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A GENERALIZATION OF MOAK'S q-LAGUERRE POLYNOMIALS

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0. Introduction. In [**6**] we studied the polynomials $\{L_n^{\alpha,M,N}(x)\}_{n=0}^{\infty}$ which are generalizations of the classical (generalized) Laguerre polynomials $\{L_n^{(\alpha)}(x)\}_{n=0}^{\infty}$. These polynomials were shown to be orthogonal on the interval $[0, \infty)$ with respect to the inner product

$$(0.1) \qquad \langle f, g\rangle = \frac{1}{\Gamma(\alpha+1)} \cdot \int_0^{\infty} x^{\alpha} e^{-x} \cdot f(x) g(x) dx + M \cdot f(0)g(0) + N \cdot f'(0)g'(0),$$

where $\alpha > -1$, $M \geqq 0$ and $N \geqq 0$. They can be defined in terms of the classical Laguerre polynomials as

$$(0.2) \qquad L_n^{\alpha,M,N}(x) = A_0 \cdot L_n^{(\alpha)}(x) - A_1 \cdot L_{n-1}^{(\alpha+1)}(x) + A_2 \cdot L_{n-2}^{(\alpha+2)}(x)$$

where

$$(0.3) \qquad \begin{cases} A_0 = 1 + M \cdot \binom{n+\alpha}{n-1} + \dfrac{n(\alpha+2)-(\alpha+1)}{(\alpha+1)(\alpha+3)} \cdot N \cdot \binom{n+\alpha}{n-2} \\ \qquad + \dfrac{M \cdot N}{(\alpha+1)(\alpha+2)} \cdot \binom{n+\alpha}{n-1}\binom{n+\alpha+1}{n-2} \\ A_1 = M \cdot \binom{n+\alpha}{n} + \dfrac{(n-1)}{(\alpha+1)} \cdot N \cdot \binom{n+\alpha}{n-1} \\ \qquad + \dfrac{2M \cdot N}{(\alpha+1)^2} \cdot \binom{n+\alpha}{n}\binom{n+\alpha+1}{n-2} \\ A_2 = \dfrac{N}{(\alpha+1)} \cdot \binom{n+\alpha}{n-1} + \dfrac{M \cdot N}{(\alpha+1)^2} \cdot \binom{n+\alpha}{n}\binom{n+\alpha+1}{n-1} \end{cases}$$

and

$$L_{-1}^{(\alpha)}(x) := 0 =: L_{-2}^{(\alpha)}(x).$$

For $N = 0$ these polynomials reduce to

$$L_n^{\alpha,M}(x) = \left[1 + M \cdot \binom{n+\alpha}{n-1}\right] \cdot L_n^{(\alpha)}(x) - M \cdot \binom{n+\alpha}{n} \cdot L_{n-1}^{(\alpha+1)}(x),$$

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which are Koornwinder's generalized Laguerre polynomials. See also [**7**], [**8**] and [**9**]. In [**7**] we found a q-analogue of the polynomials $\{L_n^{\alpha,M}(x)\}_{n=0}^{\infty}$ which can be defined by

$$(0.4)\qquad L_n^{\alpha,M}(x;\,q) = \left[1 + M\cdot\frac{(q^{\alpha+2};\,q)_{n-1}}{(q;\,q)_{n-1}}\right]\cdot L_n^{(\alpha)}(x;\,q) - M\cdot\frac{(q^{\alpha+1};\,q)_n}{(q;\,q)_n}\cdot L_{n-1}^{(\alpha+1)}(x;\,q),\quad n \geqq 1$$

where $L_n^{(\alpha)}(x;q)$ denotes the q-Laguerre polynomial described by D. S. Moak in [**10**]. See also [**7**] for more details and Section 2 of this paper for a summary. In this paper we study further generalizations of the polynomials $\{L_n^{\alpha,M}(x;\,q)\}_{n=0}^{\infty}$. These polynomials $\{L_n^{\alpha,M,N}(x;\,q)\}_{n=0}^{\infty}$ are q-analogues of the polynomials $\{L_n^{\alpha,M,N}(x)\}_{n=0}^{\infty}$ defined by (0.2) and (0.3).

1. Some basic formulas. First we summarize some definitions and formulas from the q-theory. For details the reader is referred to [**3**].

Let $0 < q < 1$. Then we define for $n = 1, 2, 3, \ldots$

$$(1.1)\qquad \begin{cases} (a;\,q)_n = (1-a)(1-aq)(1-aq^2)\cdots(1-aq^{n-1}) \\ (a;\,q)_0 = 1 \\ (a;q)_{-n} = \dfrac{1}{(aq^{-n};\,q)_n} = \dfrac{(-qa^{-1})^n\cdot q^{\binom{n}{2}}}{(qa^{-1};\,q)_n} = \dfrac{1}{(aq^{-1};\,q^{-1})_n},\ a \neq 0. \end{cases}$$

For all $\alpha \in \mathbf{C}$ we may define

$$(a;\,q)_\alpha = \frac{(a;\,q)_\infty}{(aq^\alpha;\,q)_\infty}$$

where

$$(a;\,q)_\infty = \prod_{n=0}^{\infty}(1-aq^n).$$

In [**4**] F. H. Jackson defined a q-analogue of the gamma function as

$$(1.2)\qquad \Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;\,q)_\infty}\cdot(1-q)^{1-x}.$$

Note that

$$\Gamma_q(x+1) = \frac{1-q^x}{1-q}\cdot\Gamma_q(x).$$

He also showed that $\Gamma_q(x) \longrightarrow \Gamma(x)$ as $q \longrightarrow 1^-$.

In [**1**] R. Askey proved an integral formula which is due to Ramanujan:

$$\int_0^\infty \frac{x^\alpha}{(-(1-q)x;\,q)_\infty}\,dx = \frac{\Gamma(-\alpha)\cdot\Gamma(\alpha+1)}{\Gamma_q(-\alpha)}, \quad \alpha > -1. \tag{1.3}$$

A sketch of the proof can be found in [**7**] too.

For $\alpha = k \in \mathbf{N}$ we take the limit

$$\begin{aligned}\lim_{\alpha\to k}\frac{\Gamma(-\alpha)\cdot\Gamma(\alpha+1)}{\Gamma_q(-\alpha)} &= \lim_{\alpha\to k}\frac{(-\alpha+k)\cdot\Gamma(-\alpha)}{(-\alpha+k)\cdot\Gamma_q(-\alpha)}\cdot\Gamma(\alpha+1)\\ &= \frac{(-1)^k}{k!}\cdot\frac{(q^{-k};\,q)_k\cdot\ln q^{-1}}{(1-q)^{k+1}}\cdot\Gamma(k+1)\\ &= \frac{(q;\,q)_k\cdot q^{-\binom{k+1}{2}}\cdot\ln q^{-1}}{(1-q)^{k+1}}.\end{aligned}$$

See formula (1.10.6) in [**3**] for the residue of the q-gamma function.

By using the q-binomial theorem

$$\sum_{n=0}^\infty \frac{(a;\,q)_n}{(q;\,q)_n}\cdot z^n = \frac{(az;\,q)_\infty}{(z;\,q)_\infty}, \quad |z|<1.$$

we easily see that

$$\frac{1}{(-(1-q)x;\,q)_\infty} = \sum_{n=0}^\infty \frac{(1-q)^n}{(q;\,q)_n}\cdot(-x)^n \longrightarrow e^{-x} \quad \text{as } q\longrightarrow 1^-. \tag{1.4}$$

Further we have a q-analogue of the differentiation operator:

$$D_q f(x) = \frac{f(x)-f(qx)}{(1-q)x}. \tag{1.5}$$

Observe that $D_q f(x) \longrightarrow f'(x)$ as $q \longrightarrow 1^-$ if $f'(x)$ exists and that

$$D_q[f(\gamma x)] = \gamma\cdot(D_q f)(\gamma x), \quad \gamma\in\mathbf{R}. \tag{1.6}$$

The q-product rule reads

$$D_q[f(x)g(x)] = f(qx)\cdot D_q g(x) + g(x)\cdot D_q f(x). \tag{1.7}$$

This follows immediately from the definition (1.5).

The basic hypergeometric series ${}_r\Phi_s$ is defined by

$$\begin{aligned}&{}_r\Phi_s(a_1,a_2,\ldots,a_r;\,b_1,b_2,\ldots,b_s;\,q,z)\\ &= {}_r\Phi_s\left(\left.\begin{matrix}a_1,a_2,\ldots,a_r\\ b_1,b_2,\ldots,b_s\end{matrix}\right|q,z\right)\\ &= \sum_{n=0}^\infty \frac{(a_1,a_2,\ldots,a_r;\,q)_n}{(b_1,b_2,\ldots,b_s;\,q)_n}\cdot\frac{(-1)^{(1+s-r)n}\cdot q^{(1+s-r)\binom{n}{2}}\cdot z^n}{(q;\,q)_n}\end{aligned} \tag{1.8}$$

where

$$(a_1, a_2, \ldots, a_r;\, q)_n = (a_1;\, q)_n \cdot (a_2;\, q)_n \cdots (a_r;\, q)_n.$$

Note that

$$\begin{aligned} &{}_r\Phi_s\left(\begin{matrix} q^{\alpha_1}, q^{\alpha_2}, \ldots, q^{\alpha_r} \\ q^{\beta_1}, q^{\beta_2}, \ldots, q^{\beta_s} \end{matrix}\middle|\, q, (q-1)^{1+s-r} \cdot z\right) \\ &\longrightarrow {}_rF_s\left(\begin{matrix} \alpha_1, \alpha_2, \ldots, \alpha_r \\ \beta_1, \beta_2, \ldots, \beta_s \end{matrix}\middle|\, z\right) \quad q \longrightarrow 1^-, \end{aligned}$$

where ${}_rF_s$ denotes the hypergeometric series.

We will use the following q-identities:

(1.9) $$(a^{-1} \cdot q^{1-n};\, q)_n = (-a^{-1})^n \cdot q^{-\binom{n}{2}} \cdot (a;\, q)_n,$$

(1.10) $$(a;\, q)_{n+k} = (a;\, q)_n \cdot (aq^n;\, q)_k,$$

and

(1.11) $$e_q(z) := {}_1\Phi_0(0;\, -;\, q, z) = \sum_{n=0}^{\infty} \frac{z^n}{(q;\, q)_n} = \frac{1}{(z;\, q)_\infty}, \quad |z| < 1.$$

The function $e_q(z)$ is a q-analogue of the exponential function, since

$$e_q((1-q)z) \longrightarrow e^z \quad \text{as } q \longrightarrow 1^-.$$

In (1.4) we have seen a special case of (1.11):

$$\frac{1}{(-(1-q)x;\, q)_\infty} = e_q(-(1-q)x) \longrightarrow e^{-x} \quad \text{as } q \longrightarrow 1^-.$$

And we will use one summation formula for a terminating ${}_2\Phi_1$:

(1.12) $${}_2\Phi_1(q^{-n}, b;\, c;\, q, cq^n/b) = \frac{(c/b;\, q)_n}{(c;\, q)_n}.$$

Further we have the basic bilateral series defined by

$$\begin{aligned} &{}_r\Psi_s(a_1, a_2, \ldots, a_r; b_1, b_2, \ldots, b_s; q, z) \\ &= {}_r\Psi_s\left(\begin{matrix} a_1, a_2, \ldots, a_r \\ b_1, b_2, \ldots, b_s \end{matrix}\middle|\, q, z\right) \\ &= \sum_{n=-\infty}^{\infty} \frac{(a_1, a_2, \ldots, a_r; q)_n}{(b_1, b_2, \ldots, b_s; q)_n} \cdot (-1)^{(s-r)n} \cdot q^{(s-r)\binom{n}{2}} \cdot z^n. \end{aligned}$$

In the special case $r = s = 1$ we have a summation formula, called "Ramanujan's sum" (see for instance [**1**] and [**3**]):

$$(1.13)\qquad {}_1\Psi_1(a;b;q,z) = \sum_{n=-\infty}^{\infty} \frac{(a;q)_n}{(b;q)_n} \cdot z^n$$
$$= \frac{(q, a^{-1}\cdot b, az, a^{-1}\cdot z^{-1}\cdot q; q)_\infty}{(b, a^{-1}\cdot q, z, a^{-1}\cdot z^{-1}\cdot b; q)_\infty}$$
$$\text{for } |a^{-1}\cdot b| < |z| < 1.$$

2. The q-Laguerre polynomials. In this section we state the definition and some properties of the q-Laguerre polynomials $\{L_n^{(\alpha)}(x;q)\}_{n=0}^{\infty}$ which were described by D. S. Moak in [**10**]. For more details the reader is referred to [**7**] and [**10**].

Let $\alpha > -1$ and $0 < q < 1$. The q-Laguerre polynomials $\{L_n^{(\alpha)}(x;q)\}_{n=0}^{\infty}$ are defined by

$$(2.1)\qquad L_n^{(\alpha)}(x;q) = \frac{(q^{\alpha+1};q)_n}{(q;q)_n} \cdot \sum_{k=0}^{n} \frac{(q^{-n};q)_k \cdot q^{\binom{k}{2}} \cdot (1-q)^k \cdot (q^{n+\alpha+1}\cdot x)^k}{(q^{\alpha+1};q)_k \cdot (q;q)_k},$$
$$n = 0, 1, 2, \ldots.$$

For $q \longrightarrow 1^-$ the polynomials $\{L_n^{(\alpha)}(x;q)\}_{n=0}^{\infty}$ tend to the classical Laguerre polynomials $\{L_n^{(\alpha)}(x)\}_{n=0}^{\infty}$. It is easy to see that for $n \geqq 1$

$$(2.2)\qquad D_q L_n^{(\alpha)}(x;q) = -q^{\alpha+1} \cdot L_{n-1}^{(\alpha+1)}(qx;q),$$

where D_q is the q-analogue of the differentiation operator defined by (1.5). By using (1.6) we find more general for $n \geqq k$

$$D_q^k L_n^{(\alpha)}(x;q) = (-1)^k \cdot q^{k(k+\alpha)} \cdot L_{n-k}^{(\alpha+k)}(q^k x;q).$$

Further we easily see from (2.1):

$$(2.3)\qquad L_n^{(\alpha)}(0;q) = \frac{(q^{\alpha+1};q)_n}{(q;q)_n}.$$

The q-Laguerre polynomials are orthogonal on the interval $[0,\infty)$ with respect to the weight function

$$x^\alpha \cdot e_q(-(1-q)x) = \frac{x^\alpha}{(-(1-q)x;q)_\infty}.$$

We have the following orthogonality relation (compare with (1.3));

$$(2.4)\qquad \frac{\Gamma_q(-\alpha)}{\Gamma(-\alpha)\Gamma(\alpha+1)} \cdot \int_0^\infty \frac{x^\alpha}{(-(1-q)x;q)_\infty} \cdot L_m^{(\alpha)}(x;q) L_n^{(\alpha)}(x;q)\,dx$$
$$= \frac{(q^{\alpha+1};q)_n}{(q;q)_n \cdot q^n} \cdot \delta_{mn}.$$

There is another orthogonality relation given by

$$(2.5)\quad \frac{1}{A}\cdot\sum_{k=-\infty}^{\infty}\frac{q^{k\alpha+k}}{(-c(1-q)q^k;q)_\infty}\cdot L_m^{(\alpha)}(cq^k;q)L_n^{(\alpha)}(cq^k;q) = \frac{(q^{\alpha+1};q)_n}{(q;q)_n\cdot q^n}\cdot\delta_{mn}$$

where $c>0$ is an arbitrary constant and

$$(2.6)\quad A=\sum_{k=-\infty}^{\infty}\frac{q^{k\alpha+k}}{(-c(1-q)q^k;q)_\infty} = \frac{(q,-c(1-q)q^{\alpha+1},-c^{-1}(1-q)^{-1}\cdot q^{-\alpha};q)_\infty}{(q^{\alpha+1},-c(1-q),-q\cdot c^{-1}(1-q)^{-1};q)_\infty}$$

is a normalization factor. This can be shown by using "Ramanujan's sum" (1.13). A proof of both (2.4) and (2.5) can be found in [7] and [**10**].

From the definition (2.1) we easily derive

$$(2.9)\quad L_n^{(\alpha)}(x;q)=(-1)^n\cdot q^{n(n+\alpha)}\cdot\frac{(1-q)^n}{(q;q)_n}\cdot x^n+\text{ lower order terms}$$

and the representation as basic hypergeometric series (see (1.8) for the definition):

$$L_n^{(\alpha)}(x;q)=\frac{(q^{\alpha+1};q)_n}{(q;q)_n}\cdot{}_1\Phi_1(q^{-n};q^{\alpha+1};q,-(1-q)q^{n+\alpha+1}\cdot x).$$

The q-Laguerre polynomials satisfy a second order q-difference equation:

$$(2.10)\quad x\cdot D_q^2L_n^{(\alpha)}(x;q)+\left[\frac{(1-q^{\alpha+1})}{(1-q)}-q^{\alpha+2}\cdot x\right]\cdot(D_qL_n^{(\alpha)})(qx;q) + \frac{(1-q^n)}{(1-q)}\cdot q^{\alpha+1}\cdot L_n^{(\alpha)}(qx;q)=0.$$

We remark that the brackets in $(D_qL_n^{(\alpha)})(qx;q)$ are essential in view of (1.6). Further we have a three term recurrence relation:

$$x\cdot L_n^{(\alpha)}(x;q)=-\frac{(1-q^{n+1})}{(1-q)\cdot q^{2n+\alpha+1}}\cdot L_{n+1}^{(\alpha)}(x;q) + \left[\frac{(1-q^{n+\alpha+1})}{(1-q)\cdot q^{2n+\alpha+1}}+\frac{(1-q^n)}{(1-q)\cdot q^{2n+\alpha}}\right]\cdot L_n^{(\alpha)}(x;q) - \frac{(1-q^{n+\alpha})}{(1-q)\cdot q^{2n+\alpha}}\cdot L_{n-1}^{(\alpha)}(x;q)$$

and a Christoffel–Darboux formula:

$$\text{(2.11)}\quad (x-y)\cdot\frac{(q^{\alpha+1};q)_n}{(q;q)_n}\cdot\sum_{k=0}^{n}\frac{q^k\cdot(q;q)_k\cdot L_k^{(\alpha)}(x;q)L_k^{(\alpha)}(y;q)}{(q^{\alpha+1};q)_k}$$
$$=\frac{(1-q^{n+1})}{(1-q)\cdot q^{n+\alpha+1}}\cdot[L_{n+1}^{(\alpha)}(y;q)L_n^{(\alpha)}(x;q)-L_{n+1}^{(\alpha)}(x;q)L_n^{(\alpha)}(y;q)].$$

If we set $y=qx$ and use (1.5) we obtain from (2.11):

$$\frac{(q^{\alpha+1};q)_n}{(q;q)_n}\cdot\sum_{k=0}^{n}\cdot\frac{q^k\cdot(q;q)_k\cdot L_k^{(\alpha)}(x;q)L_k^{(\alpha)}(qx;q)}{(q^{\alpha+1};q)_k}$$
$$=\frac{(1-q^{n+1})}{(1-q)\cdot q^{n+\alpha+1}}\cdot[L_{n+1}^{(\alpha)}(x;q)\cdot D_qL_n^{(\alpha)}(x;q)$$
$$-L_n^{(\alpha)}(x;q)\cdot D_qL_{n+1}^{(\alpha)}(x;q)].$$

And if we divide (2.11) by $x-y$ and let y tend to x then we find

$$\frac{(q^{\alpha+1};q)_n}{(q;q)_n}\cdot\sum_{k=0}^{n}\frac{q^k\cdot(q;q)_k\cdot\{L_k^{(\alpha)}(x;q)\}^2}{(q^{\alpha+1};q)_k}$$
$$=\frac{(1-q^{n+1})}{(1-q)\cdot q^{n+\alpha+1}}\cdot\Big[L_{n+1}^{(\alpha)}(x;q)\cdot\frac{d}{dx}L_n^{(\alpha)}(x;q)$$
$$-L_n^{(\alpha)}(x;q)\cdot\frac{d}{dx}L_{n+1}^{(\alpha)}(x;q)\Big].$$

3. Definition and some elementary properties. Now we define the polynomials $\{L_n^{\alpha,M,N}(x;q)\}_{n=0}^{\infty}$ in terms of the q-Laguerre polynomials by

$$\text{(3.1)}\quad L_n^{\alpha,M,N}(x;q)=C_0\cdot L_n^{(\alpha)}(x;q)$$
$$-C_1\cdot L_{n-1}^{(\alpha+1)}(x;q)+C_2\cdot L_{n-2}^{(\alpha+2)}(x;q)$$

where

$$\text{(3.2)}\quad\begin{cases} C_0=1+M\cdot\frac{(q^{\alpha+2};q)_{n-1}}{(q;q)_{n-1}} \\ \quad +N\cdot q^{2\alpha+3}\cdot\left\{\frac{(1-q^n)(1-q^{\alpha+2})-q(1-q)(1-q^{\alpha+1})}{(1-q^{\alpha+1})(1-q^{\alpha+3})}\right\}\cdot\frac{(q^{\alpha+3};q)_{n-2}}{(q;q)_{n-2}} \\ \quad +M\cdot N\cdot q^{2\alpha+3}\cdot\frac{(1-q)^2}{(1-q^{\alpha+1})(1-q^{\alpha+2})}\cdot\frac{(q^{\alpha+2};q)_{n-1}}{(q;q)_{n-1}}\cdot\frac{(q^{\alpha+4};q)_{n-2}}{(q;q)_{n-2}} \\ C_1=M\cdot\frac{(q^{\alpha+1};q)_n}{(q;q)_n}+N\cdot q^{2\alpha+2}\cdot\frac{(1-q^{n-1})}{(1-q^{\alpha+1})}\cdot\frac{(q^{\alpha+2};q)_{n-1}}{(q;q)_{n-1}} \\ \quad +M\cdot N\cdot q^{2\alpha+2}\cdot\frac{(1-q)(1-q^2)}{(1-q^{\alpha+1})^2}\cdot\frac{(q^{\alpha+1};q)_n}{(q;q)_n}\cdot\frac{(q^{\alpha+4};q)_{n-2}}{(q;q)_{n-2}} \\ C_2=N\cdot q^{2\alpha+2}\cdot\frac{(1-q)}{(1-q^{\alpha+1})}\cdot\frac{(q^{\alpha+2};q)_{n-1}}{(q;q)_{n-1}} \\ \quad +M\cdot N\cdot q^{2\alpha+2}\cdot\frac{(1-q)^2}{(1-q^{\alpha+1})^2}\cdot\frac{(q^{\alpha+1};q)_n}{(q;q)_n}\cdot\frac{(q^{\alpha+3};q)_{n-1}}{(q;q)_{n-1}}. \end{cases}$$

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Note that this is a q-analogue of (0.2) and (0.3).

The definition (3.2) is valid for all $n \geqq 0$, since (1.1) implies that

$$\frac{1}{(q;\,q)_{-n}} = (q^{1-n};\,q)_n = 0, \quad n = 1, 2, 3, \ldots.$$

For $N = 0$ these polynomials reduce to the polynomials $\{L_n^{\alpha,M}(x;\,q)\}_{n=0}^{\infty}$ defined by (0.4). Further we have by using (2.3) and (3.2):

$$(3.3) \qquad L_n^{\alpha,M,N}(0;\,q) = \frac{(q^{\alpha+1};\,q)_n}{(q;\,q)_n} \cdot \left[1 - N \cdot q^{2\alpha+4} \cdot \frac{(1-q)}{(1-q^{\alpha+1})} \cdot \frac{(q^{\alpha+4};\,q)_{n-2}}{(q;\,q)_{n-2}}\right]$$

and

$$(3.4) \qquad (D_q L_n^{\alpha,M,N})(0;\,q) = -q^{\alpha+1} \cdot \frac{(q^{\alpha+2};\,q)_{n-1}}{(q;\,q)_{n-1}} - M \cdot q^{\alpha+1} \cdot \frac{(1-q)}{(1-q^{\alpha+1})} \cdot \frac{(q^{\alpha+1};\,q)_n}{(q;\,q)_n} \cdot \frac{(q^{\alpha+3};\,q)_{n-1}}{(q;\,q)_{n-1}}.$$

We will prove that the polynomials $\{L_n^{\alpha,M,N}(x;\,q)\}_{n=0}^{\infty}$ defined by (3.1) and (3.2) are orthogonal on the interval $[0, \infty)$ with respect to the inner product

$$(3.5) \qquad \langle f, g\rangle = \frac{\Gamma_q(-\alpha)}{\Gamma(-\alpha)\Gamma(\alpha+1)} \cdot \int_0^{\infty} \frac{x^{\alpha}}{(-(1-q)x;\,q)_{\infty}} \cdot f(x)g(x)dx + M \cdot f(0)g(0) + N \cdot (D_q f)(0) \cdot (D_q g)(0)$$

where $\alpha > -1$, $M \geqq 0$ and $N \geqq 0$. Note that this a q-analogue of (0.1).

Further we will prove another orthogonality relation in terms of an inner product involving a bilateral series defined by

$$(3.6) \qquad [f, g] = \frac{1}{A} \cdot \sum_{k=-\infty}^{\infty} \frac{q^{k\alpha+k}}{(-c(1-q)q^k;\,q)_{\infty}} \cdot f(cq^k)g(cq^k) + M \cdot f(0)g(0) + N \cdot (D_q f)(0) \cdot (D_q g)(0)$$

where $c > 0$ is an arbitrary constant and A is defined by (2.6). Compare this with (2.4) and (2.5). The orthogonality is proved in the next section.

4. The orthogonality. In this section we will prove two orthogonality relations for the polynomials $\{L_n^{\alpha,M,N}(x;\,q)\}_{n=0}^{\infty}$ defined by (3.1) and (3.2). First we have a generalization of (2.4):

$$(4.1) \qquad \langle L_m^{\alpha,M,N}(x;\,q), L_n^{\alpha,M,N}(x;\,q)\rangle = \frac{(q^{\alpha+1};\,q)_n}{(q;\,q)_n \cdot q^n} \cdot C_0 \cdot [C_0 + q^n \cdot C_1 + q^{2n-1} \cdot C_2] \cdot \delta_{mn}$$

where the inner product $\langle , \rangle$ is defined by (3.5). A second orthogonality relation is

$$(4.2)\qquad [L_m^{\alpha,M,N}(x;q), L_n^{\alpha,M,N}(x;q)] = \frac{(q^{\alpha+1};q)_n}{(q;q)_n\cdot q^n}\cdot C_0\cdot[C_0+q^n\cdot C_1+q^{2n-1}\cdot C_2]\cdot\delta_{mn}$$

where the inner product $[,]$ is defined by (3.6).

To prove (4.1) we first show that

$$(4.3)\qquad \frac{\Gamma_q(-\alpha)}{\Gamma(-\alpha)\Gamma(\alpha+1)}\cdot\int_0^\infty \frac{x^{\alpha+m}}{(-(1-q)x;q)_\infty}\cdot L_{n-i}^{(\alpha+i)}(x;q)dx = \frac{(q^{i-m};q)_{n-i}}{(q;q)_{n-i}}\cdot\frac{(q^{\alpha+1};q)_m}{(1-q)^m}\cdot q^{-(\alpha+1)m-\binom{m}{2}}.$$

To do this we use the definition (2.1) of the q-Laguerre polynomial and the integral formula (1.3) to obtain

$$(4.4)\qquad \int_0^\infty \frac{x^{\alpha+m}}{(-(1-q)x;q)_\infty}\cdot L_{n-i}^{(\alpha+i)}(x;q)dx = \frac{(q^{\alpha+i+1};q)_{n-i}}{(q;q)_{n-i}}\cdot\sum_{k=0}^{n-i}\frac{(q^{-n+i};q)_k\cdot q^{\binom{k}{2}}\cdot(1-q)^k\cdot q^{(n+\alpha+1)k}}{(q^{\alpha+i+1};q)_k\cdot(q;q)_k} \times\int_0^\infty\frac{x^{\alpha+m+k}}{(-(1-q)x;q)_\infty}dx$$
$$= \frac{(q^{\alpha+i+1};q)_{n-i}}{(q;q)_{n-i}}\cdot\sum_{k=0}^{n-i}\frac{(q^{-n+i};q)_k\cdot q^{\binom{k}{2}}\cdot(1-q)^k\cdot q^{(n+\alpha+1)k}}{(q^{\alpha+i+1};q)_k\cdot(q;q)_k} \times\frac{\Gamma(-\alpha-m-k)\Gamma(\alpha+m+k+1)}{\Gamma_q(-\alpha-m-k)}.$$

Now we use the definition (1.2) of the q-gamma function and the identities (1.10) and (1.9) to find

$$(4.5)\qquad \frac{\Gamma_q(-\alpha)\Gamma(-\alpha-m-k)\Gamma(\alpha+m+k+1)}{\Gamma(-\alpha)\Gamma(\alpha+1)\Gamma_q(-\alpha-m-k)} = (-1)^{m+k}\cdot(1-q)^{-m-k}\cdot\frac{(q^{-\alpha-m-k};q)_\infty}{(q^{-\alpha};q)_\infty}$$
$$= (-1)^{m+k}\cdot(1-q)^{-m-k}\cdot(q^{-\alpha-m-k};q)_{m+k}$$
$$= (-1)^{m+k}\cdot(1-q)^{-m-k}\cdot(q^{-\alpha-m-k};q)_k\cdot(q^{-\alpha-m};q)_m$$
$$= (1-q)^{-m-k}\cdot q^{-(\alpha+1)m-\binom{m}{2}}\cdot q^{-(\alpha+m+1)k-\binom{k}{2}}\cdot(q^{\alpha+1};q)_m\cdot(q^{\alpha+m+1};q)_k.$$

Combining (4.4) and (4.5) we obtain by using the summation formula (1.12):

$$\frac{\Gamma_q(-\alpha)}{\Gamma(-\alpha)\Gamma(\alpha+1)}\cdot\int_0^\infty\frac{x^{\alpha+m}}{(-(1-q)x;\,q)_\infty}\cdot L_{n-i}^{(\alpha+i)}(x;\,q)dx$$
$$=\frac{(q^{\alpha+i+1};\,q)_{n-i}}{(q;\,q)_{n-i}}\cdot\sum_{k=0}^{n-i}\frac{(q^{-n+i};\,q)_k\cdot q^{\binom{k}{2}}\cdot(1-q)^k\cdot q^{(n+\alpha+1)k}}{(q^{\alpha+i+1};\,q)_k\cdot(q;\,q)_k}$$
$$\times\frac{\Gamma_q(-\alpha)\Gamma(-\alpha-m-k)\Gamma(\alpha+m+k+1)}{\Gamma(-\alpha)\Gamma(\alpha+1)\Gamma_q(-\alpha-m-k)}$$
$$=\frac{(q^{\alpha+i+1};\,q)_{n-i}}{(q;\,q)_{n-i}}\cdot\frac{(q^{\alpha+1};\,q)_m}{(1-q)^m}\cdot q^{-(\alpha+1)m-\binom{m}{2}}$$
$$\times{}_2\Phi_1\left(\begin{matrix}q^{-n+i},q^{\alpha+m+1}\\ q^{\alpha+i+1}\end{matrix}\middle|\,q,q^{n-m}\right)$$
$$=\frac{(q^{i-m};\,q)_{n-i}}{(q;\,q)_{n-i}}\cdot\frac{(q^{\alpha+1};\,q)_m}{(1-q)^m}\cdot q^{-(\alpha+1)m-\binom{m}{2}}.$$

This proves (4.3).

Now we have by using the definition (3.1) and (4.3)

$$\frac{\Gamma_q(-\alpha)}{\Gamma(-\alpha)\Gamma(\alpha+1)}\cdot\int_0^\infty\frac{x^{\alpha+m}}{(-(1-q)x;\,q)_\infty}\cdot L_n^{\alpha,M,N}(x;\,q)dx$$
$$=\frac{(q^{\alpha+1};\,q)_m}{(1-q)^m}\cdot q^{-(\alpha+1)m-\binom{m}{2}}\cdot\left[\frac{(q^{-m};\,q)_n}{(q;\,q)_n}\cdot C_0-\frac{(q^{1-m};\,q)_{n-1}}{(q;\,q)_{n-1}}\cdot C_1+\frac{(q^{2-m};\,q)_{n-2}}{(q;\,q)_{n-2}}\cdot C_2\right]. \tag{4.6}$$

This equals zero for $2\leqq m<n$. Hence

$$\langle x^m,L_n^{\alpha,M,N}(x;\,q)\rangle=0\quad\text{for }2\leqq m<n.$$

For $m=0$ and $m=1$ we find by using (4.6), (3.1), (2.3) and (2.2)

$$\langle 1,L_n^{\alpha,M,N}(x;\,q)\rangle=-C_1+\frac{(1-q^{n-1})}{(1-q)}\cdot C_2+M\cdot\left[\frac{(q^{\alpha+1};\,q)_n}{(q;\,q)_n}\cdot C_0-\frac{(q^{\alpha+2};\,q)_{n-1}}{(q;\,q)_{n-1}}\cdot C_1+\frac{(q^{\alpha+3};\,q)_{n-2}}{(q;\,q)_{n-2}}\cdot C_2\right]=0, \tag{4.7}$$

$$n\geqq 1.$$

Also for $n\geqq 2$

$$\langle x,L_n^{\alpha,M,N}(x;\,q)\rangle=\frac{(1-q^{\alpha+1})}{(1-q)}\cdot q^{-(\alpha+1)}\cdot C_2-N\cdot\left[q^{\alpha+1}\cdot\frac{(q^{\alpha+2};\,q)_{n-1}}{(q;\,q)_{n-1}}\cdot C_0-q^{\alpha+2}\cdot\frac{(q^{\alpha+3};\,q)_{n-2}}{(q;\,q)_{n-2}}\cdot C_1+q^{\alpha+3}\cdot\frac{(q^{\alpha+4};\,q)_{n-3}}{(q;\,q)_{n-3}}\cdot C_2\right]=0. \tag{4.8}$$

This proves the orthogonality. To complete the proof of (4.1) note that we have with (3.1) and (2.9):

$$(4.9)\qquad L_n^{\alpha,M,N}(x;q) = (-1)^n \cdot q^{n(n+\alpha)} \cdot \frac{(1-q)^n}{(q;q)_n} \cdot C_0 \cdot x^n + \text{ lower order terms.}$$

Now it follows from (4.6) with $m = n \geqq 2$, (4.9) and (1.9) that

$$\begin{aligned}
&\langle L_n^{\alpha,M,N}(x;q), L_n^{\alpha,M,N}(x;q)\rangle \\
&= (-1)^n \cdot q^{\binom{n}{2}} \cdot \frac{(q^{\alpha+1};q)_n}{(q;q)_n} \cdot C_0 \cdot \left[\frac{(q^{-n};q)_n}{(q;q)_n} \cdot C_0 \right.\\
&\qquad \left. - \frac{(q^{1-n};q)_{n-1}}{(q;q)_{n-1}} \cdot C_1 + \frac{(q^{2-n};q)_{n-2}}{(q;q)_{n-2}} \cdot C_2\right] \\
&= \frac{(q^{\alpha+1};q)_n}{(q;q)_n \cdot q^n} \cdot C_0 \cdot [C_0 + q^n \cdot C_1 + q^{2n-1} \cdot C_2].
\end{aligned}$$

For $n = 0$ and $n = 1$ we find the same formula by direct calculation. This proves (4.1). To see (4.2) we prove that for $m < n$:

$$(4.10)\qquad [x^m, L_n^{\alpha,M,N}(x;q)] = \langle x^m, L_n^{\alpha,M,N}(x;q)\rangle,$$

where $[,]$ denotes the inner product defined by (3.6) and $\langle , \rangle$ that defined by (3.5). By using (2.1) we find for $m \in \mathbf{N}$

$$\begin{aligned}
(4.11)\qquad &\frac{1}{A} \cdot \sum_{k=-\infty}^{\infty} \frac{q^{k\alpha+k}}{(-c(1-q)q^k;q)_\infty} \cdot (cq^k)^m \cdot L_{n-i}^{(\alpha+i)}(cq^k;q) \\
&= \frac{1}{A} \cdot \frac{(q^{\alpha+i+1};q)_{n-i}}{(q;q)_{n-i}} \cdot \sum_{j=0}^{n-i} \frac{(q^{-n+i};q)_j \cdot q^{\binom{j}{2}} \cdot (1-q)^j \cdot q^{(n+\alpha+1)j}}{(q^{\alpha+i+1};q)_j \cdot (q;q)_j} \\
&\times \sum_{k=-\infty}^{\infty} \frac{c^{m+j} \cdot q^{(\alpha+m+j+1)k}}{(-c(1-q)q^k;q)_\infty}.
\end{aligned}$$

Now we use (2.6) and (1.9) to obtain

$$\begin{aligned}
(4.12)\qquad &\frac{1}{A} \cdot \sum_{k=-\infty}^{\infty} \frac{c^{m+j} \cdot q^{(\alpha+m+j+1)k}}{(-c(1-q)q^k;q)_\infty} \\
&= c^{m+j} \cdot \frac{(q^{\alpha+1}, -c(1-q)q^{\alpha+m+j+1}, -c^{-1}(1-q)^{-1}q^{-\alpha-m-j};q)_\infty}{(q^{\alpha+m+j+1}, -c(1-q)q^{\alpha+1}, -c^{-1} \cdot (1-q)^{-1} \cdot q^{-\alpha};q)_\infty} \\
&= \frac{c^{m+j} \cdot (q^{\alpha+1};q)_{m+j} \cdot (-c^{-1} \cdot (1-q)^{-1} \cdot q^{-\alpha-m-j};q)_{m+j}}{(-c(1-q)q^{\alpha+1};q)_{m+j}} \\
&= \frac{(q^{\alpha+1};q)_{m+j}}{(1-q)^{m+j} \cdot q^{(\alpha+1)(m+j)}} \cdot q^{-\binom{m+j}{2}}.
\end{aligned}$$

So we have with (4.11), (4.12) and (1.12)

$$\frac{1}{A}\cdot\sum_{k=-\infty}^{\infty}\frac{q^{k\alpha+k}}{(-c(1-q)q^k;\,q)_\infty}\cdot(cq^k)^m\cdot L_{n-i}^{(\alpha+i)}(cq^k;\,q)$$
$$=\frac{(q^{\alpha+i+1};\,q)_{n-i}}{(q;\,q)_{n-i}}\cdot\frac{(q^{\alpha+1};\,q)_m}{(1-q)^m}\cdot q^{-(\alpha+1)m-\binom{m}{2}}$$
$$\times\,{}_2\Phi_1\left(\begin{matrix}q^{-n+i},q^{\alpha+m+1}\\ q^{\alpha+i+1}\end{matrix}\middle|\,q,q^{n-m}\right)$$
$$=\frac{(q^{i-m};\,q)_{n-i}}{(q;\,q)_{n-i}}\cdot\frac{(q^{\alpha+1};\,q)_m}{(1-q)^m}\cdot q^{-(\alpha+1)m-\binom{m}{2}},$$

which equals (4.3). This proves (4.10) and therefore (4.2).

5. Representation as basic hypergeometric series. If we write

(5.1) $$L_n^{\alpha,M,N}(x;\,q)$$
$$=\frac{(q^{\alpha+1};\,q)_n}{(q;\,q)_n}\cdot\sum_{k=0}^{n}E_k\cdot q^{\binom{k}{2}}\cdot(1-q)^k\cdot q^{(n+\alpha+1)k}\cdot\frac{x^k}{(q;\,q)_k}$$

then it follows from the definition (3.1) and (2.1) that

(5.2) $$E_k=\frac{(q^{-n};\,q)_k}{(q^{\alpha+1};\,q)_k}\cdot C_0-\frac{(1-q^n)}{(1-q^{\alpha+1})}\cdot\frac{(q^{-n+1};\,q)_k}{(q^{\alpha+2};\,q)_k}\cdot C_1$$
$$+\frac{(1-q^n)(1-q^{n-1})}{(1-q^{\alpha+1})(1-q^{\alpha+2})}\cdot\frac{(q^{-n+2};\,q)_k}{(q^{\alpha+3};\,q)_k}\cdot C_2$$
$$=\frac{(q^{-n};\,q)_k}{(q^{\alpha+1};\,q)_{k+2}}\cdot[(1-q^{k+\alpha+1})(1-q^{k+\alpha+2})\cdot C_0$$
$$=q^n\cdot(1-q^{k-n})(1-q^{k+\alpha+2})\cdot C_1$$
$$+q^{2n-1}\cdot(1-q^{k-n})(1-q^{k-n+1})\cdot C_2]$$
$$=[C_0+q^n\cdot C_1+q^{2n-1}\cdot C_2]$$
$$\times\frac{(1-q^\beta)(1-q^\gamma)}{(1-q^{\alpha+1})(1-q^{\alpha+2})}\cdot\frac{(q^{-n};\,q)_k}{(q^{\alpha+3};\,q)_k}\cdot\frac{(q^{\beta+1};\,q)_k(q^{\gamma+1};\,q)_k}{(q^\beta;\,q)_k\cdot(q^\gamma;\,q)_k}$$

for some $\beta\in\mathbf{C}$ and $\gamma\in\mathbf{C}$. Note that β and γ satisfy

$$q^\beta+q^\gamma=\frac{(q^{\alpha+1}+q^{\alpha+2})\cdot C_0+(1+q^{n+\alpha+2})\cdot C_1+(q^{n-1}+q^n)\cdot C_2}{C_0+q^n\cdot C_1+q^{2n-1}\cdot C_2}$$

and

$$q^\beta\cdot q^\gamma=\frac{q^{2\alpha+3}\cdot C_0+q^{\alpha+2}\cdot C_1+C_2}{C_0+q^n\cdot C_1+q^{2n-1}\cdot C_2}.$$

Hence with (5.1) and (5.2) we have by using definition (1.8)

$$L_n^{\alpha,M,N}(x;q) = [C_0 + q^n \cdot C_1 + q^{2n-1} \cdot C_2] \cdot \frac{(1-q^\beta)(1-q^\gamma)}{(1-q^{\alpha+1})(1-q^{\alpha+2})} \times \frac{(q^{\alpha+1};q)_n}{(q;q)_n} \cdot {}_3\Phi_3\left(\begin{matrix} q^{-n}, q^{\beta+1}, q^{\gamma+1} \\ q^{\alpha+3}, q^\beta, q^\gamma \end{matrix} \middle| q, -(1-q)\cdot q^{n+\alpha+1}\cdot x\right).$$

But in view of (3.3) we may write

$$\text{(5.3)} \quad L_n^{\alpha,M,N}(x;q) = \left[1 - N \cdot q^{2\alpha+4} \cdot \frac{(1-q)}{(1-q^{\alpha+1})} \cdot \frac{(q^{\alpha+4};q)_{n-2}}{(q;q)_{n-2}}\right] \times \frac{(q^{\alpha+1};q)_n}{(q;q)_n} \cdot {}_3\Phi_3\left(\begin{matrix} q^{-n}, q^{\beta+1}, q^{\gamma+1} \\ q^{\alpha+3}, q^\beta, q^\gamma \end{matrix} \middle| q, -(1-q)\cdot q^{n+\alpha+1}\cdot x\right).$$

Note that in the case that $-\beta \in \{0,1,2,\ldots,n\}$ or $-\gamma \in \{0,1,2,\ldots,n\}$ we have to take the analytic continuation of (5.3) in view of (5.1) and (5.2).

6. Recurrence relation. In this section we will derive a five term recurrence relation for the polynomials $\{L_n^{\alpha,M,N}(x;q)\}_{n=0}^\infty$. First we introduce the notation λ_n for

$$\text{(6.1)} \quad \lambda_n = \langle L_n^{\alpha,M,N}, L_n^{\alpha,M,N}\rangle = \frac{(q^{\alpha+1};q)_n}{(q;q)_n \cdot q^n} \cdot C_0 \cdot [C_0 + q^n \cdot C_1 + q^{2n-1}\cdot C_2].$$

Since C_0, C_1 and C_2 depend on n we will sometimes write $C_0(n)$, $C_1(n)$ and $C_2(n)$. Since $x^2 \cdot L_n^{\alpha,M,N}(x;q)$ is a polynomial of degree $n+2$ we may write

$$\text{(6.2)} \quad x^2 \cdot L_n^{\alpha,M,N}(x;q) = \sum_{k=0}^{n+2} D_k^{(n)} \cdot L_k^{\alpha,M,N}(x;q)$$

for some coefficients $D_k^{(n)} \in \mathbf{R}$, $k = 0,1,\ldots,n+2$.

It appears to be convenient, in the sequel, to set

$$\text{(6.3)} \quad \begin{cases} D_k^{(n)} = 0 & \text{if either } k < 0 \text{ or } n < 0 \\ \lambda_k = 1 & \text{if } k < 0. \end{cases}$$

By taking the inner product with $L_k^{\alpha,M,N}(x;q)$ on both sides of (6.2) we find with (6.1) and the definition of the inner product (3.5)

$$\text{(6.4)} \quad \lambda_k \cdot D_k^{(n)} = \langle L_n^{\alpha,M,N}(x;q), x^2 \cdot L_k^{\alpha,M,N}(x;q)\rangle.$$

Hence with the orthogonality property $D_k^{(n)} = 0$ for $k = 0,1,2,\ldots,n-3$. So we have with (6.2) for $n \geqq 2$

$$\text{(6.5)} \quad x^2 \cdot L_n^{\alpha,M,N}(x;q) = \sum_{k=n-2}^{n+2} D_k^{(n)} \cdot L_k^{\alpha,M,N}(x;q),$$

which is a five term recurrence relation for the polynomials $\{L_n^{\alpha,M,N}(x;q)\}_{n=0}^{\infty}$ provided that $D_{n+2}^{(n)} \neq 0$ and $D_{n-2}^{(n)} \neq 0$.

To find the coefficients $\{D_k^{(n)}\}_{k=n-2}^{n+2}$ we first note that

$$L_n^{(\alpha)}(x;q) = (-1)^n \cdot q^{n(n+\alpha)} \cdot \frac{(1-q)^n}{(q;q)_n} \cdot x^n + (-1)^{n-1} \cdot q^{(n-1)(n+\alpha-1)} \cdot \frac{(1-q^{n+\alpha})}{(1-q)} \cdot \frac{(1-q)^{n-1}}{(q;q)_{n-1}} \cdot x^{n-1} + \text{ lower order terms,}$$

which follows easily from the definition (2.1) and (1.9).

Then we have with the definition (3.1)

$$L_n^{\alpha,M,N}(x;q) = k_n \cdot x^n + k_n' \cdot x^{n-1} + \text{ lower order terms,} \tag{6.6}$$

where

$$k_n = (-1)^n \cdot q^{n(n+\alpha)} \cdot \frac{(1-q)^n}{(q;q)_n} \cdot C_0 \tag{6.7}$$

and

$$k_n' = (-1)^{n-1} \cdot q^{(n-1)(n+\alpha-1)} \cdot \frac{(1-q)^{n-1}}{(q;q)_{n-1}} \cdot \left[\frac{(1-q^{n+\alpha})}{(1-q)} \cdot C_0 - q^{n-1} \cdot C_1\right].$$

Further we have with (4.6) and (3.5) by using (1.9)

$$\begin{aligned}&\langle x^{n+1}, L_n^{\alpha,M,N}(x;q)\rangle \\ &= (-1)^n \cdot \frac{(q^{\alpha+1};q)_{n+1}}{(1-q)^{n+1}} \cdot q^{-(\alpha+1)(n+1)-n(n+2)} \\ &\times \left[\frac{(1-q^{n+1})}{(1-q)} \cdot C_0 + \frac{(1-q^n)}{(1-q)} \cdot q^{n+1} \cdot C_1 + \frac{(1-q^{n-1})}{(1-q)} \cdot q^{2n+1} \cdot C_2\right].\end{aligned}$$

Now we find $D_{n+2}^{(n)}$ by comparing the leading coefficients on both sides of (6.5):

$$D_{n+2}^{(n)} = \frac{k_n}{k_{n+2}} = \frac{(1-q^{n+1})(1-q^{n+2})}{(1-q)^2 \cdot q^{2(2n+\alpha+2)}} \cdot \frac{C_0(n)}{C_0(n+2)} \neq 0. \tag{6.8}$$

For $D_{n+1}^{(n)}$ we obtain by using (6.4) and (6.6)

$$D_{n+1}^{(n)} = \frac{k_n}{\lambda_{n+1}} \cdot \langle x^{n+2}, L_{n+1}^{\alpha,M,N}(x;q)\rangle + \frac{k_n'}{k_{n+1}}. \tag{6.9}$$

Alternatively, we compare the coefficient of x^{n+1} on both sides of (6.5) and use (6.6) and (6.8) to obtain

$$D_{n+1}^{(n)} = \frac{k_n' - k_{n+2}' \cdot D_{n+2}^{(n)}}{k_{n+1}} = \frac{k_n' \cdot k_{n+2} - k_n \cdot k_{n+2}'}{k_{n+1} \cdot k_{n+2}}.$$

From (6.4) we obtain by using (6.6), (6.7) and (6.1)

$$(6.10)\quad D_{n-2}^{(n)} = \frac{k_{n-2}\cdot\lambda_n}{k_n\cdot\lambda_{n-2}} = \frac{(1-q^{n+\alpha-1})(1-q^{n+\alpha})}{(1-q)^2\cdot q^{2(2n+\alpha-1)}} \times \frac{C_0(n)+q^n\cdot C_1(n)+q^{2n-1}\cdot C_2(n)}{C_0(n-2)+q^{n-2}\cdot C_1(n-2)+q^{2n-5}\cdot C_2(n-2)} \neq 0, \quad n \geqq 2$$

and by using (6.4) and (6.6)

$$(6.11)\quad D_{n-1}^{(n)} = \frac{k_{n-1}}{\lambda_{n-1}}\cdot\langle x^{n+1}, L_n^{\alpha,M,N}(x;q)\rangle + \frac{\lambda_n\cdot k'_{n-1}}{k_n\cdot\lambda_{n-1}}, \quad n \geqq 1.$$

To find $D_n^{(n)}$ we substitute $x=0$ in (6.5) and find

$$(6.12)\quad -L_n^{\alpha,M,N}(0;q)\cdot D_n^{(n)} = D_{n+2}^{(n)}\cdot L_{n+2}^{\alpha,M,N}(0;q) + D_{n+1}^{(n)}\cdot L_{n+1}^{\alpha,M,N}(0;q) + D_{n-1}^{(n)}\cdot L_{n-1}^{\alpha,M,N}(0;q) + D_{n-2}^{(n)}\cdot L_{n-2}^{\alpha,M,N}(0;q).$$

Hence $D_n^{(n)}$ can be computed by using (3.3), (6.8), (6.9), (6.10), (6.11) and (6.12) in the case that $L_n^{\alpha,M,N}(0;q)\neq 0$.

7. A Christoffel–Darboux type formula. From the recurrence relation (6.5) we obtain

$$(7.1)\quad (x^2-y^2)\cdot L_k^{\alpha,M,N}(x;q)L_k^{\alpha,M,N}(y;q)$$
$$= D_{k+2}^{(k)}\cdot[L_{k+2}^{\alpha,M,N}(x;q)L_k^{\alpha,M,N}(y;q) - L_{k+2}^{\alpha,M,N}(y;q)L_k^{\alpha,M,N}(x;q)]$$
$$+ D_{k+1}^{(k)}\cdot[L_{k+1}^{\alpha,M,N}(x;q)L_k^{\alpha,M,N}(y;q) - L_{k+1}^{\alpha,M,N}(y;q)L_k^{\alpha,M,N}(x;q)]$$
$$+ D_{k-1}^{(k)}\cdot[L_{k-1}^{\alpha,M,N}(x;q)L_k^{\alpha,M,N}(y;q) - L_{k-1}^{\alpha,M,N}(y;q)L_k^{\alpha,M,N}(x;q)]$$
$$+ D_{k-2}^{(k)}\cdot[L_{k-2}^{\alpha,M,N}(x;q)L_k^{\alpha,M,N}(y;q) - L_{k-2}^{\alpha,M,N}(y;q)L_k^{\alpha,M,N}(x;q)].$$

From (6.4) it follows by using (3.5)

$$\lambda_{k+2}\cdot D_{k+2}^{(k)} = \langle L_k^{\alpha,M,N}(x;q), x^2\cdot L_{k+2}^{\alpha,M,N}(x;q)\rangle = \lambda_k\cdot D_k^{(k+2)}$$

and in the same way

$$\lambda_{k+1}\cdot D_{k+1}^{(k)} = \lambda_k\cdot D_k^{(k+1)}.$$

Hence with (7.1) we obtain, by using (6.3)

$$(7.2)\quad (x^2-y^2)\cdot\sum_{k=0}^{n}\frac{L_k^{\alpha,M,N}(x;q)L_k^{\alpha,M,N}(y;q)}{\lambda_k}$$
$$= \frac{D_{n+2}^{(n)}}{\lambda_n}\cdot[L_{n+2}^{\alpha,M,N}(x;q)L_n^{\alpha,M,N}(y;q) - L_{n+2}^{\alpha,M,N}(y;q)L_n^{\alpha,M,N}(x;q)]$$
$$+ \frac{D_{n+1}^{(n)}}{\lambda_n}\cdot[L_{n+1}^{\alpha,M,N}(x;q)L_n^{\alpha,M,N}(y;q) - L_{n+1}^{\alpha,M,N}(y;q)L_n^{\alpha,M,N}(x;q)]$$
$$+ \frac{D_{n+1}^{(n-1)}}{\lambda_{n-1}}\cdot[L_{n+1}^{\alpha,M,N}(x;q)L_{n-1}^{\alpha,M,N}(y;q) - L_{n+1}^{\alpha,M,N}(y;q)L_{n-1}^{\alpha,M,N}(x;q)].$$

This can be seen as a Christoffel–Darboux type formula for the polynomials $\{L_n^{\alpha,M,N}(x;\,q)\}_{n=0}^{\infty}$. If we set $y = qx$ in (7.2) and use (1.5) we obtain:

$$(1+q)\cdot x\cdot\sum_{k=0}^{n}\frac{L_k^{\alpha,M,N}(x;\,q)L_k^{\alpha,M,N}(qx;\,q)}{\lambda_k}$$
$$=\frac{D_{n+2}^{(n)}}{\lambda_n}\cdot[L_n^{\alpha,M,N}(x;\,q)D_qL_{n+2}^{\alpha,M,N}(x;\,q)-L_{n+2}^{\alpha,M,N}(x;\,q)D_qL_n^{\alpha,M,N}(x;\,q)]$$
$$+\frac{D_{n+1}^{(n)}}{\lambda_n}\cdot[L_n^{\alpha,M,N}(x;\,q)D_qL_{n+1}^{\alpha,M,N}(x;\,q)-L_{n+1}^{\alpha,M,N}(x;\,q)D_qL_n^{\alpha,M,N}(x;\,q)]$$
$$+\frac{D_{n+1}^{(n-1)}}{\lambda_{n-1}}\cdot[L_{n-1}^{\alpha,M,N}(x;\,q)D_qL_{n+1}^{\alpha,M,N}(x;\,q)-L_{n+1}^{\alpha,M,N}(x;\,q)D_qL_{n-1}^{\alpha,M,N}(x;\,q)].$$

Moreover, if we first divide (7.2) by $x - y$, then let y tend to x, we find:

$$2x\cdot\sum_{k=0}^{n}\frac{\{L_k^{\alpha,M,N}(x;\,q)\}^2}{\lambda_k}$$
$$=\frac{D_{n+2}^{(n)}}{\lambda_n}\cdot\left[L_n^{\alpha,M,N}(x;\,q)\frac{d}{dx}L_{n+2}^{\alpha,M,N}(x;\,q)-L_{n+2}^{\alpha,M,N}(x;\,q)\frac{d}{dx}L_n^{\alpha,M,N}(x;\,q)\right]$$
$$+\frac{D_{n+1}^{(n)}}{\lambda_n}\cdot\left[L_n^{\alpha,M,N}(x;\,q)\frac{d}{dx}L_{n+1}^{\alpha,M,N}(x;\,q)-L_{n+1}^{\alpha,M,N}(x;\,q)\frac{d}{dx}L_n^{\alpha,M,N}(x;\,q)\right]$$
$$+\frac{D_{n+1}^{(n-1)}}{\lambda_{n-1}}\cdot\left[L_{n-1}^{\alpha,M,N}(x;\,q)\frac{d}{dx}L_{n+1}^{\alpha,M,N}(x;\,q)-L_{n+1}^{\alpha,M,N}(x;\,q)\frac{d}{dx}L_{n-1}^{\alpha,M,N}(x;\,q)\right].$$

8. The zeros. All sets of polynomials $\{P_n(x)\}_{n=0}^{\infty}$ which are orthogonal with respect to a positive weight function have the nice property that the n-th polynomial $P_n(x)$ has n real simple zeros, which are located in the interior of the interval of orthogonality. Our polynomials $\{L_n^{\alpha,M,N}(x;\,q)\}_{n=0}^{\infty}$ fail to have this property, but we can prove:

THEOREM 8.1. *The polynomial $L_n^{\alpha,M,N}(x;\,q)$ has n real simple zeros. At least $n-1$ of them lie in $(0,\infty)$, the interior of the interval of orthogonality.*

In other words: at most one zero of $L_n^{\alpha,M,N}(x;\,q)$ is located in the interval $(-\infty,0]$.

Proof. Suppose that $x_1, x_2, \ldots, x_k$ are all zeros of $L_n^{\alpha,M,N}(x;\,q)$ which are positive and have odd multiplicity. Define

$$p(x) = k_n\cdot(x-x_1)(x-x_2)\cdots(x-x_k)$$

where k_n is defined by (6.7). Then we have

$$p(x)\cdot L_n^{\alpha,M,N}(x;\,q)\geqq 0,\quad \forall\, x\geqq 0.$$

Now we define $h(x)$ and d so that

$$h(x) = (x + d) \cdot p(x)$$

and $(D_q h)(0) = 0$. For every polynomial $h(x)$ we have $(D_q h)(0) = h'(0)$, hence

$$0 = (D_q h)(0) = h'(0) = p(0) + d \cdot p'(0).$$

Hence

$$d = -\frac{p(0)}{p'(0)} > 0$$

since $p(0)$ and $p'(0)$ have opposite signs. This implies

$$\begin{aligned} &\langle h, L_n^{\alpha,M,N} \rangle \\ &= \frac{\Gamma_q(-\alpha)}{\Gamma(-\alpha)\Gamma(\alpha+1)} \cdot \int_0^\infty \frac{x^\alpha}{(-(1-q)x;\, q)_\infty} \cdot h(x) L_n^{\alpha,M,N}(x;\, q)dx \\ &+ M \cdot h(0) L_n^{\alpha,M,N}(0;\, q) > 0. \end{aligned}$$

Hence: degree $[h] \geqq n$, which implies that $k \geqq n - 1$. This proves the theorem.

Now we examine the non-positive zero of $L_n^{\alpha,M,N}(x;\, q)$ in somewhat greater detail. Since $0 < q < 1$ and $\alpha > -1$ we have

$$1 - q < 1 - q^n, n \geqq 2 \quad \text{and} \quad q(1 - q^{\alpha+1}) = q - q^{\alpha+2} < 1 - q^{\alpha+2}.$$

Hence

$$(1 - q^n)(1 - q^{\alpha+2}) - q(1 - q)(1 - q^{\alpha+1}) > 0, \quad n \geqq 2.$$

This together with (3.2) implies that $C_0 > 0$ for $M > 0$ and $N > 0$. So we have in view of (4.9): $L_n^{\alpha,M,N}(x;\, q) > 0$ for all $x < -B$ with $B > 0$ sufficiently large. This implies that the polynomial $L_n^{\alpha,M,N}(x;\, q)$ has a zero in $(-\infty, 0]$ if and only if $L_n^{\alpha,M,N}(0;\, q) \leqq 0$.

Then from (3.3) we must have $n \geqq 2$ and

$$(8.1) \qquad 1 - N \cdot q^{2\alpha+4} \cdot \frac{(1-q)}{(1-q^{\alpha+1})} \cdot \frac{(q^{\alpha+4};\, q)_{n-2}}{(q;\, q)_{n-2}} \leqq 0.$$

Define

$$f(n) = \frac{(q^{\alpha+4};\, q)_{n-2}}{(q;\, q)_{n-2}}.$$

Then we have

$$(8.2) \qquad f(n+1) = \frac{(q^{\alpha+4};\, q)_{n-1}}{(q;\, q)_{n-1}} = \frac{(1 - q^{n+\alpha+2})}{(1 - q^{n-1})} \cdot f(n) > f(n)$$

since $n+\alpha+2 > n+1 > n-1$. So $f(n)$ is an increasing function of n. But

$$\lim_{n\to\infty} f(n) = \frac{(q^{\alpha+4};\, q)_\infty}{(q;\, q)_\infty}.$$

Now we look at

$$F(\alpha, q, N) = 1 - N \cdot q^{2\alpha+4} \cdot \frac{(1-q)}{(1-q^{\alpha+1})} \cdot \frac{(q^{\alpha+4};\, q)_\infty}{(q;\, q)_\infty}.$$

For $\alpha = 0$ we have

$$F(0, q, N) = 1 - N \cdot \frac{q^4}{(1-q)(1-q^2)(1-q^3)}.$$

So we have for instance

$$F\left(0, \tfrac{1}{10}, 999\right) = \tfrac{791}{891} > 0.$$

This implies that we cannot guarantee the existence of a non-positive zero for n sufficiently large as in the case of the polynomials $\{L_n^{\alpha,M,N}(x)\}_{n=0}^{\infty}$ described in [**6**]. Note that

$$L_n^{\alpha,M,N}(x;\, q) \longrightarrow L_n^{\alpha,M,N}(x) \quad \text{for } q \longrightarrow 1^-.$$

But in view of (8.1) and (8.2) it is clear that if $L_n^{\alpha,M,N}(x;\, q)$ has a non-positive zero for some $n \in \mathbf{N}$, then $L_{n+1}^{\alpha,M,N}(x;\, q)$ has one too. Moreover, we have: the polynomials $\{L_n^{\alpha,M,N}(x;\, q)\}_{n=0}^{\infty}$ have a non-positive zero for all $n \geqq n_0$ if and only if $F(\alpha, q, N) < 0$. Here n_0 is the smallest n for which (8.1) holds.

In that case we can prove the following

THEOREM 8.2. *If the polynomial $L_n^{\alpha,M,N}(x;\, q)$ has a non-positive zero $-x_n$, then we have for $M > 0$:*

$$(8.3) \qquad 0 \leqq x_n < \frac{1}{2} \cdot \sqrt{\frac{N}{M}}.$$

Proof. Suppose that the polynomial $L_n^{\alpha,M,N}(x;\, q)$ has a non-positive zero $-x_n$. Then it is clear that $n \geqq 2$ and $N > 0$.

Let $x_1, x_2, \ldots, x_{n-1}$ be the positive zeros of $L_n^{\alpha,M,N}(x;\, q)$ and define

$$r(x) = (x - x_1)(x - x_2) \cdots (x - x_{n-1}).$$

Then we have in view of (6.6)

$$L_n^{\alpha,M,N}(x;\, q) = k_n \cdot r(x) \cdot (x + x_n)$$

where $x_n \geqq 0$. Since degree $[r] = n - 1$ and $(D_q r)(0) = r'(0)$ we have

$$(8.4) \qquad 0 = \langle r(x), L_n^{\alpha,M,N}(x; q)\rangle = \frac{k_n \cdot \Gamma_q(-\alpha)}{\Gamma(-\alpha)\Gamma(\alpha+1)} \cdot \int_0^\infty \frac{x^\alpha}{(-(1-q)x; q)_\infty} \cdot r^2(x) \cdot (x + x_n)dx + k_n \cdot M \cdot r^2(0) \cdot x_n + k_n \cdot N \cdot r'(0) \cdot [x_n \cdot r'(0) + r(0)].$$

Since the integral in (8.4) is positive we must have

$$M \cdot r^2(0) \cdot x_n + N \cdot r'(0) \cdot [x_n \cdot r'(0) + r(0)] < 0.$$

Hence

$$[M \cdot r^2(0) + N \cdot \{r'(0)\}^2] \cdot x_n < -N \cdot r(0)r'(0) = N \cdot |r(0)r'(0)|$$

since $r(0)$ and $r'(0)$ have opposite signs. Now it follows that

$$2\sqrt{M \cdot N} \cdot |r(0)r'(0)| \cdot x_n \leqq [M \cdot r^2(0) + N \cdot \{r'(0)\}^2] \cdot x_n < N \cdot |r(0)r'(0)|.$$

Hence

$$2\sqrt{M \cdot N} \cdot x_n < N.$$

This proves (8.3).

9. Another definition. In view of the relative simple formulas (3.3) and (3.4) we might expect that there is another definition for the polynomials $\{L_n^{\alpha,M,N}(x; q)\}_{n=0}^\infty$ which is simpler than the definition (3.1) and (3.2).

By using the same arguments as in [**7**], Section 3.7 we find the formula

$$(9.1) \qquad \frac{(1-q^n)}{(1-q)} \cdot L_n^{(\alpha)}(x; q) - \frac{(1-q^{\alpha+1})}{(1-q)} \cdot L_{n-1}^{(\alpha+1)}(x; q) = -q^{\alpha+1} \cdot x \cdot L_{n-1}^{(\alpha+2)}(x; q)$$

which is a q-analogue of formula (A.35) in [**6**] and formula (1.7.2) in [**7**]. Now it easily follows from (3.1) and (9.1) that we may define

$$(9.2) \qquad L_n^{\alpha,M,N}(x; q) = B_0 \cdot L_n^{(\alpha)}(x; q) + B_1 \cdot x \cdot L_{n-1}^{(\alpha+2)}(x; q) + B_2 \cdot x^2 \cdot L_{n-2}^{(\alpha+4)}(x; q)$$

where

$$B_2 = \frac{(1-q)^2 \cdot q^{2\alpha+5}}{(1-q^{\alpha+2})(1-q^{\alpha+3})} \cdot C_2.$$

Hence with (3.2) we have

$$B_2 = N \cdot q^{4\alpha+7} \cdot \frac{(1-q)^3}{(1-q^{\alpha+1})(1-q^{\alpha+2})(1-q^{\alpha+3})} \cdot \frac{(q^{\alpha+2};\, q)_{n-1}}{(q;\, q)_{n-1}}$$
$$+ MN \cdot q^{4\alpha+7} \cdot \frac{(1-q)^4}{(1-q^{\alpha+1})^2(1-q^{\alpha+2})(1-q^{\alpha+3})}$$
$$\times \frac{(q^{\alpha+1};\, q)_n}{(q;\, q)_n} \cdot \frac{(q^{\alpha+3};\, q)_{n-1}}{(q;\, q)_{n-1}}.$$

For B_0 we easily obtain from (3.3) and (2.3)

$$B_0 = 1 - N \cdot q^{2\alpha+4} \cdot \frac{(1-q)}{(1-q^{\alpha+1})} \cdot \frac{(q^{\alpha+4};\, q)_{n-2}}{(q;\, q)_{n-2}}. \tag{9.3}$$

To find B_1 we note that it follows from (9.2), (2.2) and (2.3) that

$$(D_q L_n^{\alpha,M,N})(0) = -q^{\alpha+1} \cdot B_0 \cdot \frac{(q^{\alpha+2};\, q)_{n-1}}{(q;\, q)_{n-1}} + B_1 \cdot \frac{(q^{\alpha+3};\, q)_{n-1}}{(q;\, q)_{n-1}}. \tag{9.4}$$

So we obtain from (3.4), (9.3) and (9.4)

$$B_1 = -M \cdot q^{\alpha+1} \cdot \frac{(1-q)}{(1-q^{\alpha+1}})) \cdot \frac{(q^{\alpha+1};\, q)_n}{(q;\, q)_n}$$
$$- N \cdot q^{3\alpha+5} \cdot \frac{(1-q)(1-q^{\alpha+2})}{(1-q^{\alpha+1})(1-q^{\alpha+3})} \cdot \frac{(q^{\alpha+3};\, q)_{n-2}}{(q;\, q)_{n-2}}.$$

Hence we have found another definition for the polynomials $\{L_n^{\alpha,M,N}(x;\, q)\}_{n=0}^{\infty}$ given by (9.2) and

$$\begin{cases} B_0 = 1 - N \cdot q^{2\alpha+4} \cdot \frac{(1-q)}{(1-q^{\alpha+1})} \cdot \frac{(q^{\alpha+4};\, q)_{n-2}}{(q;\, q)_{n-2}} \\ B_1 = -M \cdot q^{\alpha+1} \cdot \frac{(1-q)}{(1-q^{\alpha+1})} \cdot \frac{(q^{\alpha+1};\, q)_n}{(q;\, q)_n} \\ \qquad -N \cdot q^{3\alpha+5} \cdot \frac{(1-q)(1-q^{\alpha+2})}{(1-q^{\alpha+1})(1-q^{\alpha+3})} \cdot \frac{(q^{\alpha+3};\, q)_{n-2}}{(q;\, q)_{n-2}} \\ B_2 = N \cdot q^{4\alpha+7} \cdot \frac{(1-q)^3}{(1-q^{\alpha+1})(1-q^{\alpha+2})(1-q^{\alpha+3})} \cdot \frac{(q^{\alpha+2};\, q)_{n-1}}{(q;\, q)_{n-1}} \\ \qquad +MN \cdot q^{4\alpha+7} \cdot \frac{(1-q)^4}{(1-q^{\alpha+1})^2(1-q^{\alpha+2})(1-q^{\alpha+3})} \cdot \frac{(q^{\alpha+1};\, q)_n}{(q;\, q)_n} \cdot \frac{(q^{\alpha+3};\, q)_{n-1}}{(q;\, q)_{n-1}}. \end{cases} \tag{9.5}$$

The formulas in (9.5) are simpler than those of (3.2).

Note that this definition given by (9.2) and (9.5) is a q-analogue of the definition (A.33) and (A.34) in [**6**] for the polynomials $\{L_n^{\alpha,M,N}(x)\}_{n=0}^{\infty}$.

For $N = 0$ this definiton reduces to the definition

$$L_n^{\alpha,M}(x;\, q) = L_n^{(\alpha)}(x;\, q) - M \cdot q^{\alpha+1}$$
$$\times \frac{(1-q)}{(1-q^{\alpha+1})} \cdot \frac{(q^{\alpha+1};\, q)_n}{(q;\, q)_n} \cdot x \cdot L_{n-1}^{(\alpha+2)}(x;\, q)$$

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for the q-analogue of Koornwinder's generalized Laguerre polynomials which was found in [**7**].

10. A q-difference equation. In [**5**] J. Koekoek found a simple proof of a second order differential equation for the polynomials $\{L_n^{\alpha,M,N}(x)\}_{n=0}^{\infty}$ defined by (0.2) and (0.3). A similar method can be used to prove that the polynomials $\{L_n^{\alpha,M,N}(x;q)\}_{n=0}^{\infty}$ satisfy a q-difference equation of the form:

$$x\cdot P_2(x)\cdot(D_q^2L_n^{\alpha,M,N})(q^{-2}x;q)-P_1(x)\cdot(D_qL_n^{\alpha,M,N})(q^{-1}x;q)+\frac{(1-q^n)}{(1-q)}\cdot P_0(x)\cdot L_n^{\alpha,M,N}(x;q)=0, \tag{10.1}$$

where $\{P_k(x)\}_{k=0}^{2}$ are polynomials with

$$\begin{cases} P_2(x)=C_0\cdot[C_0+q^n\cdot C_1+q^{2n-1}\cdot C_2]\cdot x^2 \\ \qquad + \text{lower order terms} \\ P_1(x)=q^{n+\alpha+2}\cdot C_0\cdot[C_0+q^n\cdot C_1+q^{2n-1}\cdot C_2]\cdot x^3 \\ \qquad + \text{lower order terms} \\ P_0(x)=q^{\alpha+3}\cdot C_0\cdot[C_0+q^n\cdot C_1+q^{2n-1}\cdot C_2]\cdot x^2 \\ \qquad + \text{lower order terms.} \end{cases} \tag{10.2}$$

To prove this we start with the definition (3.1) and use (2.2) to see that

$$L_n^{\alpha,M,N}(x;q)=C_0\cdot L_n^{(\alpha)}(x;q)+q^{-\alpha-1}\cdot C_1\cdot(D_qL_n^{(\alpha)})(q^{-1}x;q)+q^{-2\alpha-4}\cdot C_2\cdot(D_q^2L_n^{(\alpha)})(q^{-2}x;q). \tag{10.3}$$

Equation (1.5) implies that

$$L_n^{(\alpha)}(q^{-1}x;q)=L_n^{(\alpha)}(x;q)+q^{-1}(1-q)x\cdot(D_qL_n^{(\alpha)})(q^{-1}x;q).$$

This together with the q-difference equation (2.10) yields

$$\begin{aligned} q^{-2}x\cdot(D_q^2L_n^{(\alpha)})(q^{-2}x;q) &= -\left[\frac{(1-q^{\alpha+1})}{(1-q)}-q^{n+\alpha}\cdot x\right]\cdot(D_qL_n^{(\alpha)})(q^{-1}x;q) \\ &\quad -\frac{(1-q^n)}{(1-q)}\cdot q^{\alpha+1}\cdot L_n^{(\alpha)}(x;q). \end{aligned} \tag{10.4}$$

Now we multiply (10.3) by x and use (10.4) to find

$$x\cdot L_n^{\alpha,M,N}(x;q)=p_0(x)\cdot L_n^{(\alpha)}(x;q)+p_1(x)\cdot(D_qL_n^{(\alpha)})(q^{-1}x;q) \tag{10.5}$$

where

$$\begin{cases} p_0(x)=C_0x-\dfrac{(1-q^n)}{(1-q)q^{\alpha+1}}\cdot C_2 \\ p_1(x)=q^{-\alpha-1}\cdot C_1\cdot x-q^{-2\alpha-2}\cdot\left[\dfrac{(1-q^{\alpha+1})}{(1-q)}-q^{n+\alpha}\cdot x\right]\cdot C_2. \end{cases} \tag{10.6}$$

We can then use the q-product rule (1.7) together with (1.6) and (10.5) to obtain

$$\begin{aligned}(10.7)\quad & q^{-1}x\cdot(D_qL_n^{\alpha,M,N})(q^{-1}x;\,q)+L_n^{\alpha,M,N}(x;\,q)\\ &=(D_qp_0)(q^{-1}x)\cdot L_n^{(\alpha)}(x;\,q)+[p_0(q^{-1}x)\\ &+(D_qp_1)(q^{-1}x)]\cdot(D_qL_n^{(\alpha)})(q^{-1}x;\,q)\\ &+q^{-1}\cdot p_1(q^{-1}x)\cdot(D_q^2L_n^{(\alpha)})(q^{-2}x;\,q).\end{aligned}$$

Now we multiply (10.7) by qx and use (10.4) and (10.5) to find

$$(10.8)\quad x^2\cdot(D_qL_n^{\alpha,M,N})(q^{-1}x;\,q)=r_0(x)\cdot L_n^{(\alpha)}(x;\,q)+r_1(x)\cdot(D_qL_n^{(\alpha)})(q^{-1}x;\,q)$$

where

$$(10.9)\quad \begin{cases} r_0(x)=qx\cdot(D_qp_0)(q^{-1}x)-\dfrac{(1-q^n)}{(1-q)}\cdot q^{\alpha+3}\cdot p_1(q^{-1}x)-q\cdot p_0(x)\\ r_1(x)=qx\cdot[p_0(q^{-1}x)+(D_qp_1)(q^{-1}x)]\\ \qquad -q^2\cdot\left[\dfrac{(1-q^{\alpha+1})}{(1-q)}-q^{n+\alpha}\cdot x\right]\cdot p_1(q^{-1}x)-q\cdot p_1(x).\end{cases}$$

In the same way we obtain from (10.8) and (10.4)

$$(10.10)\quad x^3\cdot(D_q^2L_n^{\alpha,M,N})(q^{-2}x;\,q)=s_0(x)\cdot L_n^{(\alpha)}(x;\,q)+s_1(x)\cdot(D_qL_n^{(\alpha)})(q^{-1}x;\,q)$$

where

$$(10.11)\quad \begin{cases} s_0(x)=q^3x\cdot(D_qr_0)(q^{-1}x)\\ \qquad -\dfrac{(1-q^n)}{(1-q)}\cdot q^{\alpha+5}\cdot r_1(q^{-1}x)-(1+q)q^2\cdot r_0(x)\\ s_1(x)=q^3x\cdot[r_0(q^{-1}x)+(D_qr_1)(q^{-1}x)]\\ \qquad -q^4\cdot\left[\dfrac{(1-q^{\alpha+1})}{(1-q)}-q^{n+\alpha}\cdot x\right]\cdot r_1(q^{-1}x)-(1+q)q^2\cdot r_1(x).\end{cases}$$

Elimination of $(D_qL_n^{(\alpha)})(q^{-1}x;\,q)$ in (10.5) and (10.8) gives us in view of (2.3)

$$(10.12)\quad p_0(x)\cdot r_1(x)-p_1(x)\cdot r_0(x)=x\cdot P_2(x)$$

for some polynomial $P_2(x)$. In the same way we obtain from (10.5) and (10.10)

$$(10.13)\quad p_0(x)\cdot s_1(x)-p_1(x)\cdot s_0(x)=x\cdot P_1(x)$$

for some polynomial $P_1(x)$. Using (10.6), (10.9) and (10.11) we have

$$p_0(x)=C_0\cdot x\quad\text{and}\quad r_0(x)=s_0(x)\equiv 0\quad\text{for } n=0.$$

This together with (10.8) and (10.10) yields

$$r_0(x)\cdot s_1(x) - r_1(x)\cdot s_0(x) = \frac{(1-q^n)}{(1-q)}\cdot x^2\cdot P_0(x) \tag{10.14}$$

for some polynomial $P_0(x)$.

In view of (10.5), (10.8) and (10.10) we conclude that the following determinant

$$\begin{vmatrix} x\cdot L_n^{\alpha,M,N}(x;q) & p_0(x) & p_1(x) \\ x^2\cdot (D_qL_n^{\alpha,M,N})(q^{-1}x;q) & r_0(x) & r_1(x) \\ x^3\cdot (D_q^2L_n^{\alpha,M,N})(q^{-2}x;q) & s_0(x) & s_1(x) \end{vmatrix}$$

must be zero. The first column can be divided by x, hence with (10.12), (10.13) and (10.14) we find

$$\begin{aligned} 0 &= \begin{vmatrix} L_n^{\alpha,M,N}(x;q) & p_0(x) & p_1(x) \\ x\cdot (D_qL_n^{\alpha,M,N})(q^{-1}x;q) & r_0(x) & r_1(x) \\ x^2\cdot (D_q^2L_n^{\alpha,M,N})(q^{-2}x;q) & s_0(x) & s_1(x) \end{vmatrix} \\ &= x^3\cdot P_2(x)\cdot (D_q^2L_n^{\alpha,M,N})(q^{-2}x;q) \\ &\quad - x^2\cdot P_1(x)\cdot (D_qL_n^{\alpha,M,N})(q^{-1}x;q) \\ &\quad + \frac{(1-q^n)}{(1-q)}\cdot x^2\cdot P_0(x)\cdot L_n^{\alpha,M,N}(x;q). \end{aligned}$$

So we can divide by x^2 to obtain (10.1). By using (10.6), (10.9) and (10.11) we can easily check (10.2). This proves the q-difference equation.

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