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A weighted Trudinger–Moser inequalities and applications to some weighted $(N, q)-$Laplacian equation in $\mathbb{R}^N$ with new exponential growth conditions

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In this paper, we prove some weighted sharp inequalities of Trudinger–Moser type. The weights considered here have a logarithmic growth. These inequalities are completely new and are established in some new Sobolev spaces where the norm is a mixture of the norm of the gradient in two different Lebesgue spaces. This fact allowed us to prove a very interesting result of sharpness for the case of doubly exponential growth at infinity. Some improvements of these inequalities for the weakly convergent sequences are also proved using a version of the Concentration-Compactness principle of P.L. Lions. Taking profit of these inequalities, we treat in the last part of this work some elliptic quasilinear equation involving the weighted $(N, q)-$Laplacian operator where $1 < q < N$ and a nonlinearities enjoying a new type of exponential growth condition at infinity.

Keywords: Weighted Trudinger–Moser inequality; weighted Sobolev spaces; logarithmic weights; doubly exponential growth; radial functions; concentration-compactness; weighted (N,q)–Laplacian; elliptic equation

2020 *Mathematics Subject Classification:* 26D15; 35A15; 35A23; 35B38; 35D30; 35J20; 35J62; 46E35

1. Introduction and statement of main results

In 2015, M. Calanchi and B. Ruf have established a weighted Trudinger–Moser inequality in the unit ball $\mathcal{B}$ of $\mathbb{R}^N$, $N \geqslant 2$. Such type of inequality is not new and many inequalities of Trudinger–Moser type defined in weighted Sobolev spaces have been proved; we can for example cite [**1–3**, **5**, **13–16**, **21**, **22**, **26**, **29**, **33**]. The majority of those works considered the restriction to radial functions, and in [**29**] although the weight is not necessarily radial but its growth permits to pass to the radial case through some radial rearrangement. This interest to reduce the inequality to the radial functions is mainly motivated by their ability to increase the maximal growth of the integrability. The weight that M. Calanchi and B. Ruf considered is of logarithmic type and turned out to be of great interest. More precisely, they introduced the subspace $W^{1,N}_{0,rad}(\mathcal{B}, \sigma_\beta)$ defined as the radial functions of the completion of $C_0^\infty(\mathcal{B})$ with respect to the norm

$$\|u\|_{\sigma_\beta}^N = \int_{\mathcal{B}} \sigma_\beta(x) \, |\nabla u|^N \, \mathrm{d}x,$$

where $\sigma_\beta(x) = (\log \frac{1}{|x|})^{\beta(N-1)}$ or $\sigma_\beta(x) = (\log \frac{e}{|x|})^{\beta(N-1)}$, $0 < \beta \leqslant 1$, $x \in \mathcal{B}$. In [**16**, theorem 1], M. Calanchi and B. Ruf proved the following result: for $0 < \beta < 1$, we have

- $$\int_{\mathcal{B}} \mathrm{e}^{|u|^{\frac{N'}{1-\beta}}} \, \mathrm{d}x < +\infty, \; \forall \; u \in W_{0,rad}^{1,N}(\mathcal{B}, \sigma_\beta). \tag{1.1}$$

- $$\sup \left\{ \int_{\mathcal{B}} \mathrm{e}^{\alpha |u|^{\frac{N'}{1-\beta}}} \, \mathrm{d}x, \; u \in W_{0,rad}^{1,N}(\mathcal{B}, \sigma_\beta), \; \|u\|_{\sigma_\beta} \leqslant 1 \right\} < +\infty \Leftrightarrow \alpha \leqslant \alpha_{N,\beta}, \tag{1.2}$$

  where $\alpha_{N,\beta} = N[\omega_{N-1}^{\frac{1}{N-1}}(1-\beta)]^{\frac{1}{1-\beta}}$ and ω_{N-1} is the area of the unit sphere in $\mathbb{R}^N$.

The case $N = 2$ has been considered in a previous work (see [**15**]). Note that when $\beta = 0$, (1.2) recovers the classical Trudinger–Moser inequality (see [**31**, **38**]). Next, M. Calanchi and B. Ruf considered the case when $\beta = 1$. In this case, the specific behaviour of the weight function has an impact on the corresponding embeddings. In fact, the maximal growth $\mathrm{e}^{|s|^{N'}}$ proved in the classical Trudinger–Moser inequality significantly increased such that a doubly exponential growth is now permitted. More precisely, they proved the following result given in [**16**, theorem 4]:

- $$\int_{\mathcal{B}} \mathrm{e}^{\mathrm{e}^{|u|^{N'}}} \, \mathrm{d}x < +\infty, \; \forall u \in W_{0,rad}^{1,N}(\mathcal{B}, \sigma_1), \text{ where } \sigma_1(x) = \left(\log \frac{e}{|x|} \right)^{N-1}. \tag{1.3}$$

- $$\sup \left\{ \int_{\mathcal{B}} \mathrm{e}^{a \, \mathrm{e}^{\omega_{N-1}^{\frac{1}{N-1}} |u|^{N'}}} \, \mathrm{d}x, \; u \in W_{0,rad}^{1,N}(\mathcal{B}, \sigma_1), \; \|u\|_{\sigma_1} \leqslant 1 \right\} < +\infty \Longleftrightarrow a \leqslant N. \tag{1.4}$$

The proof of (1.2) in the critical case $\alpha = \alpha_{N,\beta}$ is mainly based on a suitable change of variable combined to some integral inequality due to M.A. Leckband. In [**34**], V.H. Nguyen provided a simpler proof of (1.2) in which he proved that the function

$$\beta \longmapsto MT(N, \alpha, \beta) = \sup \left\{ \int_{\mathcal{B}} \mathrm{e}^{\alpha |u|^{\frac{N'}{1-\beta}}} \, \mathrm{d}x, \; u \in W_{0,rad}^{1,N}(\mathcal{B}, \sigma_\beta), \; \|u\|_{\sigma_\beta} \leqslant 1 \right\}$$

is decreasing on $[0, 1)$. Moreover, V.H. Nguyen proved the existence of maximizer for this inequality when β is sufficiently small. The question of the attainability of

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the inequality (1.2) has been also considered by P. Roy in [**35**] for the case $N = 2$, and in [**36**] for higher dimensions. Taking advantage of these new Trudinger–Moser inequalities defined on the unit ball $\mathcal{B}$ in $\mathbb{R}^N$, some authors studied an elliptic problem having a doubly exponential growth at infinity. It mainly consists in the following equation

$$\begin{cases} -\mathrm{div}(\sigma_1(x)|\nabla u|^{N-2}\nabla u) = f(x,u), & \text{in } \mathcal{B}, \\ u > 0, & \text{in } \mathcal{B}, \\ u = 0, & \text{on } \partial\mathcal{B}, \end{cases}$$

where the nonlinear term $f(x,u)$ is a continuous function, radial in $x \in \mathcal{B}$ and has a critical doubly exponential growth at infinity, which means that there exists a positive constant α_0 such that

$$\lim_{s\to+\infty} \frac{f(x,s)}{\mathrm{e}^{N\,\mathrm{e}^{\alpha|s|^{N'}}}} = \begin{cases} 0, & \text{for all } \alpha > \alpha_0, \\ +\infty, & \text{for all } \alpha \leqslant \alpha_0. \end{cases}$$

M. Calanchi, B. Ruf and F. Sani proved in [**17**] the existence of a nontrivial radial solution for the case $N = 2$. This result has been recently generalized by C. Zhang in [**40**] for higher dimensions. When we try to extend (1.1)–(1.4) to the whole Euclidean space $\mathbb{R}^N$, $N \geqslant 2$, we face many obstacles which mainly consist of the embedding and denseness properties of the functional space that we construct by extending the weight outside of the unit ball $\mathcal{B}$. For the first attempts, we worked with the weight $\sigma_\beta(x) = (\log \frac{e}{|x|})^{\beta(N-1)}$, $|x| < 1$. In [**6**], we considered a radial weight w_β defined by

$$w_\beta(x) = \begin{cases} \left(\log\left(\dfrac{e}{|x|}\right)\right)^{\beta(N-1)} & \text{if } |x| < 1, \\ \chi(|x|) & \text{if } |x| \geqslant 1, \end{cases} \tag{1.5}$$

where, $0 < \beta \leqslant 1$ and $\chi : [1,+\infty[\to]0,+\infty[$ is a continuous function such that $\chi(1) = 1$, $\inf\limits_{t\geqslant 1} \chi(t) > 0$. Denoted by Y_β the weighted Sobolev space

$$Y_\beta = \left\{ u \in W^{1,N}_{rad}(\mathbb{R}^N) \; ; \; \int_{\mathbb{R}^N} w_\beta(x)|\nabla u|^N \,\mathrm{d}x < +\infty \right\}$$

and we equip it with the standard Sobolev norm

$$\|u\|^N_{Y_\beta} = \int_{\mathbb{R}^N} |\nabla u|^N w_\beta(x)\,\mathrm{d}x + \int_{\mathbb{R}^N} |u|^N \,\mathrm{d}x.$$

We obtained the following extensions of (1.1) and (1.2) to the whole space $\mathbb{R}^N$: Let $N \geqslant 2$ and w_β be defined by (1.5). Then, for all $\alpha > 0$ and $u \in Y_\beta$, we have

$$\int_{\mathbb{R}^N} \left(\mathrm{e}^{\alpha|u|^{\frac{N'}{1-\beta}}} - S_{N-2}\left(\alpha\,|u|^{\frac{N'}{1-\beta}}\right)\right) \mathrm{d}x < +\infty, \tag{1.6}$$

where

$$S_{N-2}(t) = \sum_{k=0}^{N-2} \frac{t^k}{k!}, \; t \geqslant 0.$$

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Moreover, if $\alpha < \alpha_{N,\beta}$, then

$$\sup_{u\in Y_\beta,\ \|u\|_{Y_\beta}\leqslant 1} \int_{\mathbb{R}^N} \left(\mathrm{e}^{\alpha |u|^{\frac{N'}{1-\beta}}} - S_{N-2}\left(\alpha, |u|^{\frac{N'}{1-\beta}}\right)\right) \mathrm{d}x < +\infty, \tag{1.7}$$

and if $\alpha > \alpha_{N,\beta}$, then the supremum in (1.7) becomes infinite. For the value $\alpha = \alpha_{N,\beta}$, the supremum in (1.7) is not necessarily finite. However, the sharpness of the Trudinger–Moser inequality could be recovered by considering a different functional space. More precisely, for $0 < \beta < 1$, we define Y'_β as the space of all the radial functions of the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\|_{Y'_\beta} = |\nabla u|_{L^N(\mathbb{R}^N, w_\beta)} + |u|_{L^{d'_\beta}(\mathbb{R}^N)} = \left(\int_{\mathbb{R}^N} w_\beta(x)\, |\nabla u|^N \,\mathrm{d}x\right)^{\frac{1}{N}} + \left(\int_{\mathbb{R}^N} |u|^{d'_\beta}\,\mathrm{d}x\right)^{\frac{1}{d'_\beta}},$$

where $d'_\beta = \frac{N'(1-\beta)}{N'-1+\beta}$. For that space, we obtained the following sharp Trudinger–Moser inequality which can be considered as another extension of (1.2): Let $0 < \beta \leqslant \frac{1}{N'+1}$ and w_β be defined by (1.5). Then,

$$\sup_{u\in Y'_\beta,\ \|u\|_{Y'_\beta}\leqslant 1} \int_{\mathbb{R}^N} \left(\mathrm{e}^{\alpha |u|^{\frac{N'}{1-\beta}}} - S_{N-2}\left(\alpha\, |u|^{\frac{N'}{1-\beta}}\right)\right) \mathrm{d}x < +\infty \Leftrightarrow \alpha \leqslant \alpha_{N,\beta}.$$

The value $\beta = 1$ is a kind of second order limiting case. In [6], we established the following extension of (1.3) and (1.4):

- For all $\alpha > 0$ and $u \in Y_1$, there holds
$$\int_{\mathbb{R}^N} \left(\mathrm{e}^{\alpha\left(\mathrm{e}^{|u|^{N'}}-1\right)} - S_{N-2}\left(\alpha\left(\mathrm{e}^{|u|^{N'}}-1\right)\right)\right) \mathrm{d}x < +\infty. \tag{1.8}$$
- If $a \leqslant N\,\mathrm{e}^{-(\inf_{s\geqslant 1}\chi(s))^{-\frac{1}{N(N-1)}}}$, then
$$\sup_{u\in Y_1,\ \|u\|_{Y_1}\leqslant 1} \int_{\mathbb{R}^n} \left(\mathrm{e}^{a\left(\mathrm{e}^{\omega_{N-1}^{\frac{1}{N-1}}|u|^{N'}}-1\right)} - S_{N-2}\left(a\left(\mathrm{e}^{\omega_{N-1}^{\frac{1}{N-1}}|u|^{N'}}-1\right)\right)\right) \mathrm{d}x < +\infty. \tag{1.9}$$
- If $a > N\exp\left(\dfrac{1}{N-1}\displaystyle\int_0^{+\infty} \log^N(1+t)\,\mathrm{e}^{-Nt}\,\mathrm{d}t\right)$, then
$$\sup_{u\in Y_1,\ \|u\|_{Y_1}\leqslant 1} \int_{\mathbb{R}^N} \left(\mathrm{e}^{a\left(\mathrm{e}^{\omega_{N-1}^{\frac{1}{N-1}}|u|^{N'}}-1\right)} - S_{N-2}\left(a\left(\mathrm{e}^{\omega_{N-1}^{\frac{1}{N-1}}|u|^{N'}}-1\right)\right)\right) \mathrm{d}x = +\infty. \tag{1.10}$$

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Note that the previous results come as generalizations of earlier works dealing with the case $N = 2$. See [**4**, **7**]. A further interesting extensions of (1.1)–(1.4) to the whole Euclidean space $\mathbb{R}^N$ has been provided in [**8**, **10**].

In this paper, we consider two types of weights. First, for the case when $0 < \beta < 1$, we consider the weight defined by

$$w_\beta(x) = \begin{cases} (-\log(|x|))^{\beta(N-1)} & \text{if } |x| < 1, \\ \chi(|x|) & \text{if } |x| \geqslant 1, \end{cases} \tag{1.11}$$

where $\chi : [1, +\infty[\to]0, +\infty[$ is a continuous function such that $\chi(1) = 0$. Moreover, the function χ is chosen such that w_β satisfies (1.12), that is, w_β belongs to the Muckenhoupt's class A_N (we also say that w_β is an A_N-weight), that is

$$\sup \left(\frac{1}{|B|} \int_B w_\beta(x)\, \mathrm{d}x \right) \left(\frac{1}{|B|} \int_B (w_\beta(x))^{\frac{1}{1-N}}\, \mathrm{d}x \right)^{N-1} < +\infty, \tag{1.12}$$

where the supremum is taken over all balls B in $\mathbb{R}^N$. The importance of this property of the weight w_β lies in the fact that it implies that $C_0^\infty(\mathbb{R}^N)$ is dense in the space E_β (see, for instance, [**18**, **28**, **32**] and references therein). An interesting example of such a function χ is given by: $\chi(t) = \log^\gamma(t),\ \gamma > 0$ (see [**27**]). In particular, one can consider the weight

$$w_\beta(x) = |\log|x||^{\beta(N-1)},\ x \in \mathbb{R}^N \backslash \{0\}.$$

That last weight can be seen as a natural extension of $(-\log|x|)^{\beta(N-1)}$ defined on $\mathcal{B} = \left\{x \in \mathbb{R}^N,\ |x| < 1\right\}$ and considered in [**16**]. Second, for the case $\beta = 1$, we consider the weight

$$w_1(x) = \begin{cases} (1 - \log(|x|))^{N-1} & \text{if } |x| < 1, \\ \chi(|x|) & \text{if } |x| \geqslant 1, \end{cases} \tag{1.13}$$

where $\chi : [1, +\infty[\to]0, +\infty[$ is a continuous function such that $\chi(1) = 1$ and w_1 belongs to the Muckenhoupt's class A_N. Here, are some examples of such a function χ.

- χ can be any continuous and positive function such that $\chi(1) = 1$ and
$$0 < \inf_{t \geqslant 1} \chi(t) \leqslant \sup_{t \geqslant 1} \chi(t) < +\infty.$$
- $\chi(t) = t^\alpha,\ 0 < \alpha < N(N-1)$.
- $\chi(t) = 1 + \log^\gamma t,\ \gamma > 0$.

For details about these examples, we refer to [8]. Let $1 < q < N$. For $0 < \beta \leqslant 1$, denote by $E_{q,\beta}$ the weighted Sobolev space

$$E_{q,\beta} = \left\{ u \in D_r^{1,q}(\mathbb{R}^N),\ \int_{\mathbb{R}^N} w_\beta(x) \left|\nabla u\right|^N \,\mathrm{d}x < +\infty \right\},$$

where $D_r^{1,q}(\mathbb{R}^N) = \left\{ u \in L^{q^*}(\mathbb{R}^N),\ u \text{ radial},\ \int_{\mathbb{R}^N} \left|\nabla u\right|^q \,\mathrm{d}x < +\infty \right\}$ and $q^* = \frac{Nq}{N-q}$. We first equip the functional space $E_{q,\beta}$ with the norm

$$\|u\|_{E_{q,\beta}} = \left(\int_{\mathbb{R}^N} \left|\nabla u\right|^N w_\beta(x) \,\mathrm{d}x + \left(\int_{\mathbb{R}^N} \left|\nabla u\right|^q \,\mathrm{d}x \right)^{\frac{N}{q}} \right)^{\frac{1}{N}},\ u \in E_{q,\beta}.$$

The first result in the present work concerns the case $0 < \beta < 1$ and the norm $\|\cdot\|_{E_{q,\beta}}$. It is given by the following Trudinger–Moser inequality.

Theorem 1.1. *Let $0 < \beta < 1$ and w_β be defined by (1.11). Let $j_\beta = \inf\left\{ j \geqslant 1,\ j \geqslant \frac{(1-\beta)q^*}{N'} \right\}$.*

For all $\alpha > 0$ and $u \in E_{q,\beta}$, we have

$$\int_{\mathbb{R}^N} \left(\mathrm{e}^{\alpha|u|^{\frac{N'}{1-\beta}}} - S_{j_\beta-1}\left(\alpha\left|u\right|^{\frac{N'}{1-\beta}}\right) \right) \mathrm{d}x < +\infty, \tag{1.14}$$

where $S_{j_\beta-1}(t) = \sum_{j=0}^{j_\beta-1} \frac{t^j}{j!}$, $t \in [0, +\infty[$. Moreover, if $\alpha < \alpha_{N,\beta}$, then

$$\sup_{u \in E_{q,\beta},\ \|u\|_{E_{q,\beta}} \leqslant 1} \int_{\mathbb{R}^N} \left(\mathrm{e}^{\alpha|u|^{\frac{N'}{1-\beta}}} - S_{j_\beta-1}\left(\alpha\left|u\right|^{\frac{N'}{1-\beta}}\right) \right) \mathrm{d}x < +\infty, \tag{1.15}$$

and if $\alpha > \alpha_{N,\beta}$, then

$$\sup_{u \in E_{q,\beta},\ \|u\|_{E_{q,\beta}} \leqslant 1} \int_{\mathbb{R}^N} \left(\mathrm{e}^{\alpha|u|^{\frac{N'}{1-\beta}}} - S_{j_\beta-1}\left(\alpha\left|u\right|^{\frac{N'}{1-\beta}}\right) \right) \mathrm{d}x = +\infty. \tag{1.16}$$

The second result in this paper concerns the case $\beta = 1$. More precisely, we prove the following theorem:

Theorem 1.2. *Let w_1 be defined by (1.13). Let $j_1 = \inf\left\{ j \geqslant 1,\ j \geqslant \frac{q^*}{N'} \right\}$. Set*

$$C_{q,N} = \omega_{N-1}^{-\frac{1}{q}} \left(\frac{q-1}{N-q} \right)^{\frac{q-1}{q}}. \tag{1.17}$$

For all $\alpha > 0$ and $u \in E_{q,1}$, we have

$$\int_{\mathbb{R}^N} \left(\mathrm{e}^{\alpha\left(\mathrm{e}^{|u|^{N'}}-1\right)} - S_{j_1-1}\left(\alpha\left(\mathrm{e}^{|u|^{N'}}-1\right)\right) \right) \mathrm{d}x < +\infty, \tag{1.18}$$

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where $S_{j_1-1}(t)=\sum_{j=0}^{j_1-1}\frac{t^j}{j!}$, $t\in[0,+\infty[$. *Moreover, if* $\alpha\leqslant N\,\mathrm{e}^{-\omega_{N-1}^{\frac{1}{N-1}}C_{q,N}^{N'}}$, *then*

$$\sup_{u\in E_{q,1},\ \|u\|_{E_{q,1}}\leqslant 1}\int_{\mathbb{R}^N}\left(\mathrm{e}^{\alpha\left(\mathrm{e}^{\omega_{N-1}^{\frac{1}{N-1}}|u|^{N'}}-1\right)}-S_{j_1-1}\left(\alpha\left(\mathrm{e}^{\omega_{N-1}^{\frac{1}{N-1}}|u|^{N'}}-1\right)\right)\right)\mathrm{d}x<+\infty, \tag{1.19}$$

and if

$$\alpha>N\exp\left(\frac{\omega_{N-1}^{\frac{N}{q}-1}}{N-1}\left(\int_0^{+\infty}\frac{\mathrm{e}^{(q-N)t}}{(1+t)^q}\,\mathrm{d}t\right)^{\frac{N}{q}}\right),$$

then

$$\sup_{u\in E_{q,1},\ \|u\|_{E_{q,1}}\leqslant 1}\int_{\mathbb{R}^N}\left(\mathrm{e}^{\alpha\left(\mathrm{e}^{\omega_{N-1}^{\frac{1}{N-1}}|u|^{N'}}-1\right)}-S_{j_1-1}\left(\alpha\left(\mathrm{e}^{\omega_{N-1}^{\frac{1}{N-1}}|u|^{N'}}-1\right)\right)\right)\mathrm{d}x=+\infty. \tag{1.20}$$

Note that the Trudinger–Moser inequalities proved in theorems 1.1 and 1.2 are not necessarily sharp. However, as we will see, this sharpness can be recovered when we consider another norm on the space $E_{q,\beta}$ equivalent to $\|\cdot\|_{E_{q,\beta}}$ and given by:

$$\|u\|_{q,\beta}=|\nabla u|_{L^N_{w_\beta}(\mathbb{R}^N)}+|\nabla u|_{L^q(\mathbb{R}^N)}=\left(\int_{\mathbb{R}^N}|\nabla u|^N w_\beta(x)\,\mathrm{d}x\right)^{\frac{1}{N}}+\left(\int_{\mathbb{R}^N}|\nabla u|^q\,\mathrm{d}x\right)^{\frac{1}{q}},\ \text{for } 0<\beta<1,$$

and

$$\|u\|_{q,1}=\left(\int_{\mathbb{R}^N}|\nabla u|^N w_1(x)\,\mathrm{d}x\right)^{\frac{1}{N}}+\left(\int_{|x|\geqslant 1}|\nabla u|^q\,\mathrm{d}x\right)^{\frac{1}{q}},\ \text{for } \beta=1.$$

The equivalence of this norm and $\|\cdot\|_{E_{q,\beta}}$ is proved below (see remark 1.8). Using the new norm $\|\cdot\|_{q,\beta}$, we can establish the following sharp Trudinger–Moser inequalities.

THEOREM 1.3. *Let* $0<\beta<1$ *and* w_β *be defined by* (1.11). *We have,*

$$\sup_{u\in E_{q,\beta},\ \|u\|_{q,\beta}\leqslant 1}\int_{\mathbb{R}^N}\left(\mathrm{e}^{\alpha|u|^{\frac{N'}{1-\beta}}}-S_{j_\beta-1}\left(\alpha|u|^{\frac{N'}{1-\beta}}\right)\right)\mathrm{d}x<+\infty\Leftrightarrow\alpha\leqslant\alpha_{N,\beta}. \tag{1.21}$$

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THEOREM 1.4. *Let w_1 be defined by* (1.13). *We have,*

$$\sup_{u\in E_{q,1},\ \|u\|_{q,1}\leqslant 1}\int_{\mathbb{R}^N}\left(\mathrm{e}^{\alpha\left(\mathrm{e}^{\omega_{N-1}^{\frac{1}{N-1}}|u|^{N'}}-1\right)}-S_{j_1-1}\left(\alpha\left(\mathrm{e}^{\omega_{N-1}^{\frac{1}{N-1}}|u|^{N'}}-1\right)\right)\right)\mathrm{d}x$$
$$<+\infty \Leftrightarrow \alpha\leqslant N. \tag{1.22}$$

Comparing to previously cited works, there are many novelty aspects in the present work that we have to highlight. First, we are considering the case when the weight w_β, $0<\beta<1$, vanishes at $x\in\mathbb{R}^N$ such that $|x|=1$. In fact, as it was mentioned above, we did not consider such a case and we preferred take $w_\beta(x)=(1-\log|x|)^{\beta(N-1)}$, $0<|x|<1$ in such a way that $\inf_{x\in\mathbb{R}^N} w_\beta(x)>0$. The second aspect of novelty consists on taking only the gradient of the function to define the norms $\|\cdot\|_{E_{q,\beta}}$ and $\|\cdot\|_{q,\beta}$. The combination of the norms of the gradient in two different Lebesgue spaces which are $L^N_{w_\beta}(\mathbb{R}^N)$ and $L^q(\mathbb{R}^N)$ has a real impact on the obtained inequalities. At this stage, we have to mention the work [**19**] in which the authors proved that

$$\sup_{u\in E^{N,q},\|u\|_{E^{N,q}}\leqslant 1}\int_{\mathbb{R}^N}\Phi_{\alpha,j_0}(u)\,\mathrm{d}x<+\infty,\ \forall\ 0<\alpha<\alpha_N=N\omega_{N-1}^{\frac{1}{N-1}}, \tag{1.23}$$

and

$$\sup_{u\in E^{N,q},\|u\|_{E^{N,q}}\leqslant 1}\int_{\mathbb{R}^N}\Phi_{\alpha,j_0}(u)\,\mathrm{d}x=+\infty,\ \forall\ \alpha>\alpha_N, \tag{1.24}$$

where $1<q<N$, $E^{N,q}$ is defined as the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\|_{E^{N,q}}=\left(\int_{\mathbb{R}^N}|\nabla u|^N\ \mathrm{d}x+\left(\int_{\mathbb{R}^N}|\nabla u|^q\ \mathrm{d}x\right)^{\frac{N}{q}}\right)^{\frac{1}{N}},$$

and

$$\Phi_{\alpha,j_0}(u)=\mathrm{e}^{\alpha|u|^{N'}}-\sum_{j=0}^{j_0-1}\frac{\alpha^j}{j!}|u|^{jN'},\ j_0=\inf\left\{j\in\mathbb{N},\ j\geqslant\frac{q^*}{N'}\right\}.$$

So, we can clearly note that this result can be recovered when we take $\beta=0$ in theorem 1.1. In other words, our present work can be partially seen as a generalization of [**19**] (when we choose $\chi\equiv 1$). But in contrast with [**19**], we are able here to establish the sharpness of the inequality by introducing the new norm $\|\cdot\|_{q,\beta}$. Obviously, this result of sharpness also holds for (1.23) provided that we pass from the norm $\|\cdot\|_{E^{N,q}}$ to the new one given by

$$\|u\|=\left(\int_{\mathbb{R}^N}|\nabla u|^N\ \mathrm{d}x\right)^{\frac{1}{N}}+\left(\int_{\mathbb{R}^N}|\nabla u|^q\ \mathrm{d}x\right)^{\frac{1}{q}},\ u\in E^{N,q}.$$

This leads us to the next point of novelty in the present work. It mainly consists on the sharpness of the inequalities (1.21) and (1.22). Actually, we have to highlight that such a sharp inequalities have been obtained in [9] for the case when $0<\beta<1$ and its singular generalization proved in [11] for the case when $0<\beta<1$ or even when $\beta=1$ (i.e., for the doubly exponential growth case). In our present work and due to the existence of the term $|\nabla u|_{L^q(\mathbb{R}^N)}$, we are able to guarantee the same sharpness property of the inequalities for the both cases $0<\beta<1$ and also $\beta=1$. Finally, we establish an improvement of (1.15), (1.19), (1.21) and (1.22) for weakly convergent sequences in $E_{q,\beta}$, $0<\beta\leqslant 1$ with constants larger than those found in (1.15), (1.19), (1.21), and (1.22). These results are completely new. Moreover, inequality (1.25) proved below is also an improvement of the inequality proved by J.L. Carvalho, G.M. Figueiredo, M.F. Furtado, and E. Medeiros in [19]. The proof of these new results is mainly based on some version of the Concentration-Compactness principle due to P.L. Lions in [30].

THEOREM 1.5.

1. *Assume that* $0<\beta<1$. *Let* $(u_n)_n\subset E_{q,\beta}$ *and* $u\in E_{q,\beta}\backslash\{0\}$ *be such that* $\|u_n\|_{E_{q,\beta}}=1$ *and* $u_n\rightharpoonup u$ *weakly in* $E_{q,\beta}$. *Then,*

$$\sup_n\int_{\mathbb{R}^N}\left(\mathrm{e}^{p\alpha_{N,\beta}|u_n|^{\frac{N'}{1-\beta}}}-S_{j_\beta-1}\left(p\alpha_{N,\beta}\,|u_n|^{\frac{N'}{1-\beta}}\right)\right)\mathrm{d}x$$
$$<+\infty,\ \forall\ 0<p<P_{N,\beta}(u), \tag{1.25}$$

where

$$P_{N,\beta}(u)=\begin{cases}\left(\dfrac{1}{1-\|u\|^N_{E_{q,\beta}}}\right)^{\dfrac{1}{(1-\beta)(N-1)}}, & \text{if } \|u\|_{E_{q,\beta}}<1,\\ +\infty, & \text{if } \|u\|_{E_{q,\beta}}=1.\end{cases}$$

Moreover, there exist a sequence $(u_n)_n\subset E_{q,\beta}$ *and a function* $u\in E_{q,\beta}\backslash\{0\}$ *satisfying* $\|u_n\|_{E_{q,\beta}}=1$ *and* $u_n\rightharpoonup u$ *weakly in* $E_{q,\beta}$ *such that*

$$\sup_n\int_{\mathbb{R}^N}\left(\mathrm{e}^{p\alpha_{N,\beta}|u_n|^{\frac{N'}{1-\beta}}}-S_{j_\beta-1}\left(p\alpha_{N,\beta}\,|u_n|^{\frac{N'}{1-\beta}}\right)\right)\mathrm{d}x=+\infty,\ \forall\ p>P_{N,\beta}(u). \tag{1.26}$$

2. *Let* $(u_n)_n\subset E_{q,1}$ *and* $u\in E_{q,1}\backslash\{0\}$ *be such that* $\|u_n\|_{E_{q,1}}=1$ *and* $u_n\rightharpoonup u$ *weakly in* $E_{q,1}$. *Then, for all* $0<p<P_{N,1}(u)$, *we have*

$$\sup_n\int_{\mathbb{R}^N}\left(\mathrm{e}^{N\left(\mathrm{e}^{\omega_{N-1}^{\frac{1}{N-1}}p|u_n|^{N'}}-1\right)}-S_{j_1-1}\left(N\left(\mathrm{e}^{\omega_{N-1}^{\frac{1}{N-1}}p|u_n|^{N'}}-1\right)\right)\right)\mathrm{d}x<+\infty, \tag{1.27}$$

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where

$$P_{N,1}(u)=\begin{cases}\left(\dfrac{1}{1-\|u\|_{E_{q,1}}^{N}}\right)^{\dfrac{1}{N-1}}, & \text{if } \|u\|_{E_{q,1}}<1,\\ +\infty, & \text{if } \|u\|_{E_{q,1}}=1.\end{cases}$$

Moreover, there exist a sequence $(u_n)_n\subset E_{q,1}$ *and a function* $u\in E_{q,1}\backslash\{0\}$ *satisfying* $\|u_n\|_{E_{q,1}}=1$ *and* $u_n\rightharpoonup u$ *weakly in* $E_{q,1}$ *such that, for all* $\alpha>0$ *and* $p>P_{N,1}(u)$, *we have*

$$\sup_n\int_{\mathbb{R}^N}\left(\mathrm{e}^{\alpha\left(\mathrm{e}^{\omega_{N-1}^{\frac{1}{N-1}}p|u_n|^{N'}}-1\right)}-S_{j_1-1}\left(\alpha\left(\mathrm{e}^{\omega_{N-1}^{\frac{1}{N-1}}p|u_n|^{N'}}-1\right)\right)\right)\mathrm{d}x=+\infty. \tag{1.28}$$

The next result concerns the norm $\|\cdot\|_{q,\beta}$, $0<\beta\leqslant 1$, and it consists in some improvements of the inequalities (1.21) and (1.22). At first glance and in a natural way, the reader is expecting to find that these improvements can be obtained by a simple change of the norm $\|\cdot\|_{E_{q,\beta}}$, $0<\beta\leqslant 1$ which appears in the expression of $P_{N,\beta}(u)$ in theorem 1.5 by $\|\cdot\|_{q,\beta}$. But, due to the difference of the 'geometric structure' of the two norms, the situation is less easier than it seems.

THEOREM 1.6.

1. *Assume that* $0<\beta<1$. *Let* $(u_n)_n\subset E_{q,\beta}$ *and* $u\in E_{q,\beta}\backslash\{0\}$ *be such that* $\|u_n\|_{q,\beta}=1$ *and* $u_n\rightharpoonup u$ *weakly in* $E_{q,\beta}$. *Then,*

$$\sup_n\int_{\mathbb{R}^N}\left(\mathrm{e}^{p\alpha_{N,\beta}|u_n|^{\frac{N'}{1-\beta}}}-S_{j_\beta-1}\left(p\alpha_{N,\beta}\,|u_n|^{\frac{N'}{1-\beta}}\right)\right)\mathrm{d}x<+\infty,\ \forall\ 0<p<P_{N,\beta}(u), \tag{1.29}$$

where

$$P_{N,\beta}(u)=\begin{cases}\left(\dfrac{1}{1-\|u\|_{q,\beta}^{q}}\right)^{\dfrac{N'}{q(1-\beta)}}, & \text{if } \|u\|_{q,\beta}<1,\\ +\infty, & \text{if } \|u\|_{q,\beta}=1.\end{cases}$$

Moreover, there exist a sequence $(u_n)_n\subset E_{q,\beta}$ *and a function* $u\in E_{q,\beta}\backslash\{0\}$ *satisfying* $\|u_n\|_{q,\beta}=1$ *and* $u_n\rightharpoonup u$ *weakly in* $E_{q,\beta}$ *such that*

$$\sup_n\int_{\mathbb{R}^N}\left(\mathrm{e}^{p\alpha_{N,\beta}|u_n|^{\frac{N'}{1-\beta}}}-S_{j_\beta-1}\left(p\alpha_{N,\beta}\,|u_n|^{\frac{N'}{1-\beta}}\right)\right)\mathrm{d}x=+\infty,\ \forall\ p>P_{N,\beta}(u). \tag{1.30}$$

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2. • *Let* $(u_n)_n \subset E_{q,1}$ *and* $u \in E_{q,1}\backslash\{0\}$ *be such that* $\|u_n\|_{q,1} = 1$ *and* $u_n \rightharpoonup u$*weakly in* $E_{q,1}$*. Then, for all* $0 < p < P_{N,1}(u)$*, we have*

$$\sup_n \int_{\mathbb{R}^N} \left(\mathrm{e}^{N\left(\mathrm{e}^{\omega_{N-1}^{\frac{1}{N-1}} p|u_n|^{N'}}-1\right)} - S_{j_1-1}\left(N\left(\mathrm{e}^{\omega_{N-1}^{\frac{1}{N-1}} p|u_n|^{N'}}-1\right)\right)\right) \mathrm{d}x < +\infty, \tag{1.31}$$

where

$$P_{N,1}(u) = \begin{cases} \left(\dfrac{1}{1-\|u\|_{q,1}^{q}}\right)^{\dfrac{N'}{q}}, & \text{if } \|u\|_{q,1} < 1, \\ +\infty, & \text{if } \|u\|_{q,1} = 1. \end{cases}$$

Moreover, there exist a sequence $(u_n)_n \subset E_{q,1}$ *and a function* $u \in E_{q,1}\backslash\{0\}$ *satisfying* $\|u_n\|_{q,1} = 1$ *and* $u_n \rightharpoonup u$ *weakly in* $E_{q,1}$ *such that, for all* $\alpha > 0$ *and* $p > P_{N,1}(u)$*, we have*

$$\sup_n \int_{\mathbb{R}^N} \left(\mathrm{e}^{\alpha\left(\mathrm{e}^{\omega_{N-1}^{\frac{1}{N-1}} p|u_n|^{N'}}-1\right)} - S_{j_1-1}\left(\alpha\left(\mathrm{e}^{\omega_{N-1}^{\frac{1}{N-1}} p|u_n|^{N'}}-1\right)\right)\right) \mathrm{d}x = +\infty. \tag{1.32}$$

REMARK 1.7. Obviously, all the results obtained for the weight w_β given by (1.11) hold true when we take

$$w_\beta(x) = \begin{cases} (1-\log(|x|))^{\beta(N-1)} & \text{if } |x| < 1, \\ \chi(|x|) & \text{if } |x| \geqslant 1, \end{cases}$$

where $\chi : [1, +\infty[\to]0, +\infty[$ is a continuous function such that $\chi(1) = 1$ and $w_\beta \in A_N$.

In the last part of this work, we apply the Trudinger–Moser inequalities established in theorem 1.2 to study some elliptic quasilinear equation defined in $\mathbb{R}^N$ and containing a nonlinearities having a doubly exponential growth at infinity. More precisely, we prove the existence of at least one nontrivial solution to the equation

$$-\mathrm{div}\left(w_1(x)\,|\nabla u|^{N-2}\,\nabla u\right) - \Delta_q u = f(u), \text{ in } \mathbb{R}^N, \; N \geqslant 2,$$

where $f : \mathbb{R} \to \mathbb{R}$ is a continuous function enjoying a doubly exponential growth at infinity governed by the inequality (1.22). In the mathematical literature, the first equation involving an operator with non-standard growth of the type (p, N)−Laplacian with $0 < p < N$ appeared in [**39**] where the problem was studied in a bounded domain and where the nonlinear term has an exponential growth governed by the classical Trudinger–Moser inequality. In [**39**], the authors obtained an existence result via a suitable minimax argument. This work was followed by [**24**] where the Nehari manifold approach has been used to obtain an existence result always by assuming the boundedness of the domain. For equations in the entire space, we can quote the following recent works [19, 20, 25] which deal with exponential growth governed by classical non-weighted Trudinger–Moser inequality.

REMARK 1.8. We can easily show that, for $0<\beta\leqslant 1$, the norms $\|\cdot\|_{q,\beta}$ and $\|\cdot\|_{E_{q,\beta}}$ are equivalent. The case when $0<\beta<1$ is rather evident, we only prove the equivalence of the norms when $\beta=1$. For that aim, let $u\in E_{q,1}$. We have

$$\int_{|x|<1}|\nabla u|^q\,\mathrm{d}x=\int_{|x|<1}|\nabla u|^q\,w_1^{\frac{q}{N}}w_1^{-\frac{q}{N}}\,\mathrm{d}x\leqslant\left(\int_{|x|<1}|\nabla u|^N\,w_1(x)\,\mathrm{d}x\right)^{\frac{q}{N}}\left(\int_{|x|<1}w_1^{-\frac{q}{N-q}}\,\mathrm{d}x\right)^{\frac{N-q}{N}}.$$

Since $\displaystyle\int_0^1(1-\log r)^{-q(N-1)/N-q}r^{N-1}\,\mathrm{d}r<+\infty$, then $\displaystyle\int_{|x|<1}w_1^{-q/N-q}\,\mathrm{d}x<+\infty$. Consequently, there exists a positive constant M_0 such that

$$\left(\int_{|x|<1}|\nabla u|^q\,\mathrm{d}x\right)^{\frac{1}{q}}\leqslant M_0\left(\int_{|x|<1}|\nabla u|^N\,w_1\,\mathrm{d}x\right)^{\frac{1}{N}}.$$

Thus,

$$\left(\int_{|x|<1}|\nabla u|^q\,\mathrm{d}x\right)^{\frac{1}{q}}+\left(\int_{|x|\geqslant 1}|\nabla u|^q\,\mathrm{d}x\right)^{\frac{1}{q}}\leqslant(1+M_0)\,\|u\|_{q,1}\,. \tag{1.33}$$

Now, using the following elementary inequality,

$$(a+b)^\alpha\leqslant a^\alpha+b^\alpha,\ \forall\ a,b\geqslant 0,\ \forall\ 0<\alpha<1,$$

we infer from (1.33) that

$$\begin{aligned}\left(\int_{\mathbb{R}^N}|\nabla u|^q\,\mathrm{d}x\right)^{\frac{1}{q}}&=\left(\int_{|x|<1}|\nabla u|^q\,\mathrm{d}x+\int_{|x|\geqslant 1}|\nabla u|^q\,\mathrm{d}x\right)^{\frac{1}{q}}\\&\leqslant\left(\int_{|x|<1}|\nabla u|^q\,\mathrm{d}x\right)^{\frac{1}{q}}+\left(\int_{|x|\geqslant 1}|\nabla u|^q\,\mathrm{d}x\right)^{\frac{1}{q}}\\&\leqslant(1+M_0)\,\|u\|_{q,1}\,.\end{aligned}$$

Hence,

$$\left(\int_{\mathbb{R}^N}|\nabla u|^N\,w_1\,\mathrm{d}x\right)^{\frac{1}{N}}+\left(\int_{\mathbb{R}^N}|\nabla u|^q\,\mathrm{d}x\right)^{\frac{1}{q}}\leqslant(2+M_0)\,\|u\|_{q,1}\,. \tag{1.34}$$

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Now, having in mind that

$$\|u\|_{E_{q,1}} = \left(\int_{\mathbb{R}^N} |\nabla u|^N w_1 \, \mathrm{d}x + \left(\int_{\mathbb{R}^N} |\nabla u|^q \, dx\right)^{\frac{N}{q}}\right)^{\frac{1}{N}} \leqslant \left(\int_{\mathbb{R}^N} |\nabla u|^N w_1 \, \mathrm{d}x\right)^{\frac{1}{N}} + \left(\int_{\mathbb{R}^N} |\nabla u|^q \, \mathrm{d}x\right)^{\frac{1}{q}},$$

by (1.34) we obtain

$$\|u\|_{E_{q,1}} \leqslant (2 + M_0) \|u\|_{q,1}. \tag{1.35}$$

On the other hand, taking into account that the function $x \longmapsto x^{\frac{1}{N}}$ is concave on $[0, +\infty[$, we get

$$a^{\frac{1}{N}} + b^{\frac{1}{N}} \leqslant 2^{1-\frac{1}{N}} (a + b)^{\frac{1}{N}}, \ \forall \ a, b \geqslant 0.$$

It follows that,

$$\|u\|_{E_{q,1}} \geqslant 2^{\frac{1}{N}-1} \left(\left(\int_{\mathbb{R}^N} |\nabla u|^N w_1 \, \mathrm{d}x\right)^{\frac{1}{N}} + \left(\int_{\mathbb{R}^N} |\nabla u|^q \, \mathrm{d}x\right)^{\frac{1}{q}}\right) \geqslant 2^{\frac{1}{N}-1} \|u\|_{q,1}.$$

Combining that last inequality with (1.35), we deduce that

$$2^{\frac{1}{N}-1} \|u\|_{q,1} \leqslant \|u\|_{E_{q,1}} \leqslant (2 + M_0) \|u\|_{q,1}.$$

REMARK 1.9. A pertinent question is why when $\beta = 1$, we change the form of the norm $\| \cdot \|_{q,1}$ by taking only the integral over the set $\left\{x \in \mathbb{R}^N, \ |x| \geqslant 1\right\}$. In fact, one can naturally expect that this last norm takes the form

$$\|u\|_{q,1}^{(1)} = \left(\int_{\mathbb{R}^N} |\nabla u|^N w_1 \, \mathrm{d}x\right)^{\frac{1}{N}} + \left(\int_{\mathbb{R}^N} |\nabla u|^q \, \mathrm{d}x\right)^{\frac{1}{q}}.$$

Taking that last norm, we can easily adapt the proof of theorem 1.4 to prove that, if $\alpha \leqslant N$, then

$$\sup_{u \in E_{q,1}, \ \|u\|_{q,1}^{(1)} \leqslant 1} \int_{\mathbb{R}^N} \left(\mathrm{e}^{\alpha\left(\mathrm{e}^{\omega_{N-1}^{\frac{1}{N-1}} |u|^{N'}} - 1\right)} - S_{j_1 - 1}\left(\alpha \left(\mathrm{e}^{\omega_{N-1}^{\frac{1}{N-1}} |u|^{N'}} - 1\right)\right)\right) \mathrm{d}x < +\infty.$$

The problem lies in the construction of a sequence (if there exists) $(u_k)_k \subset E_{q,1}$ such that $\|u_k\|_{q,1}^{(1)} \leqslant 1$ and

$$\int_{\mathbb{R}^N} \left(\mathrm{e}^{\alpha\left(\mathrm{e}^{\omega_{N-1}^{\frac{1}{N-1}} |u_k|^{N'}} - 1\right)} - S_{j_1 - 1}\left(\alpha \left(\mathrm{e}^{\omega_{N-1}^{\frac{1}{N-1}} |u_k|^{N'}} - 1\right)\right)\right) \mathrm{d}x \to +\infty, \ k \to +\infty.$$

We do not know the existence of such a sequence.

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2. Proof of theorem 1.1

We start by proving (1.14). For that aim, fix $\alpha > 0$ and $u \in E_{q,\beta}$. We have

$$\begin{aligned}\int_{\mathbb{R}^N} \left(\mathrm{e}^{\alpha |u|^{\frac{N'}{1-\beta}}} - S_{j_\beta - 1}\left(\alpha |u|^{\frac{N'}{1-\beta}}\right)\right) \mathrm{d}x \\ = \int_{|x|\geqslant 1} \left(\mathrm{e}^{\alpha |u|^{\frac{N'}{1-\beta}}} - S_{j_\beta - 1}\left(\alpha |u|^{\frac{N'}{1-\beta}}\right)\right) \mathrm{d}x \\ + \int_{|x|<1} \left(\mathrm{e}^{\alpha |u|^{\frac{N'}{1-\beta}}} - S_{j_\beta - 1}\left(\alpha |u|^{\frac{N'}{1-\beta}}\right)\right) \mathrm{d}x. \end{aligned} \tag{2.1}$$

On the one hand, we have

$$\int_{|x|\geqslant 1} \left(\mathrm{e}^{\alpha |u|^{\frac{N'}{1-\beta}}} - S_{j_\beta - 1}\left(\alpha |u|^{\frac{N'}{1-\beta}}\right)\right) \mathrm{d}x = \sum_{j=j_\beta}^{+\infty} \frac{\alpha^j}{j!} \int_{|x|\geqslant 1} |u|^{\frac{jN'}{1-\beta}} \mathrm{d}x. \tag{2.2}$$

Since u belongs to $D^{1,q}_r(\mathbb{R}^N)$, then by the radial lemma (see [**37**, lemma 1]), we know that

$$|u(x)| \leqslant C_{q,N} |x|^{-\frac{N-q}{q}} |\nabla u|_{L^q(\mathbb{R}^N)}, \ \forall\, x \neq 0, \tag{2.3}$$

where $C_{q,N}$ is given by (1.17). For $j \geqslant j_\beta$, we have $\frac{jN'}{1-\beta} \geqslant q^*$. By (2.3), it yields

$$|u(x)|^{\frac{jN'}{1-\beta}-q^*} \leqslant C_{q,N}{}^{\frac{jN'}{1-\beta}-q^*} |\nabla u|^{\frac{jN'}{1-\beta}-q^*}_{L^q(\mathbb{R}^N)}, \ \forall\, x \in \mathbb{R}^N,\ |x| \geqslant 1. \tag{2.4}$$

By (2.4), we infer

$$\begin{aligned}\sum_{j=j_\beta}^{+\infty} \frac{\alpha^j}{j!} \int_{|x|\geqslant 1} |u|^{\frac{jN'}{1-\beta}} \mathrm{d}x &\leqslant \sum_{j=j_\beta}^{+\infty} \frac{\alpha^j}{j!} C_{q,N}{}^{\frac{jN'}{1-\beta}-q^*} |\nabla u|^{\frac{jN'}{1-\beta}-q^*}_{L^q(\mathbb{R}^N)} \int_{|x|\geqslant 1} |u|^{q^*} \mathrm{d}x \\ &\leqslant \sum_{j=j_\beta}^{+\infty} \frac{\alpha^j}{j!} C_{q,N}{}^{\frac{jN'}{1-\beta}-q^*} |\nabla u|^{\frac{jN'}{1-\beta}-q^*}_{L^q(\mathbb{R}^N)} |u|^{q^*}_{L^{q^*}(\mathbb{R}^N)} \\ &\leqslant C' \,\mathrm{e}^{\alpha C_{q,N}{}^{\frac{N'}{1-\beta}} |\nabla u|^{\frac{N'}{1-\beta}}_{L^q(\mathbb{R}^N)}}, \end{aligned} \tag{2.5}$$

where we used the continuous embedding $D^{1,q}_r(\mathbb{R}^N) \hookrightarrow L^{q^*}(\mathbb{R}^N)$. Putting (2.5) in (2.2), we obtain

$$\int_{|x|\geqslant 1} \left(\mathrm{e}^{\alpha |u|^{\frac{N'}{1-\beta}}} - S_{j_\beta - 1}\left(\alpha |u|^{\frac{N'}{1-\beta}}\right)\right) \mathrm{d}x \leqslant C' \,\mathrm{e}^{\alpha C_{q,N}{}^{\frac{N'}{1-\beta}} \|u\|^{\frac{N'}{1-\beta}}_{E_{q,\beta}}}. \tag{2.6}$$

Now, in order to estimate the second integral in (2.1), set

$$v(x) = \begin{cases} u(x) - u(e_1), & 0 \leqslant |x| < 1, \\ 0, & |x| \geqslant 1, \end{cases} \tag{2.7}$$

where $e_1 = (1, 0, \cdots, 0)$ is the first vector in the canonical basis of $\mathbb{R}^N$. Clearly, $v \in W^{1,N}_{0,rad}(\mathcal{B}, \sigma_\beta)$, with $\sigma_\beta(x) = (-\log|x|)^{\beta(N-1)},\ x \in \mathcal{B}$. An elementary calculus

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gives the following inequality: for all $\epsilon > 0$, we have

$$(a+b)^{\frac{N'}{1-\beta}} \leqslant (1+\epsilon)a^{\frac{N'}{1-\beta}} + \frac{1+\epsilon}{\left((1+\epsilon)^{\frac{1-\beta}{N'-1+\beta}}-1\right)^{\frac{N'-1+\beta}{1-\beta}}} b^{\frac{N'}{1-\beta}}, \ \forall\, a,b \geqslant 0. \tag{2.8}$$

Fix $0 < \epsilon < 1$. By (2.8), we get

$$\begin{aligned}
&\int_{|x|<1} \left(\mathrm{e}^{\alpha|u|^{\frac{N'}{1-\beta}}} - S_{j_\beta-1}\left(\alpha\,|u|^{\frac{N'}{1-\beta}}\right)\right)\mathrm{d}x \\
&\leqslant \int_{|x|<1} \mathrm{e}^{\alpha|u|^{\frac{N'}{1-\beta}}}\,\mathrm{d}x \\
&\leqslant \int_{|x|<1} \mathrm{e}^{\alpha(|v|+|u(e_1)|)^{\frac{N'}{1-\beta}}}\,\mathrm{d}x \\
&\leqslant \exp\left(\frac{\alpha(1+\epsilon)}{\left((1+\epsilon)^{\frac{1-\beta}{N'-1+\beta}}-1\right)^{\frac{N'-1+\beta}{1-\beta}}}\,|u(e_1)|^{\frac{N'}{1-\beta}}\right)\int_{|x|<1} \mathrm{e}^{\alpha(1+\epsilon)|v|^{\frac{N'}{1-\beta}}}\,\mathrm{d}x.
\end{aligned} \tag{2.9}$$

By (1.1), we know that

$$\int_{|x|<1} \mathrm{e}^{\alpha(1+\epsilon)|v|^{\frac{N'}{1-\beta}}}\,\mathrm{d}x < +\infty,$$

and by consequence (2.9) leads to

$$\int_{|x|<1} \left(\mathrm{e}^{\alpha|u|^{\frac{N'}{1-\beta}}} - S_{j_\beta-1}\left(\alpha\,|u|^{\frac{N'}{1-\beta}}\right)\right)\mathrm{d}x < +\infty. \tag{2.10}$$

Combining (2.10) and (2.6), we deduce that (1.14) holds.

Now, we prove (1.15). By (2.6), it yields

$$\sup_{u\in E_{q,\beta},\ \|u\|_{E_{q,\beta}}\leqslant 1} \int_{|x|\geqslant 1} \left(e^{\alpha_{N,\beta}|u|^{\frac{N'}{1-\beta}}} - S_{j_\beta-1}\left(\alpha_{N,\beta}\,|u|^{\frac{N'}{1-\beta}}\right)\right)\mathrm{d}x \leqslant C'\,\mathrm{e}^{\alpha_{N,\beta}C_{q,N}{}^{\frac{N'}{1-\beta}}}. \tag{2.11}$$

Next, let $\alpha < \alpha_{N,\beta}$. Clearly, there exists $\epsilon > 0$ such that $\alpha(1+\epsilon) < \alpha_{N,\beta}$. Let $u \in E_{q,\beta}$ be such that $\|u\|_{E_{q,\beta}} \leqslant 1$. Having in mind that v defined by (2.7) belongs to $W^{1,N}_{0,rad}(\mathcal{B},\sigma_\beta)$ and

$$\int_{|x|<1} |\nabla v|^N \,\sigma_\beta(x)\,\mathrm{d}x = \int_{|x|<1} |\nabla u|^N\, w_\beta(x)\,\mathrm{d}x \leqslant 1,$$

then, by the virtue of (1.2), we infer that there exists a positive constant $C_\beta > 0$ such that

$$\int_{|x|<1} \mathrm{e}^{\alpha(1+\epsilon)|v|^{\frac{N'}{1-\beta}}}\,\mathrm{d}x \leqslant \int_{|x|<1} \mathrm{e}^{\alpha_{N,\beta}|v|^{\frac{N'}{1-\beta}}}\,\mathrm{d}x \leqslant C_\beta. \tag{2.12}$$

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Moreover, by (2.3), we know that $|u(e_1)| \leqslant C_{q,N}$. Hence, by (2.9) and (2.12) we obtain

$$\int_{|x|<1} \left(\mathrm{e}^{\alpha|u|^{\frac{N'}{1-\beta}}} - S_{j_\beta - 1}\left(\alpha |u|^{\frac{N'}{1-\beta}}\right)\right) \mathrm{d}x \leqslant \exp\left(\frac{\alpha_{N,\beta} C_{q,N}^{\frac{N'}{1-\beta}}}{\left((1+\epsilon)^{\frac{1-\beta}{N'-1+\beta}} - 1\right)^{\frac{N'-1+\beta}{1-\beta}}} \right) C_\beta. \tag{2.13}$$

Combining (2.13) and (2.11), we deduce that (1.15) holds.

For $u \in E_{q,\beta}$, set $\psi(t) = \omega_{N-1}^{\frac{1}{N}} u(x)$ with $|x| = \mathrm{e}^{-t}$, $t \in \mathbb{R}$. A direct computation gives:

$$\int_{|x|<1} |\nabla u|^N w_\beta(x)\, \mathrm{d}x = \int_0^{+\infty} t^{\beta(N-1)} |\psi'(t)|^N \, \mathrm{d}t,$$

$$\int_{|x|\geqslant 1} w_\beta(x) |\nabla u|^N \, \mathrm{d}x = \int_{-\infty}^0 \chi(\mathrm{e}^{-t}) |\psi'(t)|^N \, \mathrm{d}t,$$

$$\int_{\mathbb{R}^N} |\nabla u|^q \, \mathrm{d}x = \omega_{N-1}^{1-\frac{q}{N}} \int_{-\infty}^{+\infty} |\psi'(t)|^q \, \mathrm{e}^{(q-N)t} \, \mathrm{d}t,$$

and

$$\begin{aligned} &\int_{\mathbb{R}^N} \left(\mathrm{e}^{\alpha|u|^{\frac{N'}{1-\beta}}} - S_{j_\beta-1}\left(\alpha |u|^{\frac{N'}{1-\beta}}\right)\right) \mathrm{d}x \\ &= \omega_{N-1} \int_{-\infty}^{+\infty} \left(\mathrm{e}^{\alpha \omega_{N-1}^{-\frac{1}{(N-1)(1-\beta)}} |\psi(t)|^{\frac{N'}{1-\beta}}} \right. \\ &\qquad \left. - S_{j_\beta-1}\left(\alpha \omega_{N-1}^{-\frac{1}{(N-1)(1-\beta)}} |\psi(t)|^{\frac{N'}{1-\beta}} \right) \right) \mathrm{e}^{-Nt} \, \mathrm{d}t. \end{aligned}$$

Let $\gamma > 0$ to be fixed later. Consider the sequence of test functions:

$$\psi_k(t) = \begin{cases} k^{(\beta-1)(\gamma+\frac{1}{N})+\gamma} t, & 0 \leqslant t \leqslant k^{-\gamma}, \\ k^{\frac{\beta-1}{N}} t^{1-\beta}, & k^{-\gamma} \leqslant t \leqslant k, \\ k^{\frac{1-\beta}{N'}}, & t \geqslant k, \\ 0, & t \leqslant 0. \end{cases}$$

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For $k \geqslant 1$, define $u_k \in E_{q,\beta}$ by $\psi_k(t) = \omega_{N-1}^{\frac{1}{N}} u_k(x),\ |x| = \mathrm{e}^{-t},\ t \in \mathbb{R}$. We have

$$\begin{aligned} \int_{\mathbb{R}^N} |\nabla u_k|^N w_\beta(x)\,\mathrm{d}x &= \int_0^k t^{\beta(N-1)} |\psi_k'(t)|^N \,\mathrm{d}t \\ &= \int_{k^{-\gamma}}^k \frac{(1-\beta)^N t^{-\beta}}{k^{1-\beta}}\,\mathrm{d}t + \int_0^{k^{-\gamma}} t^{\beta(N-1)} k^{N((\beta-1)(\gamma+\frac{1}{N})+\gamma)}\,\mathrm{d}t \\ &= (1-\beta)^{N-1} + k^{(\beta-1)(\gamma+1)}\left(\frac{1}{1+\beta(N-1)} - (1-\beta)^{N-1}\right). \end{aligned} \tag{2.14}$$

$$\begin{aligned} \int_{\mathbb{R}^N} |\nabla u_k|^q \,\mathrm{d}x &= \omega_{N-1}^{1-\frac{q}{N}} \frac{k^{q((\beta-1)(\gamma+\frac{1}{N})+\gamma)}}{N-q}\left(1-\mathrm{e}^{(q-N)k^{-\gamma}}\right) \\ &\quad + \omega_{N-1}^{1-\frac{q}{N}} \frac{(1-\beta)^k}{k^{\frac{(1-\beta)q}{N}}} \int_{k^{-\gamma}}^k \frac{\mathrm{e}^{(q-N)t}}{t^{q\beta}}\,\mathrm{d}t. \end{aligned} \tag{2.15}$$

Choosing γ small enough such that $(\beta-1)(\gamma+\frac{1}{N})+\gamma<0$. By this choice, we get

$$\omega_{N-1}^{1-\frac{q}{N}} \frac{k^{q((\beta-1)(\gamma+\frac{1}{N})+\gamma)}}{N-q}\left(1-\mathrm{e}^{(q-N)k^{-\gamma}}\right) \to 0,\ k \to +\infty.$$

On the other hand,

$$\frac{(1-\beta)^k}{k^{\frac{(1-\beta)q}{N}}} \int_{k^{-\gamma}}^k \frac{\mathrm{e}^{(q-N)t}}{t^{q\beta}}\,\mathrm{d}t \leqslant \frac{(1-\beta)^k}{k^{\frac{(1-\beta)q}{N}}} \frac{k^{\gamma q\beta}}{N-q}\left(\mathrm{e}^{(q-N)k^{-\gamma}} - \mathrm{e}^{(q-N)k}\right) \to 0,\ k \to +\infty.$$

Thus, by (2.15), we infer

$$\int_{\mathbb{R}^N} |\nabla u_k|^q \,\mathrm{d}x \to 0,\ k \to +\infty.$$

Now, taking into account that $(\beta-1)(\gamma+1)<0$, it follows from (2.14) that

$$\int_{\mathbb{R}^N} |\nabla u_k|^N w_\beta(x)\,\mathrm{d}x \to (1-\beta)^{N-1},\ k \to +\infty.$$

Hence,

$$\|u_k\|_{E_{q,\beta}} = \left(|\nabla u_k|^N_{L^N_{w_\beta}(\mathbb{R}^N)} + |\nabla u_k|^N_{L^q(\mathbb{R}^N)}\right)^{\frac{1}{N}} \to (1-\beta)^{\frac{1}{N'}},\ k \to +\infty.$$

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Set $\widetilde{u_k} = \frac{u_k}{\|u_k\|_{E_{q,\beta}}}$. For $\alpha > 0$, we have

$$\sup_{u\in E_{q,\beta},\ \|u\|_{E_{q,\beta}}\leqslant 1}\int_{\mathbb{R}^N}\left(\mathrm{e}^{\alpha|u|^{\frac{N'}{1-\beta}}} - S_{j_\beta-1}\left(\alpha\,|u|^{\frac{N'}{1-\beta}}\right)\right)\mathrm{d}x$$

$$\geqslant \int_{\mathbb{R}^N}\left(\mathrm{e}^{\alpha|\widetilde{u_k}|^{\frac{N'}{1-\beta}}} - S_{j_\beta-1}\left(\alpha\,|\widetilde{u_k}|^{\frac{N'}{1-\beta}}\right)\right)\mathrm{d}x$$

$$= \omega_{N-1}\int_{-\infty}^{+\infty}\left(\mathrm{e}^{\alpha\omega_{N-1}^{-\frac{1}{(N-1)(1-\beta)}}\left|\frac{\psi_k(t)}{\|u_k\|_{E_{q,\beta}}}\right|^{\frac{N'}{1-\beta}}}\right.$$

$$\left.-S_{j_\beta-1}\left(\alpha\omega_{N-1}^{-\frac{1}{(N-1)(1-\beta)}}\left|\frac{\psi_k(t)}{\|u_k\|_{E_{q,\beta}}}\right|^{\frac{N'}{1-\beta}}\right)\right)\mathrm{e}^{-Nt}\,\mathrm{d}t$$

$$\geqslant \omega_{N-1}\int_{k}^{+\infty}\left(\mathrm{e}^{\alpha\omega_{N-1}^{-\frac{1}{(N-1)(1-\beta)}}\left|\frac{\psi_k(t)}{\|u_k\|_{E_{q,\beta}}}\right|^{\frac{N'}{1-\beta}}}\right.$$

$$\left.-S_{j_\beta-1}\left(\alpha\omega_{N-1}^{-\frac{1}{(N-1)(1-\beta)}}\left|\frac{\psi_k(t)}{\|u_k\|_{E_{q,\beta}}}\right|^{\frac{N'}{1-\beta}}\right)\right)\mathrm{e}^{-Nt}\,\mathrm{d}t$$

$$= \omega_{N-1}\frac{\mathrm{e}^{-Nk}}{N}\left(\mathrm{e}^{\alpha\frac{\omega_{N-1}^{-\frac{1}{(N-1)(1-\beta)}}k}{\|u_k\|_{E_{q,\beta}}^{\frac{N'}{1-\beta}}}} - S_{j_\beta-1}\left(\alpha\frac{\omega_{N-1}^{-\frac{1}{(N-1)(1-\beta)}}k}{\|u_k\|_{E_{q,\beta}}^{\frac{N'}{1-\beta}}}\right)\right). \tag{2.16}$$

Clearly, for $0 \leqslant j \leqslant j_\beta - 1$, we have

$$\mathrm{e}^{-Nk}k^j\frac{1}{\|u_k\|_{E_{q,\beta}}^{\frac{N'}{1-\beta}}} \to 0,\ \ k\to+\infty.$$

Thus,

$$S_{j_\beta-1}\left(\alpha\frac{\omega_{N-1}^{-\frac{1}{(N-1)(1-\beta)}}k}{\|u_k\|_{E_{q,\beta}}^{\frac{N'}{1-\beta}}}\right)\to 0,\ \ k\to+\infty. \tag{2.17}$$

Moreover,

$$\mathrm{e}^{-Nk}\,\mathrm{e}^{\alpha\frac{\omega_{N-1}^{-\frac{1}{(N-1)(1-\beta)}}k}{\|u_k\|_{E_{q,\beta}}^{\frac{N'}{1-\beta}}}} = \mathrm{e}^{k\left(-N+\alpha\frac{\omega_{N-1}^{-\frac{1}{(N-1)(1-\beta)}}}{\|u_k\|_{E_{q,\beta}}^{\frac{N'}{1-\beta}}}\right)}.$$

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Having in mind that $\|u_k\|_{E_{q,\beta}} \to (1-\beta)^{1/N'}$, then

$$-N+\alpha\frac{\omega_{N-1}^{-\frac{1}{(N-1)(1-\beta)}}}{\|u_k\|_{E_{q,\beta}}^{\frac{N'}{1-\beta}}} \to -N+\alpha\frac{\omega_{N-1}^{-\frac{1}{(N-1)(1-\beta)}}}{(1-\beta)^{\frac{1}{1-\beta}}}.$$

If $\alpha > \alpha_{N,\beta}$, then

$$-N+\alpha\frac{\omega_{N-1}^{-\frac{1}{(N-1)(1-\beta)}}}{(1-\beta)^{\frac{1}{1-\beta}}} > 0.$$

Consequently,

$$\mathrm{e}^{-Nk}\,\mathrm{e}^{\alpha\frac{\omega_{N-1}^{-\frac{1}{(N-1)(1-\beta)}}k}{\|u_k\|_{E_{q,\beta}}^{\frac{N'}{1-\beta}}}} \to +\infty,\ k\to+\infty. \tag{2.18}$$

Combining (2.18) and (2.17), we deduce from (2.16) that

$$\sup_{u\in E_{q,\beta},\ \|u\|_{E_{q,\beta}}\leqslant 1}\int_{\mathbb{R}^N}\left(\mathrm{e}^{\alpha|u|^{\frac{N'}{1-\beta}}} - S_{j_\beta-1}\left(\alpha\,|u|^{\frac{N'}{1-\beta}}\right)\right)\mathrm{d}x = +\infty,\ \forall\ \alpha > \alpha_{N,\beta}.$$

We conclude that (1.16) holds.

3. Proof of theorem 1.2

Let $\alpha > 0$ and $u \in E_{q,1}$. We have

$$\int_{|x|\geqslant 1}\left(\mathrm{e}^{\alpha(\mathrm{e}^{|u|^{N'}}-1)} - S_{j_1-1}\left(\alpha(\mathrm{e}^{|u|^{N'}}-1)\right)\right)\mathrm{d}x = \sum_{j=j_1}^{+\infty}\frac{\alpha^j}{j!}\int_{|x|\geqslant 1}\left(\mathrm{e}^{|u|^{N'}}-1\right)^j\mathrm{d}x. \tag{3.1}$$

Using the monotony of the function defined on $[0,+\infty[$ by $s \longmapsto \frac{e^s-1}{s}$, from (2.3) it yields

$$\mathrm{e}^{|u(x)|^{N'}}-1 \leqslant \frac{\mathrm{e}^{C_{q,N}^{N'}|\nabla u|_{L^q(\mathbb{R}^N)}^{N'}}-1}{C_{q,N}^{N'}\,|\nabla u|_{L^q(\mathbb{R}^N)}^{N'}}\,|u(x)|^{N'},\ \forall\ x\in\mathbb{R}^N,\ |x|\geqslant 1.$$

Observe that $j \geqslant j_1 \Leftrightarrow jN' \geqslant q^*$. Thus, for $j \geqslant j_1$, we have

$$\begin{aligned}\left(\mathrm{e}^{|u(x)|^{N'}}-1\right)^j &\leqslant \left(\frac{\mathrm{e}^{C_{q,N}^{N'}|\nabla u|_{L^q(\mathbb{R}^N)}^{N'}}-1}{C_{q,N}^{N'}\,|\nabla u|_{L^q(\mathbb{R}^N)}^{N'}}\right)^j |u(x)|^{jN'}\\ &\leqslant \left(\frac{\mathrm{e}^{C_{q,N}^{N'}|\nabla u|_{L^q(\mathbb{R}^N)}^{N'}}-1}{C_{q,N}^{N'}\,|\nabla u|_{L^q(\mathbb{R}^N)}^{N'}}\right)^j |u(x)|^{jN'-q^*}\,|u(x)|^{q^*}\end{aligned}$$

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$$\leqslant \left(\frac{\mathrm{e}^{C_{q,N}^{N'}|\nabla u|_{L^q(\mathbb{R}^N)}^{N'}}-1}{C_{q,N}^{N'}\,|\nabla u|_{L^q(\mathbb{R}^N)}^{N'}}\right)^j \left(C_{q,N}\,|\nabla u|_{L^q(\mathbb{R}^N)}\right)^{jN'-q^*} |u(x)|^{q^*},$$
$$\forall\, x\in\mathbb{R}^N,\ |x|\geqslant 1.$$

Putting that last inequality in (3.1), we infer

$$\begin{aligned}
&\int_{|x|\geqslant 1}\left(\mathrm{e}^{\alpha(\mathrm{e}^{|u|^{N'}}-1)}-S_{j_1-1}\left(\alpha(\mathrm{e}^{|u|^{N'}}-1)\right)\right)\mathrm{d}x\\
&\leqslant \sum_{j=j_1}^{+\infty}\frac{\alpha^j}{j!}\left(\frac{\mathrm{e}^{C_{q,N}^{N'}|\nabla u|_{L^q(\mathbb{R}^N)}^{N'}}-1}{C_{q,N}^{N'}\,|\nabla u|_{L^q(\mathbb{R}^N)}^{N'}}\right)^j\left(C_{q,N}\,|\nabla u|_{L^q(\mathbb{R}^N)}\right)^{jN'-q^*}\int_{|x|\geqslant 1}|u|^{q^*}\,\mathrm{d}x\\
&\leqslant C''\sum_{j=j_1}^{+\infty}\frac{\alpha^j}{j!}\left(\mathrm{e}^{C_{q,N}^{N'}|\nabla u|_{L^q(\mathbb{R}^N)}^{N'}}-1\right)^j\\
&\leqslant C''\sum_{j=0}^{+\infty}\frac{\alpha^j}{j!}\left(\mathrm{e}^{C_{q,N}^{N'}\|u\|_{E_{q,1}}^{N'}}-1\right)^j\\
&= C''\exp\left(\alpha\left(\mathrm{e}^{C_{q,N}^{N'}\|u\|_{E_{q,1}}^{N'}}-1\right)\right). \qquad (3.2)
\end{aligned}$$

Next, by (2.8) one can easily deduce the following inequality:

$$(a+b)^{N'}\leqslant(1+\epsilon)a^{N'}+\frac{1+\epsilon}{\left((1+\epsilon)^{\frac{1}{N'-1}}-1\right)^{N'-1}}b^{N'},\ \forall\, a,b\geqslant 0,\ \forall\,\epsilon>0. \qquad (3.3)$$

By (3.3), we have

$$|u(x)|^{N'}\leqslant(1+\epsilon)\,|v(x)|^{N'}+\frac{1+\epsilon}{\left((1+\epsilon)^{\frac{1}{N'-1}}-1\right)^{N'-1}}\,|u(e_1)|^{N'},\ \forall\, x\in\mathbb{R}^N,\ |x|<1,$$

where v is given by (2.7). Thus,

$$\begin{aligned}
&\int_{|x|<1}\mathrm{e}^{\alpha\,\mathrm{e}^{|u|^{N'}}}\,\mathrm{d}x\\
&\leqslant\int_{|x|<1}\exp\left(\alpha\exp\left(\frac{1+\epsilon}{\left((1+\epsilon)^{\frac{1}{N'-1}}-1\right)^{N'-1}}\,|u(e_1)|^{N'}\right)\mathrm{e}^{(1+\epsilon)|v|^{N'}}\right)\mathrm{d}x\\
&=\int_{|x|<1}\exp\left(\alpha\exp\left(\frac{|u(e_1)|^{N'}}{\left(1-(1+\epsilon)^{1-N}\right)^{\frac{1}{N-1}}}\right)\mathrm{e}^{(1+\epsilon)|v|^{N'}}\right)\mathrm{d}x. \qquad (3.4)
\end{aligned}$$

Clearly, $v\in W_{0,rad}^{1,N}(\mathcal{B},\sigma_1)$. By (1.3) and (3.4), it follows

$$\int_{|x|<1}\mathrm{e}^{\alpha\mathrm{e}^{|u|^{N'}}}\,\mathrm{d}x<+\infty. \qquad (3.5)$$

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Combining (3.5) and (3.2), we easily see that (1.18) holds.

The next step in the proof of theorem 1.2 consists of proving (1.19). First, observe that by (3.2), we have

$$\sup_{u\in E_{q,1},\ \|u\|_{E_{q,1}}\leqslant 1}\int_{|x|\geqslant 1}\left(\mathrm{e}^{\alpha\left(\mathrm{e}^{\omega_{N-1}^{\frac{1}{N-1}}|u|^{N'}}-1\right)}-S_{j_1-1}\left(\alpha(\mathrm{e}^{\omega_{N-1}^{\frac{1}{N-1}}|u|^{N'}}-1)\right)\right)\mathrm{d}x$$
$$\leqslant C''\exp\left(\alpha\left(\mathrm{e}^{C_{q,N}^{N'}\omega_{N-1}^{\frac{1}{N-1}}}-1\right)\right). \tag{3.6}$$

Let $u\in E_{q,1}$ be such that $\|u\|_{E_{q,1}}\leqslant 1$. Note that if $\int_{|x|<1}|\nabla u|^N w_1(x)\,\mathrm{d}x=1$, then $\int_{\mathbb{R}^N}|\nabla u|^q\,\mathrm{d}x=0$ which implies that $u=0$. If $\int_{|x|<1}|\nabla u|^N w_1(x)\,\mathrm{d}x=0$, then $v=0$ (where v is given by (2.7)) and by consequence $u(x)=u(e_1),\ \forall\, x\in\mathbb{R}^N,\ 0<|x|<1$. Hence, by (2.3), we get

$$\int_{|x|<1}\mathrm{e}^{\alpha\,\mathrm{e}^{\omega_{N-1}^{\frac{1}{N-1}}|u(x)|^{N'}}}\mathrm{d}x=\int_{|x|<1}\mathrm{e}^{\alpha\,\mathrm{e}^{\omega_{N-1}^{\frac{1}{N-1}}|u(e_1)|^{N'}}}\mathrm{d}x\leqslant\frac{\omega_{N-1}}{N}\,\mathrm{e}^{\alpha\,\mathrm{e}^{\omega_{N-1}^{\frac{1}{N-1}}C_{q,N}^{N'}}},\ \forall\,\alpha>0.$$

Thus, without loss of generality, we can assume that

$$0<\int_{|x|<1}|\nabla u|^N w_1(x)\,\mathrm{d}x<1.$$

Choose $\epsilon>0$ such that

$$\frac{1}{(1+\epsilon)^{N-1}}=\int_{|x|<1}|\nabla u|^N w_1(x)\,\mathrm{d}x.$$

Using again (2.3), it yields

$$\begin{aligned}|u(e_1)|^N&\leqslant C_{q,N}^N\,|\nabla u|^N_{L^q(\mathbb{R}^N)}\\&=C_{q,N}^N\left(1-\int_{\mathbb{R}^N}|\nabla u|^N w_1(x)\,\mathrm{d}x\right)\\&\leqslant C_{q,N}^N\left(1-\int_{|x|<1}|\nabla u|^N w_1(x)\,\mathrm{d}x\right)\\&=C_{q,N}^N\left(1-\frac{1}{(1+\epsilon)^{N-1}}\right).\end{aligned}$$

Thus,

$$|u(e_1)|^{N'}\leqslant C_{q,N}^{N'}\left(1-\frac{1}{(1+\epsilon)^{N-1}}\right)^{\frac{1}{N-1}}. \tag{3.7}$$

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In view of (3.7) and (3.4), we infer

$$
\begin{aligned}
&\int_{|x|<1} \mathrm{e}^{\alpha\, \mathrm{e}^{\omega_{N-1}^{\frac{1}{N-1}} |u|^{N'}}}\, \mathrm{d}x \\
&\leqslant \int_{|x|<1} \exp\left(\alpha \exp\left(\frac{\omega_{N-1}^{\frac{1}{N-1}} |u(e_1)|^{N'}}{(1-(1+\epsilon)^{1-N})^{\frac{1}{N-1}}}\right) \mathrm{e}^{\omega_{N-1}^{\frac{1}{N-1}}(1+\epsilon)|v|^{N'}}\right) \mathrm{d}x \\
&\leqslant \int_{|x|<1} \exp\left(\alpha \exp\left(\frac{\omega_{N-1}^{\frac{1}{N-1}} C_{q,N}^{N'} \left(1-(1+\epsilon)^{1-N}\right)^{\frac{1}{N-1}}}{(1-(1+\epsilon)^{1-N})^{\frac{1}{N-1}}}\right) \mathrm{e}^{\omega_{N-1}^{\frac{1}{N-1}}(1+\epsilon)|v|^{N'}}\right) \mathrm{d}x. \\
&= \int_{|x|<1} \exp\left(\alpha\, \mathrm{e}^{\omega_{N-1}^{\frac{1}{N-1}} C_{q,N}^{N'}}\, \mathrm{e}^{\omega_{N-1}^{\frac{1}{N-1}}(1+\epsilon)|v|^{N'}}\right) \mathrm{d}x \\
&= \int_{|x|<1} \exp\left(\alpha\, \mathrm{e}^{\omega_{N-1}^{\frac{1}{N-1}} C_{q,N}^{N'}}\, \mathrm{e}^{\omega_{N-1}^{\frac{1}{N-1}} |\widetilde{v}|^{N'}}\right) \mathrm{d}x, \qquad (3.8)
\end{aligned}
$$

where $\widetilde{v} = (1+\epsilon)^{\frac{1}{N'}} v$. Assume that $\alpha\, \mathrm{e}^{\omega_{N-1}^{\frac{1}{N-1}} C_{q,N}^{N'}} \leqslant N$. Taking into account that $\widetilde{v} \in W_{0,rad}^{1,N}(\mathcal{B}, \sigma_1)$ and

$$
\int_{|x|<1} |\nabla \widetilde{v}|^N w_1(x)\, \mathrm{d}x = 1,
$$

then we deduce from (1.4) that

$$
\sup_{u \in E_{q,1},\ \|u\|_{E_{q,1}} \leqslant 1} \int_{|x|<1} \mathrm{e}^{\alpha\, \mathrm{e}^{\omega_{N-1}^{\frac{1}{N-1}} |u|^{N'}}}\, \mathrm{d}x < +\infty. \qquad (3.9)
$$

Plainly, (1.19) immediately follows from (3.9) and (3.6).

The end of the proof of theorem 1.2 consists of showing (1.20). For that aim, we make a change of variable similar to the case $0 < \beta < 1$. More precisely, for $u \in E_{q,1}$, set $\psi(t) = \omega_{N-1}^{\frac{1}{N}} u(x)$ with $|x| = \mathrm{e}^{-t},\ t \in \mathbb{R}$. We have

$$
\begin{aligned}
&\int_{|x|<1} |\nabla u|^N w_1(x)\, \mathrm{d}x = \int_0^{+\infty} (1+t)^{N-1} |\psi'(t)|^N\, \mathrm{d}t, \\
&\int_{|x|\geqslant 1} |\nabla u|^N w_1(x)\, \mathrm{d}x = \int_{-\infty}^{0} \chi(\mathrm{e}^{-t})\, |\psi'(t)|^N\, \mathrm{d}t, \\
&\int_{\mathbb{R}^N} |\nabla u|^q\, \mathrm{d}x = \omega_{N-1}^{1-\frac{q}{N}} \int_{-\infty}^{+\infty} |\psi'(t)|^q\, \mathrm{e}^{(q-N)t}\, \mathrm{d}t,
\end{aligned}
$$

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and

$$\int_{\mathbb{R}^N}\left(\mathrm{e}^{\alpha\left(\mathrm{e}^{\omega_{N-1}^{\frac{1}{N-1}}|u|^{N'}}-1\right)}-S_{j_1-1}\left(\alpha\left(\mathrm{e}^{\omega_{N-1}^{\frac{1}{N-1}}|u|^{N'}}-1\right)\right)\right)\mathrm{d}x$$
$$=\omega_{N-1}\int_{-\infty}^{+\infty}\left(\mathrm{e}^{\alpha\left(\mathrm{e}^{|\psi(t)|^{N'}}-1\right)}-S_{j_1-1}\left(\alpha\left(\mathrm{e}^{|\psi(t)|^{N'}}-1\right)\right)\right)\mathrm{e}^{-Nt}\,\mathrm{d}t.$$

For $k\geqslant 0$, consider the family of test functions:

$$\psi_k(t)=\begin{cases}\dfrac{\log(1+t)}{(\log(k+1))^{\frac{1}{N}}}, & 0\leqslant t\leqslant k,\\ (\log(k+1))^{\frac{1}{N'}}, & t\geqslant k,\\ 0, & t\leqslant 0,\end{cases}\tag{3.10}$$

and define $u_k\in E_{q,1}$ by $\psi_k(t)=\omega_{N-1}^{\frac{1}{N}}u_k(x)$. We have

$$\int_{\mathbb{R}^N}|\nabla u_k|^N\,w_1(x)\,\mathrm{d}x=1,$$

and

$$\int_{\mathbb{R}^N}|\nabla u_k|^q\,\mathrm{d}x=\frac{\omega_{N-1}^{1-\frac{q}{N}}}{(\log(k+1))^{\frac{q}{N}}}\int_0^k\frac{\mathrm{e}^{(q-N)t}}{(1+t)^q}\,\mathrm{d}t\to 0,\ \ k\to+\infty.$$

Observe that,

$$\lim_{k\to+\infty}\log(k+1)\left(1-\frac{1}{\|u_k\|_{E_{q,1}}^{N'}}\right)=\lim_{k\to+\infty}\log(k+1)\left(1-\frac{1}{\left(1+|\nabla u_k|_{L^q(\mathbb{R}^N)}^N\right)^{\frac{1}{N-1}}}\right)$$
$$=\lim_{k\to+\infty}\log(k+1)\left(\frac{1}{N-1}|\nabla u_k|_{L^q(\mathbb{R}^N)}^N\right).\tag{3.11}$$

We have

$$\lim_{k\to+\infty}\log(k+1)\left(\frac{1}{N-1}|\nabla u_k|_{L^q(\mathbb{R}^N)}^N\right)=\frac{\omega_{N-1}^{\frac{N}{q}-1}}{N-1}\left(\int_0^{+\infty}\frac{\mathrm{e}^{(q-N)t}}{(1+t)^q}\,\mathrm{d}t\right)^{\frac{N}{q}},$$

which, by (3.11), leads to

$$\lim_{k\to+\infty}\log(k+1)\left(1-\frac{1}{\|u_k\|_{E_{q,1}}^{N'}}\right)=\frac{\omega_{N-1}^{\frac{N}{q}-1}}{N-1}\left(\int_0^{+\infty}\frac{\mathrm{e}^{(q-N)t}}{(1+t)^q}\,\mathrm{d}t\right)^{\frac{N}{q}}.\tag{3.12}$$

Set $\widetilde{u_k} = \frac{u_k}{\|u_k\|_{E_{q,1}}}$. For $\alpha > 0$, we have

$$
\begin{aligned}
&\sup_{u\in E_{q,1},\ \|u\|_{E_{q,1}}\leqslant 1} \int_{\mathbb{R}^N} \left(\mathrm{e}^{\alpha\left(\mathrm{e}^{\omega_{n-1}^{\frac{1}{n-1}}|u|^{N'}}-1\right)} - S_{j_1-1}\left(\alpha\left(\mathrm{e}^{\omega_{n-1}^{\frac{1}{N-1}}|u|^{N'}}-1\right)\right)\right) \mathrm{d}x \\
&\geqslant \int_{\mathbb{R}^N} \left(\mathrm{e}^{\alpha\left(\mathrm{e}^{\omega_{n-1}^{\frac{1}{N-1}}|\widetilde{u_k}|^{N'}}-1\right)} - S_{j_1-1}\left(\alpha\left(\mathrm{e}^{\omega_{n-1}^{\frac{1}{n-1}}|\widetilde{u_k}|^{N'}}-1\right)\right)\right) \mathrm{d}x \\
&= \omega_{N-1} \int_{-\infty}^{+\infty} \left(\mathrm{e}^{\alpha\left(\mathrm{e}^{\left|\frac{\psi_k(t)}{\|u_k\|_{E_{q,1}}}\right|^{N'}}-1\right)} - S_{j_1-1}\left(\alpha\left(\mathrm{e}^{\left|\frac{\psi_k(t)}{\|u_k\|_{E_{q,1}}}\right|^{N'}}-1\right)\right)\right) \mathrm{e}^{-Nt}\,\mathrm{d}t \\
&\geqslant \omega_{N-1} \int_{k}^{+\infty} \left(\mathrm{e}^{\alpha\left(\mathrm{e}^{\frac{\log(k+1)}{\|u_k\|_{E_{q,1}}^{N'}}}-1\right)} - S_{j_1-1}\left(\alpha\left(\mathrm{e}^{\frac{\log(k+1)}{\|u_k\|_{E_{q,1}}^{N'}}}-1\right)\right)\right) \mathrm{e}^{-Nt}\,\mathrm{d}t \\
&= \omega_{N-1} \frac{\mathrm{e}^{-Nk}}{N} \left(\mathrm{e}^{\alpha\left(\mathrm{e}^{\frac{\log(k+1)}{\|u_k\|_{E_{q,1}}^{N'}}}-1\right)} - S_{j_1-1}\left(\alpha\left(\mathrm{e}^{\frac{\log(k+1)}{\|u_k\|_{E_{q,1}}^{N'}}}-1\right)\right)\right). \qquad (3.13)
\end{aligned}
$$

Taking into account that

$$
\begin{aligned}
-Nk + \alpha\left(\mathrm{e}^{\frac{\log(k+1)}{\|u_k\|_{E_{q,1}}^{N'}}} - 1\right) &= -N(k+1) + \alpha\,\mathrm{e}^{\frac{\log(k+1)}{\|u_k\|_{E_{q,1}}^{N'}}} - \alpha + N \\
&= (k+1)\left(-N + \alpha\,\mathrm{e}^{\frac{\log(k+1)}{\|u_k\|_{E_{q,1}}^{N'}} - \log(k+1)}\right) - \alpha + N,
\end{aligned}
$$

and using (3.12) we obtain that

$$
-Nk + \alpha\left(\mathrm{e}^{\frac{\log(k+1)}{\|u_k\|_{E_{q,1}}^{N'}}} - 1\right) \to +\infty,\ \forall\, \alpha > N \exp\left(\frac{\omega_{N-1}^{\frac{N}{q}-1}}{N-1} \left(\int_0^{+\infty} \frac{\mathrm{e}^{(q-N)t}}{(1+t)^q}\,\mathrm{d}t \right)^{\frac{N}{q}} \right).
$$

Finally, by (3.13), we conclude that, if

$$\alpha > N \exp\left(\frac{\omega_{N-1}^{\frac{N}{q}-1}}{N-1}\left(\int_0^{+\infty}\frac{\mathrm{e}^{(q-N)t}}{(1+t)^q}\,\mathrm{d}t\right)^{\frac{N}{q}}\right),$$

then

$$\sup_{u\in E_{q,1},\ \|u\|_{E_{q,1}}\leqslant 1}\int_{\mathbb{R}^N}\left(\mathrm{e}^{\alpha\left(\mathrm{e}^{\omega_{N-1}^{\frac{1}{N-1}}|u|^{N'}}-1\right)}-S_{j_1-1}\left(\alpha\left(\mathrm{e}^{\omega_{N-1}^{\frac{1}{N-1}}|u|^{N'}}-1\right)\right)\right)\mathrm{d}x=+\infty.$$

This ends the proof of theorem 1.2.

4. Proof of theorem 1.3

We claim that

$$\sup_{u\in E_{q,\beta},\ \|u\|_{q,\beta}\leqslant 1}\int_{\mathbb{R}^N}\left(\mathrm{e}^{\alpha_{N,\beta}|u|^{\frac{N'}{1-\beta}}}-S_{j_\beta-1}\left(\alpha_{N,\beta}|u|^{\frac{N'}{1-\beta}}\right)\right)\mathrm{d}x<+\infty. \tag{4.1}$$

First, observe that arguing exactly as in the proof of (2.11), one can easily show that

$$\sup_{u\in E_{q,\beta},\ \|u\|_{q,\beta}\leqslant 1}\int_{|x|\geqslant 1}\left(e^{\alpha_{N,\beta}|u|^{\frac{N'}{1-\beta}}}-S_{j_\beta-1}\left(\alpha_{N,\beta}|u|^{\frac{N'}{1-\beta}}\right)\right)\mathrm{d}x<+\infty. \tag{4.2}$$

It remains to prove that

$$\sup_{u\in E_{q,\beta},\ \|u\|_{q,\beta}\leqslant 1}\int_{|x|<1}\left(\mathrm{e}^{\alpha_{N,\beta}|u|^{\frac{N'}{1-\beta}}}-S_{j_\beta-1}\left(\alpha_{N,\beta}|u|^{\frac{N'}{1-\beta}}\right)\right)\mathrm{d}x<+\infty. \tag{4.3}$$

For that aim, let $u\in E_{q,\beta}$ be such that $\|u\|_{q,\beta}\leqslant 1$. Choose $\epsilon>0$ such that

$$(1+\epsilon)^{(1-\beta)(N-1)}=\left(\int_{|x|<1}|\nabla u|^N w_\beta(x)\,\mathrm{d}x\right)^{-1}.$$

Using inequality (2.8), it follows

$$\int_{|x|<1}\mathrm{e}^{\alpha_{N,\beta}|u|^{\frac{N'}{1-\beta}}}\,\mathrm{d}x\leqslant\exp\left(\frac{\alpha_{N,\beta}(1+\epsilon)\,|u(e_1)|^{\frac{N'}{1-\beta}}}{\left((1+\varepsilon)^{\frac{1-\beta}{N'-1+\beta}}-1\right)^{\frac{N'-1+\beta}{1-\beta}}}\right)$$
$$\int_{|x|<1}\mathrm{e}^{\alpha_{N,\beta}(1+\varepsilon)|v|^{\frac{N'}{1-\beta}}}\,\mathrm{d}x. \tag{4.4}$$

We have,

$$\int_{|x|<1}\left|\nabla((1+\epsilon)^{\frac{1-\beta}{N'}}v)\right|^N w_\beta(x)\,\mathrm{d}x=(1+\epsilon)^{(1-\beta)(N-1)}\int_{|x|<1}|\nabla u|^N w_\beta(x)\,\mathrm{d}x=1.$$

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By the virtue of (1.2), we know that there exists a constant $C_\beta > 0$ such that

$$\int_{|x|<1} \mathrm{e}^{\alpha_{N,\beta}(1+\epsilon)|v|^{\frac{N'}{1-\beta}}} \,\mathrm{d}x \leqslant C_\beta.$$

On the other hand, by (2.3), we have

$$\begin{aligned}
|u(e_1)|^{\frac{N'}{1-\beta}} &\leqslant C_{q,N}^{\frac{N'}{1-\beta}} \, |\nabla u|_{L^q(\mathbb{R}^N)}^{\frac{N'}{1-\beta}} \\
&\leqslant C_{q,N}^{\frac{N'}{1-\beta}} \left(1 - |\nabla u|_{L^N_{w_\beta}(\mathbb{R}^N)}\right)^{\frac{N'}{1-\beta}} \\
&\leqslant C_{q,N}^{\frac{N'}{1-\beta}} \left(1 - \left(\int_{|x|<1} |\nabla u|^N \, w_\beta(x)\,\mathrm{d}x\right)^{\frac{1}{N}}\right)^{\frac{N'}{1-\beta}} \\
&\leqslant C_{q,N}^{\frac{N'}{1-\beta}} \left(1 - (1+\epsilon)^{\frac{(1-\beta)(1-N)}{N}}\right)^{\frac{N'}{1-\beta}} \\
&\leqslant C_{q,N}^{\frac{N'}{1-\beta}} (1+\epsilon)^{-1} \left((1+\epsilon)^{\frac{1-\beta}{N'}} - 1\right)^{\frac{N'}{1-\beta}}.
\end{aligned}$$

Putting that last inequality in (4.4), we deduce that

$$\int_{|x|<1} \mathrm{e}^{\alpha_{N,\beta}|u|^{\frac{N'}{1-\beta}}} \,\mathrm{d}x \leqslant C_\beta \exp\left(\frac{\alpha_{N,\beta} C_{q,N}^{\frac{N'}{1-\beta}} \left((1+\epsilon)^{\frac{1-\beta}{N'}} - 1\right)^{\frac{N'}{1-\beta}}}{\left((1+\epsilon)^{\frac{1-\beta}{N'-1+\beta}} - 1\right)^{\frac{N'-1+\beta}{1-\beta}}}\right). \tag{4.5}$$

Since $\frac{N'}{1-\beta} \geqslant \frac{N'-1+\beta}{1-\beta}$, then the function defined on $]1,+\infty[$ by

$$x \longmapsto \frac{\left(x^{\frac{1-\beta}{N'}} - 1\right)^{\frac{N'}{1-\beta}}}{\left(x^{\frac{1-\beta}{N'-1+\beta}} - 1\right)^{\frac{N'-1+\beta}{1-\beta}}}$$

is bounded. In view of (4.5), we can easily conclude that (4.3) follows. Combining (4.2) and (4.3), we deduce that (4.1) holds. Finally, if $\alpha > \alpha_{N,\beta}$, we proceed exactly as in the proof of theorem (1.1) keeping the same Moser sequence to prove that

$$\sup_{u\in E_{q,\beta},\ \|u\|_{q,\beta}\leqslant 1} \int_{|x|<1} \left(\mathrm{e}^{\alpha|u|^{\frac{N'}{1-\beta}}} - S_{j_\beta-1}\left(\alpha\,|u|^{\frac{N'}{1-\beta}}\right)\right)\mathrm{d}x = +\infty.$$

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5. Proof of theorem 1.4

We claim that

$$\sup_{u\in E_{q,1},\ \|u\|_{q,1}\leqslant 1}\int_{\mathbb{R}^N}\left(\mathrm{e}^{N\left(\mathrm{e}^{\omega_{N-1}^{\frac{1}{N-1}}|u|^{N'}}-1\right)}-S_{j_1-1}\left(N(\mathrm{e}^{\omega_{N-1}^{\frac{1}{N-1}}|u|^{N'}}-1)\right)\right)dx<+\infty. \tag{5.1}$$

First, proceeding as in the proof of (3.6), one can easily see that

$$\sup_{u\in E_{q,1},\ \|u\|_{q,1}\leqslant 1}\int_{|x|\geqslant 1}\left(\mathrm{e}^{N\left(\mathrm{e}^{\omega_{N-1}^{\frac{1}{N-1}}|u|^{N'}}-1\right)}-S_{j_1-1}\left(N(\mathrm{e}^{\omega_{N-1}^{\frac{1}{N-1}}|u|^{N'}}-1)\right)\right)\mathrm{d}x<+\infty. \tag{5.2}$$

Now, let $u\in E_{q,1}$ be such that $\|u\|_{q,1}\leqslant 1$. Without loss of generality, we can assume that

$$0<\int_{|x|<1}|\nabla u|^N\,w_1(x)\,\mathrm{d}x<1.$$

Using the convexity of the function defined on $[0,+\infty[$ by $x\longmapsto \mathrm{e}^{x^{N'}}$, we can easily get the following inequality

$$\mathrm{e}^{(a+b)^{N'}}\leqslant\frac{\epsilon}{1+\epsilon}\,\mathrm{e}^{\left(\frac{1+\epsilon}{\epsilon}\right)^{N'}a^{N'}}+\frac{1}{1+\epsilon}\,\mathrm{e}^{(1+\epsilon)^{N'}b^{N'}},\ \forall\ a,b\geqslant 0,\ \forall\ \epsilon>0.$$

For v defined as in (2.7), it yields

$$\mathrm{e}^{|u(x)|^{N'}}\leqslant\frac{\epsilon}{1+\epsilon}\,\mathrm{e}^{\left(\frac{1+\epsilon}{\epsilon}\right)^{N'}|u(e_1)|^{N'}}+\frac{1}{1+\epsilon}\,\mathrm{e}^{(1+\epsilon)^{N'}|v(x)|^{N'}},\ \forall\ x\in\mathbb{R}^N,\ |x|<1.$$

Hence,

$$\begin{aligned}\int_{|x|<1}\mathrm{e}^{\alpha\,\mathrm{e}^{\omega_{N-1}^{\frac{1}{N-1}}|u|^{N'}}}\,\mathrm{d}x&\leqslant \mathrm{e}^{\alpha\frac{\epsilon}{1+\epsilon}\,\mathrm{e}^{\left(\frac{1+\epsilon}{\epsilon}\right)^{N'}\omega_{N-1}^{\frac{1}{N-1}}|u(e_1)|^{N'}}}\\ &\int_{|x|<1}\mathrm{e}^{\frac{\alpha}{1+\epsilon}\,\mathrm{e}^{(1+\epsilon)^{N'}\omega_{N-1}^{\frac{1}{N-1}}|v(x)|^{N'}}}\,\mathrm{d}x.\end{aligned} \tag{5.3}$$

Choose $\epsilon>0$ such that

$$(1+\epsilon)^N\int_{|x|<1}|\nabla u|^N\,w_1(x)\,\mathrm{d}x=1.$$

Clearly, $\widetilde{v}=(1+\epsilon)v\in W^{1,N}_{0,rad}(\mathcal{B},\sigma_1)$. We have

$$\begin{aligned}\int_{|x|<1}\mathrm{e}^{\frac{\alpha}{1+\epsilon}\,\mathrm{e}^{(1+\epsilon)^{N'}\omega_{N-1}^{\frac{1}{N-1}}|v(x)|^{N'}}}\,\mathrm{d}x&=\int_{|x|<1}\mathrm{e}^{\frac{\alpha}{1+\epsilon}\,\mathrm{e}^{\omega_{N-1}^{\frac{1}{N-1}}|\widetilde{v}(x)|^{N'}}}\,\mathrm{d}x\\ &\leqslant\int_{|x|<1}\mathrm{e}^{\alpha\,\mathrm{e}^{\omega_{N-1}^{\frac{1}{N-1}}|\widetilde{v}(x)|^{N'}}}\,\mathrm{d}x.\end{aligned}$$

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For $\alpha \leqslant N$, by (1.4) we get

$$\begin{aligned}
&\int_{|x|<1} \mathrm{e}^{\frac{\alpha}{1+\epsilon}\, \mathrm{e}^{\omega_{N-1}^{\frac{1}{N-1}} |\tilde{v}(x)|^{N'}}} \,\mathrm{d}x \\
&\leqslant \int_{|x|<1} \mathrm{e}^{N\, \mathrm{e}^{\omega_{N-1}^{\frac{1}{N-1}} |\tilde{v}(x)|^{N'}}} \,\mathrm{d}x \\
&\leqslant \sup\left\{ \int_{|x|<1} \mathrm{e}^{N\, \mathrm{e}^{\omega_{N-1}^{\frac{1}{N-1}} |z(x)|^{N'}}} \,\mathrm{d}x,\ z \in W_{0,rad}^{1,N}(\mathcal{B},\sigma_1),\ \|z\|_{\sigma_1} \leqslant 1 \right\} < +\infty. \quad (5.4)
\end{aligned}$$

On the other hand, by looking at the proof of (2.3) in [**37**], we can easily see that we have a more precise inequality, that is

$$|u(x)| \leqslant C_{q,N}\, |x|^{-\frac{N-q}{q}} \left(\int_{|z|\geqslant |x|} |\nabla u(z)|^q \,\mathrm{d}z \right)^{\frac{1}{q}},\ \forall\ x \neq 0.$$

It follows,

$$\begin{aligned}
|u(e_1)| &\leqslant C_{q,N} \left(\int_{|x|\geqslant 1} |\nabla u|^q \,\mathrm{d}x \right)^{\frac{1}{q}} \\
&\leqslant C_{q,N} \left(1 - |\nabla u|_{L^N_{w_1}(\mathbb{R}^N)} \right) \\
&\leqslant C_{q,N} \left(1 - \left(\int_{|x|<1} |\nabla u|^N w_1(x) \,\mathrm{d}x \right)^{\frac{1}{N}} \right) \\
&= C_{q,N} \left(1 - \frac{1}{1+\epsilon} \right).
\end{aligned}$$

Hence,

$$\mathrm{e}^{\alpha \frac{\epsilon}{1+\epsilon}\, \mathrm{e}^{\left(\frac{1+\epsilon}{\epsilon}\right)^{N'} \omega_{N-1}^{\frac{1}{N-1}} |u(e_1)|^{N'}}} \leqslant \mathrm{e}^{\alpha\, \mathrm{e}^{\omega_{N-1}^{\frac{1}{N-1}} C_{q,N}^{N'}}}. \quad (5.5)$$

Combining (5.5) and (5.4), we deduce from (5.3) that

$$\sup_{u\in E_{q,1},\ \|u\|_{q,1}\leqslant 1} \int_{|x|<1} \mathrm{e}^{N \mathrm{e}^{\omega_{N-1}^{\frac{1}{N-1}} |u|^{N'}}} \,\mathrm{d}x < +\infty. \quad (5.6)$$

In view of (5.6) and (5.2), we can conclude that (5.1) holds. The end of the proof gives a clear idea about the real reason of taking only the integral of $|\nabla u|^q$ over the set $\left\{x \in \mathbb{R}^N,\ |x| \geqslant 1\right\}$ in the definition of the norm $\|u\|_{q,1}$. In fact, we take the family of test functions given by (3.10) and we define as usual $u_k \in E_{q,1}$ by $u_k(x) = \omega_{N-1}^{-\frac{1}{N}} \psi_k(t),\ |x| = e^{-t},\ t \in \mathbb{R}$. Observing that $u_k(x) = 0,\ \forall\ x \in \mathbb{R}^N,\ |x| \geqslant$

1, we immediately get

$$\|u_k\|_{q,1} = \left(\int_{\mathbb{R}^N} |\nabla u_k|^N w_1 \,\mathrm{d}x\right)^{\frac{1}{N}} = \left(\int_{|x|<1} |\nabla u_k|^N w_1 \,\mathrm{d}x\right)^{\frac{1}{N}} = 1, \ \forall\ k \geqslant 0.$$

Consequently, we are returning to the case of a sequence lying in $W^{1,N}_{0,rad}(\mathcal{B}, \sigma_1)$. Therefore, the conclusion follows.

6. Proof of theorem 1.5

1. Case $0 < \beta < 1$: *Sub-case* $0 < \|u\|_{E_{q,\beta}} < 1$:

   Assume by contradiction that for some $0 < p_1 < \left(\frac{1}{1-\|u\|^N_{E_{q,\beta}}}\right)^{\frac{1}{(1-\beta)(N-1)}}$ we have

$$\sup_n \int_{\mathbb{R}^N} \left(\mathrm{e}^{p_1\alpha_{N,\beta}|u_n|^{\frac{N'}{1-\beta}}} - S_{j_\beta-1}\left(p_1\alpha_{N,\beta}\,|u_n|^{\frac{N'}{1-\beta}}\right)\right)\mathrm{d}x = +\infty. \qquad (6.1)$$

   For $L \in]0, +\infty[$ and $v \in E_{q,\beta}$, set

$$G_L(v) = \begin{cases} L, & \text{if } v > L, \\ -L, & \text{if } v < -L, \\ v, & \text{if } |v| \leqslant L, \end{cases} \quad \text{and } T_L(v) = v - G_L(v). \qquad (6.2)$$

   Plainly, there exists $\epsilon > 0$ such that

$$(p_1(1+\epsilon))^{(1-\beta)(N-1)} < \frac{1}{1-\|u\|^N_{E_{q,\beta}}}.$$

   Since $\|G_L(u)\|_{E_{q,\beta}} \to \|u\|_{E_{q,\beta}}$ as $L \to +\infty$, then one can choose L large enough such that

$$(p_1(1+\epsilon))^{(1-\beta)(N-1)} < \frac{1}{1-\|G_L(u)\|^N_{E_{q,\beta}}}. \qquad (6.3)$$

   We claim that

$$\limsup_{n\to+\infty}\left(\int_{\mathbb{R}^N} |\nabla T_L(u_n)|^N w_\beta(x)\,\mathrm{d}x + \left(\int_{\mathbb{R}^N} |\nabla T_L(u_n)|^q\,\mathrm{d}x\right)^{\frac{N}{q}}\right) < \left(\frac{1}{p_1(1+\epsilon)}\right)^{(1-\beta)(N-1)}. \qquad (6.4)$$

   Suppose that this does not hold. Then, there exists a subsequence of $(u_n)_n$ that we still denote by $(u_n)_n$ such that

$$\int_{\mathbb{R}^N} |\nabla T_L(u_n)|^N w_\beta(x)\,\mathrm{d}x + \left(\int_{\mathbb{R}^N} |\nabla T_L(u_n)|^q\,\mathrm{d}x\right)^{\frac{N}{q}} \geqslant \left(\frac{1}{p_1(1+\epsilon)}\right)^{(1-\beta)(N-1)}, \ \forall\ n \geqslant 0. \qquad (6.5)$$

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Using (6.5), it yields

$$\begin{aligned}1 &= \int_{\mathbb{R}^N} |\nabla u_n|^N w_\beta(x)\,\mathrm{d}x + \left(\int_{\mathbb{R}^N} |\nabla u_n|^q\,\mathrm{d}x\right)^{\frac{N}{q}} \\ &= \int_{\mathbb{R}^N} |\nabla T_L(u_n)|^N w_\beta(x)\,\mathrm{d}x + \int_{\mathbb{R}^N} |\nabla G_L(u_n)|^N w_\beta(x)\,\mathrm{d}x \\ &\quad + \left(\int_{\mathbb{R}^N} |\nabla T_L(u_n)|^q\,\mathrm{d}x + \int_{\mathbb{R}^N} |\nabla G_L(u_n)|^q\,\mathrm{d}x\right)^{\frac{N}{q}} \\ &\geqslant \int_{\mathbb{R}^N} |\nabla T_L(u_n)|^N w_\beta(x)\,\mathrm{d}x + \int_{\mathbb{R}^N} |\nabla G_L(u_n)|^N w_\beta(x)\,\mathrm{d}x \\ &\quad + \left(\int_{\mathbb{R}^N} |\nabla T_L(u_n)|^q\,\mathrm{d}x\right)^{\frac{N}{q}} + \left(\int_{\mathbb{R}^N} |\nabla G_L(u_n)|^q\,\mathrm{d}x\right)^{\frac{N}{q}} \\ &\geqslant \left(\frac{1}{p_1(1+\epsilon)}\right)^{(1-\beta)(N-1)} + \|G_L(u_n)\|^N_{E_{q,\beta}}. \end{aligned} \tag{6.6}$$

Clearly $G_L(u_n) \rightharpoonup G_L(u)$ weakly in $E_{q,\beta}$. Consequently, passing to the lower limit as n tends to $+\infty$ in (6.6), we obtain

$$1 \geqslant \|G_L(u)\|^N_{E_{q,\beta}} + \left(\frac{1}{p_1(1+\epsilon)}\right)^{(1-\beta)(N-1)}.$$

Thus,

$$(p_1(1+\epsilon))^{(1-\beta)(N-1)} \geqslant \frac{1}{1 - \|G_L(u)\|^N_{E_{q,\beta}}},$$

which is in contradiction with (6.3). Therefore, our claim (6.4) is true. Set

$$\Omega_{n,L} = \left\{x \in \mathbb{R}^N,\ |u_n(x)| \geqslant L\right\}.$$

By (6.4), up to a subsequence,

$$\left\|(p_1(1+\epsilon))^{\frac{(1-\beta)}{N'}} T_L(u_n)\right\|_{E_{q,\beta}} < 1,\ \forall\, n \geqslant 0.$$

We have

$$\begin{aligned}&\int_{\Omega_{n,L}} \left(\mathrm{e}^{\alpha_{N,\beta} p_1 |u_n|^{\frac{N'}{1-\beta}}} - S_{j_\beta - 1}\left(\alpha_{N,\beta} p_1 |u_n|^{\frac{N'}{1-\beta}}\right)\right)\mathrm{d}x \\ &= \int_{\Omega_{n,L}\cap\mathcal{B}} \left(\mathrm{e}^{\alpha_{N,\beta} p_1 |u_n|^{\frac{N'}{1-\beta}}} - S_{j_\beta - 1}\left(\alpha_{N,\beta} p_1 |u_n|^{\frac{N'}{1-\beta}}\right)\right)\mathrm{d}x \\ &\quad + \int_{\Omega_{n,L}\cap\mathcal{B}^c} \left(\mathrm{e}^{\alpha_{N,\beta} p_1 |u_n|^{\frac{N'}{1-\beta}}} - S_{j_\beta - 1}\left(\alpha_{N,\beta} p_1 |u_n|^{\frac{N'}{1-\beta}}\right)\right)\mathrm{d}x. \end{aligned} \tag{6.7}$$

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On the one hand, by (2.8), we get

$$\begin{aligned}
&\int_{\Omega_{n,L}\cap\mathcal{B}} \left(\mathrm{e}^{\alpha_{N,\beta}p_1|u_n|^{\frac{N'}{1-\beta}}} - S_{j_\beta-1}\left(\alpha_{N,\beta}p_1\,|u_n|^{\frac{N'}{1-\beta}}\right)\right)\mathrm{d}x\\
&\leqslant \int_{\Omega_{n,L}\cap\mathcal{B}} \mathrm{e}^{\alpha_{N,\beta}p_1|u_n|^{\frac{N'}{1-\beta}}}\,\mathrm{d}x\\
&\leqslant \int_{\Omega_{n,L}\cap\mathcal{B}} \mathrm{e}^{p_1\alpha_{N,\beta}(1+\epsilon)|u_n-L|^{\frac{N'}{1-\beta}}}\,\mathrm{e}^{p_1\alpha_{N,\beta}A(\epsilon)L^{\frac{N'}{1-\beta}}}\,\mathrm{d}x\\
&\leqslant \mathrm{e}^{p_1\alpha_{N,\beta}A(\epsilon)L^{\frac{N'}{1-\beta}}}\int_{\Omega_{n,L}\cap\mathcal{B}} \mathrm{e}^{p_1\alpha_{N,\beta}(1+\epsilon)|T_L(u_n)|^{\frac{N'}{1-\beta}}}\,\mathrm{d}x\\
&\leqslant \mathrm{e}^{p_1\alpha_{N,\beta}A(\epsilon)L^{\frac{N'}{1-\beta}}}\int_{\mathcal{B}} \mathrm{e}^{p_1\alpha_{N,\beta}(1+\epsilon)|T_L(u_n)|^{\frac{N'}{1-\beta}}}\,\mathrm{d}x, \qquad (6.8)
\end{aligned}$$

where $A(\epsilon)=\frac{1+\epsilon}{((1+\epsilon)^{\frac{1-\beta}{N'-1+\beta}}-1)^{\frac{N'-1+\beta}{1-\beta}}}$. Having in mind that $|u_n(x)| = |u_n(e_1)| \leqslant C_{q,N},\ \forall\ x\in\mathbb{R}^N,\ |x|=1$, then

$$T_L(u_n(x))=0,\ \forall\ x\in\mathbb{R}^N,\ |x|=1,\ \forall\ L>C_{q,N}.$$

Consequently, $T_L(u_n)\in W^{1,N}_{0,rad}(\mathcal{B},\sigma_\beta),\ \forall\ L>C_{q,N}$. Since

$$\int_{\mathcal{B}}\left|\nabla(p_1(1+\epsilon))^{\frac{1-\beta}{N'}}T_L(u_n)\right|^N w_\beta(x)\,\mathrm{d}x<1,$$

then by (1.2), we infer

$$\sup_n\int_{\mathcal{B}} \mathrm{e}^{p_1\alpha_{N,\beta}(1+\epsilon)|T_L(u_n)|^{\frac{N'}{1-\beta}}}\,\mathrm{d}x<+\infty,\ \forall\ L>C_{q,N}.$$

Putting that result in (6.8), we obtain

$$\begin{aligned}
&\sup_n\int_{\Omega_{n,L}\cap\mathcal{B}} \left(\mathrm{e}^{\alpha_{N,\beta}p_1|u_n|^{\frac{N'}{1-\beta}}} - S_{j_\beta-1}\left(\alpha_{N,\beta}p_1\,|u_n|^{\frac{N'}{1-\beta}}\right)\right)\mathrm{d}x\\
&<+\infty,\ \forall\ L>C_{q,N}. \qquad (6.9)
\end{aligned}$$

On the other hand, in view of (2.3), we know that

$$|u_n(x)|\leqslant C_{q,N}\,|\nabla u_n|_{L^q(\mathbb{R}^N)}\leqslant C_{q,N},\ \forall\ x\in\mathbb{R}^N,\ |x|\geqslant 1.$$

Hence,

$$\Omega_{n,L}\cap\mathcal{B}^c=\emptyset,\ \forall\ L>C_{q,N}.$$

We deduce from (6.9) and (6.7) that

$$\sup_n\int_{\Omega_{n,L}} \left(\mathrm{e}^{\alpha_{N,\beta}p_1|u_n|^{\frac{N'}{1-\beta}}} - S_{j_\beta-1}\left(\alpha_{N,\beta}p_1\,|u_n|^{\frac{N'}{1-\beta}}\right)\right)\mathrm{d}x<+\infty,\ \forall\ L>C_{q,N}. \qquad (6.10)$$

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Next, observe that

$$\begin{aligned}
&\int_{\Omega_{n,L}^c}\left(\mathrm{e}^{p_1\alpha_{N,\beta}|u_n|^{\frac{N'}{1-\beta}}}-S_{j_\beta-1}\left(\alpha_{N,\beta}p_1\left|u_n\right|^{\frac{N'}{1-\beta}}\right)\right)\mathrm{d}x\\
&=\int_{|u_n|<L}\sum_{j=j_\beta}^{+\infty}\frac{(p_1\alpha_{N,\beta})^j}{j!}\left|u_n\right|^{\frac{jN'}{1-\beta}}\,\mathrm{d}x\\
&=\int_{|u_n|<L}\sum_{j=j_\beta}^{+\infty}\frac{(p_1\alpha_{N,\beta})^j}{j!}\left|\frac{u_n}{L}\right|^{\frac{jN'}{1-\beta}}L^{\frac{jN'}{1-\beta}}\,\mathrm{d}x\\
&\leqslant c_4\sum_{j=j_\beta}^{+\infty}\frac{(p_1\alpha_{N,\beta}L^{\frac{N'}{1-\beta}})^j}{j!}\int_{|u_n|<L}\left|\frac{u_n}{L}\right|^{j_\beta\frac{N'}{1-\beta}}\,\mathrm{d}x\\
&\leqslant c_4\sum_{j=j_\beta}^{+\infty}\frac{(p_1\alpha_{N,\beta}L^{\frac{N'}{1-\beta}})^j}{j!}\int_{|u_n|<L}\left|\frac{u_n}{L}\right|^{q^*}\,\mathrm{d}x\\
&\leqslant c_4\frac{\mathrm{e}^{p_1\alpha_{N,\beta}L^{\frac{N'}{1-\beta}}}}{L^{q^*}}\int_{\mathbb{R}^N}\left|u_n\right|^{q^*}\,\mathrm{d}x\\
&\leqslant c_5\frac{\mathrm{e}^{p_1\alpha_{N,\beta}L^{\frac{N'}{1-\beta}}}}{L^{q^*}},
\end{aligned}$$

where we used the fact that $j_\beta \geqslant \frac{(1-\beta)q^*}{N'}$ together with the boundedness of the sequence $(u_n)_n$ in $L^{q^*}(\mathbb{R}^N)$. Therefore,

$$\sup_n\int_{\Omega_{n,L}^c}\left(e^{p_1\alpha_{N,\beta}|u_n|^{\frac{N'}{1-\beta}}}-S_{j_\beta-1}\left(\alpha_{N,\beta}p_1\left|u_n\right|^{\frac{N'}{1-\beta}}\right)\right)\mathrm{d}x<+\infty. \tag{6.11}$$

Combining (6.11) and (6.10), we conclude that

$$\sup_n\int_{\mathbb{R}^N}\left(\mathrm{e}^{p_1\alpha_{N,\beta}|u_n|^{\frac{N'}{1-\beta}}}-S_{j_\beta-1}\left(\alpha_{N,\beta}p_1\left|u_n\right|^{\frac{N'}{1-\beta}}\right)\right)\mathrm{d}x<+\infty,$$

which, in view of (6.1), leads to the expected contradiction. *Case* $\|u\|_{E_{q,\beta}}=1:$ Since $u_n \rightharpoonup u$ weakly in $E_{q,\beta}$ which is uniformly convex, then $u_n \to u$ strongly in $E_{q,\beta}$ (see [**12**, proposition 3.32]). We can easily adapt the arguments used in the proof of [**23**, proposition1] to deduce that there exists $v \in E_{q,\beta}$ such that, up to a subsequence, $|u_n(x)| \leqslant v(x)$ a.e. $x \in \mathbb{R}^N,\ \forall\, n$. Let $0 < p < +\infty$.

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We have

$$\begin{aligned}
&\sup_n \int_{\mathbb{R}^N} \left(\mathrm{e}^{p\alpha_{N,\beta}|u_n|^{\frac{N'}{1-\beta}}} - S_{j_\beta-1}\left(p\alpha_{N,\beta}\,|u_n|^{\frac{N'}{1-\beta}}\right)\right) \mathrm{d}x \\
&= \sup_n \sum_{j=j_\beta}^{+\infty} \int_{\mathbb{R}^N} \frac{(p\alpha_{N,\beta}\,|u_n|^{\frac{N'}{1-\beta}})^j}{j!}\,\mathrm{d}x \\
&\leqslant \sum_{j=j_\beta}^{+\infty} \int_{\mathbb{R}^N} \frac{(p\alpha_{N,\beta}\,|v|^{\frac{N'}{1-\beta}})^j}{j!}\,\mathrm{d}x \\
&= \int_{\mathbb{R}^N} \left(\mathrm{e}^{p\alpha_{N,\beta}|v|^{\frac{N'}{1-\beta}}} - S_{j_\beta-1}\left(p\alpha_{N,\beta}\,|v|^{\frac{N'}{1-\beta}}\right)\right) \mathrm{d}x < +\infty.
\end{aligned}$$

Now, we exhibit a sequence $(\xi_k)_k \subset E_{q,\beta}$ and a function $\xi \in E_{q,\beta}\backslash\{0\}$ such that $\|\xi_k\|_{E_{q,\beta}} = 1,\ \xi_k \rightharpoonup \xi$ weakly in $E_{q,\beta},\ \|\xi\|_{E_{q,\beta}} < 1$, and

$$\begin{aligned}
&\int_{\mathbb{R}^N} \left(\mathrm{e}^{\alpha_{N,\beta}p|\xi_k|^{\frac{N'}{1-\beta}}} - S_{j_\beta-1}\left(\alpha_{N,\beta}p\,|\xi_k|^{\frac{N'}{1-\beta}}\right)\right) \mathrm{d}x \\
&\quad \to +\infty,\ k \to +\infty,\ \forall\ p > P_{N,\beta}(\xi).
\end{aligned}$$

For $k \in \mathbb{N}\backslash\{0\}$ and $\lambda \in]0,+\infty[$ to be fixed later, we define the function $\psi_k : \mathbb{R} \to \mathbb{R}$ by

$$\psi_k(t) = \lambda \begin{cases} \dfrac{t^{1-\beta} - (1/2)^{1-\beta}}{(k^{1-\beta} - (1/2)^{1-\beta})^{\frac{1}{N}}}, & 1/2 \leqslant t \leqslant k, \\ \left(k^{1-\beta} - (1/2)^{1-\beta}\right)^{\frac{1}{N'}}, & t \geqslant k, \\ 0, & t \leqslant 1/2. \end{cases}$$

We also define the function $\psi : [0,+\infty[\to [0,+\infty[$ by

$$\psi(t) = a \begin{cases} 0, & t \leqslant 1/4, \\ \dfrac{t^{1-\beta} - (1/4)^{1-\beta}}{((1/2)^{1-\beta} - (1/4)^{1-\beta})^{\frac{1}{N}}}, & 1/4 \leqslant t \leqslant 1/2, \\ \left((1/2)^{1-\beta} - (1/4)^{1-\beta}\right)^{\frac{1}{N'}}, & t \geqslant 1/2, \end{cases}$$

where $a > 0$. Set $u_k(x) = \omega_{N-1}^{-\frac{1}{N}}\psi_k(t),\ u(x) = \omega_{N-1}^{-\frac{1}{N}}\psi(t),\ |x| = \mathrm{e}^{-t},\ t \in \mathbb{R}$. We have

$$\begin{aligned}
\int_{\mathbb{R}^N} |\nabla u_k|^N w_\beta(x)\,\mathrm{d}x &= \int_0^{+\infty} t^{\beta(N-1)}\,|\psi_k'(t)|^N\,\mathrm{d}t \\
&= \lambda^N \int_{1/2}^{k} \frac{t^{\beta(N-1)}(1-\beta)^N t^{-\beta N}}{k^{1-\beta} - (1/2)^{1-\beta}}\,\mathrm{d}t \\
&= \lambda^N (1-\beta)^{N-1}.
\end{aligned}$$

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Furthermore,

$$\int_{\mathbb{R}^N} |\nabla u_k|^q \, \mathrm{d}x = \omega_{N-1}^{1-\frac{q}{N}} \int_{1/2}^{k} |\psi_k'(t)|^q \, \mathrm{e}^{(q-N)t} \, \mathrm{d}t$$

$$= \frac{\lambda^q \omega_{N-1}^{1-\frac{q}{N}} (1-\beta)^q}{(k^{1-\beta} - (1/2)^{1-\beta})^{\frac{q}{N}}} \int_{1/2}^{k} \frac{\mathrm{e}^{(q-N)t}}{t^{q\beta}} \, \mathrm{d}t.$$

Clearly,

$$\int_{\mathbb{R}^N} |\nabla u_k|^q \, \mathrm{d}x \to 0, \; k \to +\infty.$$

Set $v_k = u + u_k$. It yields,

$$\|v_k\|_{E_{q,\beta}}^N = \int_{\mathbb{R}^N} |\nabla u_k|^N w_\beta(x) \, \mathrm{d}x + \int_{\mathbb{R}^N} |\nabla u|^N w_\beta(x) \, \mathrm{d}x$$

$$+ \left(\int_{\mathbb{R}^N} |\nabla u_k|^q \, \mathrm{d}x + \int_{\mathbb{R}^N} |\nabla u|^q \, \mathrm{d}x \right)^{\frac{N}{q}}$$

$$= (1-\beta)^{N-1}(a^N + \lambda^N) + \left(\int_{\mathbb{R}^N} |\nabla u_k|^q \, \mathrm{d}x + \int_{\mathbb{R}^N} |\nabla u|^q \, \mathrm{d}x \right)^{\frac{N}{q}}.$$

Consequently,

$$\|v_k\|_{E_{q,\beta}}^N \to (1-\beta)^{N-1}(a^N + \lambda^N) + \left(\int_{\mathbb{R}^N} |\nabla u|^q \, \mathrm{d}x \right)^{\frac{N}{q}}$$

$$= (1-\beta)^{N-1}\lambda^N + \|u\|_{E_{q,\beta}}^N, \; k \to +\infty,$$

where we used the fact that

$$\|u\|_{E_{q,\beta}}^N = \int_{\mathbb{R}^N} |\nabla u|^N w_\beta(x) \, \mathrm{d}x + \left(\int_{\mathbb{R}^N} |\nabla u|^q \, \mathrm{d}x \right)^{\frac{N}{q}}$$

$$= (1-\beta)^{N-1} a^N + \left(\int_{\mathbb{R}^N} |\nabla u|^q \, \mathrm{d}x \right)^{\frac{N}{q}}.$$

Choose $\lambda > 0$ and $a > 0$ such that

$$(1-\beta)^{N-1}\lambda^N + \|u\|_{E_{q,\beta}}^N = 1. \tag{6.12}$$

One can easily see that $u_k \rightharpoonup 0$ weakly in $E_{q,\beta}$. By (6.12), we derive that $\frac{v_k}{\|v_k\|_{E_{q,\beta}}} \rightharpoonup u$ weakly in $E_{q,\beta}$. Let $p > P_{N,\beta}(u)$. Then, there exists $\epsilon > 0$ such that $p = (1+\epsilon)P_{N,\beta}(u)$. We have

$$
\begin{aligned}
&\int_{\mathbb{R}^N} \left(\mathrm{e}^{\alpha_{N,\beta} p \left| \frac{v_k}{\|v_k\|_{E_{q,\beta}}} \right|^{\frac{N'}{1-\beta}}} - S_{j_\beta -1}\left(\alpha_{N,\beta} p \left| \frac{v_k}{\|v_k\|_{E_{q,\beta}}} \right|^{\frac{N'}{1-\beta}} \right) \right) \mathrm{d}x \\
&\geqslant \omega_{N-1} \int_k^{+\infty} \exp\left(\frac{\alpha_{N,\beta} p \omega_{N-1}^{-\frac{1}{(N-1)(1-\beta)}} |\psi(t)+\psi_k(t)|^{\frac{N'}{1-\beta}}}{\|v_k\|_{E_{q,\beta}}^{\frac{N'}{1-\beta}}} \right) \mathrm{e}^{-Nt}\,\mathrm{d}t \\
&\quad - \sum_{j=0}^{j_\beta -1} \frac{(\alpha_{N,\beta} p)^j}{j!} \frac{1}{\|v_k\|_{E_{q\beta}}^{\frac{jN'}{1-\beta}}} \int_{\mathbb{R}^N} |v_k|^{\frac{jN'}{1-\beta}}\,\mathrm{d}x \\
&= \omega_{N-1} \frac{\mathrm{e}^{-Nk}}{N} \exp\Bigg(\theta_k \Big(a \left((1/2)^{1-\beta} - (1/4)^{1-\beta} \right)^{\frac{1}{N'}} \\
&\quad + \lambda \left(k^{1-\beta} - (1/2)^{1-\beta} \right)^{\frac{1}{N'}} \Big)^{\frac{N'}{1-\beta}} \Bigg) \\
&\quad - \sum_{j=0}^{j_\beta -1} \frac{(\alpha_{N,\beta} p)^j}{j!} \frac{1}{\|v_k\|_{E_{q,\beta}}^{\frac{jN'}{1-\beta}}} \int_{\mathbb{R}^N} |v_k|^{\frac{jN'}{1-\beta}}\,\mathrm{d}x, \qquad (6.13)
\end{aligned}
$$

where

$$
\begin{aligned}
\theta_k = \frac{\alpha_{N,\beta} p \omega_{N-1}^{-\frac{1}{(N-1)(1-\beta)}}}{\|v_k\|_{E_{q,\beta}}^{\frac{N'}{1-\beta}}} &\to N(1-\beta)^{\frac{1}{1-\beta}} p \\
&= N(1-\beta)^{\frac{1}{1-\beta}} (1+\epsilon) P_{N,\beta}(u),\ \ k \to +\infty.
\end{aligned}
$$

On the other hand, by (6.12), we have

$$
\begin{aligned}
P_{N,\beta}(u) &= \frac{1}{\left(1 - \|u\|_{E_{q,\beta}}^N\right)^{\frac{1}{(N-1)(1-\beta)}}} = \frac{1}{\left((1-\beta)^{N-1}\lambda^N\right)^{\frac{1}{(N-1)(1-\beta)}}} \\
&= \frac{1}{(1-\beta)^{\frac{1}{1-\beta}} \lambda^{\frac{N'}{1-\beta}}}.
\end{aligned}
$$

Thus,

$$
\theta_k \left(a \left((1/2)^{1-\beta} - (1/4)^{1-\beta} \right)^{\frac{1}{N'}} + \lambda \left(k^{1-\beta} - (1/2)^{1-\beta} \right)^{\frac{1}{N'}} \right)^{\frac{N'}{1-\beta}} \underset{k\to+\infty}{\sim} (1+\epsilon)Nk.
$$

For $0 \leqslant j \leqslant j_\beta - 1$, we have

$$
\int_{\mathbb{R}^N} |v_k|^{\frac{jN'}{1-\beta}}\,\mathrm{d}x = \omega_{N-1}^{1-\frac{j}{(N-1)(1-\beta)}} \int_{1/4}^{+\infty} |\psi(t)+\psi_k(t)|^{\frac{jN'}{1-\beta}} \mathrm{e}^{-Nt}\,\mathrm{d}t.
$$

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Hence,

$$\sup_{k\geqslant 1}\frac{1}{\|v_k\|_{E_{q,\beta}}^{\frac{jN'}{1-\beta}}}\int_{\mathbb{R}^N}|v_k|^{\frac{jN'}{1-\beta}}\,\mathrm{d}x<+\infty.$$

We finally deduce from (6.13) that

$$\int_{\mathbb{R}^N}\left(\mathrm{e}^{\alpha_{N,\beta}p\left|\frac{v_k}{\|v_k\|_{E_{q,\beta}}}\right|^{\frac{N'}{1-\beta}}}-S_{j_\beta-1}\left(\alpha_{N,\beta}p\left|\frac{v_k}{\|v_k\|_{E_{q,\beta}}}\right|^{\frac{N'}{1-\beta}}\right)\right)\mathrm{d}x$$
$$\to+\infty,\ k\to+\infty.$$

2. Now, we treat the case $\beta=1$. For the first part of the proof (i.e., inequality (1.27)), we can adapt the contradiction argument used for the case $\beta<1$. But, due to the existence of some essential technical difference, we give the proof with a minimum of details. Observing that, as previously, the case $\|u\|_{E_{q,1}}=1$ can be easily studied using the uniform convexity of the functional space $E_{q,1}$, we can assume that $0<\|u\|_{E_{q,1}}<1$. Assume by contradiction that there exists $0<p_1<\left(\frac{1}{1-\|u\|_{E_{q,1}}^N}\right)^{\frac{1}{N-1}}$ such that

$$\sup_n\int_{\mathbb{R}^N}\left(\mathrm{e}^{N\left(\mathrm{e}^{\omega_{N-1}^{\frac{1}{N-1}}p_1|u_n|^{N'}}-1\right)}-S_{j_1-1}\left(N\left(\mathrm{e}^{\omega_{N-1}^{\frac{1}{N-1}}p_1|u_n|^{N'}}-1\right)\right)\right)\mathrm{d}x$$
$$=+\infty.$$

Arguing as for the case $0<\beta<1$, we can easily find $\epsilon>0$ small enough and L large enough such that

$$\limsup_{n\to+\infty}\left(\int_{\mathbb{R}^N}|\nabla T_L(u_n)|^N w_1(x)\,\mathrm{d}x+\left(\int_{\mathbb{R}^N}|\nabla T_L(u_n)|^q\,\mathrm{d}x\right)^{\frac{N}{q}}\right)$$
$$<\left(\frac{1}{p_1(1+\epsilon)^2}\right)^{N-1}. \tag{6.14}$$

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Here, the function T_L is defined by (6.2). Set again $\Omega_{n,L} = \{x \in \mathbb{R}^N,\ |u_n(x)| \geqslant L\}$. Using (3.3) and Young's inequality, it yields

$$\begin{aligned}
&\int_{\Omega_{n,L}\cap\mathcal{B}} \left(\mathrm{e}^{N\left(\mathrm{e}^{\omega_{N-1}^{\frac{1}{N-1}} p_1 |u_n|^{N'}} - 1\right)} - S_{j_1-1}\left(N\left(\mathrm{e}^{\omega_{N-1}^{\frac{1}{N-1}} p_1 |u_n|^{N'}} - 1\right)\right)\right) \mathrm{d}x \\
&\leqslant \int_{\Omega_{n,L}\cap\mathcal{B}} \mathrm{e}^{N\left(\mathrm{e}^{\omega_{N-1}^{\frac{1}{N-1}} p_1 |u_n|^{N'}} - 1\right)} \mathrm{d}x \\
&\leqslant \mathrm{e}^{-N} \int_{\Omega_{n,L}\cap\mathcal{B}} \mathrm{e}^{N \mathrm{e}^{\omega_{N-1}^{\frac{1}{N-1}} (1+\epsilon) p_1 |u_n - L|^{N'}} \mathrm{e}^{\omega_{N-1}^{\frac{1}{N-1}} p_1 A_1(\epsilon) L^{N'}}} \mathrm{d}x \\
&\leqslant \int_{\Omega_{n,L}\cap\mathcal{B}} \exp\left(N\left[\frac{\mathrm{e}^{\omega_{N-1}^{\frac{1}{N-1}} p_1 (1+\epsilon)^2 |T_L(u_n)|^{N'}}}{1+\epsilon} + \frac{\epsilon}{1+\epsilon}\mathrm{e}^{\omega_{N-1}^{\frac{1}{N-1}} p_1 \frac{1+\epsilon}{\epsilon} A_1(\epsilon) L^{N'}}\right]\right) \mathrm{d}x \\
&\leqslant \mathrm{e}^{N \mathrm{e}^{\omega_{N-1}^{\frac{1}{N-1}} p_1 \frac{1+\epsilon}{\epsilon} A_1(\epsilon) L^{N'}}} \int_{\mathcal{B}} \mathrm{e}^{N \mathrm{e}^{\omega_{N-1}^{\frac{1}{N-1}} p_1 (1+\epsilon)^2 |T_L(u_n)|^{N'}}} \mathrm{d}x, \qquad (6.15)
\end{aligned}$$

where $A_1(\epsilon) = \frac{1+\epsilon}{((1+\epsilon)^{\frac{1}{N-1}} - 1)^{N-1}}$. By (6.14), we know that

$$\int_{\mathcal{B}} \left|\nabla (p_1 (1+\epsilon)^2)^{\frac{1}{N'}} T_L(u_n)\right|^N w_1(x)\, \mathrm{d}x < 1.$$

Using (1.4), we get

$$\sup_n \int_{\mathcal{B}} \mathrm{e}^{N \mathrm{e}^{\omega_{N-1}^{\frac{1}{N-1}} p_1 (1+\epsilon)^2 |T_L(u_n)|^{N'}}} \mathrm{d}x < +\infty.$$

By (6.15), we obtain

$$\begin{aligned}
\sup_n \int_{\Omega_{n,L}\cap\mathcal{B}} \Bigg(&\mathrm{e}^{N\left(\mathrm{e}^{\omega_{N-1}^{\frac{1}{N-1}} p_1 |u_n|^{N'}} - 1\right)} \\
&- S_{j_1-1}\left(N\left(\mathrm{e}^{\omega_{N-1}^{\frac{1}{N-1}} p_1 |u_n|^{N'}} - 1\right)\right)\Bigg) \mathrm{d}x < +\infty.
\end{aligned}$$

The boundedness of the sequence

$$\int_{\Omega_{n,L}^c} \left(\mathrm{e}^{N\left(\mathrm{e}^{\omega_{N-1}^{\frac{1}{N-1}} p_1 |u_n|^{N'}} - 1\right)} - S_{j_1-1}\left(N\left(\mathrm{e}^{\omega_{N-1}^{\frac{1}{N-1}} p_1 |u_n|^{N'}} - 1\right)\right)\right) \mathrm{d}x,$$

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can be established by proceeding as for the case $0 < \beta < 1$ and the details are omitted. In order to prove (1.28), we consider the sequence of functions $\psi_k : \mathbb{R} \to \mathbb{R}$ defined by

$$\psi_k(t) = \begin{cases} \dfrac{\log(t+1) - \log(3/2)}{(\log(k+1) - \log(3/2))^{\frac{1}{N}}}, & 1/2 \leqslant t \leqslant k, \\ (\log(k+1) - \log(3/2))^{\frac{1}{N'}}, & t \geqslant k, \\ 0, & t \leqslant 1/2. \end{cases} \tag{6.16}$$

We also define the function $\psi : \mathbb{R} \to \mathbb{R}$ by

$$\psi(t) = \begin{cases} 0, & t \leqslant 1/4, \\ \dfrac{\log(t+1) - \log(5/4)}{(\log(3/2) - \log(5/4))^{\frac{1}{N}}}, & 1/4 \leqslant t \leqslant 1/2, \\ (\log(3/2) - \log(5/4))^{\frac{1}{N'}}, & t \geqslant 1/2. \end{cases}$$

Set, as for the first case, $u_k(x) = \omega_{N-1}^{-\frac{1}{N}} \psi_k(t)$ and $u(x) = \omega_{N-1}^{-\frac{1}{N}} \psi(t),\ |x| = \mathrm{e}^{-t}$. Observe that

$$\int_{\mathbb{R}^N} |\nabla u_k|^q \,\mathrm{d}x \to 0,\ k \to +\infty.$$

Set, $v_k = u + u_k$. We have,

$$\|v_k\|_{E_{q,1}}^N = \|u_k + u\|_{E_{q,1}}^N = 2 + \left(\int_{\mathbb{R}^N} |\nabla u_k|^q \,\mathrm{d}x + \int_{\mathbb{R}^N} |\nabla u|^q \,\mathrm{d}x \right)^{\frac{N}{q}}.$$

Thus,

$$\|v_k\|_{E_{q,1}}^N \to 2 + \left(\int_{\mathbb{R}^N} |\nabla u|^q \,\mathrm{d}x \right)^{\frac{N}{q}} = 1 + \|u\|_{E_{q,1}}^N.$$

Since $u_k \rightharpoonup 0$ weakly in $E_{q,1}$, then

$$\frac{v_k}{\|v_k\|_{E_{q,1}}} \rightharpoonup \widetilde{u} = \frac{u}{\left(1 + \|u\|_{E_{q,1}}^N\right)^{\frac{1}{N}}}.$$

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Let $\alpha > 0$ and $p > P_{N,1}(\widetilde{u}) = (1 + \|u\|^N_{E_{q,1}})^{\frac{1}{N-1}}$. We have

$$\int_{\mathbb{R}^N} \left(\mathrm{e}^{\alpha\left(\mathrm{e}^{\omega_{N-1}^{\frac{1}{N-1}} p\left|\frac{v_k(x)}{\|v_k\|_{E_{q,1}}}\right|^{N'}}-1\right)} - S_{j_1-1}\left(\alpha\left(\mathrm{e}^{\omega_{N-1}^{\frac{1}{N-1}} p\left|\frac{v_k(x)}{\|v_k\|_{E_{q,1}}}\right|^{N'}}-1\right)\right)\right) \mathrm{d}x$$

$$\geqslant \omega_{N-1} \int_k^{+\infty} \mathrm{e}^{\alpha\left(\mathrm{e}^{p\frac{\log(k+1)-\log(3/2)}{\left(2+\left(|\nabla u|^q_{L^q(\mathbb{R}^N)}+|\nabla u_k|^q_{L^q(\mathbb{R}^N)}\right)^{\frac{N}{q}}\right)^{\frac{1}{N-1}}}}-1\right)} \mathrm{e}^{-Nt}\,\mathrm{d}t$$

$$- \int_{\mathbb{R}^N} S_{j_1-1}\left(\alpha\left(\mathrm{e}^{\omega_{N-1}^{\frac{1}{N-1}} p\left|\frac{v_k(x)}{\|v_k\|_{E_{q,1}}}\right|^{N'}}-1\right)\right) \mathrm{d}x$$

$$= \omega_{N-1}\, \mathrm{e}^{\alpha\left(\mathrm{e}^{p\frac{\log(k+1)-\log(3/2)}{\left(2+\left(|\nabla u|^q_{L^q(\mathbb{R}^N)}+|\nabla u_k|^q_{L^q(\mathbb{R}^N)}\right)^{\frac{N}{q}}\right)^{\frac{1}{N-1}}}}-1\right)} \frac{\mathrm{e}^{-Nk}}{N}$$

$$- \int_{\mathbb{R}^N} S_{j_1-1}\left(\alpha\left(\mathrm{e}^{\omega_{N-1}^{\frac{1}{N-1}} p\left|\frac{v_k(x)}{\|v_k\|_{E_{q,1}}}\right|^{N'}}-1\right)\right) \mathrm{d}x. \tag{6.17}$$

Since $p > (1 + \|u\|^N_{E_{q,1}})^{\frac{1}{N-1}}$, then

$$\lim_{k\to+\infty}\left(p\frac{\log(k+1)-\log(3/2)}{\left(2+\left(|\nabla u|^q_{L^q(\mathbb{R}^N)}+|\nabla u_k|^q_{L^q(\mathbb{R}^N)}\right)^{\frac{N}{q}}\right)^{\frac{1}{N-1}}} - \log k\right) = +\infty.$$

It follows,

$$\alpha\left(\mathrm{e}^{p\frac{\log(k+1)-\log(3/2)}{\left(2+\left(|\nabla u|^q_{L^q(\mathbb{R}^N)}+|\nabla u_k|^q_{L^q(\mathbb{R}^N)}\right)^{\frac{N}{q}}\right)^{\frac{1}{N-1}}}} - 1\right) - Nk \to +\infty,\ k \to +\infty.$$

Finally, one can easily show that

$$\sup_{k\geqslant 1}\int_{\mathbb{R}^N} S_{j_1-1}\left(\alpha\left(\mathrm{e}^{\omega_{N-1}^{\frac{1}{N-1}} p\left|\frac{v_k(x)}{\|v_k\|_{E_{q,1}}}\right|^{N'}}-1\right)\right) \mathrm{d}x < +\infty.$$

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In view of (6.17), we deduce that (1.28) holds. This ends the proof of theorem 1.5.

7. Proof of theorem 1.6

We start with the case $0<\beta<1$. As in the proof of theorem 1.5, we argue by contradiction. So, assume that there exists $0<p_1<(\frac{1}{1-\|u\|_{q,\beta}})^{\frac{N'}{q(1-\beta)}}$ such that

$$\sup_n \int_{\mathbb{R}^N}\left(e^{p_1\alpha_{N,\beta}|u_n|^{\frac{N'}{1-\beta}}}-S_{j_\beta-1}\left(p_1\alpha_{N,\beta}\,|u_n|^{\frac{N'}{1-\beta}}\right)\right)\mathrm{d}x=+\infty.$$

The function T_L and G_L being defined by (6.2), it is easy to see that there exist $0<\epsilon<1$ small enough and $L>0$ large enough such that

$$(p_1(1+\epsilon))^{\frac{1-\beta}{N'}}<\left(1-\|G_L(u)\|_{q,\beta}^q\right)^{-\frac{1}{q}}. \tag{7.1}$$

The keystone of the proof is to establish the inequality

$$\limsup_{n\to+\infty}\|T_L(u_n)\|_{q,\beta}<\left(\frac{1}{p_1(1+\epsilon)}\right)^{\frac{1-\beta}{N'}}. \tag{7.2}$$

For that aim, we argue once again by contradiction. So, we assume that there exists a subsequence of $(u_n)_n$, still denoted by $(u_n)_n$, such that

$$\|T_L(u_n)\|_{q,\beta}\geqslant\left(\frac{1}{p_1(1+\epsilon)}\right)^{\frac{1-\beta}{N'}},\ \forall\, n\geqslant 0. \tag{7.3}$$

First, observe that the general form of (2.8) is given by the following inequality

$$(a+b)^t\leqslant(1+\delta)a^t+D_t(\delta)b^t,\ \forall\, t>1,\ \forall\, a,b\geqslant 0,\ \forall\,\delta>0,$$

where $D_t(\delta)=\frac{1+\delta}{((1+\delta)^{\frac{1}{t-1}}-1)^{t-1}}$. From that last inequality, we can easily deduce another useful inequality, that is

$$(a^t+b^t)^{\frac{1}{t}}\geqslant(1+\delta)^{-\frac{1}{t}}a+(D_t(\delta))^{-\frac{1}{t}}b,\ \forall\, t>1,\ \forall\, a,b\geqslant 0,\ \forall\,\delta>0. \tag{7.4}$$

Let $\delta>0$. Applying (7.4), we obtain

$$\begin{aligned}|\nabla u_n|_{L^N_{w_\beta}(\mathbb{R}^N)}&=\left(\int_{\mathbb{R}^N}|\nabla u_n|^N w_\beta\,\mathrm{d}x\right)^{\frac{1}{N}}\\&=\left(\int_{\mathbb{R}^N}|\nabla T_L(u_n)|^N w_\beta\,\mathrm{d}x+\int_{\mathbb{R}^N}|\nabla G_L(u_n)|^N w_\beta\,\mathrm{d}x\right)^{\frac{1}{N}}\\&=\left(|\nabla T_L(u_n)|^N_{L^N_{w_\beta}(\mathbb{R}^N)}+|\nabla G_L(u_n)|^N_{L^N_{w_\beta}(\mathbb{R}^N)}\right)^{\frac{1}{N}}\\&\geqslant(1+\delta)^{-\frac{1}{N}}\,|\nabla T_L(u_n)|_{L^N_{w_\beta}(\mathbb{R}^N)}+(D_N(\delta))^{-\frac{1}{N}}\,|\nabla G_L(u_n)|_{L^N_{w_\beta}(\mathbb{R}^N)}.\end{aligned} \tag{7.5}$$

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In a similar way, we have

$$
\begin{aligned}
|\nabla u_n|_{L^q(\mathbb{R}^N)} &= \left(\int_{\mathbb{R}^N} |\nabla u_n|^q \,\mathrm{d}x\right)^{\frac{1}{q}} \\
&= \left(\int_{\mathbb{R}^N} |\nabla T_L(u_n)|^q \,\mathrm{d}x + \int_{\mathbb{R}^N} |\nabla G_L(u_n)|^q \,\mathrm{d}x\right)^{\frac{1}{q}} \\
&= \left(|\nabla T_L(u_n)|^q_{L^q(\mathbb{R}^N)} + |\nabla G_L(u_n)|^q_{L^q(\mathbb{R}^N)}\right)^{\frac{1}{q}} \\
&\geqslant (1+\delta)^{-\frac{1}{q}} |\nabla T_L(u_n)|_{L^q(\mathbb{R}^N)} + (D_q(\delta))^{-\frac{1}{q}} |\nabla G_L(u_n)|_{L^q(\mathbb{R}^N)}. \qquad (7.6)
\end{aligned}
$$

A simple analysis shows that the function defined on $]1,+\infty[$ by

$$
t \longmapsto \left(\frac{z}{(z^{\frac{1}{t-1}}-1)^{t-1}}\right)^{-\frac{1}{t}} = \left(1 - z^{\frac{1}{1-t}}\right)^{\frac{1-t}{t}},
$$

where z is some fixed real number such that $z > 1$, is nondecreasing. Thus,

$$
(D_N(\delta))^{-\frac{1}{N}} \geqslant (D_q(\delta))^{-\frac{1}{q}}.
$$

From (7.5) and (7.6), we get

$$
\begin{aligned}
1 = \|u_n\|_{q,\beta} &= |\nabla u_n|_{L^N_{w_\beta}(\mathbb{R}^N)} + |\nabla u_n|_{L^q(\mathbb{R}^N)} \\
&\geqslant (1+\delta)^{-\frac{1}{q}} \left(|\nabla T_L(u_n)|_{L^N_{w_\beta}(\mathbb{R}^N)} + |\nabla T_L(u_n)|_{L^q(\mathbb{R}^N)}\right) \\
&\quad + (D_q(\delta))^{-\frac{1}{q}} \left(|\nabla G_L(u_n)|_{L^N_{w_\beta}(\mathbb{R}^N)} + |\nabla G_L(u_n)|_{L^q(\mathbb{R}^N)}\right) \\
&\geqslant (1+\delta)^{-\frac{1}{q}} \|T_L(u_n)\|_{q,\beta} + (D_q(\delta))^{-\frac{1}{q}} \|G_L(u_n)\|_{q,\beta}.
\end{aligned}
$$

Putting (7.3) in that last inequality and using the fact that

$$
\liminf_{n\to+\infty} \|G_L(u_n)\|_{q,\beta} \geqslant \|G_L(u)\|_{q,\beta},
$$

we infer

$$
(p_1(1+\epsilon))^{\frac{1-\beta}{N'}} \geqslant \frac{(1+\delta)^{-\frac{1}{q}}}{1 - \|G_L(u)\|_{q,\beta} (D_q(\delta))^{-\frac{1}{q}}}. \qquad (7.7)
$$

Now, consider the function defined on $]1,+\infty[$ by

$$
x \longmapsto \frac{1}{x - \|G_L(u)\|_{q,\beta} \left(x^{\frac{q}{q-1}} - 1\right)^{\frac{q-1}{q}}}.
$$

A quick analysis of this function shows that it attains its maximum at the point

$$
x_0 = \frac{1}{\left(1 - \|G_L(u)\|^q_{q,\beta}\right)^{\frac{q-1}{q}}}
$$

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and this maximum is

$$\frac{1}{\left(1-\|G_L(u)\|_{q,\beta}^q\right)^{\frac{1}{q}}}.$$

Hence,

$$\begin{aligned}
&\max_{x>1} \frac{x^{-\frac{1}{q}}}{1-\|G_L(u)\|_{q,\beta}\left(\frac{x}{(x^{\frac{1}{q-1}}-1)^{q-1}}\right)^{-\frac{1}{q}}} \\
&= \max_{x>1} \frac{1}{x^{\frac{1}{q}}-\|G_L(u)\|_{q,\beta}(x^{\frac{1}{q-1}}-1)^{\frac{q-1}{q}}} \\
&= \max_{x>1} \frac{1}{x-\|G_L(u)\|_{q,\beta}\left(x^{\frac{q}{q-1}}-1\right)^{\frac{q-1}{q}}} \\
&= \frac{1}{\left(1-\|G_L(u)\|_{q,\beta}^q\right)^{\frac{1}{q}}}.
\end{aligned}$$

Consequently, the function defined on $]0,+\infty[$ by

$$\delta \longmapsto \frac{(1+\delta)^{-\frac{1}{q}}}{1-\|G_L(u)\|_{q,\beta}(D_q(\delta))^{\frac{-1}{q}}}$$

attains its maximum at the point $\delta_0>0$ given by the identity

$$(1+\delta_0)^{\frac{1}{q}} = \frac{1}{\left(1-\|G_L(u)\|_{q,\beta}^q\right)^{\frac{q-1}{q}}},$$

and this maximum is

$$\frac{1}{\left(1-\|G_L(u)\|_{q,\beta}^q\right)^{\frac{1}{q}}}.$$

Thus, choosing $\delta=\delta_0$ in (7.7), it comes

$$(p_1(1+\epsilon))^{\frac{1-\beta}{N'}} \geqslant \frac{1}{\left(1-\|G_L(u)\|_{q,\beta}^q\right)^{\frac{1}{q}}},$$

which is in contradiction with (7.1). Therefore, (7.2) holds. The rest of the proof is similar to what has been done in the proof of theorem 1.5 (with suitable adaptation) and, in order to avoid redundancy, the details will be omitted. For the case $\beta=1$, we can easily adapt the same arguments used previously for the case $0<\beta<1$ to

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prove that there exists $\epsilon > 0$ small enough and $L > 0$ large enough such that

$$\limsup_{n\to+\infty} \|T_L(u_n)\|_{q,1} < \left(\frac{1}{p_1(1+\epsilon)^2}\right)^{\frac{1}{N'}}.$$

The rest of the proof of (1.31) is similar to the proof of (1.27) and will be omitted. The same can be said concerning (1.32) whose proof is similar to (1.28). This ends the proof of theorem 1.6.

8. Applications to some elliptic equations

In this section, we deal with the following elliptic equation:

$$-\operatorname{div}\left(w_1(x)\,|\nabla u|^{N-2}\,\nabla u\right) - \Delta_q u = f(u), \text{ in } \mathbb{R}^N, \ N \geqslant 2, \ 1 < q < N, \tag{8.1}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a continuous function such that $f(s) = 0, \ \forall\ s \leqslant 0$. Here, we assume that the weight given by (1.13) satisfies

$$\inf_{x\in\mathbb{R}^N} w_1(x) > 0.$$

By this assumption, it yields $E_{q,1} \hookrightarrow E_r^{N,q}$ with continuous embedding where $E_r^{N,q}$ is defined in [19] as the subspace of radial functions of the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$u \longmapsto \left(\int_{\mathbb{R}^N} |\nabla u|^N \, \mathrm{d}x + \left(\int_{\mathbb{R}^N} |\nabla u|^q \, \mathrm{d}x\right)^{\frac{N}{q}}\right)^{\frac{1}{N}}, \ u \in E_r^{N,q}.$$

By [19, proposition 2.1], we know that $E_r^{N,q}$ is continuously (resp. compactly) embedded into $L^t(\mathbb{R}^N)$, $\forall\ q^* \leqslant t < +\infty$ (resp. $\forall\ q^* < t < +\infty$). Consequently, the embedding $E_{q,1} \hookrightarrow L^t(\mathbb{R}^N)$ is continuous for $q^* \leqslant t < +\infty$ and compact for $q^* < t < +\infty$. We assume that f has a critical double exponential growth at infinity, that is there exists a constant $\alpha_0 > 0$ such that

$$\lim_{s\to+\infty} \frac{f(s)}{\mathrm{e}^{N\mathrm{e}^{\alpha s^{N'}}}} = \begin{cases} 0, & \text{if } \alpha > \alpha_0, \\ +\infty, & \text{if } \alpha < \alpha_0. \end{cases} \tag{8.2}$$

We also assume that: (F_1) There exists $\theta > N$ such that

$$0 < \theta F(s) = \theta \int_0^s f(t)\,\mathrm{d}t \leqslant f(s)s, \ \forall\ s > 0.$$

(F_2) There exist $C > 0, \ s_1 > 0$ and $p > \max\{q^*, N\}$ such that

$$f(s) \leqslant C s^{p-1}, \ \forall\ 0 \leqslant s \leqslant s_1.$$

(F_3) There exist $A > 0$ and $r > q$ such that

$$F(s) \geqslant A s^r, \ \forall\ s \geqslant 0.$$

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EXAMPLE 8.1. An example of a function f satisfying the conditions $(F_1)-(F_3)$ is given by: $f(s)=F'(s)$, where

$$F(s)=s^m\left(\mathrm{e}^{N\left(\mathrm{e}^{\alpha s^{N'}}-1\right)}-1\right),\ \forall\ s\geqslant 0,\ F(s)=0,\ \forall\ s\leqslant 0,$$

with $\alpha>0,\ m>\max\{N,q^*\}$. A radial weak solution of the equation (8.1) is a function $u\in E_{q,1}$ such that

$$\int_{\mathbb{R}^N} w_1(x)\left|\nabla u\right|^{N-2}\nabla u\nabla v\,\mathrm{d}x+\int_{\mathbb{R}^N}\left|\nabla u\right|^{q-2}\nabla u\nabla v\,\mathrm{d}x=\int_{\mathbb{R}^N} f(u)v\,\mathrm{d}x,\ \forall\ v\in E_{q,1}.$$

THEOREM 8.2. *Assume that $(F_1)-(F_3)$ hold. Then, there exists $A_0>0$ such that the equation (8.1) has at least one nontrivial and nonnegative radial weak solution for all $A>A_0$.*

The energy functional associated to (8.1) is

$$I(u)=\frac{1}{N}\int_{\mathbb{R}^N}\left|\nabla u\right|^N w_1(x)\,\mathrm{d}x+\frac{1}{q}\int_{\mathbb{R}^N}\left|\nabla u\right|^q\,\mathrm{d}x-\int_{\mathbb{R}^N}F(u)\,\mathrm{d}x,\ u\in E_{q,1}.$$

LEMMA 8.3. *Assume that (F_1) and (F_2) hold. Then, the functional I satisfies the $(PS)_c$ condition for all $c<\left(\frac{1}{N}-\frac{1}{\theta}\right)\min\left\{1,\frac{\omega_{N-1}}{\alpha_0^{N-1}2^{N-1}}\right\}$.*

Proof. For the simplicity in notation, set

$$\Phi_1(s)=e^s-S_{j_1-1}(s),\ \forall\ s\geqslant 0.$$

Let $(u_n)_n\subset E_{q,1}$ be a (PS) sequence of I at a level $c<\left(\frac{1}{N}-\frac{1}{\theta}\right)\min\left\{1,\frac{\omega_{N-1}}{\alpha_0^{N-1}2^{N-1}}\right\}$. It yields

$$\begin{aligned}&\theta I(u_n)-\langle I'(u_n),u_n\rangle=\theta c+o_n(1)\left\|u_n\right\|_{q,1}\\&\Updownarrow\\&\left(\frac{\theta}{N}-1\right)\int_{\mathbb{R}^N}\left|\nabla u_n\right|^N w_1(x)\,\mathrm{d}x+\left(\frac{\theta}{q}-1\right)\int_{\mathbb{R}^N}\left|\nabla u_n\right|^q\,\mathrm{d}x\\&\quad+\int_{\mathbb{R}^N}\left(f(u_n)u_n-\theta F(u_n)\right)\mathrm{d}x=\theta c+o_n(1)\left\|u_n\right\|_{q,1},\end{aligned}$$

where $o_n(1)$ stands for any sequence of nonnegative real numbers converging to zero when n tends to $+\infty$. Since $\theta>N>q$, from (F_1) we can immediately deduce that

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$(u_n)_n$ is bounded. Moreover,

$$\limsup_{n\to+\infty}\left(|\nabla u_n|^N_{L^N_{w_1}(\mathbb{R}^N)}+|\nabla u_n|^q_{L^q(\mathbb{R}^N)}\right)\leqslant \frac{\theta c}{\frac{\theta}{N}-1}=\frac{N\theta c}{\theta-N}. \tag{8.3}$$

Taking into account that $\frac{N\theta c}{\theta-N}<1$, we deduce from (8.3) that, up to a subsequence,

$$|\nabla u_n|^N_{L^N_{w_1}(\mathbb{R}^N)}+|\nabla u_n|^q_{L^q(\mathbb{R}^N)}<1.$$

Thus,

$$\begin{aligned}|\nabla u_n|^N_{L^N_{w_1}(\mathbb{R}^N)}+|\nabla u_n|^q_{L^q(\mathbb{R}^N)} &\geqslant |\nabla u_n|^N_{L^N_{w_1}(\mathbb{R}^N)}+|\nabla u_n|^N_{L^q(\mathbb{R}^N)}\\ &\geqslant 2^{1-N}\left(|\nabla u_n|_{L^N_{w_1}(\mathbb{R}^N)}+|\nabla u_n|_{L^q(\mathbb{R}^N)}\right)^N\\ &\geqslant 2^{1-N}\left\|u_n\right\|^N_{q,1}.\end{aligned}$$

Putting that last inequality in (8.3), we get

$$\limsup_{n\to+\infty}\|u_n\|^N_{q,1}\leqslant\frac{N\theta 2^{N-1}c}{\theta-N}.$$

Then, there exist $\epsilon_0,\ \epsilon_1>0$ and a subsequence of $(u_n)_n$, still denoted by $(u_n)_n$, such that

$$\|u_n\|_{q,1}<\left(\frac{\omega_{N-1}^{\frac{1}{N-1}}}{(1+\epsilon_0)(1+\epsilon_1)\alpha_0}\right)^{\frac{1}{N'}},\ \forall\ n.$$

Let $u\in E_{q,1}$ be the weak limit of $(u_n)_n$ in $E_{q,1}$. We claim that, up to a subsequence, $u_n\to u$ strongly in $E_{q,1}$. Let $t>q^*-1$. By (8.2) and (F_2), we have

$$\begin{aligned}&\left|\int_{\mathbb{R}^N} f(u_n)(u_n-u)\,\mathrm{d}x\right|\\ &\leqslant c_6\int_{\mathbb{R}^N}|u_n|^{p-1}|u_n-u|\,\mathrm{d}x\\ &\quad+c_6\int_{\mathbb{R}^N}|u_n|^{t-1}|u_n-u|\,\Phi_1\left(N\left(\mathrm{e}^{(1+\epsilon_0)\alpha_0|u_n|^{N'}}-1\right)\right)\mathrm{d}x\\ &\leqslant c_6\left(\int_{\mathbb{R}^N}|u_n|^p\,\mathrm{d}x\right)^{\frac{p-1}{p}}\left(\int_{\mathbb{R}^N}|u_n-u|^p\,\mathrm{d}x\right)^{\frac{1}{p}}\\ &\quad+c_7\left(\int_{\mathbb{R}^N}|u_n|^{t+1}\right)^{\frac{t-1}{t+1}}\left(\int_{\mathbb{R}^N}|u_n-u|^{t+1}\,\mathrm{d}x\right)^{\frac{1}{t+1}}\\ &\quad\times\left(\int_{\mathbb{R}^N}\Phi_1\left((t+1)N\left(\mathrm{e}^{(1+\epsilon_0)\alpha_0|u_n|^{N'}}-1\right)\right)\mathrm{d}x\right)^{\frac{1}{t+1}},\ \forall\ n.\end{aligned} \tag{8.4}$$

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Clearly, there exist $c_8 > 0$ and $s_2 > 0$ such that

$$\Phi_1\left((t+1)N\left(\mathrm{e}^{(1+\epsilon_0)\alpha_0|s|^{N'}}-1\right)\right) \leqslant c_8\Phi_1\left(N\left(\mathrm{e}^{(1+\epsilon_0)(1+\epsilon_1)\alpha_0|s|^{N'}}-1\right)\right),\ \forall\ |s|\geqslant s_2.$$

Hence,

$$\begin{aligned}
&\int_{\mathbb{R}^N}\Phi_1\left((t+1)N\left(\mathrm{e}^{(1+\epsilon_0)\alpha_0|u_n|^{N'}}-1\right)\right)\mathrm{d}x\\
&=\int_{\{|u_n|\geqslant s_2\}}\Phi_1\left((t+1)N\left(\mathrm{e}^{(1+\epsilon_0)\alpha_0|u_n|^{N'}}-1\right)\right)\mathrm{d}x\\
&\quad+\int_{\{|u_n|<s_2\}}\Phi_1\left((t+1)N\left(\mathrm{e}^{(1+\epsilon_0)\alpha_0|u_n|^{N'}}-1\right)\right)\mathrm{d}x\\
&\leqslant c_8\int_{\mathbb{R}^N}\Phi_1\left(N\left(\mathrm{e}^{(1+\epsilon_0)(1+\epsilon_1)\alpha_0|u_n|^{N'}}-1\right)\right)\mathrm{d}x\\
&\quad+\int_{\{|u_n|<s_2\}}\Phi_1\left((t+1)N\left(\mathrm{e}^{(1+\epsilon_0)\alpha_0|u_n|^{N'}}-1\right)\right)\mathrm{d}x.
\end{aligned}\tag{8.5}$$

Plainly, there exists a positive constant κ such that

$$\mathrm{e}^{(1+\epsilon_0)\alpha_0 s^{N'}}-1\leqslant\kappa s^{N'},\ \forall\ 0\leqslant s\leqslant s_2.$$

Then, we can derive that

$$\begin{aligned}
&\int_{\{|u_n|<s_2\}}\Phi_1\left((t+1)N\left(\mathrm{e}^{(1+\epsilon_0)\alpha_0|u_n|^{N'}}-1\right)\right)\mathrm{d}x\\
&=\sum_{j=j_1}^{+\infty}\frac{(t+1)^jN^j}{j!}\int_{|u_n|<s_2}\left(\mathrm{e}^{(1+\epsilon_0)\alpha_0|u_n|^{N'}}-1\right)^j\,\mathrm{d}x\\
&\leqslant\sum_{j=j_1}^{+\infty}\frac{(t+1)^jN^j\kappa^j}{j!}\int_{|u_n|<s_2}|u_n|^{N'j}\ \mathrm{d}x.
\end{aligned}\tag{8.6}$$

Having in mind that if $j\geqslant j_1$, then $N'j\geqslant q^*$, it follows from (8.6) that

$$\begin{aligned}
&\int_{\{|u_n|<s_2\}}\Phi_1\left((t+1)N\left(\mathrm{e}^{(1+\epsilon_0)\alpha_0|u_n|^{N'}}-1\right)\right)\mathrm{d}x\\
&\leqslant\sum_{j=j_1}^{+\infty}\frac{(t+1)^jN^j\kappa^js_2^{N'j}}{j!}\int_{|u_n|<s_2}\left|\frac{u_n}{s_2}\right|^{N'j}\ \mathrm{d}x\\
&\leqslant\sum_{j=j_1}^{+\infty}\frac{(t+1)^jN^j\kappa^js_2^{N'j}}{j!}\int_{|u_n|<s_2}\left|\frac{u_n}{s_2}\right|^{q^*}\ \mathrm{d}x\\
&\leqslant\frac{e^{(t+1)N\kappa s_2^{N'}}}{s_2^{q^*}}\int_{\mathbb{R}^N}|u_n|^{q^*}\ \mathrm{d}x.
\end{aligned}$$

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Hence, there exists a positive constant $c_9 > 0$ such that

$$\int_{\{|u_n|<s_2\}} \Phi_1\left((t+1)N\left(\mathrm{e}^{(1+\epsilon_0)\alpha_0|u_n|^{N'}}-1\right)\right)\mathrm{d}x \leqslant c_9\int_{\mathbb{R}^N}|u_n|^{q^*}\,\mathrm{d}x.$$

Putting that last inequality in (8.5), we get

$$\begin{aligned}
&\int_{\mathbb{R}^N}\Phi_1\left((t+1)N\left(\mathrm{e}^{(1+\epsilon_0)\alpha_0|u_n|^{N'}}-1\right)\right)\mathrm{d}x\\
&\leqslant c_8\int_{\mathbb{R}^N}\Phi_1\left(N\left(\mathrm{e}^{(1+\epsilon_0)(1+\epsilon_1)\alpha_0|u_n|^{N'}}-1\right)\right)\mathrm{d}x + c_9\int_{\mathbb{R}^N}|u_n|^{q^*}\,\mathrm{d}x\\
&= c_8\int_{\mathbb{R}^N}\Phi_1\left(N\left(\mathrm{e}^{(1+\epsilon_0)(1+\epsilon_1)\alpha_0\|u_n\|_{q,1}^{N'}\left|\frac{u_n}{\|u_n\|_{q,1}}\right|^{N'}}-1\right)\right)\mathrm{d}x + c_9\int_{\mathbb{R}^N}|u_n|^{q^*}\,\mathrm{d}x\\
&\leqslant c_8\int_{\mathbb{R}^N}\Phi_1\left(N\left(\mathrm{e}^{\omega_{N-1}^{\frac{1}{N-1}}\left|\frac{u_n}{\|u_n\|_{q,1}}\right|^{N'}}-1\right)\right)\mathrm{d}x + c_9\int_{\mathbb{R}^N}|u_n|^{q^*}\,\mathrm{d}x.
\end{aligned}$$

Using (1.22), it yields

$$\sup_n\int_{\mathbb{R}^N}\Phi_1\left((t+1)N\left(\mathrm{e}^{(1+\epsilon_0)\alpha_0|u_n|^{N'}}-1\right)\right)\mathrm{d}x < +\infty.$$

Now, since $t+1 > q^*$ and $p > q^*$, then the embeddings $E_{q,1} \hookrightarrow L^{t+1}(\mathbb{R}^N)$ and $E_{q,1} \hookrightarrow L^p(\mathbb{R}^N)$ are compact. It follows from (8.4) that

$$\int_{\mathbb{R}^N} f(u_n)(u_n-u)\,\mathrm{d}x \to 0,\ n\to+\infty.$$

Taking into account that

$$\begin{aligned}
\langle I'(u_n), u_n-u\rangle &= \int_{\mathbb{R}^N}|\nabla u_n|^{N-2}\nabla u_n\nabla(u_n-u)w_1(x)\,\mathrm{d}x\\
&\quad+\int_{\mathbb{R}^N}|\nabla u_n|^{q-2}\nabla u_n\nabla(u_n-u)\,\mathrm{d}x\\
&\quad-\int_{\mathbb{R}^N}f(u_n)(u_n-u)\,\mathrm{d}x\\
&\to 0,\ n\to+\infty,
\end{aligned}$$

we infer $u_n \to 0$ strongly in $E_{q,1}$. This ends the proof of lemma 8.2. □

Now, we claim that there exist $\rho_0 > 0,\ \rho_1 > 0,\ e_0 \in E_{q,1}$ such that $\|e_0\|_{q,1} > \rho_0,\ I(e_0) < 0$, and

$$I(u) \geqslant \rho_1,\ \forall\ u \in E_{q,1},\ \|u\|_{q,1} = \rho_0.$$

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Given $u \in E_{q,1}$ and $t > p$. By (8.2) and (F_2), we have

$$\begin{aligned}\int_{\mathbb{R}^N} |F(u)|\, \mathrm{d}x &\leqslant c_{10} \int_{\mathbb{R}^N} |u|^p\, \mathrm{d}x + c_{10} \int_{\mathbb{R}^N} |u|^t\, \Phi_1\left(N\left(\mathrm{e}^{2\alpha_0 |u|^{N'}} - 1\right)\right) \mathrm{d}x \\ &\leqslant c_{10} \int_{\mathbb{R}^N} |u|^p\, \mathrm{d}x + c_{11} \left(\int_{\mathbb{R}^N} |u|^{t+1}\, \mathrm{d}x\right)^{\frac{t}{t+1}} \\ &\quad \int_{\mathbb{R}^N} \Phi_1\left((t+1)N\left(\mathrm{e}^{2\alpha_0 |u|^{N'}} - 1\right)\right) \mathrm{d}x \\ &\leqslant c_{12} \|u\|_{q,1}^p + c_{12} \|u\|_{q,1}^t \int_{\mathbb{R}^N} \Phi_1\left((t+1)N\left(\mathrm{e}^{2\alpha_0 |u|^{N'}} - 1\right)\right) \mathrm{d}x \\ &\leqslant c_{12} \|u\|_{q,1}^p + c_{13} \|u\|_{q,1}^t \int_{\mathbb{R}^N} \Phi_1\left(N\left(\mathrm{e}^{3\alpha_0 |u|^{N'}} - 1\right)\right) \mathrm{d}x. \qquad (8.7)\end{aligned}$$

For $u \in E_{q,1}$ such that $\|u\|_{q,1} = \rho_0 < \min\left\{1, \left(\frac{\omega_{N-1}^{\frac{1}{N-1}}}{3\alpha_0}\right)^{\frac{1}{N'}}\right\}$, by (1.22), there exists a constant $c_{14} > 0$ such that

$$\int_{\mathbb{R}^N} \Phi_1\left(N\left(\mathrm{e}^{3\alpha_0 |u|^{N'}} - 1\right)\right) \mathrm{d}x \leqslant \int_{\mathbb{R}^N} \Phi_1\left(N\left(\mathrm{e}^{\omega_{N-1}^{\frac{1}{N-1}} \left|\frac{u}{\|u\|_{q,1}}\right|^{N'}} - 1\right)\right) \mathrm{d}x \leqslant c_{14}.$$

Putting that inequality in (8.7), for $u \in E_{q,1}$ such that $\|u\|_{q,1} = \rho_0$, it yields

$$\begin{aligned} I(u) &\geqslant \frac{1}{N} |\nabla u|^N_{L^N_{w_1}(\mathbb{R}^N)} + \frac{1}{q} |\nabla u|^q_{L^q(\mathbb{R}^N)} - c_{15}\left(\|u\|_{q,1}^p + \|u\|_{q,1}^t\right) \\ &\geqslant \frac{1}{N}\left(|\nabla u|^N_{L^N_{w_1}(\mathbb{R}^N)} + |\nabla u|^N_{L^q(\mathbb{R}^N)}\right) - c_{15}\left(\|u\|_{q,1}^p + \|u\|_{q,1}^t\right) \\ &\geqslant \frac{2^{1-N}}{N}\left(|\nabla u|_{L^N_{w_1}(\mathbb{R}^N)} + |\nabla u|_{L^q(\mathbb{R}^N)}\right)^N - c_{15}\left(\|u\|_{q,1}^p + \|u\|_{q,1}^t\right) \\ &\geqslant \frac{2^{1-N}}{N} \|u\|_{q,1}^N - c_{15}\left(\|u\|_{q,1}^p + \|u\|_{q,1}^t\right) \\ &\geqslant \frac{2^{1-N}}{N} \rho_0^N - 2c_{15}\rho_0^p. \qquad (8.8)\end{aligned}$$

Plainly, one could choose ρ_0 small enough such that $\rho_0 < \left(\frac{2^{1-N}}{2Nc_{15}}\right)^{\frac{1}{p-N}}$. By (8.8), we deduce that

$$I(u) \geqslant \frac{2^{1-N}}{N}\rho_0^N - 2c_{15}\rho_0^p = \rho_1 > 0,\ \forall\ u \in E_{q,1},\ \|u\|_{q,1} = \rho_0.$$

Now, for a fixed $\phi \in C^\infty_{0,rad}(\mathbb{R}^N)\backslash\{0\}$ such that $\phi \geqslant 0$, we have

$$I(t\phi) \to -\infty, \text{ as } t \to +\infty.$$

In fact, from the hypothesis (F_1), one can see that there exist two positive constants c_{16} and c_{17} such that

$$F(s) \geqslant c_{16}s^\theta - c_{17},\ \forall\ s \geqslant 0.$$

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Let

$$c=\inf_{\gamma\in\Gamma}\max_{0\leqslant t\leqslant 1} I(\gamma(t))\geqslant\rho_1>0,\ \text{ where } \Gamma=\{\gamma\in C([0,1],E_{q,1}),\ \gamma(0)=0,\ I(\gamma(1))<0\},$$

the mountain pass level of the functional I. In order to complete the proof of theorem 8.1, it suffices to show that there exists $A_0>0$ such that

$$c<\frac{\theta-N}{2^{N-1}N\theta}\min\left\{2^{N-1},\frac{\omega_{N-1}}{\alpha_0^{N-1}}\right\},\ \forall\ A>A_0. \tag{8.9}$$

For that aim, fix a function $\phi_0\in C^\infty_{0,rad}(\mathbb{R}^N)$ such that $\phi_0\neq 0$ and $\phi_0\geqslant 0$. By (F_3), we have

$$\begin{aligned} I(\phi_0)&=\frac{|\nabla\phi_0|^N_{L^N_{w_1}(\mathbb{R}^N)}}{N}+\frac{|\nabla\phi_0|^q_{L^q(\mathbb{R}^N)}}{q}-\int_{\mathbb{R}^N}F(\phi_0)\mathrm{d}x\\ &\leqslant\frac{|\nabla\phi_0|^N_{L^N_{w_1}(\mathbb{R}^N)}}{N}+\frac{|\nabla\phi_0|^q_{L^q(\mathbb{R}^N)}}{q}-A\,|\phi_0|^r_{L^r(\mathbb{R}^N)}. \end{aligned}$$

We infer,

$$I(\phi_0)<0,\ \forall\ A>\frac{\frac{|\nabla\phi_0|^N_{L^N_{w_1}(\mathbb{R}^N)}}{N}+\frac{|\nabla\phi_0|^q_{L^q(\mathbb{R}^N)}}{q}}{|\phi_0|^r_{L^r(\mathbb{R}^N)}}.$$

Hence, for $A>\frac{\frac{|\nabla\phi_0|^N_{L^N_{w_1}(\mathbb{R}^N)}}{N}+\frac{|\nabla\phi_0|^q_{L^q(\mathbb{R}^N)}}{q}}{|\phi_0|^r_{L^r(\mathbb{R}^N)}}$, the function $\gamma_0:[0,1]\to E_{q,1}$ defined by

$$\gamma_0(t)=t\phi_0$$

belongs to Γ. Consequently,

$$\begin{aligned} c&\leqslant\max_{0\leqslant t\leqslant 1} I(t\phi_0)\\ &\leqslant\max_{0\leqslant t\leqslant 1}\left(\frac{t^N\,|\nabla\phi_0|^N_{L^N_{w_1}(\mathbb{R}^N)}}{N}+\frac{t^q\,|\nabla\phi_0|^q_{L^q(\mathbb{R}^N)}}{q}-At^r\,|\phi_0|^r_{L^r(\mathbb{R}^N)}\right)\\ &\leqslant\max_{0\leqslant t\leqslant 1}\left(\left(\frac{|\nabla\phi_0|^N_{L^N_{w_1}(\mathbb{R}^N)}}{N}+\frac{|\nabla\phi_0|^q_{L^q(\mathbb{R}^N)}}{q}\right)t^q-At^r\,|\phi_0|^r_{L^r(\mathbb{R}^N)}\right)\\ &=\left(\frac{\frac{q}{N}\,|\nabla\phi_0|^N_{L^N_{w_1}(\mathbb{R}^N)}+|\nabla\phi_0|^q_{L^q(\mathbb{R}^N)}}{Ar\,|\phi_0|^r_{L^r(\mathbb{R}^N)}}\right)^{\frac{q}{r-q}}\left(\frac{|\nabla\phi_0|^N_{L^N_{w_1}(\mathbb{R}^N)}}{N}+\frac{|\nabla\phi_0|^q_{L^q(\mathbb{R}^N)}}{q}\right)\left(1-\frac{q}{r}\right). \end{aligned}$$

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Clearly, there exists $A_0 > \frac{\frac{|\nabla \phi_0|^N_{L^N_{w_1}(\mathbb{R}^N)}}{N} + \frac{|\nabla \phi_0|^q_{L^q(\mathbb{R}^N)}}{q}}{|\phi_0|^r_{L^r(\mathbb{R}^N)}}$ large enough such that

$$\left(\frac{\frac{q}{N}\,|\nabla\phi_0|^N_{L^N_{w_1}(\mathbb{R}^N)} + |\nabla\phi_0|^q_{L^q(\mathbb{R}^N)}}{Ar\,|\phi_0|^r_{L^r(\mathbb{R}^N)}}\right)^{\frac{q}{r-q}} \left(\frac{|\nabla\phi_0|^N_{L^N_{w_1}(\mathbb{R}^N)}}{N} + \frac{|\nabla\phi_0|^q_{L^q(\mathbb{R}^N)}}{q}\right)\left(1 - \frac{q}{r}\right)$$
$$< \frac{\theta - N}{2^{N-1}N\theta}\min\left\{2^{N-1}, \frac{\omega_{N-1}}{\alpha_0^{N-1}}\right\},\ \forall\ A > A_0.$$

This ends the proof of theorem 8.1.

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