Canad. J. Math. 2024, pp. 1–24
http://dx.doi.org/10.4153/S0008414X23000895


# Higher dimensional algebraic fiberings for pro-$p$ groups

Dessislava H. Kochloukova

*Abstract.* We prove some conditions for higher-dimensional algebraic fibering of pro-$p$ group extensions, and we establish corollaries about incoherence of pro-$p$ groups. In particular, if $1 \to K \to G \to \Gamma \to 1$ is a short exact sequence of pro-$p$ groups, such that $\Gamma$ contains a finitely generated, non-abelian, free pro-$p$ subgroup, $K$ a finitely presented pro-$p$ group with $N$ a normal pro-$p$ subgroup of $K$ such that $K/N \simeq \mathbb{Z}_p$ and $N$ not finitely generated as a pro-$p$ group, then $G$ is incoherent (in the category of pro-$p$ groups). Furthermore, we show that if $K$ is a finitely generated, free pro-$p$ group with $d(K) \geq 2$, then either $\mathrm{Aut}_0(K)$ is incoherent (in the category of pro-$p$ groups) or there is a finitely presented pro-$p$ group, without non-procyclic free pro-$p$ subgroups, that has a metabelian pro-$p$ quotient that is not finitely presented, i.e., a pro-$p$ version of a result of Bieri–Strebel does not hold.

## 1 Introduction

For a pro-$p$ group $G$, we denote by $K[[G]]$ the completed group algebra of $G$ over the ring $K$, where $K$ is the field with $p$ elements $\mathbb{F}_p$ or the ring of the $p$-adic numbers $\mathbb{Z}_p$. By definition a pro-$p$ group $G$ is of type $FP_m$ if the trivial $\mathbb{Z}_p[[G]]$-module $\mathbb{Z}_p$ has a projective resolution where all projectives in dimension $\leq m$ are finitely generated $\mathbb{Z}_p[[G]]$-modules. Note that $G$ is of type $FP_1$ if and only if $G$ is finitely generated as a pro-$p$ group. And, $G$ is of type $FP_2$ if and only if $G$ is finitely presented as a pro-$p$ group, i.e., $G \simeq F/R$, where $F$ is a free pro-$p$ group with a finite free basis $X$ and $R$ is the smallest normal pro-$p$ subgroup of $F$ that contains some fixed finite set of relations of $G$. It is interesting to note that for abstract (discrete) groups, the abstract versions of the properties $FP_2$ and finite presentability do not coincide [3].

In this paper, we develop results on algebraic fibering and coherence of pro-$p$ groups. The case of abstract groups was considered by Kochloukova and Vidussi in [13], where the authors used specific techniques from geometric group theory, namely the Bieri–Neumann–Renz–Strebel $\Sigma$-invariants. We will use the pro-$p$ version of the $\Sigma^1$-invariant, suggested in [10] for pro-$p$ metabelian groups, only in the proof of Proposition 3.4 and most of the results in this paper would have purely homological proofs. We note that the results on incoherence we obtain are quite general and in their full generality are not known for abstract groups (see Corollaries 1.3 and 1.4).

Received by the editors January 1, 2023; revised October 30, 2023; accepted December 11, 2023.
Published online on Cambridge Core December 22, 2023.
The author was partially supported by Bolsa de produtividade em pesquisa CNPq 305457/2021-7 and Projeto temático FAPESP 2018/23690-6.
AMS subject classification: **20J05**.
Keywords: Algebraic fibering, pro-$p$ groups, coherence, homological type $FP_m$.

***Theorem 1.1*** *Let* $1 \to K \to G \to \Gamma \to 1$ *be a short exact sequence of pro-p groups and let* $n_0 \geq 1$ *be an integer such that:*

1) *G and K are of type* $FP_{n_0}$*,*
2) $\Gamma^{ab}$ *is infinite,*
3) *there is a normal pro-p subgroup N of K such that* $G' \cap K \subseteq N$*,* $K/N \simeq \mathbb{Z}_p$ *and N is of type* $FP_{n_0-1}$*.*

*Then there is a normal pro-p subgroup M of G such that* $G/M \simeq \mathbb{Z}_p$*,* $M \cap K = N$*, and M is of type* $FP_{n_0}$*. Furthermore, if K, G, and N are of type* $FP_\infty$*, then M can be chosen of type* $FP_\infty$*.*

We call a discrete pro-$p$ character of $G$ a nontrivial homomorphism of pro-$p$ groups $\alpha : G \to H$ such that $H \simeq \mathbb{Z}_p$. Then Theorem 1.1 could be restated as: assume that $G$ and $K$ are of type $FP_{n_0}$, $\Gamma^{ab}$ is infinite, and there is a discrete pro-$p$ character $\alpha$ of $G$ such that $\alpha|_K \neq 0$, $\operatorname{Ker}(\alpha) \cap K = N$ is of type $FP_{n_0-1}$. Then there exists a discrete pro-$p$ character $\mu$ of $G$ such that $M = \operatorname{Ker}(\mu)$ is of type $FP_{n_0}$ and $\mu|_K = \alpha|_K$, in particular $M \cap K = N$.

There is a lot in the literature on coherent abstract groups (see, for example, [27]), but very little is known for coherent pro-$p$ groups. Similar to the abstract case, a pro-$p$ group $G$ is coherent (in the category of pro-$p$ groups) if every finitely generated pro-$p$ subgroup of $G$ is finitely presented as a pro-$p$ group, i.e., is of type $FP_2$. We generalize this concept and define that a pro-$p$ group $G$ is $n$-coherent if any pro-$p$ subgroup of $G$ that is of type $FP_n$ is of type $FP_{n+1}$. Thus, a pro-$p$ group is 1-coherent if and only if it is coherent (in the category of pro-$p$ groups).

***Corollary 1.2*** *Let K,* $\Gamma$*, and* $G = K \rtimes \Gamma$ *be pro-p groups and let* $n_0 \geq 1$ *be an integer such that:*

1) $\Gamma$ *is finitely generated free pro-p but not pro-p cyclic,*
2) *K is of type* $FP_{n_0}$*,*
3) *there is a normal pro-p subgroup N of K such that* $G' \cap K \subseteq N$*,* $K/N \simeq \mathbb{Z}_p$ *and N is of type* $FP_{n_0-1}$ *but is not of type* $FP_{n_0}$*.*

*Then there is a normal pro-p subgroup M of G such that* $G/M \simeq \mathbb{Z}_p$*,* $M \cap K = N$*, and M is of type* $FP_{n_0}$ *but is not of type* $FP_{n_0+1}$*. In particular, G is not* $n_0$*-coherent.*

As in the case of Theorem 1.1, Corollary 1.2 can be restated in terms of discrete pro-$p$ characters.

We say that a group is incoherent if it is not coherent. The following result can be deduced from Theorem 3.6, that follows from Corollary 1.2.

***Corollary 1.3*** *Let* $1 \to K \to G \to \Gamma \to 1$ *be a short exact sequence of pro-p groups such that:*

1) *K is a finitely generated pro-p group,*
2) *there is a normal pro-p subgroup N of K with* $K/N \simeq \mathbb{Z}_p$ *and N is not finitely generated,*
3) $\Gamma$ *contains a non-abelian free pro-p subgroup.*

*Then G is incoherent (in the category of pro-p groups).*

The class of pro-$p$ groups $\mathcal{L}$ was first considered by Kochloukova and Zalesskii in [14]. It contains all finitely generated free pro-$p$ groups, and its profinite version was considered by Zalesskii and Zapata in [28]. A pro-$p$ group from $\mathcal{L}$ shares many properties with an abstract limit group, in particular it is defined using extensions of centralizers. Still there are many open questions about the class of pro-$p$ groups $\mathcal{L}$. For example, by Wilton's result from [26], every finitely generated subgroup of an abstract limit group is a virtual retract, but the pro-$p$ version of this result is still an open problem.

For a pro-$p$ group $K$, we write $d(K)$ for the cardinality of a minimal set of (topological) generators, i.e., $d(K) = \dim_{\mathbb{F}_p} H_1(K, \mathbb{F}_p)$.

***Corollary 1.4*** *Let* $1 \to K \to G \to \Gamma \to 1$ *be a short exact sequence of pro-p groups such that:*

1) *$K$ is a non-abelian pro-p group from the class $\mathcal{L}$,*
2) *$\Gamma$ contains a non-abelian, free pro-p subgroup.*

*Then $G$ is incoherent (in the category of pro-p groups). In particular, if $K$ is a finitely generated free pro-p group with $d(K) \geq 2$, then $G$ is incoherent (in the category of pro-p groups).*

We note that the version of Corollary 1.4 for abstract groups is still open even when $K$ and $\Gamma$ are free, non-abelian with $K$ of rank at least 3. The same holds for Corollary 1.3.

It is known that abstract (free finite rank)-by-$\mathbb{Z}$ groups are coherent [8]. There is a conjecture suggested by Wise and independently by Kropholler and Walsh that an abstract (free of finite rank $\geq 2$)-by-(free of finite rank $\geq 2$) group is incoherent (see [15]). The conjecture was proved in [15] for a (free of rank 2)-by-(free of finite rank $\geq 2$) abstract group, with a proof that cannot be modified for pro-$p$ groups. By Corollary 1.4, a pro-$p$ version of this result holds too.

A pro-$p$ right angled Artin group (pro-$p$ RAAG) associated with a finite simplicial graph $X$ can be defined either as the pro-$p$ completion of the abstract RAAG associated with $X$ or by the same presentation as the abstract RAAG associated with $X$ but in the category of pro-$p$ groups.

Demushkin groups are some special, finitely generated, 1-related pro-$p$ groups. The pro-$p$ completion of an orientable surface group is a Demushkin pro-$p$ group, but there are Demushkin pro-$p$ groups that are not obtained this way. There are several types of Demushkin pro-$p$ groups completely described in terms of presentations in [5, 6, 17, 23]. The following corollary provides many examples where Theorem 1.1 and Corollary 1.3 apply.

***Corollary 1.5*** *Let* $1 \to K \to G \to \Gamma \to 1$ *be a short exact sequence of pro-p groups such that:*

1) *$K$ is a non-abelian pro-p RAAG or a non-soluble Demushkin group,*
2) *$\Gamma$ is a non-abelian pro-p RAAG or a non-soluble Demushkin group.*

*Then $G$ is incoherent (in the category of pro-p groups).*

For a finite rank free pro-$p$ group $F$, the structure of $\mathrm{Aut}(F)$ was studied first by Lubotsky in [18]. $\mathrm{Aut}(F)$ is a topological group with a pro-$p$ subgroup of finite index. In [9], Gordon proved that the automorphism group of an abstract free group of rank 2 is incoherent. Unfortunately we could not prove a pro-$p$ version of this result, but still it would hold if the group of outer pro-$p$ automorphisms of a free pro-$p$ group of rank 2 contains a free non-procyclic pro-$p$ subgroup. For a free abstract group $F_2$ of rank 2, we have that $\mathrm{Out}(F_2) \simeq GL_2(\mathbb{Z})$; hence, $\mathrm{Out}(F_2)$ contains a free non-cyclic abstract group. Nevertheless, the group $GL_2^1(\mathbb{Z}_p) = \mathrm{Ker}(GL_2(\mathbb{Z}_p) \to GL_2(\mathbb{F}_p))$ does not contain a free pro-$p$ non-procyclic pro-$p$ subgroup, since it is $p$-adic analytic and so there is an upper limit on the number of generators of finitely generated pro-$p$ subgroups [7]. For related results on non-existence of free pro-$p$ subgroups in matrix groups, see [1, 2, 29].

Let $G$ be a finitely generated pro-$p$ group. Define $\mathrm{Aut}_0(G) = \mathrm{Ker}(\mathrm{Aut}(G) \to \mathrm{Aut}(G/G^*))$, where $G^*$ is the Frattini subgroup of $G$. Then $\mathrm{Aut}_0(G)$ is a pro-$p$ subgroup of $\mathrm{Aut}(G)$ of finite index.

***Corollary 1.6*** *Suppose that $K$ is a finitely generated free pro-$p$ group with $d(K) \geq 2$. If $\mathrm{Out}(K)$ contains a pro-$p$ free non-procyclic subgroup, then $\mathrm{Aut}_0(K)$ is incoherent (in the category of pro-$p$ groups).*

By the Bieri–Strebel results in [4] for a finitely presented abstract group $H$ that does not contain free non-cyclic abstract subgroups, every metabelian quotient of $H$ is finitely presented. It is an open question whether a pro-$p$ version of the Bieri–Strebel result holds, i.e., whether if $G$ is a finitely presented pro-$p$ group without free non-procyclic pro-$p$ subgroups, then every metabelian pro-$p$ quotient of $G$ is finitely presented as a pro-$p$ group. Note that by the King classification of the finitely presented metabelian pro-$p$ groups in [11], every pro-$p$ quotient of a finitely presented metabelian pro-$p$ group is finitely presented pro-$p$. Using Corollary 1.6 and some ideas introduced by Romankov in [21, 22], we prove the following result.

***Corollary 1.7*** *Suppose that $K$ is a finitely generated free pro-$p$ group with $d(K) \geq 2$. Then either $\mathrm{Aut}_0(K)$ is incoherent (in the category of pro-$p$ groups) or the pro-$p$ version of the Bieri–Strebel result does not hold.*

## 2 Preliminaries

### 2.1 Homological finiteness properties of pro-$p$ groups

Let $G$ be a pro-$p$ group. By definition,

$$\mathbb{Z}_p[[G]] = \varprojlim \frac{\mathbb{Z}}{p^i\mathbb{Z}}[[G/U]],$$

where the inverse limit is over all $i \geq 1$ and $U$ open subgroups of $G$. And,

$$\mathbb{F}_p[[G]] = \mathbb{Z}_p[[G]]/p\mathbb{Z}_p[[G]] = \varprojlim \mathbb{F}_p[[G/U]],$$

where the inverse limit is over all open subgroups $U$ of $G$.

By definition, the pro-$p$ group $G$ is of type $FP_m$ if the trivial $\mathbb{Z}_p[[G]]$-module $\mathbb{Z}_p$ has a projective resolution where all projectives in dimension $\leq m$ are finitely generated $\mathbb{Z}_p[[G]]$-modules, i.e., there is an exact complex of pro-$p$ $\mathbb{Z}_p[[G]]$-modules

$$\mathcal{P} : \cdots \to P_i \to P_{i-1} \to \cdots \to P_0 \to \mathbb{Z}_p \to 0,$$

where each $P_i$ projective and for $i \leq m$ we have that $P_i$ is finitely generated.

https://doi.org/10.4153/S0008414X23000895 Published online by Cambridge University Press

Such resolutions can be used to compute the pro-$p$ homology groups $H_i(G,-)$. Suppose $V$ is a left pro-$p$ $\mathbb{Z}_p[[G]]$-module and $\mathcal{P}$ is a complex of right pro-$p$ $\mathbb{Z}_p[[G]]$-modules. Then the pro-$p$ homology group $H_i(G,V)$ is defined as $H_i(\mathcal{P}^{\text{del}} \otimes_{\mathbb{Z}_p[[G]]} V)$. If $W$ is a discrete right $G$-module, the cohomology group $H^i(G,W)$ is defined as $H^i(\text{Hom}_G(\mathcal{P}^{\text{del}}, W))$. Here, $\mathcal{P}^{\text{del}}$ denotes the deleted resolution obtained from $\mathcal{P}$ by deleting the module $\mathbb{Z}_p$ from dimension $-1$, i.e., substituting it with the zero module and $\text{Hom}_G$ denotes continuous $G$-homomorphisms. For more on homology and cohomology of pro-$p$ groups, see [20, 25].

By [10], for a pro-$p$ group, the following conditions are equivalent:

1) $G$ is of type $FP_m$;

2) $H_i(G,\mathbb{Z}_p)$ is a finitely generated (abelian) pro-$p$ group for $i \le m$;

3) $H_i(G,\mathbb{F}_p)$ is finite for $i \le m$;

4) for $K$ either $\mathbb{F}_p$ or $\mathbb{Z}_p$ and $N$ a normal pro-$p$ subgroup of $G$ such that $K[[G/N]]$ is left and right Noetherian, the homology groups $H_i(N,K)$ are finitely generated as pro-$p$ $K[[G/N]]$-modules for all $i \le m$, where the $G/N$ action is induced by the conjugation action of $G$ on $N$.

The equivalence of the above conditions is a corollary of the fact that $\mathbb{Z}_p[[G]]$ and $\mathbb{F}_p[[G]]$ are local rings. Furthermore, in 3) $H_i(G,\mathbb{F}_p)$ could be substituted with $H^i(G,\mathbb{F}_p)$.

## 2.2 The King invariant

Let $Q$ be a finitely generated abelian pro-$p$ group, and let $\mathbb{F}$ be the algebraic closure of $\mathbb{F}_p$. Denote by $\mathbb{F}[[t]]^\times$ the multiplicative group of invertible elements in $\mathbb{F}[[t]]$. Consider

$$T(Q) = \{\chi : Q \to \mathbb{F}[[t]]^\times \mid \chi \text{ is a continuous homomorphism}\},$$

where $\mathbb{F}[[t]]^\times$ is a topological group with topology induced by the topology of the ring $\mathbb{F}[[t]]$, given by the sequence of ideals $(t) \supseteq (t^2) \supseteq \cdots \supseteq (t^i) \supseteq \cdots$. Note that since $\chi$ is continuous, we have that

$$\chi(Q) \subset 1 + t\mathbb{F}[[t]].$$

For $\chi \in T(Q)$, there is a unique continuous ring homomorphism

$$\overline{\chi} : \mathbb{Z}_p[[Q]] \to \mathbb{F}[[t]]$$

that extends $\chi$.

Let $A$ be a finitely generated pro-$p$ $\mathbb{Z}_p[[Q]]$-module. In [11], King defined the following invariant:

$$\Delta(A) = \{\chi \in T(Q) \mid \text{ann}_{\mathbb{Z}_p[[Q]]}(A) \subseteq \text{Ker}(\overline{\chi})\}.$$

In [11], King used the notation $\Xi(A)$, that we here substitute by $\Delta(A)$.

Let $P$ be a pro-$p$ subgroup of $Q$. Define

$$T(Q,P) = \{\chi \in T(Q) \mid \chi(P) = 1\}.$$

https://doi.org/10.4153/S0008414X23000895 Published online by Cambridge University Press

***Theorem 2.1*** [11, Theorem B], [11, Lemma 2.5] *Let $Q$ be a finitely generated abelian pro-p group. Let $A$ be a finitely generated pro-p $\mathbb{Z}_p[[Q]]$-module.*

a) *Then $A$ is finitely generated as an abelian pro-p group if and only if $\Delta(A) = \{1\}$.*

b) *If $P$ is a pro-p subgroup of $Q$, then*

$$T(Q, P) \cap \Delta(A) = \Delta(A/[A, P]).$$

*In particular, $A$ is finitely generated as a pro-p $\mathbb{Z}_p[[P]]$-module if and only if $T(Q, P) \cap \Delta(A) = \{1\}$.*

We state the classification of the finitely presented metabelian pro-$p$ groups given by King in [11].

***Theorem 2.2*** [11] *Let $1 \to A \to G \to Q \to 1$ be a short exact sequence of pro-p groups, where $G$ is a finitely generated pro-p group and $A$ and $Q$ are abelian. Then $G$ is a finitely presented pro-p group if and only if $\Delta(A) \cap \Delta(A)^{-1} = \{1\}$.*

***Example*** Let $A = \mathbb{F}_p[[s]]$, $Q = \mathbb{Z}_p$, $G = A \rtimes Q$, where $Q = \mathbb{Z}_p$ has a generator $b$ and $b$ acts via conjugation on $A$ by multiplication with $1 + s$. Since

$$\operatorname{ann}_{\mathbb{Z}_p[[Q]]}(A) = p\mathbb{Z}_p[[Q]] \subseteq \operatorname{Ker}(\overline{\chi}) \text{ for any } \chi \in T(Q),$$

we conclude that $\Delta(A) = T(Q) = \Delta(A)^{-1}$. Hence, by Theorem 2.2, $G$ is not finitely presented (as a pro-$p$ group).

Alternatively, it could be shown by a homological argument that if $1 \to A \to G \to Q \to 1$ is a short exact sequence of pro-$p$ groups, where $G$ is finitely presented and $A$ and $Q$ are abelian, then the pro-$p$ homology group $H_2(A, \mathbb{Z}_p) \simeq A \widehat{\wedge}_{\mathbb{Z}_p} A$ is finitely generated as $\mathbb{Z}_p[[Q]]$-module, where $\widehat{\wedge}$ denotes completed exterior product. In our example, the last condition fails, so $G$ is not finitely presented (as a pro-$p$ group).

## 2.3 Demushkin pro-*p* groups

Following [24], a Demushkin group $G$ is a Poincare duality group of dimension 2, i.e., $H^i(G, \mathbb{F}_p)$ is finite for all $i$, $\dim H^2(G, \mathbb{F}_p) = 1$ and the cup product

$$H^i(G, \mathbb{F}_p) \cup H^{2-i}(G, \mathbb{F}_p) \to H^2(G, \mathbb{F}_p)$$

is a non-degenerated bilinear form for all $i \geq 0$. In particular, the cohomological dimension of $G$ is $\operatorname{cd}(G) = 2$.

There are two invariants associated with a Demushkin pro-$p$ group: the minimal number of (topological) generators $d$ and $q$ that is either a power of the prime $p$ or $\infty$. We state several results from [5, 6, 17, 23] that classify the Demushkin pro-$p$ groups. Other excellent reference for Demushkin pro-$p$ groups is [25].

***Theorem 2.3*** [5, 6] *Let $D$ be a Demushkin group with invariants $d, q$ and suppose that $q \neq 2$. Then $d$ is even and $D$ is isomorphic to $F/R$, where $F$ is a free pro-p group with basis $x_1, \ldots, x_d$ and $R$ is generated as a normal closed subgroup by*

$$x_1^q [x_1, x_2] \ldots [x_{d-1}, x_d],$$

https://doi.org/10.4153/S0008414X23000895 Published online by Cambridge University Press

*where for $q = \infty$ we define $x_1^{\infty} = 1$. Furthermore, all groups having such presentations are Demushkin.*

**Theorem 2.4** [23] *Let $D$ be a Demushkin pro-2 group with invariants $d, q$ and suppose that $q = 2$ and $d$ is odd. Then $D$ is isomorphic to $F/R$, where $F$ is a free pro-2 group with basis $x_1, \dots, x_d$ and $R$ is generated as a normal closed subgroup by*

$$x_1^2 x_2^{2^f}[x_2, x_3] \dots [x_{d-1}, x_d]$$

*for some integer $f \geq 2$ or $\infty$. Furthermore, all groups having such presentations are Demushkin.*

**Theorem 2.5** [17] *Let $D$ be a Demushkin pro-2 group with $d$ even and $q = 2$. Then $D$ is isomorphic to $F/R$, where $F$ is a free pro-2 group with basis $x_1, \dots, x_d$ and $R$ is generated as a normal closed subgroup either by*

$$x_1^{2^f+2}[x_1, x_2][x_3, x_4] \dots [x_{d-1}, x_d]$$

*for some integer $f \geq 2$ or $f = \infty$, or by*

$$x_1^2[x_1, x_2]x_3^{2^f}[x_3, x_4] \dots [x_{d-1}, x_d]$$

*for some integer $f \geq 2$ or $f = \infty$, $d \geq 4$. Furthermore, all groups having such presentations are Demushkin.*

## 3 Proofs of the main results

The following result is a pro-$p$ version of results from [12, 16], where homotopical and homological versions for discrete groups are considered.

**Lemma 3.1** *Let $n \geq 1$ be a natural number,*

$$A \hookrightarrow B \twoheadrightarrow C$$

*a short exact sequence of pro-$p$ groups with $A$ of type $FP_n$ and $C$ of type $FP_{n+1}$. Assume that there is another short exact sequence of pro-$p$ groups*

$$A \hookrightarrow B_0 \twoheadrightarrow C_0$$

*with $B_0$ of type $FP_{n+1}$ and that there is a homomorphism of pro-$p$ groups $\theta : B_0 \to B$ such that $\theta|_A = id_A$, i.e., there is a commutative diagram of homomorphisms of pro-$p$ groups*

$$\begin{array}{ccccc} A & \hookrightarrow & B_0 & \xrightarrow{\pi_0} & C_0 \\ \downarrow{\scriptstyle id_A} & & \downarrow{\scriptstyle \theta} & & \vdots{\scriptstyle \nu} \\ A & \hookrightarrow & B & \xrightarrow{\pi} & C \end{array}$$

*Then $B$ is of type $FP_{n+1}$.*

https://doi.org/10.4153/S0008414X23000895 Published online by Cambridge University Press

***Remark*** $\theta$ does not need to be injective or surjective.

**Proof** Consider the LHS spectral sequence

$$E^2_{i,j} = H_i(C_0, H_j(A, \mathbb{F}_p)) \text{ that converges to } H_{i+j}(B_0, \mathbb{F}_p).$$

Similarly there is an LHS spectral sequence

$$\widehat{E}^2_{i,j} = H_i(C, H_j(A, \mathbb{F}_p)) \text{ that converges to } H_{i+j}(B, \mathbb{F}_p).$$

Since $A$ is of type $FP_n$, we have that $H_j(A, \mathbb{F}_p)$ is finite for all $j \le n$. Then there is a pro-$p$ subgroup $C_1$ of finite index in $C$ such that $C_1$ acts trivially on $H_j(A, \mathbb{F}_p)$ for every $j \le n$. Since $C$ is of type $FP_{n+1}$, we have that $C_1$ is of type $FP_{n+1}$. Then

$$H_i(C_1, H_j(A, \mathbb{F}_p)) \simeq \oplus H_i(C_1, \mathbb{F}_p) \text{ is finite for } j \le n, i \le n+1,$$

where we have $\dim_{\mathbb{F}_p} H_j(A, \mathbb{F}_p)$ direct summands in the right-hand side of the above isomorphism. Since $C_1$ has finite index in $C$, we deduce that

$$\widehat{E}^2_{i,j} = H_i(C, H_j(A, \mathbb{F}_p)) \text{ is finite for } j \le n, i \le n+1,$$

hence by the convergence of the second spectral sequence, we obtain that

$$H_k(B, \mathbb{F}_p) \text{ is finite for } k \le n.$$

Note that we have shown that if $i + j = n + 1, i \neq 0$, then $\widehat{E}^2_{i,j}$ is finite, and hence $\widehat{E}^\infty_{i,j}$ is finite. By the convergence of the spectral sequence, there is a filtration of $H_{n+1}(B, \mathbb{F}_p)$

$$0 = F_{-1}(H_{n+1}(B, \mathbb{F}_p)) \subseteq \cdots \subseteq F_i(H_{n+1}(B, \mathbb{F}_p)) \subseteq F_{i+1}(H_{n+1}(B, \mathbb{F}_p))$$

$$\subseteq \cdots \subseteq F_{n+1}(H_{n+1}(B, \mathbb{F}_p)) = H_{n+1}(B, \mathbb{F}_p),$$

where $F_i(H_{n+1}(B, \mathbb{F}_p))/F_{i-1}(H_{n+1}(B, \mathbb{F}_p)) \simeq \widehat{E}^\infty_{i,n+1-i}$. Thus,

$$H_{n+1}(B, \mathbb{F}_p) \text{ is finite if and only if } \widehat{E}^\infty_{0,n+1} \text{ is finite.}$$

Note that since any differential that comes out from $\widehat{E}^r_{0,n+1}$ is zero, we have that $\widehat{E}^\infty_{0,n+1}$ is a quotient of $\widehat{E}^2_{0,n+1} = H_0(C, H_{n+1}(A, \mathbb{F}_p))$, and thus there is a map

$$\mu : H_0(C, H_{n+1}(A, \mathbb{F}_p)) \to H_{n+1}(B, \mathbb{F}_p)$$

with image that equals $\widehat{E}^\infty_{0,n+1}$. Thus,

$$B \text{ is of type } FP_{n+1} \text{ if and only if } \mathrm{Im}(\mu) \text{ is finite.}$$

Similarly, there is a map

$$\mu_0 : H_0(C_0, H_{n+1}(A, \mathbb{F}_p)) \to H_{n+1}(B_0, \mathbb{F}_p)$$

with image that equals $E^\infty_{0,n+1}$ and such that $B_0$ is of type $FP_{n+1}$ if and only if $\mathrm{Im}(\mu_0)$ is finite. Since $B_0$ is of type $FP_{n+1}$, we conclude that $\mathrm{Im}(\mu_0)$ is finite.

https://doi.org/10.4153/S0008414X23000895 Published online by Cambridge University Press

The naturality of the LHS spectral sequence implies that we have the commutative diagram

$$\begin{array}{ccc} H_0(C_0, H_{n+1}(A, \mathbb{F}_p)) & \xrightarrow{\rho} & H_0(C, H_{n+1}(A, \mathbb{F}_p)) \\ \mu_0 \downarrow & & \downarrow \mu \\ H_{n+1}(B_0, \mathbb{F}_p) & \xrightarrow{\rho_0} & H_{n+1}(B, \mathbb{F}_p) \end{array}$$

where the maps $\rho$ and $\rho_0$ are induced by $\nu : C_0 \to C$ and by $\theta$.

Recall that the action of $B_0$ on $A$ via conjugation induces an action of $B_0$ on $H_{n+1}(A, \mathbb{F}_p)$ where $A$ acts trivially and this induces the action of $C_0$ on $H_{n+1}(A, \mathbb{F}_p)$ that is used to define $H_0(C_0, H_{n+1}(A, \mathbb{F}_p))$. Similarly, the action of $B$ on $A$ via conjugation induces an action of $B$ on $H_{n+1}(A, \mathbb{F}_p)$ where $A$ acts trivially and this induces the action of $C$ on $H_{n+1}(A, \mathbb{F}_p)$ that is used to define $H_0(C, H_{n+1}(A, \mathbb{F}_p))$. If $\nu$ is surjective, then $\rho$ is an isomorphism; if $\nu$ is injective, then $\rho$ is surjective. Since every homomorphism $\nu$ is a composition of one epimorphism followed by one monomorphism, we conclude that $\rho$ is always surjective. Then

$$\operatorname{Im}(\mu) = \operatorname{Im}(\mu \circ \rho) = \operatorname{Im}(\rho_0 \circ \mu_0) \text{ is a quotient of } \operatorname{Im}(\mu_0).$$

Since $\operatorname{Im}(\mu_0)$ is finite, we conclude that $\operatorname{Im}(\mu)$ is finite. Hence, $B$ is of type $FP_{n+1}$ as required. ∎

***Lemma 3.2*** *Let $1 \to A \to B \to C \to 1$ be a short exact sequence of pro-p groups such that for some $m \geq 1$ we have that $A$ is of type $FP_{m-1}$ and $B$ is of type $FP_m$. Then $C$ is of type $FP_m$.*

**Proof** We induct on $m \geq 1$. The case $m = 1$ is obvious since a pro-$p$ group is of type $FP_1$ if and only if the group is finitely generated (as a pro-$p$ group).

Assume that $m > 1$ and that the result holds for $m - 1$; hence, $C$ is of type $FP_{m-1}$. Since $A$ is $FP_{m-1}$, we have that $H_j(A, \mathbb{F}_p)$ is finite for $0 \leq j \leq m - 1$. Consider the LHS spectral sequence

$$E^2_{i,j} = H_i(C, H_j(A, \mathbb{F}_p)) \text{ that converges to } H_{i+j}(B, \mathbb{F}_p).$$

By substituting if necessary $C$ with a subgroup of finite index, we can assume that $C$ acts trivially on $H_j(A, \mathbb{F}_p)$ for $0 \leq j \leq m - 1$. Then

$$E^2_{i,j} = H_i(C, H_j(A, \mathbb{F}_p)) \simeq H_i(C, \mathbb{F}_p) \widehat{\otimes} H_j(A, \mathbb{F}_p)$$

is finite for $0 \leq i, j \leq m - 1$. Then, for $r \geq 2$, consider the differential

$$d^r_{m,0} : E^r_{m,0} \to E^r_{m-r,r-1}$$

and note that either $m - r < 0$; hence, $E^2_{m-r,r-1} = 0$ or $m - r \leq m - 1, r - 1 \leq m - 1$. In all cases, $E^2_{m-r,r-1}$ is finite, and hence $E^r_{m-r,r-1}$ is finite and so $E^{r+1}_{m,0} = \operatorname{Ker}(d^r_{m,0})$ is finite if and only if $E^r_{m,0}$ is finite. Thus,

$$E^\infty_{m,0} \text{ is finite if and only if } E^2_{m,0} = H_m(C, \mathbb{F}_p) \text{ is finite.}$$

https://doi.org/10.4153/S0008414X23000895 Published online by Cambridge University Press

Finally, since $B$ is $FP_m$, we have that $H_m(B, \mathbb{F}_p)$ is finite and by the convergence of the spectral sequence $E^{\infty}_{m,0}$ is finite. Thus, we conclude that $H_m(C, \mathbb{F}_p)$ is finite. And, this together with $C$ is of type $FP_{m-1}$ implies that $C$ is $FP_m$. ∎

Recall that a pro-$p$ HNN extension is called proper if the canonical map from the base group to the pro-$p$ HNN extension is injective.

***Lemma 3.3*** *Let $G = \langle A, t \mid K^t = K\rangle$ be a proper pro-p HNN extension and m is a positive integer. Suppose that M is a normal pro-p subgroup of G such that $G/M \simeq \mathbb{Z}_p$, $K \not\subseteq M$, and $M \cap A$ is of type $FP_m$. Then the following conditions hold:*

a) *M is of type $FP_m$ if and only if $M \cap K$ is of type $FP_{m-1}$;*

b) *if M is of type $FP_{m+1}$, then $M \cap K$ is of type $FP_m$.*

**Proof** By [19, Theorem 4.1], the proper pro-$p$ HNN extension gives rise to the exact sequence of $\mathbb{F}_p[[G]]$-modules

$$0 \to \mathbb{F}_p[[G]] \otimes_{\mathbb{F}_p[[K]]} \mathbb{F}_p \to \mathbb{F}_p[[G]] \otimes_{\mathbb{F}_p[[A]]} \mathbb{F}_p \to \mathbb{F}_p \to 0. \tag{3.1}$$

Note that since $K \not\subseteq M$, we have that $M\backslash G/K = G/MK$ is a proper pro-$p$ quotient of $G/M \simeq \mathbb{Z}_p$, hence is finite. Similarly, $M\backslash G/A = G/MA$ is finite.

Note that there is an isomorphism of (left) $\mathbb{F}_p[[M]]$-modules

$$\mathbb{F}_p[[G]] \otimes_{\mathbb{F}_p[[K]]} \mathbb{F}_p \simeq (\oplus_{t\in M\backslash G/K}\mathbb{F}_p[[M]]t\mathbb{F}_p[[K]]) \otimes_{\mathbb{F}_p[[K]]} \mathbb{F}_p \simeq$$

$$\oplus_{t\in M\backslash G/K}\mathbb{F}_p[[M]] \otimes_{\mathbb{F}_p[[M\cap tKt^{-1}]]} \mathbb{F}_p.$$

Similarly, there is an isomorphism of (left) $\mathbb{F}_p[[M]]$-modules

$$\mathbb{F}_p[[G]] \otimes_{\mathbb{F}_p[[A]]} \mathbb{F}_p \simeq (\oplus_{t\in M\backslash G/A}\mathbb{F}_p[[M]]t\mathbb{F}_p[[A]]) \otimes_{\mathbb{F}_p[[A]]} \mathbb{F}_p \simeq$$

$$\oplus_{t\in M\backslash G/A}\mathbb{F}_p[[M]] \otimes_{\mathbb{F}_p[[M\cap tAt^{-1}]]} \mathbb{F}_p.$$

The short exact sequence (3.1) gives rise to a long exact sequence in pro-$p$ homology

$$\cdots \to H_{m+1}(M, \mathbb{F}_p) \to H_m(M, \mathbb{F}_p[[G]] \otimes_{\mathbb{F}_p[[K]]} \mathbb{F}_p) \to$$

$$H_m(M, \mathbb{F}_p[[G]] \otimes_{\mathbb{F}_p[[A]]} \mathbb{F}_p) \to H_m(M, \mathbb{F}_p) \to H_{m-1}(M, \mathbb{F}_p[[G]] \otimes_{\mathbb{F}_p[[K]]} \mathbb{F}_p)$$

$$\to \cdots \to H_1(M, \mathbb{F}_p[[G]] \otimes_{\mathbb{F}_p[[A]]} \mathbb{F}_p) \to H_1(M, \mathbb{F}_p) \to$$

$$H_0(M, \mathbb{F}_p[[G]] \otimes_{\mathbb{F}_p[[K]]} \mathbb{F}_p) \to H_0(M, \mathbb{F}_p[[G]] \otimes_{\mathbb{F}_p[[A]]} \mathbb{F}_p) \to H_0(M, \mathbb{F}_p) \to 0.$$

Note that

$$H_i(M, \mathbb{F}_p[[G]] \otimes_{\mathbb{F}_p[[K]]} \mathbb{F}_p) \simeq H_i(M, \oplus_{t\in M\backslash G/K}\mathbb{F}_p[[M]] \otimes_{\mathbb{F}_p[[M\cap tKt^{-1}]]} \mathbb{F}_p) \simeq$$

$$\oplus_{t\in M\backslash G/K}H_i(M, \mathbb{F}_p[[M]] \otimes_{\mathbb{F}_p[[M\cap tKt^{-1}]]} \mathbb{F}_p) \simeq \oplus_{t\in M\backslash G/K}H_i(M \cap tKt^{-1}, \mathbb{F}_p) =$$

$$\oplus_{t\in M\backslash G/K}H_i(t(M \cap K)t^{-1}, \mathbb{F}_p) \simeq \oplus_{t\in M\backslash G/K}H_i(M \cap K, \mathbb{F}_p).$$

Similarly,

$$H_i(M, \mathbb{F}_p[[G]] \otimes_{\mathbb{F}_p[[A]]} \mathbb{F}_p) \simeq \oplus_{t\in M\backslash G/A}H_i(M \cap A, \mathbb{F}_p).$$

https://doi.org/10.4153/S0008414X23000895 Published online by Cambridge University Press

Then the long exact sequence could be rewritten as

$$\cdots \to H_{m+1}(M,\mathbb{F}_p) \to \oplus_{t\in M\backslash G/K} H_m(M\cap K,\mathbb{F}_p) \to \oplus_{t\in M\backslash G/A} H_m(M\cap A,\mathbb{F}_p) \to$$

$$H_m(M,\mathbb{F}_p) \to \oplus_{t\in M\backslash G/K} H_{m-1}(M\cap K,\mathbb{F}_p) \to \cdots \to \oplus_{t\in M\backslash G/A} H_1(M\cap A,\mathbb{F}_p) \to$$

$$H_1(M,\mathbb{F}_p) \to \oplus_{t\in M\backslash G/K} H_0(M\cap K,\mathbb{F}_p) \to$$

$$\oplus_{t\in M\backslash G/A} H_0(M\cap A,\mathbb{F}_p) \to H_0(M,\mathbb{F}_p) \to 0.$$

Since $M\cap A$ is of type $FP_m$, we have that $H_i(M\cap A,\mathbb{F}_p)$ is finite for $i\le m$. Combining with $M\backslash G/A$ is finite, we conclude that

$$\oplus_{t\in M\backslash G/A} H_i(M\cap A,\mathbb{F}_p) \text{ is finite for } i\le m.$$

a) Note that $M$ is of type $FP_m$ if and only if $H_i(M,\mathbb{F}_p)$ is finite for $i\le m$. By the above long exact sequence together with the fact that $M\backslash G/K$ is finite, $H_i(M,\mathbb{F}_p)$ is finite for $i\le m$ if and only if $\oplus_{t\in M\backslash G/K} H_i(M\cap K,\mathbb{F}_p)$ is finite for $i\le m-1$, i.e., $M\cap K$ is of type $FP_{m-1}$.

b) If $M$ is of type $FP_{m+1}$, then $H_{m+1}(M,\mathbb{F}_p)$ is finite and since $H_m(M\cap A,\mathbb{Z}_p)$ is finite by the long exact sequence $H_m(M\cap K,\mathbb{F}_p)$ is finite. We already know by a) that $M\cap K$ is of type $FP_{m-1}$, and hence $M\cap K$ is of type $FP_m$. ■

Next, we prove a technical lemma that will be used in the proof of Proposition 3.5. For a pro-$p$ group $G$ with a subset $S$, denote by $\langle S\rangle$ the pro-$p$ subgroup of $G$ generated by $S$.

***Proposition 3.4*** *Let $Q=\langle x,y\rangle \simeq \mathbb{Z}_p^2$ and $A$ be a finitely generated pro-$p\mathbb{Z}_p[[Q]]$-module. Suppose that for $H=\langle x\rangle$, we have that $A$ is finitely generated as a pro-$p$ $\mathbb{Z}_p[[H]]$-module. Let $H_j=\langle xy^{-p^j}\rangle$. Then there is $j_0>0$ such that for every $j\ge j_0$, we have that $A$ is finitely generated as a pro-$p$ $\mathbb{Z}_p[[H_j]]$-module.*

**Proof** By Theorem 2.1, if $P$ is a pro-$p$ subgroup of $Q$, then $A$ is finitely generated as pro-$p$ $\mathbb{Z}_p[[P]]$-module if and only if $T(Q,P)\cap\Delta(A)=\{1\}$.

Let

$$J=\operatorname{ann}_{\mathbb{Z}_p[[Q]]}(A).$$

Since $A$ is finitely generated as a pro-$p$ $\mathbb{Z}_p[[H]]$-module for every $\chi\in T(Q,H)\backslash\{1\}$, we have that $J\notin \operatorname{Ker}(\overline{\chi})$. Since $\mathbb{Z}_p[[Q]]$ is a Noetherian ring, there is a finite subset $\Lambda$ of $J$ that generates $J$ as an ideal (abstractly or topologically is the same).

We aim to show that for sufficiently big $j$, we have $T(Q,H_j)\cap\Delta(A)=\{1\}$. Let $\mu_j\in T(Q,H_j)\backslash\{1\}$; thus, we aim to show that $\mu_j\notin\Delta(A)$. Then, by Theorem 2.1, $A$ is finitely generated as a pro-$p$ $\mathbb{Z}_p[[H_j]]$-module.

Let

$$\overline{\mu}_j : \mathbb{Z}_p[[Q]] \to \mathbb{F}[[t]]$$

be the continuous ring homomorphism induced by $\mu_j$. Since $\overline{\mu}_j(H_j)=1$, we have $\mu_j(x)=\mu_j(y^{p^j})\ne 1$. Let

$$\chi\in T(Q,H)\backslash\{1\} \text{ be such that } \chi(y)=\mu_j(y),\ \chi(x)=1$$

https://doi.org/10.4153/S0008414X23000895 Published online by Cambridge University Press

and

$$\overline{\chi} : \mathbb{Z}_p[[Q]] \to \mathbb{F}[[t]]$$

be the continuous ring homomorphism induced by $\chi$. Let

$$\lambda \in \mathbb{Z}_p[[Q]] = \mathbb{Z}_p[[t_1, t_2]], \text{ where } x = 1 + t_1, y = 1 + t_2.$$

Then, since $\chi(y) = \mu_j(y)$, $\chi(x) = 1$, we have $\chi(t_2) = 0$, and hence

$$\overline{\chi}(\lambda) = \overline{\chi}(\lambda|_{t_1=0}) = \overline{\mu}_j(\lambda|_{t_1=0}),$$

where $\lambda = \sum_{i,k\geq 0} z_{i,k} t_1^i t_2^k$, $z_{i,k} \in \mathbb{Z}_p$ and $\lambda|_{t_1=0} = \sum_{k\geq 0} z_{0,k} t_2^k$. Note that

$$\overline{\mu}_j(t_2) = \overline{\mu}_j(1 + t_2) - \overline{\mu}_j(1) = \overline{\mu}_j(y) - 1 \in 1 + t\mathbb{F}[[t]] - 1 = t\mathbb{F}[[t]].$$

Note that since $\mathbb{F}$ has characteristic $p > 0$, we have

$$1 + \overline{\mu}_j(t_1) = \overline{\mu}_j(1 + t_1) = \overline{\mu}_j(x) = \mu_j(x) =$$

$$\mu_j(y^{p^j}) = \overline{\mu}_j((1 + t_2)^{p^j}) = \overline{\mu}_j(1 + t_2^{p^j}) = 1 + \overline{\mu}_j(t_2^{p^j}).$$

Consider

$$\lambda|_{t_1=t_2^{p^j}} := \sum_{i,k\geq 0} z_{i,k} t_2^{ip^j+k} = \sum_{s\geq 0} \left( \sum_{ip^j+k=s} z_{i,k} \right) t_2^s.$$

Then, using that $\overline{\mu}_j(t_1) = \overline{\mu}_j(t_2^{p^j})$, we conclude that

$$\overline{\mu}_j(\lambda) = \overline{\mu}_j(\lambda|_{t_1=t_2^{p^j}}) = \overline{\mu}_j \left( \sum_{i,k\geq 0} z_{i,k} t_2^{ip^j+k} \right) = \overline{\mu}_j(\sum_{k\geq 0} z_{0,k} t_2^k) + \overline{\mu}_j \left( \sum_{i\geq 1,k\geq 0} z_{i,k} t_2^{ip^j+k} \right)$$

$$= \overline{\mu}_j(\lambda|_{t_1=0}) + \overline{\mu}_j \left( \sum_{i\geq 1,k\geq 0} z_{i,k} t_2^{ip^j+k} \right) = \overline{\chi}(\lambda) + \overline{\mu}_j \left( \sum_{i\geq 1,k\geq 0} z_{i,k} t_2^{ip^j+k} \right). \tag{3.2}$$

Suppose now that $\mu_j \in \Delta(A)$. Then $\overline{\mu}_j(J) = 0$, in particular $\overline{\mu}_j(\Lambda) = 0$. On the other hand, $\chi \notin \Delta(A)$; hence, $\overline{\chi}(J) \neq 0$. This is equivalent with $\overline{\chi}(\Lambda) \neq 0$. So there is $\lambda_0 \in \Lambda$ such that

$$\overline{\chi}(\lambda_0) \neq 0 = \overline{\mu}_j(\lambda_0).$$

Write as before $\lambda_0 = \sum_{i,k\geq 0} z_{i,k} t_1^i t_2^k$ where $z_{i,k} \in \mathbb{Z}_p$. Then, by (3.2),

$$0 = \overline{\mu}_j(\lambda_0) = \overline{\chi}(\lambda_0) + \overline{\mu}_j \left( \sum_{i\geq 1,k\geq 0} z_{i,k} t_2^{ip^j+k} \right).$$

So, for

$$\lambda_0|_{t_1=0} = \sum_{i\geq 0} z_i t_2^i,$$

https://doi.org/10.4153/S0008414X23000895 Published online by Cambridge University Press

where $z_i \in \mathbb{Z}_p$ and using that $\overline{\chi}(\lambda_0) = \overline{\chi}(\lambda_0|_{t_1=0})$, we have

$$-\overline{\chi}(\sum_{i\geq 0} z_i t_2^i) = \overline{\mu}_j\left(\sum_{i\geq 1, k\geq 0} z_{i,k} t_2^{ip^j+k}\right). \tag{3.3}$$

Let

$$f_0 := \overline{\mu}_j(y) - 1 = \overline{\mu}_j(t_2+1) - 1 = \overline{\mu}_j(t_2) = \mu_j(t_2) \in t\mathbb{F}[[t]]\backslash\{0\};$$

hence,

$$\overline{\chi}(t_2) = \overline{\chi}(t_2+1) - 1 = \overline{\chi}(y) - 1 = \chi(y) - 1 = \mu_j(y) - 1 = f_0.$$

Then, by (3.3), we conclude that

$$0 \neq -\sum_{i\geq 0} \overline{z}_i f_0^i = \sum_{i\geq 1, k\geq 0} \overline{z}_{i,k} f_0^{ip^j+k}, \tag{3.4}$$

where for $z \in \mathbb{Z}_p$ we denote by $\overline{z}$ the image of $z$ in $\mathbb{F}_p = \mathbb{Z}_p/p\mathbb{Z}_p$.

Consider the map

$$o : \mathbb{F}[[t]]\backslash\{0\} \to \mathbb{Z}_{\geq 0}$$

that sends $\sum_{i\geq 0} a_i t^i$ to the smallest $i_0$ such that $a_{i_0} \neq 0$, where each $a_i \in \mathbb{F}$. Define

$$d = o(f_0) \geq 1 \text{ and } k_0 = o(\sum_i \overline{z}_i t^i) \geq 0.$$

Then

$$o\left(-\sum_{i\geq 0} \overline{z}_i f_0^i\right) = dk_0. \tag{3.5}$$

By (3.4) and (3.5), there is $\overline{z}_{i,k} \neq 0$ for some $i \geq 1, k \geq 0$ such that

$$dp^j \leq d(ip^j + k) = o(f_0^{ip^j+k}) \leq dk_0, \text{ hence } p^j \leq k_0.$$

Note that $k_0$ depends only on $\lambda_0 \in \Lambda$, where $\Lambda$ is a finite set, hence it does not depend on $j$. From the very beginning, we can choose $j_0 \in \mathbb{Z}_{>0}$ such that

$$p^{j_0} > max\{\widetilde{k} = o(\sum_i \overline{m}_i t^i) \mid \widetilde{\lambda}|_{t_1=0} = \sum_i m_i t_2^i \neq 0, \widetilde{\lambda} \in \Lambda\} \geq k_0.$$

Then, for $j \geq j_0$, we get a contradiction, so $\mu_j \notin \Delta(A)$ as required. ∎

***Proposition 3.5*** *Let $G$ be a pro-$p$ group with a normal pro-$p$ subgroup $G_0$ such that $G/G_0 \simeq \mathbb{Z}_p^2$. Let $S$ be a normal pro-$p$ subgroup of $G$ such that $G/S \simeq \mathbb{Z}_p$, $G_0 \subseteq S$, and $S$ is of type $FP_m$ for some $m \geq 1$. Then there is a normal pro-$p$ subgroup $S_0$ of $G$ such that $G/S_0 \simeq \mathbb{Z}_p$, $S \neq S_0$, $G_0 \subseteq S_0$, and $S_0$ is of type $FP_m$.*

**Proof** Note that since $S$ is a pro-$p$ group of type $FP_m$ and $G/S \simeq \mathbb{Z}_p$ is a pro-$p$ group of type $FP_\infty$, hence of type $FP_m$, we can conclude that $G$ is a pro-$p$ group of type $FP_m$. Set

$$Q = G/G_0 = \langle x, y\rangle \text{ and } H = S/G_0 = \langle x\rangle.$$

https://doi.org/10.4153/S0008414X23000895 Published online by Cambridge University Press

Since $Q = G/G_0$ is a finitely generated abelian pro-$p$ group, hence $\mathbb{Z}_p[[Q]]$ is left and right Noetherian and $G$ is of type $FP_m$, we conclude that

$$A_i = H_i(G_0, \mathbb{Z}_p) \text{ is finitely generated as a pro-}p\ \mathbb{Z}_p[[Q]]\text{-module for } i \leq m.$$

Since $S$ is a pro-$p$ group of type $FP_m$, we conclude that

$$A_i \text{ is finitely generated as a pro-}p\ \mathbb{Z}_p[[H]]\text{-module for } i \leq m.$$

Then, by Proposition 3.4, for sufficiently big $j$, we have that $A_i$ is finitely generated as a pro-$p$ $\mathbb{Z}_p[[H_j]]$-module, where $H_j = \langle xy^{-p^j} \rangle \leq Q$, for every $i \leq m$. Then we define $S_0$ as the preimage in $G$ of one such $H_j$. ■

We recall the statement of Theorem 1.1.

***Theorem 1.1*** *Let $1 \to K \to G \to \Gamma \to 1$ be a short exact sequence of pro-p groups and $n_0 \geq 1$ be an integer such that:*
*1) G and K are of type $FP_{n_0}$,*
*2) $\Gamma^{ab}$ is infinite,*
*3) there is a normal pro-p subgroup N of K such that $G' \cap K \subseteq N$, $K/N \simeq \mathbb{Z}_p$ and N is of type $FP_{n_0-1}$.*
*Then there is a normal pro-p subgroup M of G such that $G/M \simeq \mathbb{Z}_p$, $M \cap K = N$, and M is of type $FP_{n_0}$. Furthermore, if K, G, and N are of type $FP_\infty$, then M can be chosen of type $FP_\infty$.*

**Proof of Theorem 1.1** Consider a commutative diagram

$$\begin{array}{ccccc} K & \hookrightarrow & \Pi & \twoheadrightarrow & F_n \\ {\scriptstyle id_K}\downarrow & & {\scriptstyle \pi}\downarrow & & \downarrow \\ K & \hookrightarrow & G & \twoheadrightarrow & \Gamma \end{array}$$

where the lines are short exact sequences of pro-$p$ groups, $F_n$ is the free pro-$p$ group with a free basis $s_1, \dots, s_n$, and the vertical maps are surjective homomorphisms of pro-$p$ groups with the most left map being the identity map.

Define

$$\Pi = \Pi_1 \coprod_K \Pi_2 \coprod_K \dots \coprod_K \Pi_n,$$

where $\coprod_K$ is the amalgamated free product with amalgam $K$ in the category of pro-$p$ groups, and each

$$\Pi_i = K \rtimes \langle s_i \rangle, \ \langle s_i \rangle \simeq \mathbb{Z}_p.$$

Note that since $K$ is normal in $\Pi$ and $\Pi/K \simeq \Pi_1/K \coprod \Pi_2/K \coprod \cdots \coprod \Pi_n/K$ is a free pro-$p$ product, we conclude that $\Pi_1 \coprod_K \Pi_2 \coprod_K \cdots \coprod_K \Pi_i$ embeds in $\Pi$ for every $1 \leq i \leq n$.

Recall that $\Gamma^{ab}$ is infinite; hence, the image in $\Gamma^{ab}$ of at least one $\pi(s_i)$ has infinite order. Without loss of generality, we can assume that the image of $\pi(s_1)$ in $\Gamma^{ab}$ has infinite order. In particular, the restriction map

$$\pi|_{\Pi_1} : \Pi_1 \to \pi(\Pi_1)$$

is an isomorphism.

Note that $[K, s_1] \subseteq G' \cap K \subseteq N$, hence $\Pi_1' \subseteq N$. We have $N \subseteq K \subseteq \Pi_1$ where $K/N \simeq \mathbb{Z}_p$, $\Pi_1/K \simeq \mathbb{Z}_p$, this together with the inclusion $\Pi_1' \subseteq N$ implies that $\Pi_1/N \simeq \mathbb{Z}_p^2$.

By assumption, $K$ is of type $FP_{n_0}$. By Proposition 3.5, there is $S_0$ a normal pro-$p$ subgroup of $\Pi_1$ such that

$$N \subseteq S_0, S_0 \text{ is of type } FP_{n_0}, S_0 \neq K \text{ and } \Pi_1/S_0 \simeq \mathbb{Z}_p.$$

Let

$$\mu : G \to \mathbb{Z}_p$$

be a homomorphism of pro-$p$ groups such that

$$\mathrm{Ker}(\mu \circ \pi) \cap \Pi_1 = S_0, \text{ i.e., } \mathrm{Ker}(\mu) \cap \pi(\Pi_1) = \pi(S_0).$$

This is possible since $\Pi_1/N \simeq \mathbb{Z}_p^2$ is abelian and $G' \cap K \subseteq N \subseteq S_0$. Note that $K \nsubseteq S_0$, hence $\mu(K) \neq 0$.

Consider the epimorphism of pro-$p$ groups

$$\chi = \mu \circ \pi : \Pi \to \mathbb{Z}_p.$$

Note that

$$\chi(K) \neq 0, \mathrm{Ker}(\chi) \cap \Pi_1 = S_0 \text{ is of type } FP_{n_0}$$

and

$$\mathrm{Ker}(\chi) \cap K = S_0 \cap K = N \text{ is of type } FP_{n_0-1}.$$

Then we view $\Pi_1 \coprod_K \Pi_2$ as a proper HNN extension

$$\langle \Pi_1, s_2 \mid K^{s_2} = K \rangle$$

with a pro-$p$ base group $\Pi_1$, associated pro-$p$ subgroup $K$ and stable letter $s_2$. Then, by Lemma 3.3(a),

$$\mathrm{Ker}(\chi) \cap (\Pi_1 \coprod_K \Pi_2) \text{ is of type } FP_{n_0}.$$

We view $\Pi_1 \coprod_K \Pi_2 \coprod_K \Pi_3$ as a proper HNN extension with a base pro-$p$ group $\Pi_1 \coprod_K \Pi_2$, associated pro-$p$ subgroup $K$ and stable letter $s_3$ then by Lemma 3.3(a)

$$\mathrm{Ker}(\chi) \cap \left(\Pi_1 \coprod_K \Pi_2 \coprod_K \Pi_3\right) \text{ is of type } FP_{n_0}.$$

Then, repeating this argument several times, we deduce that $\mathrm{Ker}(\chi)$ is of type $FP_{n_0}$.

By construction, $\mathrm{Ker}(\mu)$ is a quotient of $\mathrm{Ker}(\chi)$. If $n_0 = 1$, then $\mathrm{Ker}(\chi)$ is finitely generated (as a pro-$p$ group), then any pro-$p$ quotient of $\mathrm{Ker}(\chi)$ is finitely

https://doi.org/10.4153/S0008414X23000895 Published online by Cambridge University Press

generated (as a pro-$p$ group). In particular, $\mathrm{Ker}(\mu)$ is finitely generated (as a pro-$p$ group).

Now, for the general case, i.e., $n_0 \geq 2$, we will apply Lemma 3.1. Write $\widetilde{\mathrm{Ker}(\chi)}$ for the image of $\mathrm{Ker}(\chi)$ in $F_n$ and $\widetilde{\mathrm{Ker}(\mu)}$ for the image of $\mathrm{Ker}(\mu)$ in $\Gamma$. By construction,

$$\mathrm{Ker}(\chi) \cap K = N = \mathrm{Ker}(\mu) \cap K.$$

By assumption, $N$ is of type $FP_{n_0-1}$ and we have already shown that $\mathrm{Ker}(\chi)$ is of type $FP_{n_0}$. By construction, $\mu(K) \neq 0$, hence $K.\mathrm{Ker}(\mu) \neq \mathrm{Ker}(\mu)$ and since $G/\mathrm{Ker}(\mu) \simeq \mathbb{Z}_p$, we deduce that $K.\mathrm{Ker}(\mu)$ has finite index in $G$ and so $\widetilde{\mathrm{Ker}(\mu)}$ has finite index in $\Gamma$.

By Lemma 3.2, since in the short exact sequence of pro-$p$ groups

$$1 \to K \to G \to \Gamma \to 1$$

the pro-$p$ groups $G$ and $K$ are of type $FP_{n_0}$ (it suffices that $K$ is of type $FP_{n_0-1}$), we deduce that $\Gamma$ is of type $FP_{n_0}$. Then $\widetilde{\mathrm{Ker}(\mu)}$ is a pro-$p$ group of type $FP_{n_0}$. Then we can apply Lemma 3.1 for the commutative diagram

$$\begin{array}{ccccc} N = \mathrm{Ker}(\chi) \cap K & \hookrightarrow & \mathrm{Ker}(\chi) & \twoheadrightarrow & \widetilde{\mathrm{Ker}(\chi)} \\ \downarrow{\scriptstyle id_N} & & \downarrow{\scriptstyle \pi|_{\mathrm{Ker}(\chi)}} & & \downarrow \\ N = \mathrm{Ker}(\mu) \cap K & \hookrightarrow & \mathrm{Ker}(\mu) & \twoheadrightarrow & \widetilde{\mathrm{Ker}(\mu)} \end{array}$$

to deduce that $\mathrm{Ker}(\mu)$ is a pro-$p$ group of type $FP_{n_0}$. Finally, we set $M = \mathrm{Ker}(\mu)$. ■

**Proof of Corollary 1.2** We define $M$ as in the proof of Theorem 1.1 for $\Gamma = F_n$ and $\pi$ the identity map, $\mu = \chi$. Thus, $M = \mathrm{Ker}(\chi) = \mathrm{Ker}(\mu)$ is a normal pro-$p$ subgroup of $G$, $G/M \simeq \mathbb{Z}_p$ and $M$ is of type $FP_{n_0}$. We view

$$G = \Pi = \Pi_1 \coprod_K \Pi_2 \coprod_K \dots \coprod_K \Pi_n$$

as a proper HNN extension with a base pro-$p$ subgroup

$$A = \Pi_1 \coprod_K \Pi_2 \coprod_K \dots \coprod_K \Pi_{n-1},$$

associated pro-$p$ subgroup $K$ and stable letter $s_n$. By the proof of Theorem 1.1,

$$A \cap M = A \cap \mathrm{Ker}(\chi) \text{ is of type } FP_{n_0}.$$

Suppose that $M$ is of type $FP_{n_0+1}$. By Lemma 3.3(b), $N = M \cap K$ is of type $FP_{n_0}$, a contradiction. Hence, $M$ is not of type $FP_{n_0+1}$. This completes the proof of the corollary. ■

***Theorem 3.6*** *Let $G = K \rtimes \Gamma$ be a pro-$p$ group such that:*

1) *$K$ is a finitely generated pro-$p$ group, i.e., is of type $FP_1$,*

2) *there is a normal pro-$p$ subgroup $N$ of $K$ with $K/N \simeq \mathbb{Z}_p$ and $N$ is not finitely generated,*

3) *$\Gamma$ a finitely generated free pro-$p$ group with $d(\Gamma) \geq 2$.*

*Then $G$ is incoherent (in the category of pro-$p$ groups).*

https://doi.org/10.4153/S0008414X23000895 Published online by Cambridge University Press

**Proof** We claim that there is a finitely generated non-procyclic pro-$p$ subgroup $\Gamma_0$ of $\Gamma$ such that $\Gamma_0$ acts trivially on the abilianization $K^{ab} = K/K'$ via conjugation. Indeed, let $T = \text{tor}(K/K')$ be the torsion pro-$p$ subgroup of $K^{ab}$. Then $V = K^{ab}/T \simeq \mathbb{Z}_p^d$, where $d \geq 1$. Note that the conjugation action of $\Gamma$ on $V \simeq \mathbb{Z}_p^d$ induces a homomorphism

$$\rho : \Gamma \to GL_d(\mathbb{Z}_p).$$

Note that $\text{Im}(\rho)$ is a pro-$p$ subgroup of $GL_d(\mathbb{Z}_p)$, hence is $p$-adic analytic and there is an upper bound on the number of generators of any finitely generated pro-$p$ subgroup of $\text{Im}(\rho)$ [7]. Hence, $\rho$ is not injective. Alternatively, we can use the main result of [1] to deduce that $\rho$ is not injective. Thus, $\text{Ker}(\rho)$ is a nontrivial normal pro-$p$ subgroup of $\Gamma$ and we can choose $\Gamma_0$ any non-procyclic finitely generated pro-$p$ subgroup of $\text{Ker}(\rho)$.

Set $G_0 = K \rtimes \Gamma_0$. Then, by Corollary 1.2, there is a normal pro-$p$ subgroup $M$ of $G_0$ such that $G_0/M \simeq \mathbb{Z}_p$, $M \cap K = N$, and $M$ is $FP_1$ but not $FP_2$, i.e., it is finitely generated as a pro-$p$ group, but is not finitely presented as a pro-$p$ group. Thus, $G_0$ and hence $G$ are incoherent (in the category of pro-$p$ groups). This completes the proof. ■

**Proof of Corollary 1.3** Let $\Gamma_0$ be a finitely generated, free non-abelian pro-$p$ subgroup of $\Gamma$. Consider the preimage $G_0$ of $\Gamma_0$ in $G$, i.e., there is a short exact sequence

$$1 \to K \to G_0 \to \Gamma_0 \to 1.$$

Note that $G_0 = K \rtimes \Gamma_0$, then by Theorem 3.6, $G_0$ is not coherent. ■

## 4 More results on coherence

In this section, we show some applications of Corollary 1.3.

We recall the definition of the class of pro-$p$ groups $\mathcal{L}$. It uses the extension of centralizer construction. We define inductively the class $\mathcal{G}_n$ of pro-$p$ groups by setting $\mathcal{G}_0$ as the class of all finitely generated free pro-$p$ groups and a group $G_n \in \mathcal{G}_n$ if there is a decomposition

$$G_n = G_{n-1} \coprod_C A,$$

where $G_{n-1} \in \mathcal{G}_{n-1}$, $C$ is self-centralized procyclic subgroup of $G_{n-1}$, and $A$ is a finitely generated free abelian pro-$p$ group such that $C$ is a direct summand of $A$. The class $\mathcal{L}$ is defined as the class of all finitely generated pro-$p$ subgroups $G$ of $G_n$, where $G_n$ runs through all pro-$p$ groups in $\mathcal{G}_n$ for $n \geq 0$. The minimal $n$ such that $G \leq G_n \in \mathcal{G}_n$ is called the weight of $G$.

***Proposition 4.1*** [14] *Let $K \in \mathcal{L}$ be a nontrivial pro-p group. Then $K^{ab} = K/K'$ is infinite.*

**Proof of Corollary 1.4** By Proposition 4.1, $K^{ab}$ is infinite. Let $N$ be a normal pro-$p$ subgroup of $K$ such that $K/N \simeq \mathbb{Z}_p$. By part (4) from the main theorem of [14], we have that $N$ is not finitely generated as a pro-$p$ group. Then we can apply Corollary 1.3. This completes the proof. ■

https://doi.org/10.4153/S0008414X23000895 Published online by Cambridge University Press

***Definition*** Given a finite simplicial graph $X$, the pro-$p$ RAAG associated with $X$ is the pro-$p$ group defined by the presentation in the category of pro-$p$ groups

$$\langle V(X) \mid [v, w] = 1 \text{ if } v, w \text{ are adjacent in } X \rangle,$$

where $V(X)$ is the set of vertices of $X$.

***Lemma 4.2*** *Let G be a Demushkin pro-p group such that $d(G) = 2$. Then G is soluble and has Euler characteristic 0.*

**Proof** The classification of Demushkin groups has several cases described in Theorems 2.3–2.5. In the case of 2-generated Demushkin group, we have a 1-relation presentation with a relation of the type $[x_1, x_2]$ or $x_1^q[x_1, x_2]$, where $q$ is a power of $p$ or of the type $2^f + 2$, $p = 2$. In all these cases, the group is soluble, since it is $\langle x_1 \rangle \rtimes \langle x_2 \rangle$ and has zero Euler characteristic. ■

**Proof of Corollary 1.5** We claim that $\Gamma$ has a free non-abelian pro-$p$ subgroup $F$. Suppose first that $\Gamma$ is a pro-$p$ RAAG. Let $v_1, v_2$ be vertices of the graph that defines the pro-$p$ RAAG $\Gamma$ that are not adjacent. Then the pro-$p$ subgroup $F$ of $\Gamma$ generated by $v_1$ and $v_2$, is a retract of $\Gamma$, hence it is non-abelian free pro-$p$.

If $\Gamma$ is a non-soluble Demushkin group, then every pro-$p$ subgroup of infinite index in $\Gamma$ has cohomological dimension 1, so is free pro-$p$ [24, Ex. 5b), p. 44]. Furthermore, the abelianization of $\Gamma$ is infinite, so we can set $F$ to be a normal pro-$p$ subgroup of $\Gamma$ such that $\Gamma/F \simeq \mathbb{Z}_p$.

We claim that there is a normal pro-$p$ subgroup $N$ of $K$ such that $N$ is not finitely generated (as a pro-$p$ group) and $K/N \simeq \mathbb{Z}_p$. Suppose first that $K$ is a non-abelian pro-$p$ RAAG. Let $w_1$ and $w_2$ be vertices of the graph that defines the pro-$p$ RAAG $K$ that are not adjacent. Then the pro-$p$ subgroup $F_0$ of $K$ generated by $w_1$ and $w_2$ is non-abelian free pro-$p$ and it is a retract of $K$. Note that any normal pro-$p$ subgroup $S$ of $F_0$ such that $F_0/S \simeq \mathbb{Z}_p$ is not finitely generated (as a pro-$p$ group), hence any preimage $N$ of $S$ in $K$ is not finitely generated (as a pro-$p$ group).

Suppose that $K$ is a non-soluble Demushkin group. Note that $K/K'$ is infinite and let $N$ be a normal pro-$p$ subgroup of $K$ such that $K/N \simeq \mathbb{Z}_p$. If $N$ is finitely generated (as a pro-$p$ subgroup), together with the fact that $N$ is free pro-$p$, we conclude that $N$ has finite Euler characteristic $\chi(N)$, $\chi(K/N) = \chi(\mathbb{Z}_p) = 0$ and

$$0 = \chi(N)\chi(K/N) = \chi(K) = 1 - d(K) + 1 = 2 - d(K),$$

so $d(K) = 2$. But by Lemma 4.2 2-generated Demushkin groups are soluble, a contradiction.

Finally, by Corollary 1.3, $G$ is not coherent. ■

***Proposition 4.3*** *Let $1 \to K \to G \to \mathbb{Z}_p^m \to 1$ be a short exact sequence of pro-p groups with K free pro-p or Demushkin pro-p group. Then G is coherent (as a pro-p group).*

**Proof** Let $H$ be a finitely generated, pro-$p$ subgroup of $G$. We aim to prove that $H$ is $FP_\infty$, in particular is finitely presented.

Consider the short exact sequence of pro-$p$ groups

$$1 \to K_0 \to H \to Q \to 1,$$

where $Q$ is a pro-$p$ subgroup of $\mathbb{Z}_p^m$, so is finitely generated, abelian, and $K_0 = K \cap H$. Recall that an infinite index subgroup in a Demushkin group is free pro-$p$, in particular $K_0$ is either free pro-$p$ or Demushkin. In the latter, $K_0$ is $FP_\infty$, hence $H$ is $FP_\infty$. Thus, we can assume from now on that $K_0$ is free pro-$p$.

Consider the LHS spectral sequence

$$E^2_{i,j} = H_i(Q, H_j(K_0, \mathbb{F}_p)) \text{ that converges to } H_{i+j}(H, \mathbb{F}_p).$$

Note that $E^2_{i,j} = 0$ for $j \geq 2$ and

$$d^k_{i,j} : E^k_{i,j} \to E^k_{i-k,j+k-1} \text{ for } k \geq 2.$$

Thus, $E^3_{i,j} = E^\infty_{i,j}$ and

$$E^3_{i,0} = \mathrm{Ker}(d^2_{i,0}) \text{ is finite}$$

since $E^2_{i,0}$ is finite for all $i$. Note that

$$E^3_{i,1} = \mathrm{Coker}(d^2_{i+2,0}) = E^2_{i,1}/\mathrm{Im}(d^2_{i+2,0}) \text{ is finite if and only if } E^2_{i,1} \text{ is finite,}$$

since $E^2_{i+2,0}$ is finite for all $i$.

Since $H$ is finitely generated and by the convergence of the spectral sequence, we have that

$$E^3_{0,1} = E^\infty_{0,1} \text{ is finite,}$$

hence

$$E^2_{0,1} = H_0(Q, H_1(K_0, \mathbb{F}_p)) \text{ is finite.}$$

Since $\mathbb{F}_p[[Q]]$ is a local ring, this implies that $H_1(K_0, \mathbb{F}_p)$ is finitely generated as a pro-$p$ $\mathbb{F}_p[[Q]]$-module. Then $E^2_{j,1}$ is finite for $j \geq 0$.

By the convergence of the spectral sequence, there is an exact sequence for $i \geq 2$

$$0 \to E^\infty_{i-1,1} \to H_i(H, \mathbb{F}_p) \to E^\infty_{i,0} \to 0$$

with both $E^\infty_{i-1,1}$ and $E^\infty_{i,0}$ finite. Hence, $H_i(H, \mathbb{F}_p)$ is finite for $i \geq 2$ and $H$ is of type $FP_\infty$, in particular is finitely presented as a pro-$p$ group. ■

## 5 Proofs of Corollaries 1.6 and 1.7

**Proof of Corollary 1.6** Let $F$ be a finitely generated free non-procyclic pro-$p$ group that embeds as a closed subgroup of $\mathrm{Out}(K)$. Note that $G = K \rtimes F$ is a pro-$p$ group that embeds as a closed subgroup of $\mathrm{Aut}(K)$ and by Theorem 3.6 $G$ is incoherent (in the category of pro-$p$ groups). Finally, $G_0 = G \cap \mathrm{Aut}_0(G)$ is a pro-$p$ subgroup of finite index in $G$, hence $G_0$ is incoherent (in the category of pro-$p$ groups). ■

**Proof of Corollary 1.7** We recall first some results from [18]. Let $G$ be a finitely generated pro-$p$ group and $\mathrm{Aut}(G)$ denote all continuous automorphisms of $G$ (which coincide with the abstract automorphisms of $G$). Denote $\mathrm{Inn}(G)$ the group of the internal automorphisms. The group $\mathrm{Aut}(G)$ is a profinite group. ■

https://doi.org/10.4153/S0008414X23000895 Published online by Cambridge University Press

***Lemma 5.1*** [18] a) *Let $G$ be a finitely generated pro-p group and $G^*$ be the Frattini subgroup of $G$, i.e., the intersection of all maximal open subgroups of $G$. Then* $\mathrm{Ker}(\mathrm{Aut}(G) \to \mathrm{Aut}(G/G^*))$ *is a pro-p subgroup of* $\mathrm{Aut}(G)$ *of finite index.*

b) *Let $F$ be a finitely generated free pro-p group and $N$ be a characteristic pro-p subgroup of $F$. Then the map* $\mathrm{Aut}(F) \to \mathrm{Aut}(F/N)$*, obtained by taking the induced automorphisms, is surjective.*

We set $\mathrm{Aut}_0(G) = \mathrm{Ker}(\mathrm{Aut}(G) \to \mathrm{Aut}(G/G^*))$ and $\mathrm{Out}_0(G) = \mathrm{Aut}_0(G)/\mathrm{Inn}(G)$.

***Lemma 5.2*** *Suppose $K$ is a finitely generated, free pro-p group, $d(K) = n \geq 2$, and $M$ is the maximal pro-p metabelian quotient of $K$. Then* $\mathrm{Out}(M)$ *contains a finitely generated pro-p subgroup $H$ such that $H$ has a metabelian pro-p quotient that is not finitely presented (as a pro-p group).*

**Lemma 5.2 implies Corollary 1.7:** If $\mathrm{Out}(K)$ contains a pro-$p$ free non-procyclic subgroup, we can apply Corollary 1.6. Then we can assume that $\mathrm{Out}(K)$ does not contain a pro-$p$ free non-procyclic subgroup. We can further assume that the pro-$p$ version of the Bieri–Strebel result holds; otherwise, Corollary 1.7 holds, i.e., if a finitely presented pro-$p$ group does not contain a free non-procyclic pro-$p$ subgroup, then any metabelian pro-$p$ quotient of that group is a finitely presented pro-$p$ group.

Let $H$ be a pro-$p$ subgroup of $\mathrm{Out}(M)$ as in Lemma 5.2. Since $\mathrm{Aut}_0(M)$ has finite index in $\mathrm{Aut}(M)$, without loss of generality, we can assume that $H \subseteq \mathrm{Out}_0(M)$. The epimorphism of pro-$p$ groups $\mathrm{Aut}_0(K) \to \mathrm{Aut}_0(M)$ induces an epimorphism of pro-$p$ groups $\mathrm{Out}_0(K) \to \mathrm{Out}_0(M)$. Then there is a finitely generated pro-$p$ subgroup $\widetilde{H}$ of $\mathrm{Out}_0(K)$ that maps surjectively to $H$, in particular $\widetilde{H}$ has a metabelian pro-$p$ quotient that is not finitely presented (as a pro-$p$ group). Then, by the previous considerations, $\widetilde{H}$ is not a finitely presented pro-$p$ group.

Note that $\mathrm{Inn}(K) \simeq K$. Consider the short exact sequence

$$1 \to K \to \mathrm{Aut}_0(K) \to \mathrm{Out}_0(K) \to 1,$$

and let $H_0$ be the preimage of $\widetilde{H}$ in $\mathrm{Aut}_0(K)$. Then there is a short exact sequence

$$1 \to K \to H_0 \to \widetilde{H} \to 1$$

of pro-$p$ groups. Since $K$ is a finitely generated pro-$p$ group, we have that $H_0$ is a finitely generated pro-$p$ group and $H_0$ is not finitely presented; otherwise, $\widetilde{H}$ would be a finitely presented pro-$p$ group, a contradiction. Thus, $\mathrm{Aut}_0(K)$ is incoherent (in the category of pro-$p$ groups).

**Proof of Lemma 5.2** Here, we use significantly ideas introduced in [21]. We fix $x_1, x_2, \ldots, x_n$ a generating set of $M$. Define

$$\mathrm{IAut}(M) = \{\varphi \in \mathrm{Aut}(M) \mid \varphi \text{ induces on } M/M' \text{ the identity map}\},$$

where $\mathrm{Aut}(M)$ denotes continuous automorphisms of $M$. In fact, every abstract automorphism of a finitely generated pro-$p$ group is a continuous one. Then there is a short exact sequence of profinite groups

$$1 \to \mathrm{IAut}(M) \to \mathrm{Aut}(M) \to \mathrm{Aut}(M^{ab}) = GL_n(\mathbb{Z}_p) \to 1.$$

https://doi.org/10.4153/S0008414X23000895 Published online by Cambridge University Press

By [21], there is a Bachmut embedding $\beta$ of $\mathrm{IAut}(M)$ in $GL_n(\mathbb{Z}_p[[M^{ab}]])$, where $M^{ab}$ is the maximal abelian pro-$p$ quotient of $M$. By [21], where $\mathrm{Aut}(M)$ acts on the right,

$\beta(\varphi) = \left(\frac{\partial}{\partial x_j}(x_i^\varphi)\right)$ and $\frac{\partial}{\partial x_j} : M \to \mathbb{Z}_p[[M^{ab}]]$ are the Fox derivatives defined by

$$\frac{\partial}{\partial x_j}(1) = 0,\ \frac{\partial}{\partial x_j}(g_1 g_2) = \frac{\partial}{\partial x_j}(g_1) + \overline{g}_1 \frac{\partial}{\partial x_j}(g_2),\ \frac{\partial}{\partial x_j}(x_i) = \delta_{i,j},$$

where $\overline{g}_1$ is the image of $g_1 \in M$ in $M^{ab}$, $\delta_{i,j}$ is the Kronecker symbol.

Set $s_i$ for the image of $x_i - 1$ in $\mathbb{Z}_p[[M^{ab}]]$; thus,

$$\mathbb{Z}_p[[M^{ab}]] \simeq \mathbb{Z}_p[[s_1, s_2, \dots, s_n]].$$

Define

$$\det(\varphi) = \det(\beta(\varphi)).$$

By [21],

$$\det(\mathrm{IAut}(M)) = 1 + \Delta =: P$$

is a multiplicative abelian group, where $\Delta$ is the unique maximal ideal of $\mathbb{Z}_p[[M^{ab}]]$, and the $GL_n(\mathbb{Z}_p)$-action via conjugation on the abelianization of $\mathrm{IAut}(M)$ induces an action on $\det(\mathrm{IAut}(M)) = P$. Then we have a short exact sequence of profinite groups

$$1 \to P \to \mathrm{Aut}(M)/\mathrm{Ker}(\det) \to GL_n(\mathbb{Z}_p) \to 1.$$

Consider the pro-$p$ group

$$GL_n^1(\mathbb{Z}_p) = \mathrm{Ker}(GL_n(\mathbb{Z}_p) \to GL_n(\mathbb{F}_p)).$$

Let $Q$ be the maximal pro-$p$ quotient of $P$ that has exponent $p$. Then there is a pro-$p$ subgroup $T$ of $\mathrm{Aut}(M)/\mathrm{Ker}(\det)$ and a short exact sequence of pro-$p$ groups

$$1 \to P \to T \to GL_n^1(\mathbb{Z}_p) \to 1$$

and a pro-$p$ quotient $T_0$ of $T$ together with a short exact sequence of pro-$p$ groups

$$1 \to Q \to T_0 \to GL_n^1(\mathbb{Z}_p) \to 1.$$

By [22, (7)],

$$P^p \cap (1 + p\Delta) = 1 + p^2\Delta,$$

and for $\delta \in \Delta$ using $[\delta]$ for the image of $1 + p\delta$ in $Q$, we have that

$$[\delta_1][\delta_2] = [\delta_1 + \delta_2].$$

Thus, the multiplicative subgroup of $Q$ generated by $\{[\delta] \mid \delta \in \Delta\}$ could be identified with the additive group $\Delta/p\Delta$. Furthermore, using the long exact sequence in homology for the short exact sequence

$$0 \to \Delta \to \mathbb{Z}_p[[s_1, s_2, \dots, s_n]] \to \mathbb{F}_p \to 0,$$

https://doi.org/10.4153/S0008414X23000895 Published online by Cambridge University Press

we have a long exact sequence

$$0 = \mathrm{Tor}_1^{\mathbb{Z}_p}(\mathbb{Z}_p[[s_1, s_2, \dots, s_n]], \mathbb{F}_p) \to \mathrm{Tor}_1^{\mathbb{Z}_p}(\mathbb{F}_p, \mathbb{F}_p) \to$$
$$\Delta/p\Delta \to \mathbb{F}_p[[s_1, s_2, \dots, s_n]] \to \mathbb{F}_p \to 0.$$

Note that $\mathrm{Tor}_1^{\mathbb{Z}_p}(\mathbb{F}_p, \mathbb{F}_p) \simeq \mathbb{F}_p$ and thus we have a short exact sequence of additive pro-$p$ groups

$$0 \to \mathbb{F}_p \to \Delta/p\Delta \xrightarrow{\nu} \Omega \to 0,$$

where $\Omega$ is the augmentation ideal of $\mathbb{F}_p[[s_1, s_2, \dots, s_n]]$ and for the canonical projection

$$\pi : \Delta \to \Delta/p\Delta,$$

the composition map $\nu \circ \pi : \Delta \to \Omega$ is the restriction of the map $\mathbb{Z}_p[[s_1, s_2, \dots, s_n]] \to \mathbb{F}_p[[s_1, s_2, \dots, s_n]]$ that reduces coefficients mod $p$. Actually, $\mathrm{Ker}(\nu) = \mathbb{F}_p$ is generated as an additive group by $p + p\Delta$.

Consider now $\varphi_2 \in \mathrm{Aut}(M)$ given by

$$\varphi_2 = \rho^p, \text{ where } \rho(x_1) = x_1x_2, \rho(x_k) = x_k \text{ for } 2 \le k \le n$$

and $\varphi_1 \in \mathrm{IAut}(M)$ such that

$$\det(\beta(\varphi_1)) = 1 + ps_1.$$

Note that $\varphi_1$ is not uniquely determined and that the image of $\varphi_2$ in $GL_n(\mathbb{Z}_p)$ is in $GL_n^1(\mathbb{Z}_p)$. Hence, the profinite subgroup $\Gamma$ of $\mathrm{Aut}(M)$ generated by $\varphi_1, \varphi_2$ is in fact a pro-$p$ group. Let

$$\Gamma_0 = \langle \psi_1, \psi_2 \rangle$$

be the image of $\Gamma$ in $T_0$, where $\psi_i$ is the image of $\varphi_i$ in $T_0$. Thus, $\Gamma_0$ is a pro-$p$ group.

By [21, Proposition 4.4], for every $\varphi \in \mathrm{IAut}(M)$ for $\varphi' = \rho^{-1}\varphi\rho$, $h' = \det(\beta(\varphi'))$ and $h = \det(\beta(\varphi))$, we have that $h'$ is obtained from $h$ applying the substitution $s_1 \to s_1 + s_2 + s_1s_2$. Then the action of $\psi_2$ on $\psi_1 = [s_1]$ by conjugations is induced by applying the substitution $s_1 \to s_1 + s_2 + s_1s_2$ exactly $p$-times, i.e., we apply the substitution

$$s_1 \to (1 + s_1)(1 + s_2)^p - 1.$$

Similarly, the action of $\psi_2^k$ on $\psi_1 = [s_1]$ by conjugation is induced by applying the substitution $s_1 \to s_1 + s_2 + s_1s_2$ exactly $pk$-times, and thus gives the substitution $s_1 \to (1 + s_1)(1 + s_2)^{pk} - 1$.

Let $A$ be the normal pro-$p$ subgroup of $\Gamma_0$ generated by $\psi_1$. Thus, $A$ can be identified with an additive subgroup of $\Delta/p\Delta$ and

$$\nu(A) \subseteq \nu(\Delta/p\Delta) = \Omega =$$
$$s_1\mathbb{F}_p[[s_1, s_2, \dots, s_n]] + s_2\mathbb{F}_p[[s_1, s_2, \dots, s_n]] + \cdots + s_n\mathbb{F}_p[[s_1, s_2, \dots, s_n]].$$

The previous paragraph shows that

$$\{(1 + s_1)(1 + s_2)^{pk} - 1 \mid k \ge 0\} \subseteq \nu(A),$$

in particular $\nu(A)$ and $A$ are infinite.

https://doi.org/10.4153/S0008414X23000895 Published online by Cambridge University Press

Note that $\Gamma_0 \simeq A \rtimes D$, where $D \simeq \mathbb{Z}_p$ is generated by $\psi_2$. We view $A$ as an $\mathbb{F}_p[[t]]$-module via the conjugation action of $\psi_2 = 1 + t$, where $\mathbb{F}_p[[t]] \simeq \mathbb{F}_p[[D]]$. Furthermore, $A$ is a pro-$p$ cyclic $\mathbb{F}_p[[t]]$-module, with a generator $\psi_1$. Since every proper $\mathbb{F}_p[[t]]$-module quotient of $\mathbb{F}_p[[t]]$ is a finite additive group and $A$ is infinite, we deduce that $A \simeq \mathbb{F}_p[[t]]$. Then, by the example after Theorem 2.2, $\Gamma_0$ is a metabelian pro-$p$ group that is not finitely presented.

Note that the image $W$ of $M \simeq \text{Inn}(M)$ in $T_0$ is inside $Q$ and since $M$ is a finitely generated pro-$p$ group and $Q$ is an abelian pro-$p$ group of finite exponent $p$, then $W$ and consequently $\Gamma_0 \cap W$ are finite. Since $\Gamma_0 \cap W$ is finite, $\Gamma_0/(\Gamma_0 \cap W)$ is not a finitely presented pro-$p$ group. Actually examining the structure of $\Gamma_0$, it is easy to see that any finite normal subgroup of $\Gamma_0$ is trivial, in particular $\Gamma_0 \cap W = 1$. Finally, $\Gamma_0 \simeq \Gamma_0/(\Gamma_0 \cap W)$ is a metabelian pro-$p$ quotient of a 2-generated pro-$p$ group $H \leq \text{Out}(M)$. This completes the proof of the lemma. ∎

**Acknowledgments** The author is thankful to the anonymous referee for pointing out a gap in the proof of Proposition 3.4 that was subsequently corrected.

## References


[1] Y. Barnea and M. Larsen, *A non-abelian free pro-p group is not linear over a local field*. J. Algebra **214**(1999), no. 1, 338–341.

[2] D. E.-C. Ben-Ezra and E. Zelmanov, *On pro-2 identities of 2×2 linear groups*. Trans. Amer. Math. Soc. **374**(2021), no. 6, 4093–4128.

[3] M. Bestvina and N. Brady, *Morse theory and finiteness properties of groups*. Invent. Math. **129**(1997), 445–470.

[4] R. Bieri and R. Strebel, *Valuations and finitely presented metabelian groups*. Proc. Lond. Math. Soc. **41**(1980), 439–464.

[5] S. Demushkin, *On the maximal p-extension of a local field*. Izv. Akad. Nauk, USSR Math. Ser. **25**, (1961), 329—346.

[6] S. Demushkin, *On 2-extensions of a local field*. Sibirsk. Mat. Z. **4**, (1963) 951—955.

[7] J. Dixon, M. P. F. Du Sautoy, A. Mann, and D. Segal, *Analytic pro-p groups*, London Mathematical Society Lecture Note Series, 157, Cambridge University Press, Cambridge, 1991.

[8] M. Feign and M. Handel, *Mapping tori of free group automorphisms are coherent*. Ann. Math. **149**(1999), 1061–1077.

[9] C. McA. Gordon, *Artin groups, 3-manifolds and coherence*. Bol. Soc. Mat. Mexicana (3) **10** (2004), 193—198.

[10] J. D. King, *Homological finiteness conditions for pro-p groups*. Comm. Algebra **27**(1999), no. 10, 4969–4991.

[11] J. D. King, *A geometric invariant for metabelian pro-p groups*. J. Lond. Math. Soc. (2) **60**(1999), no. 1, 83–94.

[12] D. H. Kochloukova and F. F. Lima, *Homological finiteness properties of fibre products*. Q. J. Math. **69**(2018), no. 3, 835–854.

[13] D. H. Kochloukova and S. Vidussi, *Higher dimensional algebraic fiberings of group extensions*. J. Lond. Math. Soc. (2) **108**(2023), no. 3, 978–1003.

[14] D. H. Kochloukova and P. A. Zalesskii, *On pro-p analogues of limit groups via extensions of centralizers*. Math. Z. **267**(2011), nos. 1–2, 109–128.

[15] R. Kropholler and G. Walsh, *Incoherence and fibering of many free-by-free groups*. Ann. Inst. Fourier (Grenoble) **72**(2022), no. 6, 2385–2397.

[16] B. Kuckuck, *Subdirect products of groups and the $n-(n+1)-(n+2)$ conjecture*. Q. J. Math. **65**(2014), 1293–1318.

[17] J. P. Labute, *Classification of Demushkin groups*. Canad. J. Math. **19**(1967), 106–132.

[18] A. Lubotzky, *Combinatorial group theory for pro-p-groups*. J. Pure Appl. Algebra **25**(1982), 311–325.

[19] L. Ribes and P. Zalesskii, *Pro-p trees and applications*. In: New horizons in pro-p groups, Progress in Mathematics, 184, Birkhauser, Boston, MA, 2000, pp. 75–119.
[20] L. Ribes and P. Zalesskii, *Profinite groups*, 2nd ed., Springer, Berlin, 2010.
[21] V. A. Romankov, *Generators for the automorphism groups of free metabelian pro-p groups*. Sibirsk. Mat. Zh. **33**(1992), no. 5, 145–158.
[22] V. A. Romankov, *Infinite generation of groups of automorphisms of free pro-p-groups*. Sibirsk. Mat. Zh. **34**(1993), no. 4, 153—159.
[23] J.-P. Serre, *Structure de certains pro-p-groupes*. Seminaire Bourbaki **1962**/63(1971), no. 252, 357—364.
[24] J.-P. Serre, *Galois cohomology*, Springer, Berlin–Heidelberg, 1997.
[25] J. S. Wilson, *Profinite groups*, London Mathematical Society Monographs. New Series, 19, The Clarendon Press and Oxford University Press, New York, 1998.
[26] H. Wilton, *Hall's theorem for limit groups*. Geom. Funct. Anal. **18**(2008), no. 1, 271–303.
[27] D. T. Wise, *An invitation to coherent groups*. In: What's next?—the mathematical legacy of William P. Thurston, Annals of Mathematics Studies, 205, Princeton University Press, Princeton, NJ, 2020, pp. 326–414.
[28] P. Zalesskii and T. Zapata, *Profinite extensions of centralizers and the profinite completion of limit groups*. Rev. Mat. Iberoam. **36**(2020), no. 1, 61–78.
[29] A. N. Zubkov, *Nonrepresentability of a free nonabelian pro-p-group by second-order matrices*. Sibirsk. Mat. Zh. **28**(1987), no. 5, 64–69.

*Department of Mathematics, State University of Campinas (UNICAMP), Campinas, SP, Brazil*
*e-mail*: desi@unicamp.br