A SPECIAL SIMPLEX †

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1. Preliminaries

Let $S = A_0 \cdots A_n$ be an n-simplex and A_{ih} the foot of its altitude from its vertex A_i to its opposite prime face S_i; O, G the circumcentre and centroid of S and O_i, G_i of S_i. Representing the position vector of a point P, referred to O, by $\boldsymbol{p}$, Coxeter [2] defines the *Monge point* M of S collinear with O and G by the relation

$$(1) \qquad (n-1)\boldsymbol{m} = (n+1)\boldsymbol{g} = n\boldsymbol{g}_i + \boldsymbol{a}_i,$$

so that the Monge point M_i of S_i is given by

$$(2) \qquad (n-2)\boldsymbol{m}_i = n\boldsymbol{g}_i - 2\boldsymbol{o}_i.$$

If the $n+1$ vectors $\boldsymbol{a}_i$ are related by

$$(3) \qquad \sum u_i \boldsymbol{a}_i = 0, \quad \sum u_i = u,$$

and $\boldsymbol{o}_i$ be given by

$$(4) \qquad \boldsymbol{o}_i = \sum c_j \boldsymbol{a}_j \quad (j \neq i), \quad \sum c_j = 1,$$

A_{ih} is given by

$$(5) \qquad \begin{aligned} u_i(\boldsymbol{a}_{ih} - \boldsymbol{a}_i) = p_i \boldsymbol{o}_i, \text{ i.e. } u_i \boldsymbol{a}_{ih} &= u_i \boldsymbol{a}_i + p_i \boldsymbol{o}_i \\ &= -\sum u_j \boldsymbol{a}_j + p_i \sum c_j \boldsymbol{a}_j \\ &= \sum (p_i c_j - u_j) \boldsymbol{a}_j. \end{aligned}$$

Since A_{ih} lies in S_i,

$$(6) \qquad u_i = p_i \sum c_j - \sum u_j = p_i - (u - u_i), \text{ i.e. } p_i = u.$$

If T_i be a point on $M_i A_{ih}$ such that

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$$
(7)\quad \begin{aligned}(n-1)\boldsymbol{t}_i &= (n-2)\boldsymbol{m}_i+\boldsymbol{a}_{ih}\\ &= n\boldsymbol{g}_i-2\boldsymbol{o}_i+\boldsymbol{a}_i+v_i\boldsymbol{o}_i \quad (v_i = u/u_i)\\ &= (n+1)\boldsymbol{g}-(2-v_i)\boldsymbol{o}_i\\ &= (n-1)\boldsymbol{m}-(2-v_i)\boldsymbol{o}_i,\end{aligned}
$$

i.e.

$$
(8)\qquad (n-1)(\boldsymbol{m}-\boldsymbol{t}_i) = (2-v_i)\boldsymbol{o}_i
$$

That is, MT_i is parallel to $\boldsymbol{oo}_i$ or normal to S_i at T_i. Or, the normals to the prime faces S_i of S at their points T_i concur at M. In fact, this property of M has been used to prove by induction [3] that an S-point $\boldsymbol{S}$ of S lies at M. Thus $M = \boldsymbol{S}, M_i = \boldsymbol{S}_i$ or

$$
(9)\qquad \boldsymbol{m} = \boldsymbol{s},\ \boldsymbol{m}_i = \boldsymbol{s}_i.
$$

2. Special simplex

If the diametric opposite B_i of a vertex A_i of the simplex S lie in its prime face S_i, S is said to be *special* [1], and denoted by (S_i), with A_i, S_i, A_iA_{ih} as its *special vertex, face and altitude.*

From (3) then follows that

$$
(10)\qquad u_i\boldsymbol{b}_i = u_i(-\boldsymbol{a}_i) = \sum u_j\boldsymbol{a}_j,\ u_i = \sum u_j = u-u_i = u/2.
$$

Hence, from the relations (1), (5)—(9), we get

$$
(11)\qquad (n-1)\boldsymbol{s} = n\boldsymbol{g}_i-(-\boldsymbol{a}_i) = n\boldsymbol{g}_i-\boldsymbol{b}_i,
$$

$$
(12)\qquad \boldsymbol{a}_{ih}-\boldsymbol{a}_i = \boldsymbol{a}_{ih}+\boldsymbol{b}_i = 2\boldsymbol{o}_i,
$$

$$
(13)\qquad \boldsymbol{s} = \boldsymbol{t}_i,
$$

$$
(14)\qquad (n-1)\boldsymbol{s} = (n-2)\boldsymbol{s}_i+\boldsymbol{a}_{ih}.
$$

Thus follows

THEOREM 1 (see Figure). *An n-simplex S becomes special,* (S_i), *if and only if* (i) *its special altitude is twice the distance of its circumcentre from its special face* S_i *and* (ii) *the foot* A_{ih} *of the same lies on its circumhypersphere as the symmetric of the diametric opposite* B_i *of its special vertex* A_i *w.r.t. the circumcentre* O_i *of* S_i; *an alternative condition is that* (iii) *its S-point lies in* S_i *on the join of the centroid* G_i *of* S_i *to* B_i *and on that of the S-point of* S_i *to* A_{ih} *dividing them in the ratios* $-1 : n$, $1 : (n-2)$ *respectively.*

3. $3(n+1)$-point-sphere

(a) With every simplex S is associated its $3(n+1)$-point-sphere (O')

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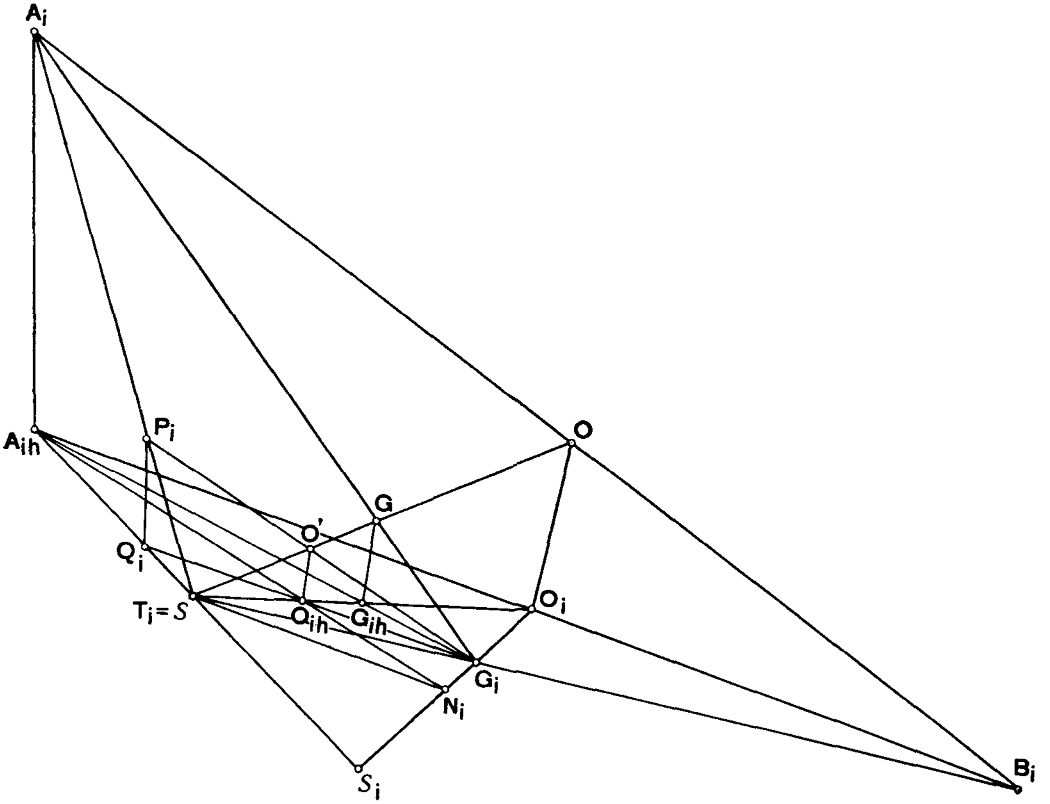

homothetic to its circumhypersphere (O) w.r.t. its centroid G[3] and its S-point S, the homothetic ratios being $-1:n$ and $1:n$ respectively. (O') passes through the $n+1$ centroids G_i of its prime faces S_i through the $n+1$ points P_i on the joins of $\boldsymbol{S}$ to its vertices A_i such that

(15) $$n\boldsymbol{p}_i = (n-1)\boldsymbol{s}+\boldsymbol{a}_i,$$

and through their $n+1$ projections Q_i in S_i. $G_iO'P_i$ is a diameter of (O'), O' being its centre, such that

(16) $$n\boldsymbol{o}' = (n-1)\boldsymbol{s}.$$

Thus G_i and the S-point $\boldsymbol{S}_i$ are the pair of homothetic centres of the $(n-2)$-sphere section (O_i) of (O) by S_i (circumscribing S_i) and the $(3n)$point-sphere (N_i) of S_i, the homothetic ratios being $1:(1-n)$ and $1:(n-1)$ respectively, so that

(17) $$(n-1)\boldsymbol{n}_i = (n-2)\boldsymbol{s}_i+\boldsymbol{o}_i,$$

where N_i is the centre of (N_i).

Now, if S becomes special, (S_i), the foot A_{ih} of its special altitude lies on (O_i) by Theorem 1 (ii) and therefore $\boldsymbol{S}$ lies on (N_i) by the relation (14). Thus follows

THEOREM 2. *The S-point of a special simplex lies on the* $(3n)$*point-sphere of its special face.*

(b) If O_{ih} be the projection of O' in S_i, $Q_iO_{ih}G_i$ is a diameter of the $(n-2)$-sphere section (O_{ih}) of (O') by S_i such that

$$(18) \qquad 2\boldsymbol{o}_{ih} = \boldsymbol{g}_i+\boldsymbol{q}_i,$$

and from (16) we have

$$(19) \qquad n\boldsymbol{o}_{ih} = (n-1)\boldsymbol{s}+\boldsymbol{o}_i.$$

Now from (11), (14)—(19) we have

$$(20) \qquad n(\boldsymbol{p}_i-\boldsymbol{o}') = \boldsymbol{a}_i$$

$$(21) \qquad (n-1)(\boldsymbol{s}-\boldsymbol{n}_i) = \boldsymbol{a}_{ih}-\boldsymbol{o}_i$$

$$(22) \qquad n\boldsymbol{o}_{ih} = (n-1)\boldsymbol{n}_i+\boldsymbol{a}_{ih}$$

$$(23) \qquad n(\boldsymbol{q}_i-\boldsymbol{o}_{ih}) = n\boldsymbol{o}_{ih}-n\boldsymbol{g}_i = \boldsymbol{o}_i-\boldsymbol{a}_i.$$

Thus follows

THEOREM 3 (see Figure). *The ratio of the radius of the* $(n-2)$*-sphere section* (O_{ih}) *of the* $3(n+1)$*-point-sphere* (O') *of a special simplex* (S_i) *by its special face* S_i *to that of the* $(3n)$*-point-sphere* (N_i) *of* S_i *is equal to* $(n-1):n$*, and the foot* A_{ih} *of its special altitude lies at the external centre of similitude of* (O_{ih}) *and* (N_i)*.*

4. Doubly special simplex

(a) If the foot A_{jh} of the altitude of a special simplex (S_i) from its vertex A_j other than A_i also lie on its circumhypersphere (O), the simplex becomes *doubly special*, and is denoted by (S_{ij}), with A_iA_j and its opposite $(n-2)$-face S_{ij} as its other *special elements.*

If $\boldsymbol{S}$ be the S-point of (S_{ij}), $\boldsymbol{S}_j$ of its special face S_j, and T_j a point on $\boldsymbol{S}_jA_{jh}$ such that

$$(24) \qquad (n-1)\boldsymbol{t}_j = (n-2)\boldsymbol{s}_j+\boldsymbol{a}_{jh},$$

we have, as in (13),

$$(25) \qquad \boldsymbol{s} = \boldsymbol{t}_j.$$

Thus follows

THEOREM 4. *An n-simplex S becomes doubly special* (S_{ij})*, if and only*

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if the joins of the S-points M_i, M_j of its special faces S_i, S_j to the feet A_{ih}, A_{jh} of its respectively special altitudes meet at its S-point in its special $(n-2)$-face in such a way that $A_{ih}A_{jh}$ is parallel to M_iM_j and equal to $(n-2)$ times M_iM_j.

(b) If A_{ihj}, A_{jhi}, T_{is}, T_{js} be the projections of A_{ih}, A_{jh}, S_i,S_j in S_{ij} and S_{ij} be its S-point, then A_{ihj} A_{jhi} are the feet of the altitudes of S_j, S_i to it, and, by Theorem 4, $T_{is}A_{ihj}$, $T_{js}A_{jhi}$ meet at the S-point S of (S_{ij}).

By definition of S-points (§ 1) we have

$$(26) \qquad (n-3)s_{ij} = (n-2)t_{is}-a_{jhi} = (n-2)t_{js}-a_{ihj},$$

$$(27) \qquad (n-1)s = (n-2)t_{is}+a_{ihj} = (n-2)t_{js}+a_{jhi},$$

and therefore

$$(28) \qquad (n-1)s = (n-3)s_{ij}+2u_{ij},\ 2u_{ij} = a_{ihj}+a_{jhi}.$$

Thus follows

THEOREM 5. *The S-point of a doubly special n-simplex (S_{ij}) lies on the join of the S-point of its special $(n-2)$-face S_{ij} to the midpoint of the segment between the feet of the altitudes of its special faces to S_{ij} and divides the same in the ratio $2:(n-3)$.*

(c) If G_{ij}, O_{ij} be the centroid and circumcentre of S_{ij}, and G^{ij} the midpoint of A_iA_j, we have by Coxeter's definition of Monge point (§ 1)

$$(29) \qquad (n-1)s = (n+1)g = (n-1)g_{ij}+2g^{ij},$$

$$(30) \qquad (n-3)s_{ij} = (n-1)g_{ij}-2o_{ij}.$$

From (28) — (30) we have

$$(31) \qquad 2(u_{ij}-o_{ij}) = (n-1)(s-g_{ij}) = 2g^{ij}.$$

Here we may observe that O, G^{ij} project in S_{ij} into O_{ij}, U_{ij}. Thus follows

THEOREM 6. *The join of the midpoint of the special edge of a doubly special n-simplex (S_{ij}) to its circumcentre projects in its special $(n-2)$-face S_{ij} into the same length parallel and equal to $(n-1)/2$ times that of its s-point to the centroid of S_{ij}.*

(d) If A_iA_j of (S_{ij}) be normal to S_{ij}, (S_{ij}) becomes *biorthocentric* [3] with *biorthocentre* H_{ij} (say); at this point its two special altitudes concur with its special *bialtitude* h_{ij} to A_iA_j in such a way that h_{ij} meets S_{ij} at

$$(32) \qquad A_{ihj} = U_{ij} = A_{jhi}.$$

Thus Theorem 5 becomes

THEOREM 7. *If the simplex (S_{ij}) be also biorthocentric, with the common* *perpendicular secant h_{ij} of its special edge and $(n-2)$-face S_{ij} as its special bialtitude, its S-point lies on the join of the S-point of S_{ij} to the foot therein of h_{ij} and divides the same in the ratio $2:(n-3)$.*

5. $(n-1)$ply special simplex

(a) We may consider an *r-ply special* simplex having r special vertices and therefore r special faces opposite them in the above manner for all values of $r > 2$. But $r = n-1$ $(n > 3)$ forms an interesting case and we develop its theory as follows.

Let the $n-1$ vertices of an n-simplex S other than A_k, A_l be all special, let S be denoted by (S^{kl}), and let A_kA_l and its opposite $(n-2)$-face S_{kl} be called its *principal* elements. Thus, from Theorem 1 (iii) follows

THEOREM 8. *An n-simplex S is $(n-1)$ ply special (S^{kl}), if and only if its S-point lies on its principal edge; the $n-1$ joins of the feet of its $n-1$ special altitudes to the S-points of its corresponding special faces then concur on the principal edge*

(b) If G_{kl} be the centroid of S_{kl}, from (3), (10) we have for (S^{kl})

$$2(u_k\mathbf{a}_k+u_l\mathbf{a}_l)+(n-1)\mathbf{g}_{kl} = 0, \tag{33}$$

$$2(u_k+u_l)+(n-1)u = 2u, \text{ or } 2(u_k+u_l) = (3-n)u, \tag{34}$$

and therefore

$$(n-3)\mathbf{r}_{kl} = (n-1)\mathbf{g}_{kl}, \tag{35}$$

where

$$(u_k+u_l)\mathbf{r}_{kl} = u_k\mathbf{a}_k+u_l\mathbf{a}_l. \tag{36}$$

Again, similar to (30) we have

$$(n-3)\mathbf{s}_{kl} = (n-1)\mathbf{g}_{kl}-2\mathbf{o}_{kl}. \tag{37}$$

Hence follows

THEOREM 9. *The join of the circumcentre 0 of an $(n-1)$ply special n-simplex (S^{kl}) to the centroid G_{kl} of its principal $(n-2)$-face S_{kl} meets its principal edge A_kA_l in a point R_{kl} such that G_{kl} divides OR_{kl} in the ratio $(n-3):2$ and R_{kl} projects into the S-point $\mathbf{S}_{kl}$ of S_{kl} which then lies on the projection of A_kA_l in S_{kl}.*

COROLLARY. *The circumcentre of an $(n-1)$-ply special n-simplex lies in its principal $(n-2)$-face, if and only if $n = 3$.* (That is, a tetrahedron is doubly special, if and only if one of its principal edges is a circum-diameter,

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and consequently its Monge point lies at the midpoint of its opposite principal edge [1].)

(c) By relation of the type (31), the join of the S-point $\boldsymbol{S}$ of any simplex to the centroid G_{kl} of any $(n-2)$-face S_{kl} is always parallel to that of the midpoint of its opposite edge A_kA_l to its circumcentre O and therefore perpendicular to A_kA_l. That is

$$(38) \qquad (\boldsymbol{a}_k-\boldsymbol{a}_l)\cdot(\boldsymbol{s}-\boldsymbol{g}_{kl}) = 0.$$

Now let (S^{kl}) be biorthcentric (§ 4d) such that A_kA_l is perpendicular to S_{kl} and therefore to every line therein, in particular to the join of G_{kl} to its S-point $\boldsymbol{S}_{kl}$. That is,

$$(39) \qquad (\boldsymbol{a}_k-\boldsymbol{a}_l)\cdot(\boldsymbol{s}_{kl}-\boldsymbol{g}_{kl}) = 0.$$

Through A_kA_l then passes a unique plane normal to S_{kl} meeting it in a point U_{kl} (say). That is, every point on A_kA_l projects in S_{kl} into U_{kl} which then coincides with $\boldsymbol{S}_{kl}$ by Theorem 9, so that $\boldsymbol{SS}_{kl}$ is normal to S_{kl}.

Again from (38) — (39) we have

$$(40) \qquad (\boldsymbol{a}_k-\boldsymbol{a}_l)\cdot(\boldsymbol{s}-\boldsymbol{s}_{kl}) = 0.$$

Hence follows

THEOREM 10. *If the $(n-1)$ply special n-simplex (S^{kl}) be also biorthocentric with its principal edge A_kA_l perpendicular to its principal $(n-2)$-face S_{kl}, its S-point and that of S_{kl} lie at the feet of its special bialtitude h_{kl} to A_kA_l, and consequently the S-points of its 2 non-special faces lie on their respective altitudes to S_{kl}.*

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