

## BETTI NUMBERS FOR CERTAIN COHEN–MACAULAY TANGENT CONES

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### Abstract

We compute Betti numbers for a Cohen–Macaulay tangent cone of a monomial curve in the affine 4-space corresponding to a pseudo-symmetric numerical semigroup. As a byproduct, we also show that for these semigroups, being of homogeneous type and homogeneous are equivalent properties.

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### 1. Introduction

Let  $S = \langle n_1, \dots, n_k \rangle = \{u_1 n_1 + \dots + u_k n_k \mid u_i \in \mathbb{N}\}$  be a numerical semigroup generated by the positive integers  $n_1, \dots, n_k$  with  $\gcd(n_1, \dots, n_k) = 1$ . For a field  $K$ , let  $A = K[X_1, X_2, \dots, X_k]$  and let  $K[S]$  be the semigroup ring  $K[t^{n_1}, t^{n_2}, \dots, t^{n_k}]$  of  $S$ . Then  $K[S] \simeq A/I_S$ , where  $I_S$  is the kernel of the surjection  $\phi_0 : A \rightarrow K[S]$ , associating  $X_i$  to  $t^{n_i}$ . If  $C_S$  is the affine curve with parameterisation

$$X_1 = t^{n_1}, X_2 = t^{n_2}, \dots, X_k = t^{n_k}$$

corresponding to  $S$  and  $1 \notin S$ , then the curve is singular at the origin. The smallest minimal generator of  $S$  is called the *multiplicity* of  $C_S$ . To understand this singularity, it is natural to study algebraic properties of the local ring  $R_S = K[[t^{n_1}, \dots, t^{n_k}]]$  with the maximal ideal  $\mathfrak{m} = \langle t^{n_1}, \dots, t^{n_k} \rangle$  and its associated graded ring

$$gr_{\mathfrak{m}}(R_S) = \bigoplus_{i=0}^{\infty} \mathfrak{m}^i / \mathfrak{m}^{i+1} \cong A/I_S^*,$$

where  $I_S^* = \langle f^* \mid f \in I_S \rangle$  with  $f^*$  denoting the least homogeneous summand of  $f$ . When  $K$  is algebraically closed,  $K[S]$  is the coordinate ring of the monomial curve  $C_S$  and

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$gr_m(R_S)$  is the coordinate ring of its tangent cone. A natural set of invariants for these coordinate rings is the Betti sequence. We refer to Stamate’s survey [12] for a comprehensive literature on this subject. The Betti sequence  $\beta(M) = (\beta_0, \dots, \beta_{k-1})$  of an  $A$ -module  $M$  is the sequence consisting of the ranks of the free modules in a minimal free resolution  $\mathbf{F}$  of  $M$ , where

$$\mathbf{F} : 0 \longrightarrow A^{\beta_{k-1}} \longrightarrow \dots \longrightarrow A^{\beta_1} \longrightarrow A^{\beta_0}.$$

When  $\beta(A/I_S^*) = \beta(K[S])$ , the semigroup  $S$  is said to be of homogeneous type as defined in [6]. In particular, if a semigroup is of homogeneous type then the Betti sequence of its Cohen–Macaulay tangent cone can be obtained from a minimal free resolution of  $K[S]$ . To take advantage of this idea, Jafari and Zarzuela Armengou introduced the concept of a *homogeneous* semigroup in [8]. When the multiplicity of a monomial curve corresponding to a homogeneous semigroup is  $n_i$ , homogeneity guarantees the existence of a minimal generating set for  $I_S$  whose image under the map

$$\pi_i : A \rightarrow \bar{A} = K[X_1, \dots, \bar{X}_i, \dots, X_k]$$

is homogeneous, where  $\pi(X_i) = \bar{X}_i = 0$  and  $\pi(X_j) = X_j$  for  $i \neq j$ . Together with the assumption of a Cohen–Macaulay tangent cone, this property is inherited by a standard basis of  $I_S$  and the authors of [8] were able to prove that  $S$  is of homogeneous type. The converse is not true in general: there exists a 3-generated numerical semigroup with a complete intersection tangent cone which is of homogeneous type but not homogeneous; see [8, Example 3.19]. They also ask in [8, Question 4.22] if there are 4-generated semigroups of homogeneous type which are not homogeneous having noncomplete intersection tangent cones. Since homogeneous-type semigroups have Cohen–Macaulay tangent cones, we restrict our attention to monomial curves having Cohen–Macaulay tangent cones in this article.

The problem of determining the Betti sequence for the tangent cone (see [12, Problem 9.9]) was studied for 4-generated symmetric monomial curves by Mete and Zengin [10]. In this paper, we focus on the next interesting case of 4-generated pseudo-symmetric monomial curves. Using the standard bases we obtained in [11], we determine the Betti sequence for the tangent cone, addressing [12, Problem 9.9] for 4-generated pseudo-symmetric monomial curves having Cohen–Macaulay tangent cones, and prove that being homogeneous and being of homogeneous type are equivalent, answering [8, Question 4.22]. So, in most cases, there is no 4-generated pseudo-symmetric numerical semigroup of homogeneous type which is not homogeneous. Before we state our main result, let us recall from [9] that a 4-generated semigroup  $S = \langle n_1, n_2, n_3, n_4 \rangle$  is pseudo-symmetric if and only if there are integers  $\alpha_i > 1$ , for  $1 \leq i \leq 4$ , and  $\alpha_{21} > 0$  with  $\alpha_{21} < \alpha_1 - 1$  such that

$$\begin{aligned} n_1 &= \alpha_2 \alpha_3 (\alpha_4 - 1) + 1, \\ n_2 &= \alpha_{21} \alpha_3 \alpha_4 + (\alpha_1 - \alpha_{21} - 1)(\alpha_3 - 1) + \alpha_3, \\ n_3 &= \alpha_1 \alpha_4 + (\alpha_1 - \alpha_{21} - 1)(\alpha_2 - 1)(\alpha_4 - 1) - \alpha_4 + 1, \\ n_4 &= \alpha_1 \alpha_2 (\alpha_3 - 1) + \alpha_{21} (\alpha_2 - 1) + \alpha_2. \end{aligned}$$

TABLE 1. Examples of each case.

$\alpha_{21}$	$\alpha_1$	$\alpha_2$	$\alpha_3$	$\alpha_4$	$n_1$	$n_2$	$n_3$	$n_4$	$\beta_0$	$\beta_1$	$\beta_2$	$\beta_3$
2	5	3	2	2	7	12	13	22	1	5	6	2
2	4	4	2	4	25	19	22	26	1	5	6	2
2	4	4	2	5	33	23	28	26	1	5	7	3
2	5	4	2	4	25	20	35	30	1	6	9	4
1	3	2	3	3	13	14	9	15	1	5	6	2
3	6	3	4	6	61	82	51	63	1	6	8	3
1	3	2	2	4	13	11	12	9	1	5	6	2
1	4	2	2	4	13	12	19	11	1	5	7	3

Then the toric ideal  $I_S$  is given by  $I_S = \langle f_1, f_2, f_3, f_4, f_5 \rangle$  with

$$f_1 = X_1^{\alpha_1} - X_3 X_4^{\alpha_4 - 1}, \quad f_2 = X_2^{\alpha_2} - X_1^{\alpha_{21}} X_4, \quad f_3 = X_3^{\alpha_3} - X_1^{\alpha_1 - \alpha_{21} - 1} X_2,$$

$$f_4 = X_4^{\alpha_4} - X_1 X_2^{\alpha_2 - 1} X_3^{\alpha_3 - 1}, \quad f_5 = X_1^{\alpha_{21} + 1} X_3^{\alpha_3 - 1} - X_2 X_4^{\alpha_4 - 1}.$$

The Betti sequence of  $K[S]$  for a 4-generated pseudo-symmetric semigroup is  $\beta(K[S]) = (1, 5, 6, 2)$  by [1]. Hence,  $S$  is of homogeneous type if and only if the Betti sequence of the tangent cone is also  $\beta(A/I_S^*) = (1, 5, 6, 2)$ . We refer the reader to [3] for the Betti sequence of  $K[S]$  for 4-generated almost-symmetric semigroups.

Our main result is as follows.

**THEOREM 1.1.** *Let  $S$  be a 4-generated pseudo-symmetric semigroup with a Cohen–Macaulay tangent cone. Then the Betti sequence  $\beta(A/I_S^*)$  of the tangent cone is:*

- $\beta(A/I_S^*) = (1, 5, 6, 2)$  if  $n_1$  is the multiplicity;
- $\beta(A/I_S^*) = (1, 5, 6, 2)$  if  $n_2$  is the multiplicity and  $\alpha_1 = \alpha_4$ ;  
 $\beta(A/I_S^*) = (1, 5, 7, 3)$  if  $n_2$  is the multiplicity and  $\alpha_1 < \alpha_4$ ;  
 $\beta(A/I_S^*) = (1, 6, 9, 4)$  if  $n_2$  is the multiplicity and  $\alpha_1 > \alpha_4$ ;
- $\beta(A/I_S^*) = (1, 5, 6, 2)$  if  $n_3$  is the multiplicity and  $\alpha_2 = \alpha_{21} + 1$ ;  
 $\beta(A/I_S^*) = (1, 6, 8, 3)$  if  $n_3$  is the multiplicity and  $\alpha_2 < \alpha_{21} + 1$ ;
- $\beta(A/I_S^*) = (1, 5, 6, 2)$  if  $n_4$  is the multiplicity and  $\alpha_3 = \alpha_1 - \alpha_{21}$ ;  
 $\beta(A/I_S^*) = (1, 5, 7, 3)$  if  $n_4$  is the multiplicity and  $\alpha_3 < \alpha_1 - \alpha_{21}$ .

We illustrate in Table 1 that there are pseudo-symmetric monomial curves with Cohen–Macaulay tangent cones in all of these cases.

We make repeated use of the following effective result as in [7, 8, 12] in order to reduce the number of cases for determining the Betti numbers of the tangent cones.

**LEMMA 1.2.** *Assume that the multiplicity of the monomial curve  $C_S$  is  $n_i$ . Suppose that the  $K$ -algebra homomorphism  $\pi_i : A \rightarrow \bar{A} = K[X_1, \dots, \bar{X}_i, \dots, X_k]$  is defined by  $\pi_i(X_i) = \bar{X}_i = 0$  and  $\pi_i(X_j) = X_j$  for  $i \neq j$ , and set  $\bar{I} = \pi_i(I_S^*)$ . If the tangent cone  $\text{gr}_m(R_S)$  is Cohen–Macaulay, then the Betti sequences of  $\text{gr}_m(R_S)$  and of  $\bar{A}/\bar{I}$  are the same.*

**PROOF.** If the tangent cone  $gr_m(R_S)$  is Cohen–Macaulay, then  $X_i$  is regular on  $A/I_S^*$ . The result follows from the well-known fact that Betti sequences are the same up to a regular sequence.  $\square$

Therefore, the problem of determining the Betti sequence of the tangent cone is reduced to computing the Betti sequence of the ring  $\bar{A}/\bar{I}$ . In all proofs about the minimal free resolution of  $\bar{A}/\bar{I}$  we use the following criterion by Buchsbaum–Eisenbud to confirm the exactness, leaving the not so difficult task of checking if it is a complex to the reader.

**THEOREM 1.3 [2, Corollary 2].** *Let*

$$0 \longrightarrow F_{k-1} \xrightarrow{\phi_{k-1}} \cdots \xrightarrow{\phi_2} F_1 \xrightarrow{\phi_1} F_0$$

*be a complex of free modules over a Noetherian ring  $A$ . Let  $\text{rank}(\phi_i)$  be the size of the largest nonzero minor of the matrix describing  $\phi_i$  and let  $I(\phi_i)$  be the ideal generated by the minors of maximal rank. Then the complex is exact if and only if:*

- (a)  $\text{rank}(\phi_{i+1}) + \text{rank}(\phi_i) = \text{rank}(F_i)$ ; and
- (b)  $I(\phi_i)$  contains an  $A$ -sequence of length  $i$

for  $1 \leq i \leq k - 1$ .

The structure of the paper is as follows. We treat the cases where  $S$  is homogeneous in the next section and, when  $S$  is not homogeneous, we find the minimal free resolution of the ring  $\bar{A}/\bar{I}$  in each subsequent section, completing the proof of Theorem 1.1 by virtue of Lemma 1.2. We refer the reader to [4] for the basics of commutative algebra as we use SINGULAR [5] in our computations.

## 2. Homogeneous cases

In this section, we characterise which pseudo-symmetric 4-generated semigroups are homogeneous. We start by recalling basic definitions from [8]. The Apéry set of  $S$  with respect to  $s \in S$  is defined to be  $AP(S, s) = \{x \in S \mid x - s \notin S\}$  and the set of lengths of  $s$  in  $S$  is

$$L(s) = \left\{ \sum_{i=1}^k u_i \mid s = \sum_{i=1}^k u_i n_i, u_i \geq 0 \right\}.$$

Note that  $L(s)$  is the set of standard degrees of monomials  $X_1^{u_1} \cdots X_k^{u_k}$  of  $S$ -degree  $\text{deg}_S(X_1^{u_1} \cdots X_k^{u_k}) = s$ . A subset  $T \subset S$  is said to be homogeneous if either it is empty or  $L(s)$  is a singleton for all  $s$  with  $0 \neq s \in T$ . If  $n_i$  is the smallest among  $n_1, n_2, \dots, n_k$ , the semigroup  $S$  is said to be *homogeneous* if the Apéry set  $AP(S, n_i)$  is homogeneous.

**PROPOSITION 2.1.** *Let  $S$  be a 4-generated pseudo-symmetric numerical semigroup. Then  $S$  is homogeneous if and only if:*

- $n_1$  is the multiplicity; or
- $n_2$  is the multiplicity and  $\alpha_1 = \alpha_4$ ; or

- $n_3$  is the multiplicity and  $\alpha_2 = \alpha_{21} + 1$ ; or
- $n_4$  is the multiplicity and  $\alpha_3 = \alpha_1 - \alpha_{21}$ .

**PROOF.** By [8, Corollary 3.10],  $S$  is homogeneous if and only if there exists a set  $E$  of minimal generators for  $I_S$  such that every nonhomogeneous element of  $E$  has a term that is divisible by  $X_i$  when  $n_i$  is the multiplicity. Şahin and Şahin [11, Corollary 2.4] states that indispensable binomials of  $I_S$  are  $\{f_1, f_2, f_3, f_4, f_5\}$  if  $\alpha_1 - \alpha_{21} > 2$  and are  $\{f_1, f_2, f_3, f_5\}$  if  $\alpha_1 - \alpha_{21} = 2$ . Therefore, they must appear in every minimal generating set. Let us take  $E = \{f_1, \dots, f_5\}$  in order to prove sufficiency of the conditions.

- Since each  $f_j$  ( $j = 1, \dots, 5$ ) has a term that is divisible by  $X_1$ , when  $n_1$  is the multiplicity,  $S$  is always homogeneous.
- The only binomial in  $E$  that has no monomial term divisible by  $X_2$  is  $f_1$ . Hence, when  $n_2$  is the multiplicity and  $\alpha_1 = \alpha_4$ , it follows that  $f_1$  and thus  $S$  is homogeneous.
- The only binomial in  $E$  that has no monomial term divisible by  $X_3$  is  $f_2$ . Hence, when  $n_3$  is the multiplicity and  $\alpha_2 = \alpha_{21} + 1$ ,  $f_2$  and thus  $S$  is homogeneous.
- Similarly, only  $f_3$  has no monomial term that is divisible by  $X_4$  and it is homogeneous when  $\alpha_3 = \alpha_1 - \alpha_{21}$ . Hence,  $S$  is homogeneous if  $n_4$  is the multiplicity.

For the necessity of these conditions, recall that  $f_1, f_2$  and  $f_3$  are indispensable, so they must be homogeneous when the multiplicity is  $n_2, n_3$  and  $n_4$ , respectively.  $\square$

### 3. The proof when the multiplicity is $n_1$

If the tangent cone is Cohen–Macaulay and the semigroup is homogeneous, it is known that the semigroup is of homogeneous type. When  $n_1$  is the multiplicity, the pseudo-symmetric semigroup is always homogeneous by Proposition 2.1 and hence the Betti sequence is  $(1, 5, 6, 2)$  in this case.

### 4. The proof when the multiplicity is $n_2$

Let  $n_2$  be the multiplicity and suppose that the tangent cone is Cohen–Macaulay. If  $\alpha_1 = \alpha_4$ , then the Betti sequence is  $(1, 5, 6, 2)$  by Proposition 2.1. We treat the cases  $\alpha_1 < \alpha_4$  and  $\alpha_1 > \alpha_4$  separately.

**4.1. The proof in the case  $\alpha_1 < \alpha_4$ .** In this case,  $\{f_1, f_2, f_3, f_4, f_5\}$  is a standard basis of  $I_S$  by [11, Lemma 3.8]. Since  $\bar{I}$  is the image of  $I_S^*$  under the map  $\pi_2$  sending only  $X_2$  to 0, it follows that  $\bar{I}$  is generated by

$$G_* = \{X_1^{\alpha_1}, X_1^{\alpha_{21}} X_4, X_3^{\alpha_3}, X_4^{\alpha_4}, X_1^{\alpha_{21}+1} X_3^{\alpha_3-1}\}.$$

We prove the claim by demonstrating that the complex

$$0 \longrightarrow A^3 \xrightarrow{\phi_3} A^7 \xrightarrow{\phi_2} A^5 \xrightarrow{\phi_1} A \longrightarrow 0$$

is a minimal free resolution of  $\bar{A}/\bar{I}$  by virtue of Lemma 1.2, where

$$\phi_1 = [X_1^{\alpha_1} \quad X_1^{\alpha_{21}} X_4 \quad X_3^{\alpha_3} \quad X_4^{\alpha_4} \quad X_1^{\alpha_{21}+1} X_3^{\alpha_3-1}],$$

$$\phi_2 = \begin{bmatrix} 0 & X_4 & 0 & 0 & X_3^{\alpha_3-1} & 0 & 0 \\ 0 & -X_1^{\alpha_1-\alpha_{21}} & X_1 X_3^{\alpha_3-1} & X_4^{\alpha_4-1} & 0 & 0 & -X_3^{\alpha_3} \\ X_1^{\alpha_{21}+1} & 0 & 0 & 0 & 0 & X_4^{\alpha_4} & X_1^{\alpha_{21}} X_4 \\ 0 & 0 & 0 & -X_1^{\alpha_{21}} & 0 & -X_3^{\alpha_3} & 0 \\ -X_3 & 0 & -X_4 & 0 & -X_1^{\alpha_1-\alpha_{21}-1} & 0 & 0 \end{bmatrix}$$

and

$$\phi_3 = \begin{bmatrix} -X_4 & 0 & 0 \\ 0 & X_3^{\alpha_3-1} & 0 \\ X_3 & X_1^{\alpha_1-\alpha_{21}-1} & 0 \\ 0 & 0 & -X_3^{\alpha_3} \\ 0 & -X_4 & 0 \\ 0 & 0 & X_1^{\alpha_{21}} \\ X_1 & 0 & -X_4^{\alpha_4-1} \end{bmatrix}.$$

It is easy to check that  $\text{rank } \phi_1 = 1$ ,  $\text{rank } \phi_2 = 4$ ,  $\text{rank } \phi_3 = 3$ . So, we show that  $I(\phi_i)$  contains a regular sequence of length  $i$  for all  $i = 1, 2, 3$ . Since this is obvious for  $i = 1$ , we only discuss the other cases. For the matrix  $\phi_2$ , the 4-minor corresponding to the rows 1, 2, 4, 5 and columns 1, 5, 6, 7 is computed to be  $-X_3^{3\alpha_3}$ . Similarly, the 4-minor corresponding to the rows 2, 3, 4, 5 and columns 1, 2, 4, 5 is  $X_1^{2\alpha_1}$ . As these minors are relatively prime, the ideal  $I(\phi_2)$  contains a regular sequence of length 2. The 3-minor of  $\phi_3$  corresponding to the rows 1, 5, 7 is  $-X_4^{1+\alpha_4}$ , to the rows 2, 3, 4 is  $X_3^{2\alpha_3}$  and to the rows 3, 6, 7 is  $X_1^{\alpha_1}$ . As they are powers of different variables, they constitute a regular sequence of length 3.

**4.2. The proof in the case  $\alpha_1 > \alpha_4$ .** In this case, a standard basis of  $I_S$  is  $\{f_1, f_2, f_3, f_4, f_5, f_6 = X_1^{\alpha_1+\alpha_{21}} - X_2^{\alpha_2} X_3 X_4^{\alpha_4-2}\}$  by [11, Lemma 3.8]. Since  $\bar{I}$  is the image of  $I_S^*$  under the map  $\pi_2$  sending only  $X_2$  to 0, it follows that  $\bar{I}$  is generated by

$$G_* = \{X_3 X_4^{\alpha_4-1}, X_1^{\alpha_{21}} X_4, X_3^{\alpha_3}, X_4^{\alpha_4}, X_1^{\alpha_{21}+1} X_3^{\alpha_3-1}, X_1^{\alpha_1+\alpha_{21}}\}.$$

We prove the claim by demonstrating that the complex

$$0 \longrightarrow A^4 \xrightarrow{\phi_3} A^9 \xrightarrow{\phi_2} A^6 \xrightarrow{\phi_1} A \longrightarrow 0$$

is a minimal free resolution of  $\bar{A}/\bar{I}$  by virtue of Lemma 1.2, where

$$\phi_1 = [X_3 X_4^{\alpha_4-1} \quad X_1^{\alpha_{21}} X_4 \quad X_3^{\alpha_3} \quad X_4^{\alpha_4} \quad X_1^{\alpha_{21}+1} X_3^{\alpha_3-1} \quad X_1^{\alpha_1+\alpha_{21}}],$$

$\phi_2$  is given by

$$\begin{bmatrix} -X_4 & 0 & 0 & 0 & 0 & X_1^{\alpha_{21}} & 0 & X_3^{\alpha_3-1} & 0 \\ 0 & 0 & -X_1^{\alpha_1} & -X_1 X_3^{\alpha_3-1} & -X_4^{\alpha_4-1} & -X_3 X_4^{\alpha_4-2} & 0 & 0 & X_3^{\alpha_3} \\ 0 & -X_1^{\alpha_{21}+1} & 0 & 0 & 0 & 0 & 0 & -X_4^{\alpha_4-1} & -X_1^{\alpha_{21}} X_4 \\ X_3 & 0 & 0 & 0 & X_1^{\alpha_{21}} & 0 & 0 & 0 & 0 \\ 0 & X_3 & 0 & X_4 & 0 & 0 & -X_1^{\alpha_1-1} & 0 & 0 \\ 0 & 0 & X_4 & 0 & 0 & 0 & X_3^{\alpha_3-1} & 0 & 0 \end{bmatrix}$$

and

$$\phi_3 = \begin{bmatrix} 0 & -X_1^{\alpha_{21}} & 0 & 0 \\ X_4 & 0 & 0 & 0 \\ 0 & 0 & -X_3^{\alpha_3-1} & 0 \\ X_3 & 0 & X_1^{\alpha_1-1} & 0 \\ 0 & X_3 & 0 & 0 \\ 0 & -X_4 & 0 & -X_3^{\alpha_3-1} \\ 0 & 0 & X_4 & 0 \\ 0 & 0 & 0 & X_1^{\alpha_{21}} \\ X_1 & 0 & 0 & -X_4^{\alpha_4-2} \end{bmatrix}.$$

It is easy to check that  $\text{rank } \phi_1 = 1, \text{rank } \phi_2 = 5, \text{rank } \phi_3 = 4$ . So, we show that  $I(\phi_i)$  contains a regular sequence of length  $i$  for all  $i = 1, 2, 3$ . Since this is obvious for  $i = 1$ , we only discuss the other cases. For the matrix  $\phi_2$ , the 5-minor corresponding to the rows 1, 2, 3, 5, 6 and columns 1, 3, 4, 5, 8 is computed to be  $-X_4^{1+2\alpha_4}$ . Similarly, the 5-minor corresponding to the rows 1, 2, 4, 5, 6 and columns 1, 2, 7, 8, 9 is  $-X_3^{3\alpha_3}$ . As these minors are powers of different variables, the ideal  $I(\phi_2)$  contains a regular sequence of length 2. The 4-minor of  $\phi_3$  corresponding to the rows 1, 4, 8, 9 is  $X_1^{2\alpha_{21}+\alpha_1}$ , to the rows 3, 4, 5, 6 is  $X_3^{2\alpha_3}$  and to the rows 2, 6, 7, 9 is  $-X_4^{1+\alpha_4}$ . As they are powers of different variables, they constitute a regular sequence of length 3.

### 5. The proof when the multiplicity is $n_3$

Suppose that the tangent cone is Cohen–Macaulay. If  $\alpha_2 = \alpha_{21} + 1$ , then the Betti sequence is  $(1, 5, 6, 2)$  by Proposition 2.1. If  $\alpha_2 < \alpha_{21} + 1$ , then by [11, Lemma 3.12] a minimal standard basis for  $I_S$  is either  $\{f_1, f_2, f_3, f_4, f_5, f_6 = X_1^{\alpha_1-1} X_4 - X_2^{\alpha_2-1} X_3^{\alpha_3}\}$  or  $\{f_1, f_2, f_3, f'_4 = X_4^{\alpha_4} - X_2^{\alpha_2-2} X_3^{2\alpha_3-1}, f_5, f_6\}$ . Since  $\pi_3$  sends only  $X_3$  to 0, it follows that in both cases the ideal  $\tilde{I} = \pi_3(I_S^*)$  is generated by

$$G_* = \{X_1^{\alpha_1}, X_2^{\alpha_2}, X_1^{\alpha_1-\alpha_{21}-1} X_2, X_4^{\alpha_4}, X_2 X_4^{\alpha_4-1}, X_1^{\alpha_1-1} X_4\}.$$

We prove the claim by demonstrating that the complex

$$0 \longrightarrow A^3 \xrightarrow{\phi_3} A^8 \xrightarrow{\phi_2} A^6 \xrightarrow{\phi_1} A \longrightarrow 0$$

is a minimal free resolution of  $\bar{A}/\bar{I}$  by virtue of Lemma 1.2, where

$$\phi_1 = [X_1^{\alpha_1} \quad X_2^{\alpha_2} \quad X_1^{\alpha_1-\alpha_{21}-1}X_2 \quad X_4^{\alpha_4} \quad X_2X_4^{\alpha_4-1} \quad X_1^{\alpha_1-1}X_4],$$

$$\phi_2 = \begin{bmatrix} 0 & -X_4 & 0 & 0 & 0 & 0 & X_2 & 0 \\ 0 & 0 & X_1^{\alpha_1-\alpha_{21}-1} & 0 & -X_4^{\alpha_4-1} & 0 & 0 & 0 \\ -X_4^{\alpha_4-1} & 0 & -X_2^{\alpha_2-1} & 0 & 0 & -X_1^{\alpha_{21}}X_4 & -X_1^{\alpha_{21}+1} & 0 \\ 0 & 0 & 0 & X_2 & 0 & 0 & 0 & X_1^{\alpha_1-1} \\ X_1^{\alpha_1-\alpha_{21}-1} & 0 & 0 & -X_4 & X_2^{\alpha_2-1} & 0 & 0 & 0 \\ 0 & X_1 & 0 & 0 & 0 & X_2 & 0 & -X_4^{\alpha_4-1} \end{bmatrix}$$

and

$$\phi_3 = \begin{bmatrix} 0 & -X_2^{\alpha_2-1} & -X_1^{\alpha_{21}}X_4 \\ -X_2 & 0 & 0 \\ 0 & X_4^{\alpha_4-1} & 0 \\ 0 & 0 & -X_1^{\alpha_1-1} \\ 0 & X_1^{\alpha_1-\alpha_{21}-1} & 0 \\ X_1 & 0 & X_4^{\alpha_4-1} \\ -X_4 & 0 & 0 \\ 0 & 0 & X_2 \end{bmatrix}.$$

It is easy to check that  $\text{rank } \phi_1 = 1, \text{rank } \phi_2 = 5, \text{rank } \phi_3 = 3$ . So, we show that  $I(\phi_i)$  contains a regular sequence of length  $i$  for all  $i = 1, 2, 3$ . Since this is obvious for  $i = 1$ , we only discuss the other cases. For the matrix  $\phi_2$ , the 5-minor corresponding to the rows 1, 2, 3, 5, 6 and columns 1, 2, 4, 5, 8 is computed to be  $-X_4^{3\alpha_4-1}$ . Similarly, the 5-minor corresponding to the rows 2, 3, 4, 5, 6 and columns 1, 2, 3, 7, 8 is  $-X_1^{3\alpha_1-\alpha_{21}-1}$ . As these minors are powers of different variables, the ideal  $I(\phi_2)$  contains a regular sequence of length 2. The 3-minor of  $\phi_3$  corresponding to the rows 1, 2, 8 is  $-X_2^{\alpha_2+1}$ , to the rows 3, 6, 7 is  $-X_4^{2\alpha_4-1}$  and to the rows 4, 5, 6 is  $X_1^{2\alpha_1-\alpha_{21}-1}$ . As they are powers of different variables, they constitute a regular sequence of length 3.

### 6. The proof when the multiplicity is $n_4$

Suppose that the tangent cone is Cohen–Macaulay. If  $\alpha_3 = \alpha_1 - \alpha_{21}$ , then the Betti sequence is  $(1, 5, 6, 2)$  by Proposition 2.1. If  $\alpha_3 < \alpha_1 - \alpha_{21}$ , then a minimal standard basis for  $I_S$  is  $\{f_1, f_2, f_3, f_4, f_5\}$  by [11, Lemma 3.17]. Since  $\bar{I} = \pi_4(I_S^*)$ , under the map  $\pi_4$  sending only  $X_4$  to 0, it is generated by

$$G_* = \{X_1^{\alpha_1}, X_2^{\alpha_2}, X_3^{\alpha_3}, X_1X_2^{\alpha_2-1}X_3^{\alpha_3-1}, X_1^{\alpha_{21}+1}X_3^{\alpha_3-1}\}.$$

We prove the claim by demonstrating that the complex

$$0 \longrightarrow A^3 \xrightarrow{\phi_3} A^7 \xrightarrow{\phi_2} A^5 \xrightarrow{\phi_1} A \longrightarrow 0$$



is a minimal free resolution of  $\bar{A}/\bar{I}$  by virtue of Lemma 1.2, where

$$\phi_1 = \begin{bmatrix} X_1^{\alpha_1} & X_2^{\alpha_2} & X_3^{\alpha_3} & X_1 X_2^{\alpha_2-1} X_3^{\alpha_3-1} & X_1^{\alpha_{21}+1} X_3^{\alpha_3-1} \end{bmatrix},$$

$$\phi_2 = \begin{bmatrix} 0 & X_2^{\alpha_2} & 0 & 0 & X_3^{\alpha_3-1} & 0 & 0 \\ 0 & -X_1^{\alpha_1} & -X_1 X_3^{\alpha_3-1} & 0 & 0 & 0 & -X_3^{\alpha_3} \\ -X_1^{\alpha_{21}+1} & 0 & 0 & 0 & 0 & -X_1 X_2^{\alpha_2-1} & X_2^{\alpha_2} \\ 0 & 0 & X_2 & -X_1^{\alpha_{21}} & 0 & X_3 & 0 \\ X_3 & 0 & 0 & X_2^{\alpha_2-1} & -X_1^{\alpha_1-\alpha_{21}-1} & 0 & 0 \end{bmatrix}$$

and

$$\phi_3 = \begin{bmatrix} 0 & -X_2^{\alpha_2-1} & 0 \\ 0 & 0 & -X_3^{\alpha_3-1} \\ -X_3 & 0 & X_1^{\alpha_1-1} \\ 0 & X_3 & X_1^{\alpha_1-\alpha_{21}-1} X_2 \\ 0 & 0 & X_2^{\alpha_2} \\ X_2 & X_1^{\alpha_{21}} & 0 \\ X_1 & 0 & 0 \end{bmatrix}.$$

It is easy to check that  $\text{rank } \phi_1 = 1, \text{rank } \phi_2 = 4, \text{rank } \phi_3 = 3$ . So, we show that  $I(\phi_i)$  contains a regular sequence of length  $i$  for all  $i = 1, 2, 3$ . Since this is obvious for  $i = 1$ , we only discuss the other cases. For the matrix  $\phi_2$ , the 4-minor corresponding to the rows 1, 3, 4, 5 and columns 2, 3, 4, 7 is computed to be  $X_2^{3\alpha_2}$ . Similarly, the 4-minor corresponding to the rows 2, 3, 4, 5 and columns 1, 2, 4, 5 is  $-X_1^{2\alpha_1+\alpha_{21}}$ . As these minors are relatively prime, the ideal  $I(\phi_2)$  contains a regular sequence of length 2. The 3-minor of  $\phi_3$  corresponding to the rows 1, 5, 6 is  $-X_2^{2\alpha_2}$ , to the rows 2, 3, 4 is  $X_3^{1+\alpha_3}$  and to the rows 3, 6, 7 is  $-X_1^{\alpha_1+\alpha_{21}}$ . As they are powers of different variables, they constitute a regular sequence of length 3.

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### References

- [1] V. Barucci, R. Fröberg and M. Şahin, ‘On free resolutions of some semigroup rings’, *J. Pure Appl. Algebra* **218**(6) (2014), 1107–1116.
- [2] D. Buchsbaum and D. Eisenbud, ‘What makes a complex exact?’, *J. Algebra* **25** (1973), 259–268.
- [3] K. Eto, ‘Almost Gorenstein monomial curves in affine four space’, *J. Algebra* **488** (2017), 362–387.
- [4] G.-M. Greuel and G. Pfister, *A Singular Introduction to Commutative Algebra* (Springer, Berlin–Heidelberg, 2002).
- [5] G.-M. Greuel, G. Pfister and H. Schönemann, ‘SINGULAR 2.0. A computer algebra system for polynomial computations’, Centre for Computer Algebra, University of Kaiserslautern, 2001, <http://www.singular.uni-kl.de>.
- [6] J. Herzog, M. E. Rossi and G. Valla, ‘On the depth of the symmetric algebra’, *Trans. Amer. Math. Soc.* **296**(2) (1986), 577–606.

- [7] J. Herzog and D. I. Stamate, ‘On the defining equations of the tangent cone of a numerical semigroup ring’, *J. Algebra* **418** (2014), 8–28.
- [8] R. Jafari and S. Zarzuela Armengou, ‘Homogeneous numerical semigroups’, *Semigroup Forum* (2018), doi:10.1007/s00233-018-9941-6.
- [9] J. Komeda, ‘On the existence of Weierstrass points with a certain semigroup’, *Tsukuba J. Math.* **6**(2) (1982), 237–270.
- [10] P. Mete and E. E. Zengin, ‘Minimal free resolutions of the tangent cones of Gorenstein monomial curves’, arXiv:1801.04956.
- [11] M. Şahin and N. Şahin, ‘On pseudo symmetric monomial curves’, *Comm. Algebra* **46**(6) (2018), 2561–2573.
- [12] D. I. Stamate, ‘Betti numbers for numerical semigroup rings’, in: *Multigraded Algebra and Applications – NSA 24, 2016*, Springer Proceedings in Mathematics and Statistics, 238 (eds. V. Ene and E. Miller) (Springer, Cham, 2018).

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