THOSE STIRLING NUMBERS AGAIN

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1. Introduction. In his book [1] Combinatorial Analysis, J. Riordan (p. 32) refers to the continual rediscovery of the Stirling numbers. The author of this note has been surprised on many occasions by the number of different environments in which these numbers make a natural appearance and, in fact, this article is concerned with just such an occurrence. The connection is made in a study of the exponential generating function of n^r. A curious by-product is the following formula for f(n), the number of distinct equivalence relations amongst a set of n elements;

$$f(n) = e^{-1} \{1 + \frac{2^{n-1}}{1!} + \frac{3^{n-1}}{1!} + \ldots + \frac{(r+1)^{n-1}}{r!} + \ldots\}.$$

2. The exponential generating function for n^r. Let $\phi(r, x)$ be the exponential generating function for n^r, namely,

$$\phi(r, x) = 1 + \frac{2^r x}{1!} + \frac{3^r x^2}{2!} + \ldots + \frac{n^r x^{n-1}}{(n-1)!} + \ldots .$$

Then $\phi(0, x) = e^x$, and $\phi(r+1, x) = \frac{d}{dx}(x\phi(r, x))$.

It is easily verified that $\phi(r, x) = P_r(x)e^x$ where $P_r(x)$ is a polynomial in x of degree r. From the relation $\phi(r+1, x) = \frac{d}{dx}(x\phi(r, x))$ one obtains

$$(1) \qquad P_{r+1}(x) = (1 + x)P_r(x) + xP'_r(x).$$

The first few of these polynomials are

$$P_0(x) = 1$$

$$P_1(x) = 1 + x$$

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$$P_2(x) = 1 + 3x + x^2$$
$$P_3(x) = 1 + 7x + 6x^2 + x^3$$
$$P_4(x) = 1 + 15x + 25x^2 + 10x^3 + x^4$$
$$P_5(x) = 1 + 31x + 90x^2 + 65x^3 + 15x^4 + x^5.$$

On putting

$$P_n(x) = S(n+1, 1) + S(n+1, 2)x + \dots + S(n+1, r+1)x^r + \dots$$
$$+ S(n+1, n+1)x^n$$

and using (1), one obtains

$$S(n+1, r+1) = S(n, r) + (r+1)\, S(n, r+1)$$

together with $S(n+1, 1) = S(n+1, n+1) = 1$. These relations are in fact the well known relations for the Stirling numbers of the second kind. An explicit formula for $S(n, r)$ is

$$S(n, r) = \frac{\Delta^r 0^n}{r!}$$

as given in [1; p. 33].

It is well known (e.g. see [2]) that the number of distinct equivalence relations on n elements $f(n)$ is given by

$$f(n) = \Sigma_{r=1}^{n} S(n, r).$$

Hence

$$f(n) = P_{n-1}(1) = e^{-1} \Sigma_{r=0}^{\infty} \frac{(r+1)^{n-1}}{r!}.$$

Before the author became aware that the coefficients of $P_n(x)$ were Stirling numbers, he had computed an explicit formula for these numbers which appears to be different from any published formulas. This is now given. It may be verified by induction using the recurrence formula for $S(n, r)$. Let $T_{n,r}$ be the sum of the geometric progression $1 + r + r^2 + \dots + r^{n-1}$, i.e. $T_{n,r} = \frac{r^n - 1}{r - 1}$ and denote as usual the binomial coefficients by $\binom{n}{r}$. Then

$$S(n, r) = \frac{1}{(r-2)!}\left\{ T_{n-1,r} - \binom{r-2}{1} T_{n-1,r-1} + \dots + (-1)^k \binom{r-2}{k} T_{n-1,r-k} + \dots + (-1)^{r-2} T_{n-1,2} \right\}.$$

REFERENCES

1. J. Riordan, An Introduction to Combinatorial Analysis, (New York, 1958).

2. N. S. Mendelsohn, Applications of combinatorial formulae to generalizations of Wilson's theorem, Canad. J. Math. 1 (1949) 328-336.

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