi(a_1), \ldots, \phi(a_{n-1}); x, a_1, \ldots, a_{n-1})$$

where

$$a_k = 1 \text{ if } k \leq p, \; a_k = 0 \text{ if } k > p, \; p = 0, \ldots, n-1$$

Hence, if $\underline{S}$ is a Hausdorff space and ϕ is continuous then ϕ is uniquely determined by the values $\phi(0)$ and $\phi(1)$.

If $\underline{I}_n$ is the set of $\underline{n}$-ary fractions in $[0, 1]$ and (using the notation $H(u_1, \ldots, u_n; x_1, \ldots, x_n) = H(u_i; x_i)_i$):

(i) $H(u_i; x_i)_i$ is symmetric in u_j and u_k, x_j and x_k;

(ii) $H(u; x)_i = u$;

(iii) $H(H(u_{ij}; x_{ij})_j; \sum_j x_{ij}/n)_i$ is symmetric in u_{kq} and u_{qk}, x_{kq} and x_{qk};

(iv) $H(u_i; x_i)_i$ is a one-to-one function of u_j; and $\phi: \underline{I}_n \to \underline{S}$, then (2) is equivalent to (1).

If (i) to (iv) hold, $\underline{S}$ is a Hausdorff space, $\underline{H}$ is continuous, and $\phi : [0, 1] \to \underline{S}$ is continuous then (1) is equivalent to (2).

If $\underline{S} = \underline{C}$ (the complex plane); (i) to (iv) hold; $\underline{H}$ is continuous;

(v) there exists $r \in (0, 1)$ such that

$$|H(u_i; x_i)_i - H(v_i; x_i)_i| \leq r \sum_1^n |u_i - v_i|;$$

then there exists exactly one bounded solution $\phi: [0, 1] \to C$ of (1) for any given $\phi(0)$ and $\phi(1)$; this ϕ is continuous.

If $\underline{I}$ is a convex subset of $\underline{C}$; $\underline{H}: \underline{C}^2 \times \underline{I}^2 \to C$ is continuous; (i) to (v) hold with $\underline{n} = 2$; there exists a continuous function $\underline{K}: \underline{C}^2 \times \underline{I}^2 \to \underline{C}$ such that

$$K(H(u, v; x, y), v; x, y) = u,$$

and a_1, a_2, a_3 are non-collinear points in $\underline{I}$; then there exists exactly one locally bounded solution $\phi: \underline{I} \to \underline{C}$ of (1) for any given $\phi(a_1)$, $\phi(a_2)$ and $\phi(a_3)$; this ϕ is continuous.

68.26 M.E. Muldoon (York University)
Singular Integrals whose Kernels Involve Certain Sturm-Liouville Functions

We prove results of the form

$$(1) \qquad \lim_{\nu\to\infty} \int_a^b f(t)\, w(\nu, t)dt = \frac{2}{3} f(0-) + \frac{1}{3} f(0+),$$

where $a < 0 < b$ and, for each $\nu > 0$, $w(\nu, t)$ is the solution of

$$(2) \qquad d^2w/dt^2 = [\nu^2 t + q(t)]w$$

which satisfies

$$\lim_{t\to\infty} w(\nu, t)\, t^{-1/4} \exp(\tfrac{2}{3}\nu t^{3/2}) = \tfrac{1}{2}\pi^{-1/2}\nu^{1/2}.$$

We assume that $q(t)$ is continuous for $t \geqq a$ and that $\int_0^\infty |t|^{-1/2}|q(t)|dt$ exists. It can be shown that (1) holds if

(i) $f \varepsilon BV[a, 0]$

(ii) $f(1+)$ exists, and

(iii) $f \varepsilon L[0, b]$.

We consider applications of (1) to integrals involving various special functions which satisfy equations of the type (2). One such application is to a singular integral (involving the Bessel function J_ν) considered by L. Lorch and P. Szego.

We show that hypothesis (i) may be replaced by

(i') $f \varepsilon C[a, 0]$,

if the limit relation in (1) is interpreted in the sense of Cesàro (C, k) summability, for $k > \frac{1}{2}$, but that this is not so in the case of (C, k) summability, for $0 \leqq k \leqq 1/2$.

68.27 Charles Ford (University of Toronto)
A Note on the Schur Index of a Group Representation

Let G be a group and χ an irreducible character of G. Let $m(\chi)$ be the Schur index of χ over the rational field. For p a prime divisor of $m(\chi)$ let m_p be the p-part of $m(\chi)$. There is a theorem of Brauer which shows that m_p is the Schur index of an irreducible character ξ of a p-elementary subgroup of G. A p-elementary group is the semi-direct product $A \cdot P$ of a (normal) cyclic p'-group A and a p-group P.

It is shown that if $m(\xi) \neq 1, 2$, then there exists a prime q dividing $|A|$ for which $m(\xi) \mid (q-1)$. Thus for the character χ of G, if $m_p \neq 1, 2$, there is a prime divisor q of $|G|$ for which $m_p \mid (q-1)$.

The problem is first reduced by finding a factor group P' of a subgroup of P such that the group $A \cdot P'$ contains a cyclic, normal, self-centralizing subgroup (which is, in general, larger than A) and $A \cdot P'$ has a faithful character ξ' for which $m(\xi') = m(\xi)$. Then the theory of crossed products and factor sets is used to prove the result.

A splitting field F is found for a representation affording ξ'. The dimension $(F : Q(\xi'))$ is a p-power which divides $q-1$.

The only exceptional case occurs when $p = 2$ and A is trivial. In this case we can have $m(\xi') = 2$, and then p' must contain a generalized quaternion subgroup.

68.28 E. Barbeau (University of Toronto)
A generalization of the Algebra of Functions of Bounded Variation

Let $(E, \leq)$ be a compact Hausdorff space with a partial ordering, and A the convex cone of continuous non-negative increasing functions on E. Then A is uniformly closed and contains the product of any pair of its elements, so that its linear hull V is an algebra. With the Schaefer norm, whose unit ball is the absolutely convex hull of the set $\{f : f \in A, \|f\|_\infty \leq 1\}$, V is a Banach algebra. If E is the closed unit interval with the usual ordering, one obtains the Banach algebra of functions of bounded variation in this way.

When A separates points of E, the algebra shares with the algebra of functions of bounded variation the property that every primary ideal is maximal. However, an example is given to show that, in general, not every closed ideal is the intersection of maximal ones. For, let E be the points of the closed unit square $[0, 1]^2$ whose abscissae are either 0 or reciprocals of integers, and let $(x, y) \leq (u, v)$ if and only if $x = u$ and $y \leq v$. Then the ideal of functions in V which vanish in a neighbourhood of the edge $\{(x, y) : (x, y) \in E, \ x = 0\}$ is not norm dense in the ideal of functions vanishing on the edge.

68.29 R. Blum (University of Saskatchewan)
On a Generalization of Steiner's Quartic with 3 Cusps

Let $\Gamma \equiv \phi(u, v) + (u^2 + v^2)^n = 0$, where u, v are the non-homogeneous Hesse-coordinates and $\phi(u, v)$ is a homogeneous polynomial of degree $2n + 1$ in u and v, be the equation of a class curve in the euclidean plane. It is shown that its $2n + 1$ cusps, which do not lie on the line at infinity, can be obtained as the intersection of Γ with a curve whose equation is $\Delta \equiv \psi(u, v) + (u^2 + v^2)^{n-1} = 0$, where $\psi(u, v)$ is a homogeneous polynomial of degree $2n - 1$ in u, v.

If we impose upon the cuspidal tangents the condition that they be concurrent (which is identically satisfied when $n = 1$) Γ admits $2n + 1$ axes of symmetry and, therefore, its cusps are the vertices of a regular $(2n + 1)$ - gon. The case $n = 1$ yields Steiner's quartic with 3 cusps.

Considerations of duality lead to the following property: If the $2n + 1$ points of inflection of the curve $C \equiv \phi(x, y) + (x^2 + y^2)^n = 0$ are collinear (on a line ℓ) then the lines joining them with the origin form equal angles. However, there is no symmetry of the curve itself except when $\ell = \ell_\infty$.

68.30 J. Poland (Carleton University) and A.H. Rhemtulla
A Generalization of Hamiltonian Groups

Groups whose subgroups are all normal are called Hamiltonian groups, and their structure is well-known. Now, the core H_G of a subgroup H of the group G is defined as the intersection of the conjugates of H, or alternatively as the maximal subgroup of H which is normal in G. Dr. A.H. Rhemtulla (University of Alberta, Edmonton) and the author have been considering using the concept of core to obtain the following generalization of the Hamiltonian groups: let $\mathfrak{X}$ be a class of finite groups; we call a finite group G an $\mathfrak{X}$- Core group if $H/H_G \in \mathfrak{X}$ for all subgroups H of G (finite Hamiltonian groups are $\{1\}$ - Core groups). We first took $\mathfrak{X} = \mathfrak{A}$, the class of finite abelian groups. Our major result is: $\mathfrak{A}$ - Core groups have nilpotent derived group. We are now in the process of extending this result; for example, if $\mathfrak{X}$ is closed under the operation of taking factor groups, then if G is a solvable $\mathfrak{X}$ - Core group and Fit(G) is the Fitting subgroup, then $G/\mathrm{Fit}(G) \in \mathfrak{X}$. Our second major result is that if $\mathfrak{S}$ is the class of finite simple groups and G is an $\mathfrak{S}$ - Core group, then G is solvable and $G/Z_\infty(G)$ has abelian Sylow subgroups ($Z_\infty(G)$ is the hypercenter of G). This result does not hold for $\mathfrak{A}$- Core groups, where we originally conjectured it. (A summary of the earlier results was presented to the mini-conference on group theory held at the University of Manitoba at the end of the 1968 Summer Research Institute which the authors both attended.)

68.31 E. Hotzel
Remarks on Simple Cancellative Semigroups

(1) Any simple cancellative semigroup which contains a minimal right ideal is a group.

(2) Any simple cancellative semigroup which is finitely generated as a right ideal is a group.

The first statement follows from well known facts about simple and right simple semigroups (cf. A.H. Clifford and G.B. Preston: The algebraic theory of semigroups, Providence, (1961, 1967) Vol. II, Lemma 8.13, and Vol. I, § 1.11). The second one is obtained by observing that a cancellative semigroup S of the form $S = a_1 S \cup a_2 S \cup \ldots \cup a_n S$ contains an idempotent. A simple cancellative semigroup containing an idempotent is easily seen to be a group (Clifford and Preston, Vol. I, p. 51, ex. 11).

THEOREM Any cancellative semigroup without idempotents can be embedded in a simple cancellative semigroup without idempotents.

If S is a cancellative idempotent-free semigroup then a simple cancellative semigroup containing S can be obtained as the union of a chain $S = S_o \subseteq S_1 \subseteq S_2 \subseteq \ldots$ of cancellative idempotent-free semigroups. Take S_{i+1} to be essentially the semigroup which is generated by the set $X_i = U_i \cup S_i \cup V_i$ under the relations $rs = t$, if $rs = t$ in S_i, and $u_{s,t} s v_{s,t} = t$ $(s, t \in S_i)$ where U_i, V_i are sets such that U_i, S_i, V_i are pairwise disjoint and where $(s, t) \to u_{s,t}$, $(s, t) \to v_{s,t}$ are one-to-one mappings from $S_i \times S_i$ onto U_i and V_i, respectively.

It can be seen by the Theorem in connexion with the Malcev example of a cancellative semigroup which is not embeddable in a group (A.I. Malcev, Math. Ann. 113, (1937) 686-691) that not every simple cancellative semigroup can be embedded in a group (cf. Clifford and Preston, Vol. I, p. 51).

68.32 C. Davis (University of Toronto)
An Inequality for Hilbert-space Operators

The 'shell' of an operator A on Hilbert space $\mathcal{H}$ is the set of triples $(||Ax||^2, x^*Ax, ||x||^2)$ as x ranges over non-zero elements of $\mathcal{H}$. Two such triples being identified if they differ only by multiplication by a positive number, the shell can be represented as a point-set $s(A)$ in real 3-space. Many relationships are known (Ch. Davis, Acta Sci. Math. Szeged 29 (1968), 69-86) between properties of A and geometric properties of $s(A)$. In the course of studying the question (still unsolved) of characterizing those point-sets which can be $s(A)$ for some A, the author found the following curious theorem: Assume $||A|| = 1$. Let $\pi/2 > \nu \geqq \arcsin(-1/3)$. Then for every $\epsilon > 0$ there exists non-zero $x \in \mathcal{H}$ for which

$$(1+\epsilon)2\sqrt{2}\cos\nu\,|x^*Ax| \geqq (1+\sin\nu)||x||^2 + (1-3\sin\nu)||Ax||^2 .$$

(The case $\nu = \arcsin(1/3)$ is a previously known result. The limiting case $\nu \to \pi/2$ is trivial.) In this note the theorem is related to properties of the shell.

68.33 W. Kahan and C. Davis (University of Toronto)
The Rotation of Eigenvectors by a Perturbation

When a Hermitian linear operator A is slightly perturbed, by how much can its invariant subspaces change? Given some approximations to a cluster of neighbouring eigenvalues and to the corresponding eigenvectors of a real symmetric matrix, and given a lower bound $\delta > 0$ for the gap that separates the cluster from all other eigenvalues, how much can the subspace spanned by the eigenvectors differ from that spanned by our approximations? These questions are closely related: both are investigated here. First, the difference between the two subspaces is characterized in terms of certain angles through which one subspace must be rotated in order most directly to reach the other. The angles constitute the spectrum of a Hermitian operator Θ, with which is associated a commuting skew-Hermitian operator $J = -J^3$; the unitary operator that differs least from the identity and rotates one subspace into the other turns out to be $\exp(J\Theta)$. These operators unify the treatment of natural geometric, operator-theoretic and error-analytic questions concerning those subspaces. Given the gap δ, and given bounds upon either the perturbation (1st question) or a computable residual (2nd question), we obtain sharp bounds upon trigonometric functions of Θ. For example, let one subspace be the invariant subspace of A associated with that part of A's spectrum in some interval, let the other subspace be the invariant subspace of $A + H$ associated with that part of $(A + H)$'s spectrum lying no further than δ from the same interval, and let Θ be the angle-operator "between" the subspaces; then $\delta||\sin\Theta|| \leqq ||H||$ for every unitary-invariant operator norm $||\ldots||$.

68.34 P. Rosenthal and H. Radjavi (University of Toronto)
Matrices for Operators and Generators of $B(\mathcal{H})$

Let $\mathcal{H}$ be a separable complex Hilbert space.

THEOREM. *If A is a bounded operator on $\mathcal{H}$ and is not a multiple of the identity, then there exists an orthonormal basis $\{e_n\}$ for $\mathcal{H}$ such that every entry in the matrix of A with respect to $\{e_n\}$ is non-zero.*

The proof of this theorem is very elementary.

COROLLARY 1. *If A is not a multiple of the identity then there exists a compact Hermitian operator K such that A and K have no common invariant subspaces.*

Corollary 1 follows immediately from the Theorem. A theorem of Arveson's then gives

COROLLARY 2. *If A is not a multiple of the identity then there exists a compact Hermitian operator K such that the weakly closed algebra generated by A and K is $B(\mathcal{H})$, (the algebra of all bounded operators on $\mathcal{H}$).*

In case $\mathcal{H}$ is finite-dimensional we get

COROLLARY 3. *If A is an $n \times n$ matrix that is not a multiple of I then there exists a Hermitian matrix K such that every $n \times n$ matrix is a polynomial in A and K.*

68.35 PL. Kannapan and S. Kurepa (Universities of Waterloo and Zagreb)
Some Relation between Additive Functions

Concerning the Cauchy functional equation

$$(1) \qquad f(x+y) = f(x) + f(y), \quad x, y \in R \quad (R, \text{ real numbers})$$

I. Halperin has raised a question, whether $f: R \to R$, satisfying (1) and $f(x) = x^2 f(\frac{1}{x})$, $x \neq 0$, is necessarily continuous or not? Answer to this and some generalizations were given by many including the second author. In this direction, the following more general problem will be of interest.

Problem. Let $U_i(x)$ be rational functions in X, P_i be continuous on R except at the singular points of $U_i(x)$, and f_i be additive on R such that

(A) $$\sum_{i=1}^{n} P_i(x)\, f_i(U_i(x)) = 0,$$

for all x on which P_i's are defined. Whether the f_i's are continuous or derivatives (i.e. f_i's satisfy $f(xy) = xf(y) + yf(x)$) or functions obtained from these two?

The following results are established:

THEOREM 1. <u>Let</u> $f\ (\neq 0)$ <u>and</u> g <u>be additive functions from</u> R <u>into</u> R <u>and satisfy</u>

(2) $$f(x^n) = P(x)\ g(x^m)$$

for all $x \neq 0$, with P from $R_o = R - \{0\}$ into R as a continuous function such that $P(1) = 1$, m and n are integers. Then $P(x) = x^{n-m}$; further, if $F(x) = f(x) - f(1)x$ and $G(x) = g(x) - g(1)x$, then F and G are derivatives, and $nf(x) = mG(x)$, except when

(i) $n = 0$, $m = 0$, in which case there is nothing to prove,

(ii) $n = m$, in which case $f = g$,

(iii) $n = 0$, $m \gtrless 0$ in which case $G = 0$ is a derivative and f is arbitrary, and

(iv) $m = 0$, $n \gtrless 0$ in which case $F = 0$ is a derivative and g is arbitrary. Conversely, if F and G are derivatives on R and $f(x) = ax + F(x)$, $g(x) = ax + G(x)$, where a is any real number, $mG(x) = nF(x)$ and $P(x) = x^{n-m}$, where m and n are integers, then f, g and P satisfy (2) for all $x \in R$.

THEOREM 2. <u>Let</u> $F\ (\neq 0)$ <u>and</u> $g\ (\neq 0)$ <u>be real additive functions and</u> P, <u>a continuous function on</u> $R - \{a, b\} = R^*$, $(a \neq b)$ <u>into</u> R, <u>such that</u>

(26) $$f\left(\frac{1}{x-a}\right) = P(x)\ g\left(\frac{1}{x-b}\right), \quad \text{for all } x \in R^* .$$

Then $f(x) = Ag(x)$ for all real x, A a real constant. Further, f and g are continuous if and only if $P(x)$ is proportional to $\frac{x-b}{x-a}$. If f and g are not continuous, then

(i) $P(x) = A \dfrac{(x-b)^2}{(x-a)^2}$, ($A$, a non-zero constant).

https://doi.org/10.1017/S0008439500029866 Published online by Cambridge University Press

(ii) $h(x) = g(x) - g(1)x$ is a derivative on R, and

(iii) $h(a-b) = g(1)(a-b)$.

Conversely, if P, f, g, h and numbers A, a and b satisfy conditions (i), (ii), (iii) and $f = A g$, then (26) holds true.

68.36 P. Greiner (University of Toronto)
An Asymptotic Expansion for the Heat Equation

Let M be a compact n-dimensional C^{∞} manifold without boundary. Let E and F be two C^{∞} complex vectorbundles of fiber dimension N over M. Let $P(x, D)$ be a strongly elliptic smooth linear partial differential operator of order m sending C^{∞} sections of E into C^{∞} sections of F. Denote by dx a density over M and by $(\cdot, \cdot)$ hermitian structures over E and F. These can be introduced locally and extended to all of M by a partition of unity. The heat operator is given by

$$(1) \qquad L = \frac{\partial}{\partial t} + P(x, D)$$

defined on $C^{\infty}(E) \times C_0^{\infty}(R)$.

Now let Ω be a precompact submanifold of M with a C^{∞} boundary ω. Let $G_1, \ldots, G_\mu$ be bundles on ω and let $B_1, \ldots, B_\mu$ be boundary differential operators of the form

$$(2) \qquad B_j u = \sum_{k=0}^{m-1} B_{jk} \gamma_k u ,$$

where B_{jk} is a differential operator from E to G_j in ω of order $m_j - k$ and $\gamma_k u$ is the k-th normal derivative of u valued on ω. Our principal assumption is that B is elliptic with respect to $P(x, D) + i\tau$ for all τ with $\operatorname{Im} \tau \leq 0$. For example, this is satisfied by all strongly elliptic operators $P(x, D)$ with Dirichlet boundary conditions.

Let $G(t, x, y)$ be the Green's matrix for the boundary problem (L, B) and let

$$(3) \qquad G(t) = \int_M \operatorname{Trace}\ G(t, x, x)dx .$$

Then we have

THEOREM. Let (L, B) be the above boundary problem. Then

(4) $$G(t) \sim C_0 t^{-n/m} + C_1 t^{-n/m+1/m} + C_2 t^{-n/m+2/m} + \ldots$$

as $t \downarrow 0$. Furthermore the coefficients C_j, $j = 0, 1, 2, \ldots,$ can be evaluated explicitly in terms of the coefficients of $P(x, D)$ and B.

A consequence of the theorem is the following:

COROLLARY. Let $(-\Delta, D)$ be the negative Laplacian with Dirichlet boundary conditions in some precompact domain Ω in in the plane with smooth boundary ω. Let $\mu_0, \mu_1, \mu_2, \ldots$ be the eigenvalues for $(-\Delta, D)$. Then

(5) $$\sum_{j=0}^{\infty} e^{-\mu_j t} = \frac{|\Omega|}{4\pi t} - \frac{|\omega|}{4\sqrt{2\pi t}} + \frac{1}{6}(1-h) + 0(\sqrt{t})$$

as $t \downarrow 0$, where $|\Omega|$ and $|\omega|$ denote the area and length of Ω and ω, respectively, and h is the number of holes in Ω.

This corollary was the motivation for the investigation. It was originally conjectured by Kac and proved by McKean and Singer.

68.37 C.Y. Chan (University of Toronto)
A Two-Phase Stefan Problem with Arbitrary Rate of Liquid Removal

The two-phase Stefan problem is the problem of solving two heat equations in two regions separated by an unknown moving surface which must also be determined. In general, the interface is not necessarily monotonic. This is actually the basic difference between the single-phase and the two-phase problems.

Physically, we can think of our problem as a finite slab of solid in contact with its liquid of finite length; heat is taken away from the free end of the solid at a rate $f_1(t)$ while the temperature $f_2(t)$ is specified at the free end, $a(t)$, of the liquid which is removed at an arbitrarily prescribed rate. Mathematically, the problem is formulated as follows: Find $0 < s(t) < 1$, $u_1(x, t)$ and $u_2(x, t)$ such that

$$\kappa_1 u_{1_{xx}}(x,t) = u_{1_t}(x,t) \quad \text{for} \quad 0 < x < s(t), \; t > 0,$$

$$u_1(x,0) = \phi_1(x) \quad \text{where} \quad \phi_1(x) \le 0, \; 0 \le x \le b, \quad \text{and} \quad \phi_1(b) = 0$$

$$k_1 u_{1_x}(0,t) = f_1(t) \quad \text{where} \quad f_1(t) \ge 0, \; t > 0,$$

$$L\rho\, ds(t)/dt = k_1 u_{1_x}(s(t),t) - k_2 u_{2_x}(s(t),t)$$

$$\text{where} \quad s(0) = b > 0, \; 0 < s(t) < a(t) \le 1, \; t > 0,$$

$$u_1(s(t),t) = 0 = u_2(s(t),t) \quad \text{for} \quad t > 0,$$

$$\kappa_2 u_{2_{xx}}(x,t) = u_{2_t}(x,t) \quad \text{for} \quad s(t) < x < a(t) \le 1, \; t > 0,$$

$$u_2(x,0) = \phi_2(x) \quad \text{where} \quad \phi_2(x) \ge 0, \; 0 < b \le x \le 1,$$

$$\text{and} \quad \phi_2(b) = 0, \; \phi_2(1) = f_2(0),$$

$$u_2(a(t),t) = f_2(t) \quad \text{where} \quad f_2(t) \ge 0, \; t > 0, \quad \text{and} \quad a(0) = 1.$$

κ_i $(i = 1, 2)$ denote the respective diffusivities of the two phases; k_i $(i = 1, 2)$ denote the respective conductivities; L is the latent heat; ρ is the density of the solid and the liquid; $x = s(t)$ is the unknown free boundary, and $u_i(x,t)$ $(i = 1, 2)$ are the respective temperatures.

THEOREM. *If* $\phi_1(x)$ $(0 \le x \le b)$, $\phi_2(x)$ $(b \le x \le 1)$, $f_2(t)$ $(0 \le t < \infty)$ *and* $a(t)$ $(0 \le t < \infty)$ *are continuously differentiable, and* $f_1(t)$ $(0 \le t < \infty)$ *is continuous, then there exists one and only one solution* $u_1(x,t)$, $u_2(x,t)$ *and* $s(t)$ *of the problem for all* $t < \infty$.

68.38 W.A. O'N. Waugh (University of Toronto)

Transformation of a Birth Process into a Poisson Process

Let $\{Z_t : t > 0\}$ be the Markovian pure birth process with linear birth rates, that is

$$P\{Z_{t+\delta t} = j+1 \mid Z_t = j\} = j\lambda\,\delta t + o(\delta t) \qquad j = 1, 2, \ldots$$

where $Z_0 = 1$ and where all other transitions have a probability that is $o(\delta t)$. It is known that $Z_t e^{-\lambda t} \to W$ which is a random variable having density e^{-W}. Let the time during which $Z_t = j$ ("sojourn time") be X_j. Then $\sum_{i=1}^{\infty} (X_i - \xi X_i) (= S)$ is convergent. We prove two theorems: (1) $W = \exp\{-\lambda S - \gamma\}$, where γ is Euler's constant; (2) Where $T_n = X_1 + \ldots + X_n$ and $T_n^* = W(e^{\lambda T_n} - 1)$, then, given S, or equivalently W, the joint distribution of $T_1^*, \ldots, T_n^*$ is that of the first m epochs in a Poisson stream of rate 1. Theorem (2) has been proved by analytic methods by D.G. Kendall and the purpose of this latter part of the present work is to show the connection with the theory of random series.

68.39 M.T. Wasan (Queen's University)
Sufficient Conditions for a First Passage Time Process to be that of Brownian Motion

Let

$$T_X = \inf\{ t > 0 \mid X(t) > x\}$$

where X and T_X are respectively the state and passage time variables of the strong Markov process and are both random variables, and let

$$F(t, x; x + \Delta x) = \Pr[T_{X+\Delta X} \leq \tau \mid T_X = t]$$

be the transition probability distribution such that when the state variable take the value $x + \Delta x$, the passage time $T_{x+\Delta x}$ takes a value less than or equal to τ given that the time variable T_X takes the value t when the state variable takes the value x. We denote the transition density function of $F(t_0, x_0, t, x)$ by $f(t_0, X_0; t, x)$ and when $t_0 = 0$, $X_0 = 0$ by $f(t, x)$.

Now we assume the following set of conditions.

(a) $$\lim_{\Delta x \to 0} \frac{1}{\Delta x} \int_{|\tau - t| > \delta} d_\tau F(t, x; \tau, x + \Delta x) = 0 \text{ for any } \delta = 0$$

(b) $$\lim_{\Delta x \to 0} \frac{1}{\Delta x} \int_{|\tau - t| < \delta} (\tau - t) d_\tau F(t, x; \tau, x + \Delta x) = 1$$

(c) $$\lim_{\Delta x \to 0} \frac{1}{\Delta x} \int_{|\tau - t| \leq \delta} (\tau - t)^2 d_\tau F(t, x; \tau, x + \Delta x) = 1 .$$

THEOREM. Let the strong Markov process $X(T)$ satisfy the conditions (a), (b) and (c). Further, let us assume that the transition density function $f(t, x)$ exists and is such that the derivatives

$$(1) \qquad \frac{\partial f}{\partial x}, \quad \frac{\partial f}{\partial t} \quad \text{and} \quad \frac{\partial^2 f}{\partial t^2}$$

exist and are continuous. Then $f(t, x)$ satisfies the differential equation

$$(2) \qquad \frac{\partial f}{\partial x} = -\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial t^2},$$

when $\int_0^\infty f(t, x)\,dx = 1$, $f(t, x) > 0$ for $x > 0$ and $t > 0$ $f(\infty, x) = 0$,

$f(t, \infty) = 0$ and $f(0, 0) = 1$. Then it is shown that

$$f(t, x) = \frac{t}{\sqrt{2\pi x^3}}\, e^{-\frac{(x-t)^2}{2x}} \qquad x > 0, \quad t > 0$$

$$= 0 \qquad\qquad \text{otherwise}.$$

Now we give another set of conditions which leads to the same density for a stochastic process $\{X(t), t \geq 0\}$.

(i) Let $X(t) = \delta + X(t - \delta - W(\delta))$

(approximately for small δ and for the paths of $W(\delta)$ which do not reach the line $y = t - x$ in the interval $(0, \delta)$) where δ is any positive number and $W(\delta)$ is a Brownian motion process such that $E(W(\delta)) = 0$ and $\mathrm{Var}[W(\delta)] = \delta$.

(ii) $X(t) = X_1(t/n) + X_2(t/n) + \ldots + X_n(t/n)$ i.e. $X(t)$ is infinitely divisible process.

With the help of this density a stochastic process is defined and its existence is proved. Furthermore, in a series of papers its properties are investigated.

68.40 J. Csima (McMaster University)
Extremal Multidimensional Stochastic Matrices and Patterns

This paper deals with combinatorial properties of multidimensional matrices. The results are related to the Jurkat-Ryser classification problem of multidimensional stochastic matrices and extremal stochastic matrices. The main tool in this paper is a covering technique, developed earlier by the author for the purpose of dealing with Latin squares and multidimensional (0, 1)-matrices.

The order in which matrices and their patterns are introduced is important. First, patterns are defined as sets of d-tuples. Then restricted patterns and critical patterns are defined by simple covering criteria. After all this is done, matrices are defined and patterns are associated with them. This way a clear-cut distinction is established and maintained between those pattern properties that depend on the definition of a stochastic class and those that do not.

For multidimensional matrices König's theorem is not true in the sense that the covering number (of degree e) does not necessarily equal the term rank (of degree e) of the matrix. This spoils the possibility of trivial generalizations of important two-dimensional theorems.

The results of the paper include the proof that stochastic patterns are restricted and that only extremal matrices can have critical patterns. Among other things it is shown that the covering number of stochastic matrices of dimension d, degree e and order n is exactly n^{d-e}, and that extremal stochastic matrices are either permutation matrices or else have term rank less than n^{d-e}. An extremal 3-dimensional line-stochastic matrix is constructed which is not a permutation matrix.

Multidimensional (not necessarily stochastic) matrices are also dealt with, and higher dimensional analogues of 3-dimensional theorems of Jurkat and Ryser are given.