OUTLINE OF AN INTRODUCTION TO MATHEMATICAL LOGIC IV *)

Abraham Robinson

(received October 2, 1958)

13. The extended completeness theorem of the predicate calculus of the first order. In section 12, we developed a deductive theory of the first order predicate calculus, while in section II we dealt with the semantic theory of that calculus. We now have to consider the connection between these two theories. We recall that a sentence X can be satisfied by a structure M only if X is defined in M. Given a sentence X (a set of sentences K) we shall say that the structure M is a model of X (of K) if X is (all the sentences of K are) satisfied by M.

We may first ask the question whether a provable sentence (a theorem) is by necessity satisfied by all structures in which it is defined. A check on the rules 12.1-12.6 shows that this is indeed the case although for some of the rules the detailed argument is cumbersome.

Conversely it is natural to enquire whether a sentence X which is satisfied by all models in which it is defined is necessarily provable within the deductive theory of section 12. The answer to this question is in the affirmative and is known as "Godel's completeness theorem". It is an immediate consequence of the following "Extended completeness theorem of the first order predicate calculus".

13.1. Any consistent set of sentences in the first order predicate calculus possesses a model.

We say that a set of sentences, K, is consistent if it is not contradictory. K is contradictory if it contains a finite subset $\{X_1, \ldots, X_n\}$, $n \geqslant 1$, such that

$$13.2. \qquad [X_1 \wedge [X_2 \wedge \ldots \wedge X_n] \ldots] \supset Z$$

*) Errata for part III, Can. Math. Bull. 1 (1958), 193-208.
pg. 198, line 2 from below, for "prime" read "atomic".
pg. 200, line 3 from below, for "in X" read "in X or Y".

Can. Math. Bull., vol. 2, no. 1, Jan. 1959

is provable for arbitrary Z. By arguments used previously, an equivalent condition is that there exists a particular sentence Z of the form $Y \wedge \sim Y$ such that 13.2 is provable.

We will prove 13.1 presently. To deduce from it Gödel's completeness theorem (see above) we suppose that X is not provable. If so then $K = \{ \sim X \}$ must be consistent for if K were contradictory then $\sim X \supset X$, i.e. $\sim[\sim X] \vee X$, and hence $X \vee X$ and hence X would be provable, contrary to assumption. Thus, by 13.1, K possesses a model M. M satisfies $\sim X$ and hence does not satisfy X although X is defined in it. This proves Gödel's completeness theorem.

As another corollary of 13.1 we have the fact, mentioned earlier, that if X eq Y and if M is a structure in which both X and Y are defined then X and Y either both hold, or both do not hold, simultaneously in M.

We first prove 13.1 for the case that no quantifiers occur in K. Thus the sentences of K all are obtained by the use of propositional connectives from atomic formulae which contain only n- place relations ($n \geqslant 0$) and individual constants (but no variables). Let P be the set of atomic formulae which occur in K. To every X in P (which may occur more than once in the sentences of K) we select a variable P_X of the propositional calculus such that different P_X correspond to different X. To every sentence Y of K we now define a formula f(Y) of the propositional calculus by the following:

$$f([Y]) = P_Y \qquad \text{if } Y \text{ is atomic},$$

$$f([Y \vee Z]) = f(Y) \vee f(Z),$$

$$f([\sim Y]) = \sim f(Y).$$

Thus, Y is obtained from f(Y) simply by replacing the propositional variables (e.g. P_X) by the corresponding atomic formula (X), and by switching over to the square bracket convention for the predicate calculus.

Let $K' = f(Y)$ be the set of formulae obtained in this way from the sentences Y of K. Then the rule 12.1 shows immediately that if K' is contradictory so is K.

https://doi.org/10.4153/CMB-1959-008-2 Published online by Cambridge University Press

Suppose then that K is consistent, in agreement with the hypothesis of 13.1. Then K' is consistent and so, according to 10.1 there exists an admissible valuation, W, say, for the variables of K'. Suppose first that there are individual constants which occur in the sentences of K(i.e. that some of the relations which occur in K have a positive number of places). In that case we define a model M of K as follows. The set of individual constants of M is the set of individual constants which occur in K. The relations of M are the relations which occur in K. Now consider any expression $R(a_1, \ldots, a_n)$, $n \geqslant o$, where R and $a_1, \ldots, a_n$ are relations and constants of M respectively. If this expression occurs among the atomic formulae of K then we define that it holds or does not hold in M according as the corresponding propositional variable in K' obtains the value T or F in the valuation W. If $R(a_1, \ldots, a_n)$ does not occur among the atomic formulae of K then we define (arbitrarily) that it holds in M. Since W is an admissible valuation it follows that with these definitions M becomes a model of K.

Suppose next that K does not contain any individual constants. In this case we define the set of relations of M (all of 0 places) as above, while the set of constants of M is now empty. This is a degenerate case in which all we can say of the relations of M is that they either hold or do not hold in M. The definition given for the previous case is still applicable, but except for a slight difference in the formal framework the result provided by 13.1 can, in this case, be identified with 10.1.

Suppose now that at least one of the sentences of K includes a quantifier. In this case we first replace every sentence of K by the corresponding sentence in prenex normal form (see section 12 above). Since the resulting sentence is equivalent to the original sentence (eq) and contains the same relations and individual constants, any structure satisfying it will satisfy also the original sentence. Thus, we may as well suppose from the outset that the sentences of K are in prenex normal form. We may also suppose that K contains at least one relation with a positive number of places, for otherwise the quantifiers which occur in K can all be omitted, yielding one of the cases treated previously.

The following auxiliary consideration is required at this point. Let X be a sentence in prenex normal form, e.g.

13.3.

$$X = (\exists y_1)(x_1)(\exists y_2)(x_2)(\exists y_3)(\exists y_4)\, Z(x_1, x_2, y_1, y_2, y_3, y_4),$$

https://doi.org/10.4153/CMB-1959-008-2 Published online by Cambridge University Press

where Z is a formula which is free of quantifiers and containing the variables $x_1, \ldots, y_4$. (It will be convenient though not essential to assume that these variables occur effectively in Z.) Let M be a structure which satisfies X. Then (presupposing the validity of the axiom of choice) the semantic interpretation of X implies that we may select in M a constant φ_1 and functions $y_2 = \varphi_2(x_1)$, $y_3 = \varphi_3(x_1, x_2)$, $y_4 = \varphi_4(x_1, x_2)$, with arguments x_1, x_2 which vary over all individual constants of M such that

$$13.4. \quad Z(x_1, x_2, \varphi_1, \varphi_2(x_1), \varphi_3(x_1, x_2), \varphi_4(x_1, x_2))$$

hold in M for all values of x_1, x_2. Note that 13.4 is not formulated within the language of our calculus, but is transformed into a sentence of the calculus whenever we replace x_1, x_2 by arbitrary individual constants of M and $\varphi_2(x_1)$, $\varphi_3(x_1, x_2)$, $\varphi_4(x_1, x_2)$ by the corresponding functional values of φ_2, φ_3, φ_4. It will be seen that 13.4 is obtained formally from 13.3 by removing the quantifiers and by replacing each variable in Z which is associated with an existential quantifier by a function symbol which includes as arguments the variables in universal quantifiers which precede the existential quantifier in question. Even φ_o may be regarded as a special case of this procedure (number of variables = zero).

The same procedure can be applied to an arbitrary sentence X which is in prenex normal form. The result — which as stated is not in general a sentence within the calculus — will be said to be the Herbrand transform of X, H(X).

Considering again the particular sentence given by 13.3, we see that, conversely, if the functions φ_1, φ_2, φ_3, φ_4, are definable in a structure M so that 13.4 is satisfied when we substitute arbitrary individual constants of M for x_1 and x_2, then 13.3 is satisfied by M. Thus, the satisfiability of 13.4 in the indicated sense is equivalent to the satisfiability of 13.3 and, more generally, the satisfiability of a given sentence is equivalent to the satisfiability of its Herbrand transform.

We return to the proof of 13.1 under the stated assumptions. Supposing that K is consistent, we shall establish the existence of a structure M which is a model of K.

https://doi.org/10.4153/CMB-1959-008-2 Published online by Cambridge University Press

Let $K' = \{H(X)\}$ be the set of Herbrand transforms of the sentences X of K, where it is understood that different function symbols are used for different sentences of K. Let Φ be the set of function symbols (including individual constants and including in particular the individual constants which occur in the original K). If Φ does not include any individual constants by virtue of this definition then we include in it an arbitrary individual constant, c.

By a term we mean any of the individual constants of Φ, as well as any expression obtained by the repeated application of the function symbols of Φ to these individual constants. Thus if K contains the sentence 13.3, the φ_1, $\varphi_2(\varphi_1)$, $\varphi_3(\varphi_1, \varphi_2(\varphi_1))$, etc. are terms.

Let Ψ be the set of all terms obtained in this way. For every $t \in \Psi$, we select an individual constant c_t subject to the two conditions that different c_t correspond to different t, and that $c_t = t$ whenever t is an individual constant which occurs in K. Let Ω be the set of constants c_t obtained in this way. Ω is not empty.

We now define an infinite sequence of sentences $\{K_n\}$, $n = 0, 1, 2, \ldots$ inductively as follows.

$K_o = K$.

In order to define K_1, consider all $X \in K_o$ which begin with an existential quantifier, $(\exists y)$ say. Include in K_1 the sentences which are obtained from K_o by deleting $(\exists y)$ and by replacing the variable y in the remaining formula by the individual constant c_t where t is the constant (here-function of zero variables) which corresponds to $(\exists y)$ in H(X). Thus, if X is given by 13.3, H(X) is given by 13.4 and $y = y_1$, $t = \varphi_1$. To these sentences, add all sentences of K_o.

Then $K_o \subseteq K_1$.

K_2 is obtained next in the following way. Consider all $X \in K_1$ which begin with a universal quantifier, (y) say. Include in K_2 all sentences which are obtained from X by deleting (y) and by replacing the variable y in the remaining formula by an individual constant which occurs in K_1. If there is no such constant, include the sentence obtained from X by deleting (y) and by replacing y by the individual constant c introduced above. In either case add to the sentences obtained in this way all sentences of K_1.

Then $K_1 \subseteq K_2$.

Suppose now that we have already defined the sets K_o, $K_1, \ldots, K_n$, $n \geq 2$. In order to define K_{n+1} we distinguish between even and odd suffixes n.

If n is even, we make the inductive assumption, satisfied for $n = 2$, that any $X \in K_n$ is either included in K or that it has been obtained from a sentence X_o of K by deleting a number of the leading quantifiers and by replacing the corresponding variables in the remaining formula by certain elements of Ω. Now consider any $X \in K_n$ which begins with an existential quantifier, $(\exists y)$ say, and suppose that the universal quantifiers which were deleted in passing from X to X_o are $(x_1), \ldots, (x_k)$.

Suppose that in X, the variables $x_1, \ldots, x_k$ are replaced by individual constants $c_{t_1}, \ldots, c_{t_k}$, such that at least one of these constants occurs for the first time in K_{n-1}, and hence does not occur in K_{n-2} Suppose further that the function symbol corresponding to $(\exists y)$ in H(X) is $\psi(x_1, \ldots, x_k)$. We then include in K_{n+1} the sentence which is obtained from X by deleting $(\exists y)$ and by replacing y in the remaining formula everywhere by $c_{\psi(t_1, \ldots, t_k)}$. (We note that the same $X \in K_n$ may arise from different $X \in K$, leading to different elements of K_{n+1}.) In addition, we include in K_{n+1} all elements of K_n so that $K_n \subseteq K_{n+1}$.

If n is odd, consider all $X \in K_n$ which begin with a universal quantifier, (y) say. Include in K_{n+1} all sentences which are obtained from X by deleting (y) and by replacing the variable y in the remaining formula by an individual constant which occurs in K_n . To these add all elements of K_n. Then $K_n \subseteq K_{n+1}$. It will be seen that K_{n+1} as defined, possesses the inductive property assumed for the definition of K_{n+1} for even n.

Now let $H = K_o \cup K_1 \cup K_2 \cup \ldots$. We propose to show that H is consistent. Since $K_o \subseteq K_1 \subseteq K_2 \ldots$ it is sufficient to show that K_n is consistent for all n. $K_o = K$ is consistent by assumption. We assume that K_{2j} is consistent for some $j \geq 0$ and we prove that in this case K_{2j+1} and K_{2j+2} must be consistent as well.

https://doi.org/10.4153/CMB-1959-008-2 Published online by Cambridge University Press

Suppose on the contrary that K_{2j+1} is contradictory. Then there exist sentences

$$X_1, \ldots, X_k \in K_{2j}, \qquad Y_1, \ldots, Y_m \in K_{2j+1} - K_{2j}$$

such that

$$13.5.\ [Y_1 \wedge [\ldots \wedge [Y_m \wedge [\ldots \wedge [X_1 \wedge [\ldots \wedge X_k] \ldots] \supset Z$$

is provable for some Z which is of the form $V \wedge \sim V$ where V is a sentence of which we may assume that it does not have any individual constants in common with H, and where $m \geq 1$ (while $k \geq 0$) since K_{2j} is assumed consistent. Now the sentences $Y_1, \ldots, Y_m$ are obtained from sentences of K_{2j} of the form

$$V_j = (\exists w_j) Z_j, \qquad j = 1, \ldots, m,$$

respectively by deleting the quantifiers $(\exists w_j)$ and by replacing the w_j in Z_j by different constants which did not appear in $X_1, \ldots, X_k, \ V_1, \ldots, V_m, Z$.

The repeated application of 12.5 in conjunction with 12.1 now shows that since 13.5 is provable, so is the sentence

$$13.6.\ [V_1 \wedge [\ldots \wedge [V_m \wedge [\ldots [X_1 \wedge [\ldots \wedge X_k] \ldots] \supset Z .$$

But the sentences in the implicans of 13.6 all belong to K_{2j} and so 13.6 entails that K_{2j} is contradictory and this is contrary to assumption. Accordingly K_{2j+1} is consistent.

Suppose next that K_{2j+2} is contradictory. Then there exist

$$X_1, \ldots, X_k \in K_{2j+1}, Y_1, \ldots, Y_m \in K_{2j+2} - K_{2j+1}, m \geq 1$$

such that 13.5 is provable for some Z as described above. Now the sentences $Y_1, \ldots, Y_m$ are obtained from sentences of K_{2j} of the form

$$V_j = (w_j) Z_j \quad , \quad j = 1, \ldots, m,$$

respectively, by deleting the (w_j) and by replacing the variables w_j in Z_j by certain constants. Hence by 12.2, the sentences

https://doi.org/10.4153/CMB-1959-008-2 Published online by Cambridge University Press

$$13.7. \qquad V_j \supset Y_j, \qquad j = 1, \ldots, m$$

are provable. The application of the rules of the propositional calculus (i.e. of 12.1) to 13.5 in conjunction with 13.7 now shows that the sentence 13.6 is provable and this is again contrary to the supposition that K_{2j+1} is consistent. Thus, H is consistent.

Let H' be the set of sentences of H which are free of quantifiers. Then H' is consistent and hence, possesses a model M, as constructed previously. Notice that by that construction the individual constants of M coincide with the individual constants of H and hence constitute a subset of Ω.

Let X be any sentence of K and let $X' = H(X)$ be its Herbrand transform. Suppose that $x_1, \ldots, x_k$ are the variables which appear in X under the sign of universal quantification while $y_1, \ldots, y_m$ are the variables which are quantified in X existentially. Let the corresponding function symbols in X' be $\varphi_1, \ldots, \varphi_m$. Then we have to show that we can define $\varphi_1, \ldots, \varphi_m$ on the set of individual constants of M, C, say in such a way that X is satisfied by M for all values of $x_1, \ldots, x_m$ in C.

Suppose that $\varphi_j = \varphi_j\,(x_1, \ldots, x_\ell)$, $1 \leqslant j \leqslant m$, $0 \leqslant \ell \leqslant k$. We have to define this function for all values of $x_1, \ldots, x_\ell$ in C i.e. for certain $x_1 = c_{t_1}, \ldots, x_\ell = c_{t_\ell}$ where $t_1, \ldots, t_\ell$ are terms (elements of Ψ). A suitable definition is

$$13.8. \qquad \varphi_j(c_{t_1}, \ldots, c_{t_\ell}) = c_{\varphi_j(t_1, \ldots, t_\ell)}.$$

(Note that on the left hand side of 13.8, φ_j really denotes a function while on the right hand side it is a symbol which yields a term when combined with $t_1, \ldots, t_\ell$.)

Consider for example the sentence X given by 13.3 with the Herbrand transform 13.4. Defining $\varphi_1, \ldots, \varphi_4$ by 13.8, we have to show that

$$13.9. \qquad Z(c_{t_1}, c_{t_2}, c_{\varphi_1}, c_{\varphi_2(t_1)}, c_{\varphi_3(t_1, t_2)}, c_{\varphi_4(t_1, t_2)})$$

is satisfied by M for all terms t_1, t_2 such that c_{t_1}, c_{t_2} belong to C. But by virtue of our construction of H, 13.9 actually occurs in H' for all such t_1, t_2, and so M satisfies 13.9 and hence 13.3. The same argument applies to general $X \in K$. This completes the proof of 13.1.

Bearing in mind the definition of consistency we obtain the following immediate corollary of 13.1.

13.10. Let K be a set of sentences such that every finite subset of K possesses a model. Then K also possesses a model.

This corollary contains no reference to the deductive concepts of section 12 above and it may be expected that such concepts can be eliminated also from its proof. It is indeed possible to construct a variant of our proof of 13.1 which establishes 13.10 by means of semantic concepts alone.

A host of interesting applications of 13.10 to modern Algebra has been shown by Malcev, Henkin and the present author. For some of them, the reader is referred to the author's monograph Complete Theories (published in the series Studies in Logic and the Foundations of Mathematics, Amsterdam 1956).

Another corollary of 13.1, or rather of its proof, is the famous

13.11. <u>Theorem of Löwenheim-Skolem</u>. Suppose that the number of elements of K does not exceed $\aleph_0$ (K is finite or countable). Then if K possesses a model at all, it also possesses a model the number of whose individual constants is finite or countable.

Indeed, since K possesses a model it must be consistent. The model M constructed in the proof of 13.1 then satisfies the conclusion of 13.11.

14. <u>Conclusion</u>. We have now developed our subject as far as intended. It should be clear to the reader that we have reached only the end of the beginning. There are further developments in many directions. Of these, we may mention first the predicate calculi of higher order in which relations can be quantified and may appear as arguments of other relations. Other investigations are concerned with questions of constructivity in Mathematics, of

algorithms and decision methods, and are intimately connected with the concepts of recursive functions and predicates. This is a subject which is related to the theory of modern computers (usually electronic) and it has both influenced that theory and been influenced by it. Again, there is the possibility of replacing the set of two truth values on which Logic is based customarily (i.e. "true" and "false") by a greater (even infinite) number of truth values. This leads to the theory, or theories, of many-valued Logics. Finally, in addition to the pursuit of various ramifications at the top, we may wish to investigate more closely the logical and philosophical foundations of the entire subject. One respect in which our approach has been less subtle than is sometimes held to be necessary is that we have not distinguished between a name (e.g. symbol) and the object denoted by it. In general there is indeed a difference between these-a rose by any other name, etc. However while a complete disregard for this distinction can lead to unpleasant results in some connections, it is not difficult to justify the practice adopted in these notes as far as they went.

Hebrew University,
Jerusalem, Israel.