The boundary motive: definition and basic properties

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Abstract

We introduce the notion of the boundary motive of a scheme X over a perfect field. By definition, it measures the difference between the motive $M_{gm}(X)$ and the motive with compact support $M^c_{gm}(X)$, as defined and studied by Voevodsky *et al.* in *Cycles, transfers, and motivic homology theories* (Annals of Mathematics Studies, vol. 143 (Princeton University Press, Princeton, NJ, 2000)). We develop three tools to compute the boundary motive in terms of the geometry of a compactification of X: co-localization; invariance under abstract blow-up; and analytical invariance. We then prove auto-duality of the boundary motive of a smooth scheme X. As a formal consequence of this, and of co-localization, we obtain a fourth computational tool, namely localization for the boundary motive. In a sequel to this work (J. Wildeshaus, *On the boundary motive of a Shimura variety*, Prépublications du Laboratoire d'Analyse, Géométrie et Applications de l'Université Paris 13, no. 2004-23), these tools will be applied to Shimura varieties.

Contents

Introduction

In this paper, we define the boundary motive $\partial M_{gm}(X)$ of a scheme X over a perfect field k. By its very construction, it is part of an exact triangle

$$\partial M_{gm}(X) \longrightarrow M_{gm}(X) \longrightarrow M^c_{gm}(X) \longrightarrow \partial M_{gm}(X)[1],$$

where $M_{gm}(X)$ and $M^c_{gm}(X)$ denote the motive of X and its motive with compact support, respectively. The exact triangle is in the triangulated category $DM^{eff}_-(k)$ of effective motivic complexes which, together with $M_{gm}(X)$ and $M^c_{gm}(X)$, was defined in [VSF00]. We refer to § 1 for a review

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of these constructions. We expect this exact triangle to be of a certain interest. First, it induces long exact sequences for motivic homology and cohomology. More generally, any exact functor to a triangulated category D will induce long exact sequences of Ext-groups in D which are *a priori* compatible with the sequence in motivic homology, respectively, cohomology. Second, the exact triangle itself can be employed to construct explicit extensions of objects in $DM_-^{\mathrm{eff}}(k)$, i.e. classes in motivic cohomology.

Note that most of the existing attempts to prove the Beilinson or Bloch–Kato conjectures on special values of L-functions necessitate the explicit construction of elements in motivic cohomology. Furthermore, the realizations (Betti, de Rham, étale, ...) of these elements can often be constructed out from the cohomology of non-compact varieties, using the respective realizations of our exact triangle

$$\partial M_{\mathrm{gm}}(X) \longrightarrow M_{\mathrm{gm}}(X) \longrightarrow M_{\mathrm{gm}}^{\mathrm{c}}(X) \longrightarrow \partial M_{\mathrm{gm}}(X)[1].$$

This approach is clearly present, for example, in Harder's work on special values [Har93]. Our definition may thus be seen as an attempt to give a rigorous motivic meaning to these constructions.

In order to efficiently apply our new notion, one is naturally led to look for means to identify the boundary motive. We develop three tools to compute $\partial M_{\mathrm{gm}}(X)$: co-localization (§ 3); invariance under abstract blow-up (§ 4); and analytical invariance (§ 5). In a sequel to this work [Wil04], they will be applied to Shimura varieties, yielding in particular a motivic version of Pink's theorem on higher direct images of étale sheaves in the Baily–Borel compactification.

All our tools are based on the identification of $\partial M_{\mathrm{gm}}(X)$ with the motive associated to the diagram of schemes

$$\begin{array}{ccc} \emptyset & \longrightarrow & X \\ \downarrow & & \downarrow \\ \partial\overline{X} & \longrightarrow & \overline{X} \end{array}$$

for a compactification $\overline{X}$ of X, with $\partial\overline{X} := \overline{X} - X$ (see Proposition 2.4 for the precise statement). Let us insist that the definition of $\partial M_{\mathrm{gm}}(X)$ does not involve such a compactification.

Co-localization is the motivic analogue of the dual of the localization spectral sequence associated to a stratification of $\partial\overline{X}$. In the context of Betti cohomology (say), a good stratification $\partial\overline{X} = \coprod Y_{\mathrm{m}}$ induces a spectral sequence converging to the cohomology of $\partial\overline{X}$, and with E_2-terms given by cohomology with compact support of the Y_{m}. If one wants to express the analogue of the E_2-terms as motives rather than motives with compact support, one is led to imitate the dual spectral sequence, whose E_2-terms are equal to the usual cohomology of Y_{m}, but with coefficients given by the exceptional inverse images of the coefficients on $\partial\overline{X}$. This explains our choice of notation for the motives 'with coefficients' occurring in §§ 3–5, and more importantly, the precise nature of their behaviour. It turns out that in order to obtain a clean statement, co-localization (Theorem 3.4) has to be formulated more generally for diagrams of the type

$$\begin{array}{ccc} Y' - Y & \longrightarrow & W - Y \\ \downarrow & & \downarrow \\ Y' & \longrightarrow & W \end{array}$$

and for stratifications of Y, where $Y \hookrightarrow Y' \hookrightarrow W$ are closed immersions of not necessarily proper schemes over k. This diagram should be seen as modeling cohomology of the 'motivic sheaf' $i_Y^! j_! \mathbb{Z}$, where $i_Y : Y \hookrightarrow W$ and $j : W - Y' \hookrightarrow W$. Due to the lack of sufficient covariance properties of the functor $M_{\mathrm{gm}}^{\mathrm{c}}$, it does not seem obvious to geometrically model cohomology with compact support of $i_Y^* j_* \mathbb{Z}$. We refer to Definition 7.1 for a partial remedy to this problem.

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Given the sheaf-theoretical point of view, invariance under abstract blow-up and analytical invariance come as no surprise: in the context of complex spaces (say), the cohomology of $i_Y^! j_! \mathbb{Z}$ can be computed after proper base change to a second diagram as above, provided that this base change induces an isomorphism above the complement $W - Y'$. Theorem 4.1 is the motivic analogue of this invariance. Verdier's theory of the specialization functor (see e.g. the overview in [Ver85, §1]) shows that the sheaf $i_Y^! j_! \mathbb{Z}$ on Y can be computed on the normal cone of Y along W. Since analytically isomorphic situations near Y lead to the same normal cone, this implies that $i_Y^! j_! \mathbb{Z}$ can be computed with respect to a second set of closed immersions $Y \hookrightarrow Y_2' \hookrightarrow W_2$, provided that the formal completions of W_2, respectively of Y_2', along Y agree with those of W, respectively of Y', along Y. Theorem 5.1 states that the motivic analogue of this latter statement holds. While co-localization and invariance under abstract blow-up are direct consequences of the material contained in [VSF00], the proof of analytical invariance in addition uses the full force of Artin approximation.

In §6, we generalize duality for bivariant cycle cohomology [VSF00, Theorem IV.7.4], in order to establish an important structural property of the boundary motive $\partial M_{\mathrm{gm}}(X)$ of a smooth scheme X of pure dimension n: it is canonically isomorphic to its own dual $\partial M_{\mathrm{gm}}(X)^*$, twisted by n and shifted by $2n - 1$ (Theorem 6.1). As one formal consequence of this auto-duality, and of co-localization, we obtain a fourth tool to compute $\partial M_{\mathrm{gm}}(X)$, namely localization (§7) in the context of stratifications $\partial \overline{X} = \coprod Y_{\mathrm{m}}$, for a compactification $\overline{X}$ of a smooth scheme X (with $\partial \overline{X} = \overline{X} - X$ as above).

Sections 3–6 are logically independent of each other. Section 1 serves as a basis for all that is to follow, and §7 uses everything said before. All results from §§4, 6 and 7 require resolution of singularities for the base field k.

1. Notation and conventions

Our main and almost only reference is the book [VSF00]. When citing a result from its Chapter n, we shall precede the numbering used in [VSF00] by N, where N is the symbol representing n in the Roman number system. For example: Proposition 3.1.3 from Chapter 5 from [VSF00] will be cited as [VSF00, Proposition V.3.1.3].

We follow the notation of [VSF00]. Fix a perfect base field k. Denote by Sch/k the category of schemes which are separated and of finite type over k, and by Sm/k the full sub-category of objects which are smooth over k. Recall the definition of the category $SmCor(k)$ [VSF00, p. 190]: its objects are those of Sm/k. Morphisms from Y to X are given by the group $c(Y, X)$ of *finite correspondences* from Y to X, defined as the free Abelian group on the symbols (Z), where Z runs through the integral closed sub-schemes of $Y \times_k X$ which are finite over Y and surjective over a connected component of Y. Note for later use that the definition of $c(Y, X)$ still makes sense when $X \in Sch/k$ is not necessarily smooth. The category $Shv_{\mathrm{Nis}}(SmCor(k))$ of *Nisnevich sheaves with transfers* [VSF00, Definition V.3.1.1] is the category of those contravariant additive functors from $SmCor(k)$ to Abelian groups, whose restriction to Sm/k is a sheaf for the Nisnevich topology. This category is Abelian [VSF00, Theorem V.3.1.4]. Inside the derived category $D^-(Shv_{\mathrm{Nis}}(SmCor(k)))$ of complexes bounded from above, one defines the full sub-category $DM_-^{\mathrm{eff}}(k)$ of *effective motivic complexes* over k [VSF00, p. 205] as the one consisting of objects whose cohomology sheaves are *homotopy invariant* [VSF00, Definition V.3.1.10]. Since k is supposed to be perfect, this sub-category is triangulated [VSF00, Proposition V.3.1.13]. According to [VSF00, Proposition V.3.2.3], the inclusion of $DM_-^{\mathrm{eff}}(k)$ into $D^-(Shv_{\mathrm{Nis}}(SmCor(k)))$ admits a left adjoint $\mathbf{R}C$, which is induced from the functor

$$\underline{C}_* : Shv_{\mathrm{Nis}}(SmCor(k)) \longrightarrow C^-(Shv_{\mathrm{Nis}}(SmCor(k)))$$

which maps F to the simple complex associated to the *singular simplicial complex* [VSF00, p. 207]. Its nth term (in homological numbering) $\underline{C}_n(F)$ sends X to $F(X \times_k \Delta^n)$.

One defines two functors L and L^c from Sch/k to $Shv_{\mathrm{Nis}}(SmCor(k))$ [VSF00, pp. 223–224]: the functor L associates to X the Nisnevich sheaf with transfers $c(\bullet, X)$. The functor L^c maps X to

$$Y \longmapsto z(Y, X),$$

$z(Y, X)$ being defined as the free Abelian group on the symbols (Z), where Z runs through the integral closed sub-schemes of $Y \times_k X$ which are quasi-finite over Y and dominant over a connected component of Y. One defines the *motive* $M_{\mathrm{gm}}(X)$ of $X \in Sch/k$ as $\mathbf{R}C(L(X))$, and the *motive with compact support* $M_{\mathrm{gm}}^c(X)$ as $\mathbf{R}C(L^c(X))$.

For certain applications, it is of interest to enlarge the domain of the functor L: denote by Sch^∞/k the category of schemes which are separated and *locally* of finite type over k. The functor L extends, with the same definition of $c(\bullet, X)$ as above. This identifies $L(X)$ with the filtered direct limit of the $L(U)$, with U running through the open sub-schemes of X which are of finite type over k. This observation also allows us to use certain results from [VSF00] for the motives $M_{\mathrm{gm}}(X) := \mathbf{R}C(L(X)) \in DM_-^{\mathrm{eff}}(k)$, with X in Sch^∞/k.

We shall also use another, more geometric approach to motives, i.e. the one developed in [VSF00, § V.2.1]. There, the triangulated category $DM_{\mathrm{gm}}^{\mathrm{eff}}(k)$ of *effective geometrical motives* over k is defined. There is a canonical full triangulated embedding of $DM_{\mathrm{gm}}^{\mathrm{eff}}(k)$ into $DM_-^{\mathrm{eff}}(k)$ [VSF00, Theorem V.3.2.6], which maps the geometrical motive of $X \in Sm/k$ [VSF00, Definition V.2.1.1] to $M_{\mathrm{gm}}(X)$. Using this embedding, we consider $M_{\mathrm{gm}}(X)$ as an object of $DM_{\mathrm{gm}}^{\mathrm{eff}}(k)$. Finally, the category $DM_{\mathrm{gm}}(k)$ of *geometrical motives* over k is obtained from $DM_{\mathrm{gm}}^{\mathrm{eff}}(k)$ by inverting the *Tate motive* $\mathbb{Z}(1)$ [VSF00, p. 192]. All four categories $DM_{\mathrm{gm}}^{\mathrm{eff}}(k)$, $DM_{\mathrm{gm}}(k)$, $D^-(Shv_{\mathrm{Nis}}(SmCor(k)))$, and $DM_-^{\mathrm{eff}}(k)$ are tensor triangulated, and admit unit objects [VSF00, Proposition V.2.1.3, Corollary V.2.1.5, p. 206, Theorem V.3.2.6]. These tensor structures are such that for all $X, Y \in Sm/k$, one has

$$M_{\mathrm{gm}}(X) \otimes M_{\mathrm{gm}}(Y) = M_{\mathrm{gm}}(X \times_k Y)$$

in $DM_{\mathrm{gm}}^{\mathrm{eff}}(k)$, and

$$L(X) \otimes L(Y) = L(X \times_k Y)$$

in $D^-(Shv_{\mathrm{Nis}}(SmCor(k)))$. The unit object of $DM_{\mathrm{gm}}^{\mathrm{eff}}(k)$ is $M_{\mathrm{gm}}(\mathrm{Spec}\, k)$, and that of $D^-(Shv_{\mathrm{Nis}}(SmCor(k)))$ is $L(\mathrm{Spec}\, k)$. Both of them are denoted by $\mathbb{Z}(0)$. For $M \in DM_{\mathrm{gm}}(k)$ and $n \in \mathbb{Z}$, write $M(n)$ for the tensor product $M \otimes \mathbb{Z}(n)$. The three functors $DM_{\mathrm{gm}}^{\mathrm{eff}}(k) \hookrightarrow DM_-^{\mathrm{eff}}(k)$, $DM_{\mathrm{gm}}^{\mathrm{eff}}(k) \to DM_{\mathrm{gm}}(k)$ and $\mathbf{R}C : D^-(Shv_{\mathrm{Nis}}(SmCor(k))) \to DM_-^{\mathrm{eff}}(k)$ are compatible with the tensor structure. (By contrast, the embedding of $DM_-^{\mathrm{eff}}(k)$ into $D^-(Shv_{\mathrm{Nis}}(SmCor(k)))$ is not; see [VSF00, Remark on p. 206].) According to [VSF00, Theorem V.4.3.1], the functor $DM_{\mathrm{gm}}^{\mathrm{eff}}(k) \to DM_{\mathrm{gm}}(k)$ is a full triangulated embedding if k admits resolution of singularities.

CONVENTION 1.1. Whenever we speak about resolution of singularities, it will be taken in the sense of [VSF00, Definition IV.3.4].

CONVENTION 1.2. We shall use the same symbol for $M_{\mathrm{gm}}(X) \in DM_-^{\mathrm{eff}}(k)$ and for its canonical representative $\underline{C}_*(L(X))$ in $C^-(Shv_{\mathrm{Nis}}(SmCor(k)))$; similarly for $M_{\mathrm{gm}}^c(X)$. Whenever we speak about cones of morphisms between motives, we mean the class of the cone of the morphism between the canonical representatives. For a commutative diagram

$$\mathfrak{X} \;=\; \begin{array}{ccc} X' & \longrightarrow & X_1 \\ \downarrow & & \downarrow \\ X_2 & \longrightarrow & X'' \end{array}$$

in Sch^∞/k, define its motive $M_{\mathrm{gm}}(\mathfrak{X}) \in DM_-^{\mathrm{eff}}(k)$ as $\mathbf{R}C$ applied to the simple complex $sL(\mathfrak{X})$

associated to $L(\mathfrak{X})$, i.e. to the complex

$$L(X') \longrightarrow L(X_1) \oplus L(X_2) \longrightarrow L(X''),$$

which we normalize by assigning degree zero to the component $L(X'')$. A similar construction is possible for commutative diagrams in Sch^∞/k of 'dimension' greater than two, provided that there are not more than two schemes on any of the lines in the diagram.

2. Definition of the boundary motive

Let $X \in Sch/k$. Note that the inclusion $c(\bullet, X) \hookrightarrow z(\bullet, X)$ induces a monomorphism

$$\iota_X : L(X) \hookrightarrow L^c(X).$$

DEFINITION 2.1. The *boundary motive* of X is defined as

$$\partial M_{\mathrm{gm}}(X) := \mathbf{R}C(\operatorname{Coker} \iota_X)[-1].$$

Note that there is a canonical quasi-isomorphism

$$\operatorname{Cone}(M_{\mathrm{gm}}(X) \longrightarrow M_{\mathrm{gm}}^c(X)) \longrightarrow \partial M_{\mathrm{gm}}(X)[1],$$

where the cone is to be understood as in Convention 1.2. We have the following.

PROPOSITION 2.2. *There is an exact triangle*

$$\partial M_{\mathrm{gm}}(X) \longrightarrow M_{\mathrm{gm}}(X) \longrightarrow M_{\mathrm{gm}}^c(X) \longrightarrow \partial M_{\mathrm{gm}}(X)[1]$$

in $DM_-^{\mathrm{eff}}(k)$.

COROLLARY 2.3. *Assume that k admits resolution of singularities. Then the boundary motive* $\partial M_{\mathrm{gm}}(X)$ *belongs to* $DM_{\mathrm{gm}}^{\mathrm{eff}}(k)$.

Proof. This follows from Proposition 2.2, the fact that the embedding of $DM_{\mathrm{gm}}^{\mathrm{eff}}(k)$ into $DM_-^{\mathrm{eff}}(k)$ is triangulated, and [VSF00, Corollary V.4.1.4 and Corollary V.4.1.6]. □

The definition of the boundary motive does not involve a compactification of X. However, the tools to compute $\partial M_{\mathrm{gm}}(X)$ which we shall develop in the sequel are based on the following.

PROPOSITION 2.4. *Let $\overline{X}$ be a compactification of $X \in Sch/k$, and define $\partial\overline{X}$ as the complement $\overline{X} - X$, equipped with the reduced scheme structure. There is a canonical morphism*

$$\operatorname{Cone}\Big(M_{\mathrm{gm}}\Big(X \coprod \partial\overline{X}\Big) \to M_{\mathrm{gm}}(\overline{X})\Big) \longrightarrow \partial M_{\mathrm{gm}}(X)[1]$$

in $C^-(Shv_{\mathrm{Nis}}(SmCor(k)))$. *It becomes an isomorphism in* $DM_-^{\mathrm{eff}}(k)$ *if k admits resolution of singularities.*

Proof. Consider the exact sequences

$$0 \longrightarrow L(X) \longrightarrow L^c(X) \longrightarrow \operatorname{Coker} \iota_X \longrightarrow 0$$

and

$$0 \longrightarrow L^c(\partial\overline{X}) \longrightarrow L^c(\overline{X}) \longrightarrow L^c(X).$$

Observe that since $\partial\overline{X}$ and $\overline{X}$ are proper, we have $L(\partial\overline{X}) = L^c(\partial\overline{X})$ and $L(\overline{X}) = L^c(\overline{X})$. The monomorphism $L(X) \hookrightarrow L^c(X)$ factors through $L(\overline{X})$. Hence the exact sequences induce a monomorphism between the quotient $L(\overline{X})/L(X \coprod \partial\overline{X})$ and $\operatorname{Coker} \iota_X$, whose cokernel is identical to that of the restriction $L(\overline{X}) = L^c(\overline{X}) \to L^c(X)$. According to [VSF00, Proposition V.4.1.5], this latter cokernel has trivial image under $\mathbf{R}C$ if k admits resolution of singularities. □

3. Co-localization

Consider the geometric situation of Proposition 2.4: let $X \in Sch/k$, choose a compactification $\overline{X}$ of X, and define $\partial \overline{X} := \overline{X} - X$. In this section, we develop the motivic analogue of the dual of the localization spectral sequence associated to a stratification of $\partial \overline{X}$. It turns out to be useful to consider a more general geometric situation: fix closed immersions $Y \hookrightarrow Y' \hookrightarrow W$ in Sch^∞/k. Write j for the open immersion of $W - Y'$, and i_Y for the closed immersion of Y into W. Denote the following commutative diagram by $\mathfrak{Y}$.

$$
\begin{array}{ccc}
Y' - Y & \longrightarrow & W - Y \\
\downarrow & & \downarrow \\
Y' & \longrightarrow & W
\end{array}
$$

DEFINITION 3.1. The *motive of Y with coefficients in $i_Y^! j_! \mathbb{Z}$* is defined as

$$M_{\mathrm{gm}}(Y, i_Y^! j_! \mathbb{Z}) := M_{\mathrm{gm}}(\mathfrak{Y})$$

(see Convention 1.2).

Remark 3.2.

(a) Note that $M_{\mathrm{gm}}(Y, i_Y^! j_! \mathbb{Z})$ lies in $DM_{\mathrm{gm}}^{\mathrm{eff}}(k)$ if W and Y' are in Sm/k.

(b) Assume that k admits resolution of singularities. Then $M_{\mathrm{gm}}(Y, i_Y^! j_! \mathbb{Z})$ lies in $DM_{\mathrm{gm}}^{\mathrm{eff}}(k)$ if W is in Sch/k [VSF00, Corollary V.4.1.4].

Remark 3.3. If $Y = Y'$, i.e. if i_Y is complementary to j, then $\mathfrak{Y}$ acquires the form

$$
\begin{array}{ccc}
\emptyset & \longrightarrow & W - Y \\
\downarrow & & \downarrow \\
Y & \longrightarrow & W
\end{array}
$$

and we have

$$M_{\mathrm{gm}}(Y, i_Y^! j_! \mathbb{Z}) = \mathrm{Cone}\Big(M_{\mathrm{gm}}\big((W - Y) \textstyle\coprod Y\big) \to M_{\mathrm{gm}}(W)\Big).$$

If in addition W is proper, then Proposition 2.4 relates $M_{\mathrm{gm}}(Y, i_Y^! j_! \mathbb{Z})$ to $\partial M_{\mathrm{gm}}(W - Y)[1]$.

Now assume a given filtration

$$\emptyset = \mathfrak{F}_{-1} Y \subset \mathfrak{F}_0 Y \subset \cdots \subset \mathfrak{F}_d Y = Y$$

of Y by closed sub-schemes. It induces a stratification of Y by locally closed sub-schemes $Y_{\mathrm{m}} := \mathfrak{F}_{\mathrm{m}} Y - \mathfrak{F}_{\mathrm{m}-1} Y$, for $m = 0, \ldots, d$. Define W^m as the complement of $\mathfrak{F}_{\mathrm{m}-1} Y$ in W. This gives a descending partial filtration of W by open sub-schemes. Note in particular that we have $W^0 = W$ and $W^{d+1} = W - Y$. Write $i_{Y_{\mathrm{m}}}$ for the closed immersion of Y_{m} into W^m. By abuse of notation, we use the letter j to also denote the open immersions of $W - Y'$ into W^m.

THEOREM 3.4 (Co-localization). *There is a canonical chain of morphisms*

$$M^{d+1} = 0 \xrightarrow{\gamma^d} M^d \xrightarrow{\gamma^{d-1}} M^{d-1} \xrightarrow{\gamma^{d-2}} \cdots \xrightarrow{\gamma^0} M^0 = M_{\mathrm{gm}}(Y, i_Y^! j_! \mathbb{Z})$$

in $C^-(Shv_{\mathrm{Nis}}(SmCor(k)))$. For each $m \in \{0, \ldots, d\}$, there is a canonical isomorphism

$$\mathrm{Cone}\, \gamma^m \cong M_{\mathrm{gm}}(Y_{\mathrm{m}}, i_{Y_{\mathrm{m}}}^! j_! \mathbb{Z})$$

and hence a canonical exact triangle

$$M_{\mathrm{gm}}(Y_{\mathrm{m}}, i_{Y_{\mathrm{m}}}^! j_! \mathbb{Z})[-1] \longrightarrow M^{m+1} \xrightarrow{\gamma^m} M^m \longrightarrow M_{\mathrm{gm}}(Y_{\mathrm{m}}, i_{Y_{\mathrm{m}}}^! j_! \mathbb{Z})$$

in $DM_-^{\mathrm{eff}}(k)$. In particular, all the M^m represent objects in $DM_-^{\mathrm{eff}}(k)$. If k admits resolution of singularities, then all the M^m represent objects in $DM_{\mathrm{gm}}^{\mathrm{eff}}(k)$.

Proof. Consider the induced filtration $Y'^m := Y' - \mathfrak{F}_{m-1}Y = W^m \cap Y'$, and define $M^m := M_{\mathrm{gm}}(\mathfrak{Y}^m)$, where $\mathfrak{Y}^m$ is the following commutative diagram.

$$
\begin{array}{ccc}
Y' - Y & \longrightarrow & W - Y \\
\downarrow & & \downarrow \\
Y'^m & \longrightarrow & W^m
\end{array}
$$

Now note that $M_{\mathrm{gm}}(Y_{\mathrm{m}}, i^!_{Y_{\mathrm{m}}} j_! \mathbb{Z})$ is the motive associated to the following diagram.

$$
\begin{array}{ccc}
Y'^{m+1} & \longrightarrow & W^{m+1} \\
\downarrow & & \downarrow \\
Y'^m & \longrightarrow & W^m
\end{array}
$$

$\square$

COROLLARY 3.5. *Assume that k admits resolution of singularities. In the above situation, assume that $Y = \partial \overline{X} := \overline{X} - X$, with $X \in Sch/k$, and $\overline{X}$ a compactification of X. Write $(\partial \overline{X})_{\mathrm{m}} := Y_{\mathrm{m}}$. Then there is a canonical chain of morphisms*

$$
M^{d+1} = 0 \xrightarrow{\gamma^d} M^d \xrightarrow{\gamma^{d-1}} M^{d-1} \xrightarrow{\gamma^{d-2}} \cdots \xrightarrow{\gamma^0} M^0 = \partial M_{\mathrm{gm}}(X)[1]
$$

in $C^-(Shv_{\mathrm{Nis}}(SmCor(k)))$. For each $m \in \{0, \ldots, d\}$, there is a canonical exact triangle

$$
M_{\mathrm{gm}}((\partial \overline{X})_{\mathrm{m}}, i^!_{(\partial \overline{X})_{\mathrm{m}}} j_! \mathbb{Z})[-1] \to M^{m+1} \xrightarrow{\gamma^m} M^m \to M_{\mathrm{gm}}((\partial \overline{X})_{\mathrm{m}}, i^!_{(\partial \overline{X})_{\mathrm{m}}} j_! \mathbb{Z})
$$

in $DM_{\mathrm{gm}}(k)$. In particular, all the M^m are in $DM_{\mathrm{gm}}^{\mathrm{eff}}(k)$.

Proof. This follows from Proposition 2.4 and Theorem 3.4. $\square$

Of particular interest is the case when $X \in Sm/k$, and $\overline{X}$ is a smooth compactification of X, such that $\partial \overline{X}$ is a normal crossing divisor. Then $d < \dim X$ and $(\partial \overline{X})_{\mathrm{m}}$ equals the geometric locus of points lying on exactly $d + 1 - m$ local irreducible components of $\partial \overline{X}$. Since analytical invariance will permit us to give a good description of the $M_{\mathrm{gm}}((\partial \overline{X})_{\mathrm{m}}, i^!_{(\partial \overline{X})_{\mathrm{m}}} j_! \mathbb{Z})$, we postpone the discussion of this case until § 8.

4. Invariance under abstract blow-up

Fix a proper morphism $\pi : W_1 \to W_2$ and closed immersions $Y_2 \hookrightarrow Y'_2 \hookrightarrow W_2$ in Sch^∞/k. Write $Y_1 \hookrightarrow Y'_1 \hookrightarrow W_1$ for the base change of these immersions, j_{m} for the open immersion of $W_{\mathrm{m}} - Y'_{\mathrm{m}}$ and $i_{Y_{\mathrm{m}}}$ for the closed immersion of Y_{m} into W_{m}. We obtain the following diagram.

$$
\begin{array}{ccc}
Y_1 \overset{i_{Y_1}}{\hookrightarrow} W_1 & \overset{j_1}{\hookleftarrow} & W_1 - Y'_1 \\
\downarrow \quad \pi \downarrow & & \downarrow \pi \\
Y_2 \overset{i_{Y_2}}{\hookrightarrow} W_2 & \overset{j_2}{\hookleftarrow} & W_2 - Y'_2
\end{array}
$$

THEOREM 4.1 (Invariance under abstract blow-up). *Assume that*

$$
\pi : W_1 - Y'_1 \longrightarrow W_2 - Y'_2
$$

is an isomorphism. If k admits resolution of singularities, then the map

$$
\pi : M_{\mathrm{gm}}(Y_1, i^!_{Y_1} j_{1!} \mathbb{Z}) \longrightarrow M_{\mathrm{gm}}(Y_2, i^!_{Y_2} j_{2!} \mathbb{Z})
$$

is an isomorphism.

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Recall that $M_{gm}(Y_m, i^!_{Y_m} j_{m!} \mathbb{Z})$ is associated to the following diagram.

$$
\begin{array}{ccc}
Y'_m - Y_m & \longrightarrow & W_m - Y_m \\
\downarrow & & \downarrow \\
Y'_m & \longrightarrow & W_m
\end{array}
$$

Since the two rows are of the same nature, Theorem 4.1 is a formal consequence of the following.

THEOREM 4.2. *Consider closed immersions $Y'_1 \hookrightarrow W_1$ and $Y'_2 \hookrightarrow W_2$ in Sch^∞/k, and a proper morphism $\pi : W_1 \to W_2$ identifying Y'_1 with the fibre product $W_1 \times_{W_2} Y'_2$, and inducing an isomorphism from $W_1 - Y'_1$ to $W_2 - Y'_2$. If k admits resolution of singularities, then the monomorphism*

$$
\pi : L(W_1)/L(Y'_1) \longrightarrow L(W_2)/L(Y'_2)
$$

in $Shv_{Nis}(SmCor(k))$ induces an isomorphism $\mathbf{R}C(\pi)$ in $DM^{eff}_-(k)$.

Proof (see [VSF00, Proposition V.4.1.3] and its proof). The sequence

$$
0 \longrightarrow L(Y'_1) \longrightarrow L(W_1) \oplus L(Y'_2) \longrightarrow L(W_2)
$$

is exact and the quotient $L(W_2)/(L(W_1) + L(Y'_2))$ satisfies the condition of [VSF00, Theorem V.4.1.2]. □

5. Analytical invariance

Fix closed immersions $Y \hookrightarrow Y'_1 \hookrightarrow W_1$ and $Y \hookrightarrow Y'_2 \hookrightarrow W_2$ in Sch^∞/k. Write j_m for the open immersion of $W_m - Y'_m$, and $i_{Y,m}$ for the closed immersion of Y into W_m. The aim of this section is to prove the following.

THEOREM 5.1 (Analytical invariance). *Assume a given isomorphism*

$$
f : (W_1)_Y \xrightarrow{\sim} (W_2)_Y
$$

of formal completions along Y inducing an isomorphism $(Y'_1)_Y \cong (Y'_2)_Y$, and compatible with the immersions $i_{Y,m}$ of Y. Then f induces an isomorphism

$$
M_{gm}(Y, i^!_{Y,1} j_{1!} \mathbb{Z}) \xrightarrow{\sim} M_{gm}(Y, i^!_{Y,2} j_{2!} \mathbb{Z})
$$

in $DM^{eff}_-(k)$.

Remark 5.2. Using Proposition 2.4 together with Remark 3.3, we deduce the following statement from Theorem 5.1, assuming that k admits resolution of singularities. Let $\overline{X}_m$ be a compactification of $X_m \in Sch/k$, $m = 1, 2$, and set $\partial \overline{X}_m := \overline{X}_m - X_m$ (with the reduced scheme structure). Assume that there is an isomorphism $\partial \overline{X}_1 \cong \partial \overline{X}_2$, which can be extended to an isomorphism between the formal completions of $\overline{X}_m$ along $\partial \overline{X}_m$. Then $\partial M_{gm}(X_1)$ and $\partial M_{gm}(X_2)$ are isomorphic.

Note, however, that in practice it may not always be possible to identify the formal completion of $\partial \overline{X}$ along a given compactification $\overline{X}$ of a scheme X. Actually, one might control the formal completion of an abstract blow-up of each stratum belonging to a stratification of $\partial \overline{X}$. In order to compute $\partial M_{gm}(X)$ in such a situation, one first applies co-localization with respect to the stratification, then uses invariance under abstract blow-up for each stratum, and finally analytical invariance as stated in the above generality.

The main technical ingredient of the proof of Theorem 5.1 is the following consequence of Artin approximation.

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THEOREM 5.3 (Artin). *Let S be the spectrum of a field or of an excellent Dedekind domain, let W_1 and W_2 be two S-schemes which are locally of finite type, let Y_m be closed sub-schemes of W_m, $y_m \in Y_m$, and let $\mathfrak{a}_m$ be the ideal of Y_m in the Henselization $\mathcal{O}_{W_m,y_m}^h$ of the local ring $\mathcal{O}_{W_m,y_m}$. Denote by $\widehat{\mathcal{O}}_{W_m,y_m}$ the completion of $\mathcal{O}_{W_m,y_m}^h$ with respect to $\mathfrak{a}_m$, for $m = 1, 2$.*

(a) *If $\widehat{\mathcal{O}}_{W_1,y_1} \cong \widehat{\mathcal{O}}_{W_2,y_2}$ over $\mathcal{O}_S$, then y_1 and y_2 have a common Nisnevich neighbourhood: there exists an S-scheme W', a point $y' \in W'$ and étale morphisms $W' \to W_m$ mapping y' to y_m, for $m = 1, 2$, which identify the residue fields $\kappa(y_1) \cong \kappa(y') \cong \kappa(y_2)$.*

(b) *Assume that in addition we are given an isomorphism $Y_1 \cong Y_2$. Assume that the isomorphism in (a) maps the completed ideal $\hat{\mathfrak{a}}_1$ isomorphically to $\hat{\mathfrak{a}}_2$, and that the induced isomorphism $\mathcal{O}_{Y_1,y_1}^h \cong \mathcal{O}_{Y_2,y_2}^h$ is compatible with the given isomorphism $Y_1 \cong Y_2$. Then the Nisnevich neighbourhood W' in (a) can be chosen such that in addition*

$$Y' := W' \times_{W_1} Y_1 = W' \times_{W_2} Y_2$$

as sub-schemes of W', and the induced étale morphisms $Y' \to Y_m$ are compatible with the isomorphism $Y_1 \cong Y_2$.

Proof. This is a variant of [Art69, Corollary 2.6]. In fact, the results stated in [Art69, §2] are the translations of the main results of §1 only in the case when the ideal of definition is the maximal ideal of the point in question. In order to deduce the variant from [Art69, Theorem 1.12], one faithfully imitates the proof of [Art69, Corollary 2.6]. □

COROLLARY 5.4. *With S, W_m and Y_m as in Theorem 5.3, assume a given isomorphism $Y_1 \cong Y_2$ which extends to an isomorphism*

$$f : (W_1)_{Y_1} \xrightarrow{\sim} (W_2)_{Y_2}$$

of formal completions. Then there are Nisnevich coverings $\mathfrak{W}_m$ of W_m of the form

$$\mathfrak{W}_1 = \{W_i' \mid i \in I\} \coprod \{W_1 - Y_1\}, \quad \mathfrak{W}_2 = \{W_i' \mid i \in I\} \coprod \{W_2 - Y_2\}$$

(with the same W_i') such that for any $i \in I$, one has

$$Y_i' := W_i' \times_{W_1} Y_1 = W_i' \times_{W_2} Y_2$$

as sub-schemes of W_i', and the induced étale morphisms $Y_i' \to Y_m$ are compatible with the isomorphism $Y_1 \cong Y_2$.

We turn to the proof of Theorem 5.1. Recall that $M_{gm}(Y, i_{Y,m}^! j_{m!}\,\mathbb{Z})$ is associated to the following diagram.

$$
\begin{array}{ccc}
Y_m' - Y & \longrightarrow & W_m - Y \\
\downarrow & & \downarrow \\
Y_m' & \longrightarrow & W_m
\end{array}
$$

Since the two columns are of the same nature, Theorem 5.1 is a formal consequence of parts (a) and (b) of the following.

THEOREM 5.5. *Consider closed immersions $Y \hookrightarrow W_1$ and $Y \hookrightarrow W_2$ in Sch^∞/k which extend to an isomorphism*

$$f : (W_1)_Y \xrightarrow{\sim} (W_2)_Y$$

of formal completions along Y.

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(a) *There is an isomorphism*

$$L(W_1)/L(W_1 - Y) \xrightarrow{\sim} L(W_2)/L(W_2 - Y)$$

   of Nisnevich sheaves with transfers, and depending only on f.

(b) *The isomorphism $L(W_1)/L(W_1 - Y) \xrightarrow{\sim} L(W_2)/L(W_2 - Y)$ is compatible with restriction of the W_m to sub-schemes containing Y.*

(c) *The isomorphism $L(W_1)/L(W_1 - Y) \xrightarrow{\sim} L(W_2)/L(W_2 - Y)$ is compatible with restriction of Y.*

Proof. Choose Nisnevich coverings

$$\mathfrak{W}_m = \{W_i' \mid i \in I\} \coprod \{W_m - Y\}$$

as in Corollary 5.4. Set

$$W' := \coprod_{i \in I} W_i'$$

and write α_m for the coproduct of the étale morphisms from the W_i' to W_m, for $m = 1, 2$. By Corollary 5.4, we have

$$Y' := W' \times_{W_1} Y = W' \times_{W_2} Y$$

and α_1 and α_2 coincide on Y'. Using [VSF00, Proposition V.3.1.3], we see that we have exact sequences

$$\frac{L(W' \times_{W_1} W')}{L((W' - Y') \times_{W_1} (W' - Y'))} \xrightarrow{pr_*^1 - pr_*^2} \frac{L(W')}{L(W' - Y')} \xrightarrow{\alpha_{1*}} \frac{L(W_1)}{L(W_1 - Y)} \longrightarrow 0,$$

$$\frac{L(W' \times_{W_2} W')}{L((W' - Y') \times_{W_2} (W' - Y'))} \longrightarrow \frac{L(W')}{L(W' - Y')} \xrightarrow{\alpha_{2*}} \frac{L(W_2)}{L(W_2 - Y)} \longrightarrow 0,$$

of Nisnevich sheaves with transfers. Let us show that the map α_{2*} is zero on the image of $L(W' \times_{W_1} W')/L((W' - Y') \times_{W_1} (W' - Y'))$. We imitate the proof of [VSF00, Proposition II.4.3.9]. It is sufficient to show the following claim.

CLAIM 5.6. *For any local Henselian scheme S which is smooth over k, the composition*

$$c(S, W' \times_{W_1} W') \xrightarrow{pr_*^1 - pr_*^2} c(S, W') \xrightarrow{\alpha_{2*}} \frac{c(S, W_2)}{c(S, W_2 - Y)}$$

is trivial.

Note that the presheaves $U \mapsto c(U, T)$, for $T \in Sch^\infty/k$, can be extended in an obvious way to the category of smooth k-schemes which are not necessarily of finite type. For the proof of Claim 5.6, we shall repeatedly apply the following principle, valid since S is Henselian: for any $T \in Sch^\infty/k$, the support of any element of $c(S, T)$ is a disjoint union of local Henselian schemes. This principle allows us to consider only cycles in $c(S, W' \times_{W_1} W')$ of the form (Z), where Z is a local Henselian sub-scheme of $S \times_k (W' \times_{W_1} W')$. Without loss of generality, we may assume that the closed point of Z lies over Y', hence over Y. Write $pr_*^l(Z) = n^l \cdot (Z^l)$, with local Henselian sub-schemes Z^l of $S \times_k W'$, for $l = 1, 2$. We have $\alpha_{1*} pr_*^1 = \alpha_{1*} pr_*^2$, hence the $\alpha_{1*}(Z^l)$ are multiples of (Z_1) for *one* local Henselian sub-scheme (Z_1) of $S \times_k W_1$. In order to show the analogous statement for the $\alpha_{2*}(Z^l)$, note first that the closed point y of Z_1 belongs to $S \times_k Y$. The support of

$$\alpha_{2*}(pr_*^1 - pr_*^2)(Z) = \alpha_{2*}(n^1 \cdot (Z^1) - n^2 \cdot (Z^2))$$

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is a disjoint union of local Henselian schemes, parametrized by their closed points. But since α_1 and α_2 coincide on Y', this support must be local, and we have indeed

$$\alpha_{2*}(pr_*^1 - pr_*^2)(Z) = r \cdot (Z_2)$$

for a local Henselian sub-scheme (Z_2) of $S \times_k W_2$, whose closed point is y. In order to show that $r = 0$, consider the commutative diagram

$$\begin{array}{ccc}
& c(S, W') & \\
{\scriptstyle \alpha_{1*}} \swarrow & & \searrow {\scriptstyle \alpha_{2*}} \\
c(S, W_1) & & c(S, W_2) \\
{\scriptstyle w_{1*}} \searrow & & \swarrow {\scriptstyle w_{2*}} \\
& c(S, \operatorname{Spec} k) &
\end{array}$$

with w_{m} denoting the structure morphism of W_{m}. On the one hand,

$$w_{2*}\alpha_{2*}(pr_*^1 - pr_*^2)(Z) = w_{1*}\alpha_{1*}(pr_*^1 - pr_*^2)(Z) = w_{1*}(0) = 0.$$

On the other hand, $w_{2*}(Z_2)$ is non-zero since Z_2 is finite over S. Hence r must indeed be zero, and thus

$$\alpha_{2*}(pr_*^1 - pr_*^2)(Z) = 0.$$

This shows that α_{2*} is zero on the image of $pr_*^1 - pr_*^2$. By symmetry, we see that the identity on $L(W')/L(W' - Y')$ factors to give an isomorphism

$$L(W_1)/L(W_1 - Y) \xrightarrow{\sim} L(W_2)/L(W_2 - Y).$$

In order to prove that it does not depend on the choice of the Nisnevich coverings $\mathfrak{W}_{\mathrm{m}}$ as in Corollary 5.4, use the fact that the system of such coverings is filtering. $\qquad\square$

6. Auto-duality

Throughout this section, we assume that k admits resolution of singularities. Under this assumption, $DM_{\mathrm{gm}}(k)$ is a rigid tensor triangulated category [VSF00, Theorem V.4.3.7 1 and 2]. In particular, there exists an internal Hom functor

$$\underline{Hom} : DM_{\mathrm{gm}}(k) \times DM_{\mathrm{gm}}(k) \longrightarrow DM_{\mathrm{gm}}(k).$$

Writing $M^* := \underline{Hom}(M, \mathbb{Z}(0))$, we thus have $M = (M^*)^*$ for all $M \in DM_{\mathrm{gm}}(k)$.

Now fix $X \in Sm/k$, and assume that X is of pure dimension n. According to [VSF00, Theorem V.4.3.7 3], there is a canonical isomorphism

$$\mu_X : M_{\mathrm{gm}}^{\mathrm{c}}(X) \xrightarrow{\sim} M_{\mathrm{gm}}(X)^*(n)[2n],$$

hence by duality, a canonical isomorphism

$$\nu_X := \mu_X^*(n)[2n] : M_{\mathrm{gm}}(X) \xrightarrow{\sim} M_{\mathrm{gm}}^{\mathrm{c}}(X)^*(n)[2n].$$

The aim of this section is to prove the following.

THEOREM 6.1 (Auto-duality). *There exists a canonical isomorphism*

$$\eta_X : \partial M_{\mathrm{gm}}(X) \xrightarrow{\sim} \partial M_{\mathrm{gm}}(X)^*(n)[2n-1].$$

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It fits into the following morphism of exact triangles.

$$
\begin{array}{ccc}
\partial M_{\mathrm{gm}}(X) & \xrightarrow{\eta_X} & \partial M_{\mathrm{gm}}(X)^*(n)[2n-1] \\
{\scriptstyle\alpha}\big\downarrow & & \big\downarrow{\scriptstyle\gamma^*(n)[2n]} \\
M_{\mathrm{gm}}(X) & \xrightarrow{\nu_X} & M_{\mathrm{gm}}^{\mathrm{c}}(X)^*(n)[2n] \\
{\scriptstyle\beta}\big\downarrow & & \big\downarrow{\scriptstyle\beta^*(n)[2n]} \\
M_{\mathrm{gm}}^{\mathrm{c}}(X) & \xrightarrow{\mu_X} & M_{\mathrm{gm}}(X)^*(n)[2n] \\
{\scriptstyle\gamma}\big\downarrow & & \big\downarrow{\scriptstyle\alpha^*(n)[2n]} \\
\partial M_{\mathrm{gm}}(X)[1] & \xrightarrow{\eta_X[1]} & \partial M_{\mathrm{gm}}(X)^*(n)[2n]
\end{array}
$$

Furthermore, it is itself auto-dual in the sense that the equality

$$\eta_X = \eta_X^*(n)[2n-1]$$

holds.

First observe that the existence of *some* isomorphism η_X fitting into the above diagram of exact triangles is a consequence of the axioms of triangulated categories, and the commutativity of the following diagram.

$$
\begin{array}{ccc}
M_{\mathrm{gm}}(X) & \xrightarrow{\nu_X} & M_{\mathrm{gm}}^{\mathrm{c}}(X)^*(n)[2n] \\
{\scriptstyle\beta}\big\downarrow & & \big\downarrow{\scriptstyle\beta^*(n)[2n]} \\
M_{\mathrm{gm}}^{\mathrm{c}}(X) & \xrightarrow{\mu_X} & M_{\mathrm{gm}}(X)^*(n)[2n]
\end{array}
$$

Thus, the point of auto-duality is that η_X can be defined canonically, and that this definition is itself auto-dual.

For the proof of Theorem 6.1, observe that by adjunction, the construction of η_X is equivalent to the construction of a pairing

$$(\, \bullet \, , \, \bullet \,) : \partial M_{\mathrm{gm}}(X) \otimes \partial M_{\mathrm{gm}}(X)[1] \longrightarrow \mathbb{Z}(n)[2n].$$

We are thus led to investigate morphisms in $DM_{\mathrm{gm}}^{\mathrm{eff}}(k)$ whose target is $\mathbb{Z}(n)[2n]$. The statement we are aiming for is Theorem 6.14. It is a generalization of [VSF00, Corollary V.4.2.5]. In the above geometric context, it implies that certain codimension n-cycles on the self-product $\overline{X} \times_k \overline{X}$, where $\overline{X}$ is a smooth compactification of X, yield morphisms of the type of $(\, \bullet \, , \, \bullet \,)$. In order to prepare Theorem 6.14, we need to prove a variant of duality for bivariant cycle cohomology (Theorem 6.11). Let us start by recalling the definition of certain variants of the Nisnevich sheaves with transfers $L^{\mathrm{c}}(V) = z(\, \bullet \, , V)$ (see [VSF00, p. 228]).

DEFINITION 6.2. Let $V \in Sch/k$, $W \in Sm/k$ and $r \geqslant 0$.

(a) The Nisnevich sheaf with transfers $z_{\mathrm{equi}}(V, r)$ associates to $U \in Sm/k$ the free Abelian group on the symbols (Z), where Z runs through the integral closed sub-schemes of $U \times_k V$ which are equi-dimensional of relative dimension r over U and dominant over a connected component of U.

(b) The Nisnevich sheaf with transfers $z_{\mathrm{equi}}(W, V, r)$ maps $U \in Sm/k$ to $z_{\mathrm{equi}}(V, r)(U \times_k W)$.

Note that the sheaves $z_{\mathrm{equi}}(W, V, r)$ are contravariant in the first variable. Now recall [VSF00, Corollary V.4.1.8].

PROPOSITION 6.3. *The object* $\mathbb{Z}(n)[2n]$ *of* $DM_{\mathrm{gm}}^{\mathrm{eff}}(k) \subset DM_-^{\mathrm{eff}}(k)$ *is represented by the complex*

$$M_{\mathrm{gm}}^{\mathrm{c}}(\mathbb{A}^n) = \underline{C}_*(z(\,\bullet\,,\mathbb{A}^n)) = \underline{C}_*(z_{\mathrm{equi}}(\mathbb{A}^n,0)).$$

More generally, if $W \in Sm/k$, then the complex

$$\underline{C}_*(z_{\mathrm{equi}}(W,\mathbb{A}^n,0))$$

represents the functor

$$\underline{Hom}_{DM_-^{\mathrm{eff}}(k)}(M_{\mathrm{gm}}(W),\mathbb{Z}(n)[2n])$$

on $DM_-^{\mathrm{eff}}(k)$ [VSF00, Corollary V.4.2.7]. We need a variant of this statement as follows.

DEFINITION 6.4. Let $V \in Sch/k$, $W \in Sm/k$ and $r \geqslant 0$. Assume further that $Y \subset W$ is a closed sub-scheme.

(a) Define

$$M_{\mathrm{gm}}(W/Y) := \underline{C}_*(L(Y \to W)),$$

where $L(Y \to W)$ is the complex given by $L(Y)$ in degree -1 and $L(W)$ in degree zero, the differential being induced by the immersion of Y.

(b) Assume in addition that arbitrary intersections of the components Y_j of Y are smooth. Define

$$z_{\mathrm{equi}}(Y^\bullet \to W, V, r)$$

as the complex of Nisnevich sheaves with transfers whose zeroth component is $z_{\mathrm{equi}}(W,V,r)$, and whose mth component, for $m \geqslant 1$, is the direct sum of the $z_{\mathrm{equi}}(Y_J,V,0)$, for all m-fold intersections Y_J of the Y_j. The differentials are induced by contravariance of the sheaves $z_{\mathrm{equi}}(\,\bullet\,,V,r)$.

Of course, $M_{\mathrm{gm}}(W/Y)$ is the same thing as M_{gm} evaluated at the diagram

$$Y \longrightarrow W$$

(see Convention 1.2).

PROPOSITION 6.5. *Let* $W \in Sm/k$. *Assume further that* $Y \subset W$ *is a closed sub-scheme such that arbitrary intersections of the components of* Y *are smooth. Then the complex*

$$\underline{C}_*(z_{\mathrm{equi}}(Y^\bullet \to W, \mathbb{A}^n, 0))$$

represents the functor

$$\underline{Hom}_{DM_-^{\mathrm{eff}}(k)}(M_{\mathrm{gm}}(W/Y),\mathbb{Z}(n)[2n])$$

on $DM_-^{\mathrm{eff}}(k)$. *This identification is compatible with passage from the pair* $Y \subset W$ *to* $Y' \subset U$, *for open sub-schemes* U *of* W, *and closed sub-schemes* Y' *of* $Y \cap U$ *such that arbitrary intersections of the components* Y'_j *of* Y' *are smooth.*

CONVENTION 6.6. By over-simplification of language, we shall refer to the last compatibility statement in Proposition 6.5 as 'compatibility with restriction of W and Y'.

Proof of Proposition 6.5. First, observe that the canonical morphism $M_{\mathrm{gm}}(Y^\bullet) \to M_{\mathrm{gm}}(Y)$ is an isomorphism. This follows from induction on the number of components Y_j. The induction step is provided by [VSF00, Proposition V.4.1.3]. Similarly, $M_{\mathrm{gm}}(Y^\bullet \to W) \to M_{\mathrm{gm}}(W/Y)$ is an isomorphism.

As in the proof of [VSF00, Proposition V.4.2.8], one has a canonical morphism of complexes *can* from $\underline{C}_*(z_{\mathrm{equi}}(Y^\bullet \to W, \mathbb{A}^n, 0)) = \underline{C}_*(p_* p^*(L^{\mathrm{c}}(\mathbb{A}^n)))$ to $Rp_*(p^* \underline{C}_*(L^{\mathrm{c}}(\mathbb{A}^n)))$, where p denotes

the structure morphism of the diagram $Y^\bullet \to W$. By an obvious generalization of the last part of [VSF00, Proposition V.3.2.8], one has

$$Rp_*(p^* \underline{C}_*(L^c(\mathbb{A}^n))) = \underline{Hom}(M_{\mathrm{gm}}(Y^\bullet \to W), \mathbb{Z}(n)[2n])$$

in $DM_-^{\mathrm{eff}}(k)$. To check that *can* is an isomorphism, one uses the spectral sequences on both its source and target, associated to the stupid filtration (*filtration bête*) of $Y^\bullet \to W$. Result [VSF00, Corollary V.4.2.7] shows that *can* is an isomorphism on the E_1-terms of this spectral sequence. $\square$

Remark 6.7. In the situation of Proposition 6.5, assume in addition that W is of dimension at most n. By [VSF00, Corollary V.4.3.6], the object $M_{\mathrm{gm}}(W/Y)^*(n)[2n]$ of $DM_{\mathrm{gm}}(k)$ belongs to $DM_{\mathrm{gm}}^{\mathrm{eff}}(k)$, and its image under the embedding into $DM_-^{\mathrm{eff}}(k)$ equals $\underline{Hom}_{DM_-^{\mathrm{eff}}(k)}(M_{\mathrm{gm}}(W/Y), \mathbb{Z}(n)[2n])$. It is thus represented by the complex $\underline{C}_*(z_{\mathrm{equi}}(Y^\bullet \to W, \mathbb{A}^n, 0))$.

Given an object F of $Shv_{\mathrm{Nis}}(SmCor(k))$, denote by $\underline{h}^l(F)$ the lth cohomology object of the complex $\underline{C}_*(F)$. Thus, the lth cohomology object of $\underline{C}_*(z_{\mathrm{equi}}(Y^\bullet \to W, \mathbb{A}^n, 0))(\mathrm{Spec}\, k)$ equals $\underline{h}^l(z_{\mathrm{equi}}(Y^\bullet \to W, \mathbb{A}^n, 0))(\mathrm{Spec}\, k)$.

COROLLARY 6.8. *In the situation of Proposition 6.5, there is a canonical isomorphism*

$$c_{W/Y} : \underline{h}^l(z_{\mathrm{equi}}(Y^\bullet \to W, \mathbb{A}^n, 0))(\mathrm{Spec}\, k) \overset{\sim}{\longrightarrow} \mathrm{Hom}(M_{\mathrm{gm}}(W/Y), \mathbb{Z}(n)[2n + l]).$$

Here Hom *denotes morphisms in* $DM_{\mathrm{gm}}^{\mathrm{eff}}(k)$. *The isomorphism is compatible with restriction of W and Y in the sense of Convention 6.6.*

Proof. First, we have

$$\mathrm{Hom}(M_{\mathrm{gm}}(W/Y), \mathbb{Z}(n)[2n + l]) = \mathrm{Hom}_{DM_-^{\mathrm{eff}}(k)}(M_{\mathrm{gm}}(W/Y), \mathbb{Z}(n)[2n + l])$$

since the embedding of $DM_{\mathrm{gm}}^{\mathrm{eff}}(k)$ into $DM_-^{\mathrm{eff}}(k)$ is full. Adjointness of $\otimes$ and $\underline{Hom}$ implies that $\mathrm{Hom}_{DM_-^{\mathrm{eff}}(k)}(M_{\mathrm{gm}}(W/Y), \mathbb{Z}(n)[2n + l])$ equals

$$\mathrm{Hom}_{DM_-^{\mathrm{eff}}(k)}(M_{\mathrm{gm}}(\mathrm{Spec}\, k), \underline{Hom}_{DM_-^{\mathrm{eff}}(k)}(M_{\mathrm{gm}}(W/Y), \mathbb{Z}(n)[2n + l])).$$

By Proposition 6.5, this group equals

$$\mathrm{Hom}_{DM_-^{\mathrm{eff}}(k)}(\mathbf{R}C(L(\mathrm{Spec}\, k)), \mathbf{R}C(z_{\mathrm{equi}}(Y^\bullet \to W, \mathbb{A}^n, 0))[l]).$$

Now use the fact that $\mathbf{R}C$ is left adjoint to the inclusion of $DM_-^{\mathrm{eff}}(k)$ into $D^-(Shv_{\mathrm{Nis}}(SmCor(k)))$. $\square$

For later use, we also note a consequence of the special case $Y = \emptyset$.

COROLLARY 6.9. *Let $W \in Sm/k$. Then the functor on open sub-schemes of W,*

$$U \longmapsto \underline{C}_*(z_{\mathrm{equi}}(U, \mathbb{A}^n, 0)),$$

satisfies the Mayer–Vietoris property in the following sense. Given an equality $U = U_1 \cup U_2$ of open sub-schemes of W, the exact sequence

$$0 \to z_{\mathrm{equi}}(U, \mathbb{A}^n, 0) \to z_{\mathrm{equi}}(U_1, \mathbb{A}^n, 0) \oplus z_{\mathrm{equi}}(U_2, \mathbb{A}^n, 0) \to z_{\mathrm{equi}}(U_1 \cap U_2, \mathbb{A}^n, 0)$$

in $Shv_{\mathrm{Nis}}(SmCor(k))$ induces an exact triangle

$$\mathbf{R}C(z_{\mathrm{equi}}(U, \mathbb{A}^n, 0)) \longrightarrow \mathbf{R}C(z_{\mathrm{equi}}(U_1, \mathbb{A}^n, 0)) \oplus \mathbf{R}C(z_{\mathrm{equi}}(U_2, \mathbb{A}^n, 0))$$
$$\longrightarrow \mathbf{R}C(z_{\mathrm{equi}}(U_1 \cap U_2, \mathbb{A}^n, 0)) \longrightarrow \mathbf{R}C(z_{\mathrm{equi}}(U, \mathbb{A}^n, 0))[1]$$

in $DM_-^{\mathrm{eff}}(k)$.

Proof. This follows from Proposition 6.5 and the Mayer–Vietoris property for the functor $U \mapsto M_{\mathrm{gm}}(U)$ [VSF00, Proposition V.4.1.1]. $\square$

We need to find a way to efficiently generate elements in the group

$$\underline{h}^0(z_{\mathrm{equi}}(Y^\bullet \to W, \mathbb{A}^n, 0))(\operatorname{Spec} k).$$

DEFINITION 6.10. Let $W \in Sm/k$, and assume that W is of pure dimension m. Fix a closed sub-scheme $Y \subset W$ and an object V in Sch/k. Let $r \geqslant 0$. Define the sub-sheaf

$$z_{\mathrm{equi}}(W \times_k V, m + r)_Y \subset z_{\mathrm{equi}}(W \times_k V, m + r)$$

as follows. Map $U \in Sm/k$ to the free Abelian group on the symbols (Z), where Z runs through those generators of $z_{\mathrm{equi}}(W \times_k V, m + r)(U)$ such that for any geometric point

$$\operatorname{Spec}(\overline{k}) \longrightarrow U \times_k Y \hookrightarrow U \times_k W$$

of $U \times_k Y$, the sub-scheme

$$Z \times_{U \times_k W} \operatorname{Spec}(\overline{k})$$

of $V_{\overline{k}}$ is empty or of dimension r. The sheaf of Abelian monoids

$$z_{\mathrm{equi}}^{\mathrm{eff}}(W \times_k V, m + r)_Y \subset z_{\mathrm{equi}}(W \times_k V, m + r)_Y$$

is defined as the intersection of $z_{\mathrm{equi}}(W \times_k V, m+r)_Y$ with the monoid of effective cycles $z_{\mathrm{equi}}^{\mathrm{eff}}(W \times_k V, m + r)$ in $z_{\mathrm{equi}}(W \times_k V, m + r)$.

One checks that $z_{\mathrm{equi}}(W \times_k V, m + r)_Y$ and $z_{\mathrm{equi}}^{\mathrm{eff}}(W \times_k V, m + r)_Y$ inherit the transfers from $z_{\mathrm{equi}}(W \times_k V, m+r)$. If one imposes the defining condition on all geometric points of $U \times_k W$ instead of just those of $U \times_k Y$, then one obtains $z_{\mathrm{equi}}(W, V, r)(U)$. Hence $z_{\mathrm{equi}}(W, V, r)$ is a sub-sheaf of $z_{\mathrm{equi}}(W \times_k V, m + r)_Y$:

$$\mathcal{D} : z_{\mathrm{equi}}(W, V, r) \longhookrightarrow z_{\mathrm{equi}}(W \times_k V, m + r)_Y.$$

Define the natural inclusion

$$\iota : z_{\mathrm{equi}}(W \times_k V, m + r)_Y \longhookrightarrow z_{\mathrm{equi}}(W \times_k V, m + r).$$

The Moving Lemma [VSF00, Theorem IV.6.3] implies (see [VSF00, Lemma IV.6.6]) that if both W and V are smooth and projective, then both $\mathbf{R}C(\iota)$ and $\mathbf{R}C(\iota) \circ \mathbf{R}C(\mathcal{D})$ are isomorphisms. Hence $\mathbf{R}C(\mathcal{D})$ is an isomorphism if both W and V are smooth and projective. Our aim is to prove this statement under less restrictive hypotheses on W and V. Our result is a variant of duality for bivariant cycle cohomology [VSF00, Theorem IV.7.4].

THEOREM 6.11. *Let $W \in Sm/k$ be quasi-projective and of pure dimension m. Let $Y \subset W$ be a closed sub-scheme and $V \in Sch/k$. Let $r \geqslant 0$. Then the inclusion*

$$\mathcal{D} : z_{\mathrm{equi}}(W, V, r) \longhookrightarrow z_{\mathrm{equi}}(W \times_k V, m + r)_Y$$

induces an isomorphism $\mathbf{R}C(\mathcal{D})$. It is compatible with restriction of W and Y in the sense of Convention 6.6.

Proof. We shall follow faithfully the strategy of [VSF00, pp. 172–176]. Fix compactifications $\overline{W}$ of W and $\overline{V}$ of V with a smooth and projective $\overline{W}$. For any proper $\overline{V}$-scheme $\overline{T}$, define the morphism

$$\alpha_{\overline{T}} : z_{\mathrm{equi}}(\overline{W} \times_k \overline{T}, m + r) \longrightarrow z_{\mathrm{equi}}(\overline{W} \times_k \overline{V}, m + r) \longrightarrow z_{\mathrm{equi}}(W \times_k V, m + r)$$

as the composition of proper push-forward with restriction [VSF00, pp. 141–142]. Similarly, define the variant on effective cycle sheaves

$$\alpha_{\overline{T}}^{\mathrm{eff}} : z_{\mathrm{equi}}^{\mathrm{eff}}(\overline{W} \times_k \overline{T}, m + r) \longrightarrow z_{\mathrm{equi}}^{\mathrm{eff}}(W \times_k V, m + r).$$

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Denote by $\Psi_{\overline{T}} \subset z_{\text{equi}}(\overline{W} \times_k \overline{T}, m + r)$ the sub-sheaf of Abelian groups generated by $(\alpha_{\overline{T}}^{\text{eff}})^{-1}$ $(z_{\text{equi}}^{\text{eff}}(W \times_k V, m + r)_Y)$. We have

$$\Psi_{\overline{V}} = \alpha_{\overline{V}}^{-1}(z_{\text{equi}}(W \times_k V, m + r)_Y).$$

We claim that the inclusion

$$\iota_{\overline{T}} : \Psi_{\overline{T}} \longhookrightarrow z_{\text{equi}}(\overline{W} \times_k \overline{T}, m + r)$$

induces a quasi-isomorphism $\underline{C}_*(\iota_{\overline{T}})$. In order to prove this claim, imitate the proof of [VSF00, Proposition IV.7.3]. One uses the Moving Lemma that we have already cited. It is here that the projectivity assumption on $\overline{W}$ enters.

Next, one imitates the proof of [VSF00, Theorem IV.7.4], using the above instead of [VSF00, Proposition IV.7.3], to see that the inclusion

$$\iota : z_{\text{equi}}(W \times_k V, m + r)_Y \longhookrightarrow z_{\text{equi}}(W \times_k V, m + r)$$

induces a quasi-isomorphism $\underline{C}_*(\iota)$. The same observation, applied to the case where $Y = W$ implies that the composition $\mathbf{R}C(\iota) \circ \mathbf{R}C(\mathcal{D})$ is an isomorphism. (This is, of course, the original statement of [VSF00, Theorem IV.7.4].) $\square$

We unite the assumptions from Proposition 6.5 and Definition 6.10. Hence $W \in Sm/k$ is of pure dimension m, and $Y \subset W$ is a closed sub-scheme such that arbitrary intersections of the components of Y are smooth. Observe that the condition on elements in $z_{\text{equi}}(W \times_k \mathbb{A}^n, m)_Y$ ensures, in particular, that they intersect properly with the components $Y_j \times_k \mathbb{A}^n$ of $Y \times_k \mathbb{A}^n$. This allows us to define an inverse image δ making the following diagram commutative.

$$
\begin{array}{ccc}
z_{\text{equi}}(W, \mathbb{A}^n, 0) & \stackrel{\mathcal{D}}{\longhookrightarrow} & z_{\text{equi}}(W \times_k \mathbb{A}^n, m)_Y \\
\delta \downarrow & & \downarrow \delta \\
\bigoplus_j z_{\text{equi}}(Y_j, \mathbb{A}^n, 0) & \stackrel{\mathcal{D}}{\longhookrightarrow} & \bigoplus_j z_{\text{equi}}(Y_j \times_k \mathbb{A}^n, \dim Y_j)
\end{array}
$$

The group in the lower right-hand corner has to be modified if Y_j has several components of different dimension. But in fact, $z_{\text{equi}}(W \times_k \mathbb{A}^n, m)_Y$ is defined such that δ not only exists but maps $z_{\text{equi}}(W \times_k \mathbb{A}^n, m)_Y$ to $\oplus_j z_{\text{equi}}(Y_j, \mathbb{A}^n, 0)$. Thus we may enlarge the complex

$$z_{\text{equi}}(Y^\bullet \to W, \mathbb{A}^n, 0)$$

by replacing its zeroth component $z_{\text{equi}}(W, \mathbb{A}^n, 0)$ with $z_{\text{equi}}(W \times_k \mathbb{A}^n, m)_Y$:

$$z_{\text{equi}}(Y^\bullet \to W, \mathbb{A}^n, 0) \subset z_{\text{equi}}(Y^\bullet \to W, \mathbb{A}^n, 0)'.$$

When W is quasi-projective, then Theorem 6.11 shows that this inclusion of complexes induces an isomorphism after application of $\mathbf{R}C$. In the general case, there is a (mainly notational) complication since we do not know whether the functor on open sub-schemes of W,

$$U \longmapsto \underline{C}_*(z_{\text{equi}}(U \times_k \mathbb{A}^n, m)_Y),$$

satisfies the Mayer–Vietoris property. Therefore, we fix an additional geometric datum, namely a finite open covering $\{W_\alpha\}$ of W by quasi-projective schemes. Consider the covering of Y induced by $\{W_\alpha\}$, and define

$$z_{\text{equi}}(Y^\bullet \to W, \mathbb{A}^n, 0)''$$

as the simple complex of the double Čech complex associated to this covering and $z_{\text{equi}}(Y^\bullet \to W, \mathbb{A}^n, 0)'$. For example, the components of degree 0 and 1 are

$$\bigoplus_\alpha z_{\text{equi}}(W_\alpha \times_k \mathbb{A}^n, m)_Y$$

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and

$$\bigoplus_{\alpha,j} z_{\text{equi}}(W_\alpha \cap Y_j, \mathbb{A}^n, 0) \oplus \bigoplus_{\alpha_1 \neq \alpha_2} z_{\text{equi}}((W_{\alpha_1} \cap W_{\alpha_2}) \times_k \mathbb{A}^n, m)_Y.$$

Consider the natural inclusion

$$\mathcal{D} : z_{\text{equi}}(Y^\bullet \to W, \mathbb{A}^n, 0) \hookrightarrow z_{\text{equi}}(Y^\bullet \to W, \mathbb{A}^n, 0)',$$

and use the same symbol for the composition of $\mathcal{D}$ with the co-augmentation

$$z_{\text{equi}}(Y^\bullet \to W, \mathbb{A}^n, 0)' \longrightarrow z_{\text{equi}}(Y^\bullet \to W, \mathbb{A}^n, 0)''.$$

COROLLARY 6.12. *The morphism*

$$\mathcal{D} : z_{\text{equi}}(Y^\bullet \to W, \mathbb{A}^n, 0) \longrightarrow z_{\text{equi}}(Y^\bullet \to W, \mathbb{A}^n, 0)''$$

induces an isomorphism $\mathbf{R}C(\mathcal{D})$. *In particular, different choices of coverings* $W = \cup_\alpha W_\alpha$ *give rise to the same object* $\mathbf{R}C(z_{\text{equi}}(Y^\bullet \to W, \mathbb{A}^n, 0)'')$. *The isomorphism* $\mathbf{R}C(\mathcal{D})$ *is compatible with restriction of* W *and* Y *in the sense of Convention 6.6.*

Proof. This follows from Theorem 6.11 and Corollary 6.9. □

DEFINITION 6.13. Define the sub-sheaf with transfers

$$z_{\text{equi}}(W, m - n)_Y \subset z_{\text{equi}}(W, m - n)$$

as the sub-sheaf of cycles having empty intersection with Y.

Note that $z_{\text{equi}}(W, m - n)_Y$ behaves contravariantly with respect to restriction of W and Y. Flat pull-back defines a morphism

$$p_{\mathbb{A}^n}^* : z_{\text{equi}}(W, m - n)_Y[0] \longrightarrow z_{\text{equi}}(Y^\bullet \to W, \mathbb{A}^n, 0)'.$$

Its composition with the co-augmentation

$$z_{\text{equi}}(Y^\bullet \to W, \mathbb{A}^n, 0)' \longrightarrow z_{\text{equi}}(Y^\bullet \to W, \mathbb{A}^n, 0)''$$

induces a morphism on the level of $\underline{h}^0(\operatorname{Spec} k)$, denoted by the same symbol $p_{\mathbb{A}^n}^*$. Putting everything together, we obtain the following.

THEOREM 6.14. *Let* $W \in Sm/k$ *be of pure dimension* m *and let* $Y \subset W$ *be a closed sub-scheme such that arbitrary intersections of the components of* Y *are smooth. Then there is a unique morphism*

$$cyc_{W/Y} : \underline{h}^0(z_{\text{equi}}(W, m - n)_Y)(\operatorname{Spec} k) \longrightarrow \operatorname{Hom}(M_{\text{gm}}(W/Y), \mathbb{Z}(n)[2n])$$

making the following diagram commute.

$$
\begin{array}{ccc}
\underline{h}^0(z_{\text{equi}}(W, m-n)_Y)(\operatorname{Spec} k) & \xrightarrow{\;p_{\mathbb{A}^n}^*\;} & \underline{h}^0(z_{\text{equi}}(Y^\bullet \to W, \mathbb{A}^n, 0)'')(\operatorname{Spec} k) \\
{\scriptstyle cyc_{W/Y}}\Big\downarrow & & \cong\Big\uparrow{\scriptstyle \mathcal{D}} \\
\operatorname{Hom}(M_{\text{gm}}(W/Y), \mathbb{Z}(n)[2n]) & \xleftarrow[\cong]{\;c_{W/Y}\;} & \underline{h}^0(z_{\text{equi}}(Y^\bullet \to W, \mathbb{A}^n, 0))(\operatorname{Spec} k)
\end{array}
$$

Here Hom *denotes morphisms in* $DM_{\text{gm}}^{\text{eff}}(k)$. *The morphism* $cyc_{W/Y}$ *is compatible with restriction of* W *and* Y *in the sense of Convention 6.6.*

Proof. Apply Corollaries 6.8 and 6.12. □

Remark 6.15. Another type of compatibility property with respect to change of W and Y is useful. Assume a given second pair $Y_1 \subset W_1$ of schemes satisfying the hypotheses of Theorem 6.14, and

a proper morphism $W_1 \to W$ identifying Y_1 with the fibre product $W_1 \times_W Y$, and inducing an isomorphism from $W_1 - Y_1$ to $W - Y$. Theorem 4.2 tells us that

$$M_{\mathrm{gm}}(W_1/Y_1) \longrightarrow M_{\mathrm{gm}}(W/Y)$$

is an isomorphism. On the other hand, we clearly have

$$z_{\mathrm{equi}}(W, m - n)_Y = z_{\mathrm{equi}}(W_1, m - n)_{Y_1}.$$

It is easy to see that in this situation, the diagram

$$\begin{array}{ccc}
\underline{h}^0(z_{\mathrm{equi}}(W, m - n)_Y)(\operatorname{Spec} k) & \xrightarrow{=} & \underline{h}^0(z_{\mathrm{equi}}(W_1, m - n)_{Y_1})(\operatorname{Spec} k) \\
cyc_{W/Y} \downarrow & & cyc_{W_1/Y_1} \downarrow \\
\operatorname{Hom}(M_{\mathrm{gm}}(W/Y), \mathbb{Z}(n)[2n]) & \xrightarrow{\cong} & \operatorname{Hom}(M_{\mathrm{gm}}(W_1/Y_1), \mathbb{Z}(n)[2n])
\end{array}$$

commutes.

When Y is empty, then [VSF00, Corollary V.4.2.5] tells us that $cyc_W := cyc_{W/\emptyset}$ is an isomorphism. We have not attempted to prove whether the analogous statement for $cyc_{W/Y}$ is true when Y is non-empty.

Proof of Theorem 6.1. We start by introducing the notation $Cyc^n(W)_Y$ for the group $z_{\mathrm{equi}}(W, m - n)_Y(\operatorname{Spec} k)$, when W and $Y \subset W$ are as before. By definition, $Cyc^n(W)_Y$ is the group of codimension n cycles on W not meeting Y. Write $Cyc^n(W)$ for the group of all codimension n cycles on W. Our proof relies on the following principles, which are consequences of Theorem 6.14 and of the definition of the tensor structure on $DM_-^{\mathrm{eff}}(k)$.

(A) For $i = 1, 2$, let $W_i \in Sm/k$ be of pure dimension, with closed sub-schemes $Y_i \subset W_i$, such that arbitrary intersections of the components of the Y_i are smooth. Then any

$$c \in Cyc^n(W_1 \times_k W_2)_{Y_1 \times_k W_2 \cup W_1 \times_k Y_2}$$

defines a pairing

$$(\bullet , \bullet)_c : M_{\mathrm{gm}}(W_1/Y_1) \otimes M_{\mathrm{gm}}(W_2/Y_2) \longrightarrow \mathbb{Z}(n)[2n],$$

or equivalently, a morphism

$$\varepsilon_c : M_{\mathrm{gm}}(W_1/Y_1) \longrightarrow M_{\mathrm{gm}}(W_2/Y_2)^*(n)[2n]$$

in $DM_{\mathrm{gm}}(k)$. The morphism ε_c is induced by a morphism of Nisnevich sheaves

$$e_{c'} : c(\bullet , W_1)/c(\bullet , Y_1) \longrightarrow z_{\mathrm{equi}}(Y_2^\bullet \to W_2, \mathbb{A}^n, 0)$$

defined as follows. Use Corollary 6.12 to move the pull-back $p_{\mathbb{A}^n}^*(c)$ into a cycle

$$c' \in z_{\mathrm{equi}}(W_1 \times_k W_2, \mathbb{A}^n, 0)(\operatorname{Spec} k)$$

having empty intersection with $Y_1 \times_k W_2 \cup W_1 \times_k Y_2$. Given $U \in Sm/k$ and $\mathcal{Z} \in c(U, W_1)$, pull back $\mathcal{Z}$ to W_2, giving $\mathcal{Z}_{W_2} \in c(U \times_k W_2, W_1)$. Similarly, pull back c' to U, giving c'_U. Now consider the cycle

$$c' \cap \mathcal{Z}_{W_2} := Cor_{W_1 \times_k W_2 \times_k \mathbb{A}^n / W_1 \times_k W_2}(c'_U \otimes \mathcal{Z}_{W_2}),$$

where $Cor_{W_1 \times_k W_2 \times_k \mathbb{A}^n / W_1 \times_k W_2}$ is the correspondence homomorphism from [VSF00, §II.3.7]. By [VSF00, Corollary II.3.7.5], we have

$$c' \cap \mathcal{Z}_{W_2} \in z_{\mathrm{equi}}(W_2, W_1 \times_k \mathbb{A}^n, 0)(U).$$

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Furthermore, finiteness of $\mathcal{Z}$ over U implies that $c' \cap \mathcal{Z}_{W_2}$ is finite over $U \times_k W_2 \times_k \mathbb{A}^n$. Push-forward via the projection p_2 to this product then yields

$$e_{c'}(\mathcal{Z}) := p_{2\,*}(c' \cap \mathcal{Z}_{W_2}) \in z_{\mathrm{equi}}(W_2, \mathbb{A}^n, 0)(U).$$

In fact, $e_{c'}(\mathcal{Z})$ lies in $z_{\mathrm{equi}}(Y_2^\bullet \to W_2, \mathbb{A}^n, 0)(U)$ since c' has empty intersection with $W_1 \times_k Y_2$. Furthermore, $e_{c'}(\mathcal{Z})$ only depends on the class of $\mathcal{Z}$ modulo $c(U, Y_1)$ since c' has empty intersection with $Y_1 \times_k W_2$.

(B) Let $W \in Sm/k$ be of pure dimension, with a closed sub-scheme $Y \subset W$, such that arbitrary intersections of the components of Y are smooth. Let

$$c \in Cyc^n(W \times_k W)_{Y \times_k W \cup W \times_k Y}.$$

Then a sufficient condition for the pairing

$$(\, \bullet \,, \, \bullet \,)_c : M_{\mathrm{gm}}(W/Y) \otimes M_{\mathrm{gm}}(W/Y) \longrightarrow \mathbb{Z}(n)[2n]$$

to be symmetric or, equivalently, for the morphism

$$\varepsilon_c : M_{\mathrm{gm}}(W/Y) \longrightarrow M_{\mathrm{gm}}(W/Y)^*(n)[2n]$$

to be auto-dual (i.e. $\varepsilon_c = \varepsilon_c^*(n)[2n]$), is the symmetry of the cycle c in $Cyc^n(W \times_k W)$.

(C) For $i = 1, \ldots, 4$, let $W_i \in Sm/k$ be of pure dimension, with closed sub-schemes $Y_i \subset W_i$, such that arbitrary intersections of the components of the Y_i are smooth. For $i = 1, 2$, let $j_i : W_i \hookrightarrow W_{i+2}$ be an open immersion mapping Y_i into Y_{i+2}. Let

$$c_{1,4} \in Cyc^n(W_1 \times_k W_4)_{Y_1 \times_k W_4 \cup W_1 \times_k Y_4}$$

and

$$c_{3,2} \in Cyc^n(W_3 \times_k W_2)_{Y_3 \times_k W_2 \cup W_3 \times_k Y_2}.$$

Then a sufficient condition for the diagram

$$
\begin{array}{ccc}
M_{\mathrm{gm}}(W_1/Y_1) & \xrightarrow{\;\varepsilon_{c_{1,4}}\;} & M_{\mathrm{gm}}(W_4/Y_4)^*(n)[2n] \\
{\scriptstyle j_1}\Big\downarrow & & \Big\downarrow{\scriptstyle j_2^*(n)[2n]} \\
M_{\mathrm{gm}}(W_3/Y_3) & \xrightarrow{\;\varepsilon_{c_{3,2}}\;} & M_{\mathrm{gm}}(W_2/Y_2)^*(n)[2n]
\end{array}
$$

to commute is the equality of cycles

$$(j_1, \mathrm{id}_{W_2})^* c_{3,2} = (\mathrm{id}_{W_1}, j_2)^* c_{1,4}$$

in $Cyc^n(W_1 \times_k W_2)$.

Furthermore, using the compatibility of moving cycles with correspondence homomorphisms and direct images, and [VSF00, Proposition II.3.7.6], one sees the following principle.

(D) For $i = 1, 2$, let $W_i \in Sm/k$ be of pure dimension m_i, with a closed sub-scheme $Y_1 \subset W_1$, such that arbitrary intersections of the components of Y_1 are smooth. Assume that $m_2 \geqslant n$ and that

$$c \in Cyc^n(W_1 \times_k W_2)_{Y_1 \times_k W_2} \cap z_{\mathrm{equi}}(W_1, W_2, m_2 - n)(\mathrm{Spec}\, k).$$

Then the morphism

$$\varepsilon_c : M_{\mathrm{gm}}(W_1/Y_1) \longrightarrow M_{\mathrm{gm}}(W_2)^*(n)[2n]$$

associated to c is also induced by the composition of the morphism of Nisnevich sheaves

$$e'_c : c(\, \bullet \,, W_1)/c(\, \bullet \,, Y_1) \longrightarrow z_{\mathrm{equi}}(W_2 \times_k \mathbb{A}^n, m_2)$$

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with the inverse of the isomorphism $\mathbf{R}C(\mathcal{D})$ from Theorem 6.11, where e'_c is defined as follows. Given $U \in Sm/k$ and $\mathcal{Z} \in c(U, W_1)$, pull back $\mathcal{Z}$ to $\mathbb{A}^n$, giving $\mathcal{Z}_{\mathbb{A}^n} \in c(U \times_k \mathbb{A}^n, W_1)$. Similarly, pull back c to $U \times_k \mathbb{A}^n$, giving $c_{U \times_k \mathbb{A}^n}$. Now consider the cycle

$$c \cap \mathcal{Z}_{\mathbb{A}^n} := Cor_{W_1 \times_k W_2 \times_k \mathbb{A}^n / W_1 \times_k \mathbb{A}^n}(c_{U \times_k \mathbb{A}^n} \otimes \mathcal{Z}_{\mathbb{A}^n}).$$

By [VSF00, Corollary II.3.7.5], we have

$$c \cap \mathcal{Z}_{\mathbb{A}^n} \in z_{\mathrm{equi}}(\mathbb{A}^n, W_1 \times_k W_2, m_2 - n)(U).$$

Furthermore, finiteness of $\mathcal{Z}$ over U implies that $c \cap \mathcal{Z}_{\mathbb{A}^n}$ is finite over $U \times_k W_2 \times_k \mathbb{A}^n$. Push-forward via the projection p_2 to this product then yields

$$e'_c(\mathcal{Z}) := p_{2*}(c \cap \mathcal{Z}_{\mathbb{A}^n}) \in z_{\mathrm{equi}}(\mathbb{A}^n, W_2, m_2 - n)(U),$$

and the latter group is contained in $z_{\mathrm{equi}}(W_2 \times_k \mathbb{A}^n, m_2)(U)$. Observe that $e'_c(\mathcal{Z})$ only depends on the class of $\mathcal{Z}$ modulo $c(U, Y_1)$ since c has empty intersection with $Y_1 \times_k W_2$.

Choose and fix a smooth compactification $\overline{X}$ of X such that $\partial\overline{X} := \overline{X} - X$ is a normal crossing divisor with smooth irreducible components. By [VSF00, Proposition V.4.1.5], we have a canonical isomorphism between $M_{\mathrm{gm}}(\overline{X} / \partial\overline{X})$ and $M_{\mathrm{gm}}^{\mathrm{c}}(X)$. Applying principle (A), we see that the diagonal

$$\Delta \in Cyc^n(\overline{X} \times_k X)_{\partial\overline{X} \times_k X}$$

induces a morphism

$$\varepsilon_\Delta : M_{\mathrm{gm}}^{\mathrm{c}}(X) \longrightarrow M_{\mathrm{gm}}(X)^*(n)[2n].$$

Principle (D) and the proof of [VSF00, Theorem V.4.3.7] show that this is the morphism μ_X. To say that

$$
\begin{array}{ccc}
M_{\mathrm{gm}}(X) & \xrightarrow{\nu_X = \mu_X^*(n)[2n]} & M_{\mathrm{gm}}^{\mathrm{c}}(X)^*(n)[2n] \\
\beta \downarrow & & \downarrow \beta^*(n)[2n] \\
M_{\mathrm{gm}}^{\mathrm{c}}(X) & \xrightarrow{\mu_X} & M_{\mathrm{gm}}(X)^*(n)[2n]
\end{array}
$$

commutes is equivalent to saying that the pairing

$$M_{\mathrm{gm}}(X) \otimes M_{\mathrm{gm}}(X) \xrightarrow{\mathrm{id} \otimes \beta} M_{\mathrm{gm}}(X) \otimes M_{\mathrm{gm}}^{\mathrm{c}}(X) \xrightarrow{(\bullet, \bullet)_\Delta} \mathbb{Z}(n)[2n]$$

is symmetric. By principle (B), this is indeed the case since the restriction of the cycle Δ to $X \times_k X$ is symmetric.

Now for the construction of

$$\eta_X : \partial M_{\mathrm{gm}}(X) \longrightarrow \partial M_{\mathrm{gm}}(X)^*(n)[2n - 1].$$

Recall that by Proposition 2.4, there is a canonical isomorphism

$$\mathrm{Cone}\left(M_{\mathrm{gm}}\left(X \coprod \partial\overline{X}\right) \to M_{\mathrm{gm}}(\overline{X})\right) \xrightarrow{\sim} \partial M_{\mathrm{gm}}(X).$$

From this, one deduces that $\partial M_{\mathrm{gm}}(X) \otimes \partial M_{\mathrm{gm}}(X)[1]$ is represented by the complex $\underline{C}_*(s\mathcal{L}_*)$, where $\mathcal{L}_*$ is the complex concentrated in degrees -1, 0 and 1:

$$L(X \times_k X) \oplus L(X \times_k \partial\overline{X}) \oplus L(\partial\overline{X} \times_k X) \oplus L(\partial\overline{X} \times_k \partial\overline{X})$$

$$\downarrow$$

$$L(\overline{X} \times_k X) \oplus L(X \times_k \overline{X}) \oplus L(\overline{X} \times_k \partial\overline{X}) \oplus L(\partial\overline{X} \times_k \overline{X})$$

$$\downarrow$$

$$L(\overline{X} \times_k \overline{X})$$

with the differentials being induced by the inclusions. In particular, one sees that the complex $\mathcal{L}'_*$ concentrated in degrees -1 and 0, that is

$$L(X \times_k X) \oplus L(X \times_k \partial\overline{X}) \oplus L(\partial\overline{X} \times_k X)$$

$$\downarrow$$

$$L(\overline{X} \times_k X) \oplus L(X \times_k \overline{X}),$$

is a quotient of $\mathcal{L}_*$. By the Mayer–Vietoris property for the functor M_{gm} [VSF00, Proposition V.4.1.1], $\underline{C}_*(\mathcal{L}'_*)$ is canonically quasi-isomorphic to the complex $\underline{C}_*(\mathcal{L}''_*)$, where $\mathcal{L}''_*$ is the complex concentrated in degrees -1 and 0:

$$L(\overline{X} \times_k \overline{X} - \partial\overline{X} \times_k \partial\overline{X} - X \times_k X)$$

$$\downarrow$$

$$L(\overline{X} \times_k \overline{X} - \partial\overline{X} \times_k \partial\overline{X}).$$

This shows that there is a canonical morphism from $\partial M_{\mathrm{gm}}(X) \otimes \partial M_{\mathrm{gm}}(X)[1]$ to

$$M_{\mathrm{gm}}((\overline{X} \times_k \overline{X} - \partial\overline{X} \times_k \partial\overline{X})/(\overline{X} \times_k \overline{X} - \partial\overline{X} \times_k \partial\overline{X} - X \times_k X)).$$

Applying principle (A), we see that the diagonal

$$\Delta \in Cyc^n(\overline{X} \times_k \overline{X} - \partial\overline{X} \times_k \partial\overline{X})_{\overline{X} \times_k \overline{X} - \partial\overline{X} \times_k \partial\overline{X} - X \times_k X}$$

induces a morphism

$$\varepsilon_\Delta : \partial M_{\mathrm{gm}}(X) \longrightarrow \partial M_{\mathrm{gm}}(X)^*(n)[2n-1].$$

We define this to be the morphism η_X. Principle (B) shows that η_X is auto-dual. In order to see that η_X fits into a morphism of exact triangles

$$
\begin{array}{ccc}
\partial M_{\mathrm{gm}}(X) & \xrightarrow{\eta_X} & \partial M_{\mathrm{gm}}(X)^*(n)[2n-1] \\
\alpha \downarrow & & \downarrow \gamma^*(n)[2n] \\
M_{\mathrm{gm}}(X) & \xrightarrow{\nu_X} & M_{\mathrm{gm}}^c(X)^*(n)[2n] \\
\beta \downarrow & & \downarrow \beta^*(n)[2n] \\
M_{\mathrm{gm}}^c(X) & \xrightarrow{\mu_X} & M_{\mathrm{gm}}(X)^*(n)[2n] \\
\gamma \downarrow & & \downarrow \alpha^*(n)[2n] \\
\partial M_{\mathrm{gm}}(X)[1] & \xrightarrow{\eta_X[1]} & \partial M_{\mathrm{gm}}(X)^*(n)[2n]
\end{array}
$$

apply principle (C). By [VSF00, Theorem V.4.3.7 3], μ_X and ν_X are isomorphisms, and hence so is η_X. In order to check that η_X does not depend on the choice of the smooth compactification $\overline{X}$, use Remark 6.15 together with the fact that the system of such compactifications is filtering. $\square$

Fix a compactification $\overline{X}$ of X (which may be non-smooth) and set $\partial\overline{X} := \overline{X} - X$. Denote by j the open immersion of X and by $i_{\partial\overline{X}}$ for the closed immersion of $\partial\overline{X}$ into $\overline{X}^m$. We are thus in the situation considered in §3, with $Y = Y' = \partial\overline{X}$ and $W = \overline{X}$. By Proposition 2.4, we have a canonical isomorphism

$$M_{\mathrm{gm}}(\partial\overline{X}, i^!_{\partial\overline{X}} j_! \mathbb{Z}) \xrightarrow{\sim} \partial M_{\mathrm{gm}}(X)[1].$$

By duality, this gives

$$\alpha_X : \partial M_{\mathrm{gm}}(X)^*[-1] \xrightarrow{\sim} M_{\mathrm{gm}}(\partial\overline{X}, i^!_{\partial\overline{X}} j_! \mathbb{Z})^*.$$

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Preceding $\alpha_X(n)[2n]$ with the auto-duality η_X from Theorem 6.1, we obtain the following.

COROLLARY 6.16. *There is a canonical isomorphism*

$$\partial M_{\mathrm{gm}}(X) \overset{\sim}{\longrightarrow} M_{\mathrm{gm}}(\partial \overline{X}, i_{\partial \overline{X}}^! j_! \mathbb{Z})^*(n)[2n].$$

7. Localization

Throughout this section, we assume that k admits resolution of singularities. Fix closed immersions $Y \hookrightarrow Y' \hookrightarrow W$ in Sch/k. Denote by j the open immersion of $W - Y'$ and by i_Y the closed immersion of Y into W.

DEFINITION 7.1. Assume that $W - Y' \in Sm/k$.

(a) If $W - Y'$ is of pure dimension n, we put

$$M_{\mathrm{gm}}^{\mathrm{c}}(Y, i_Y^* j_* \mathbb{Z}) := M_{\mathrm{gm}}(Y, i_Y^! j_! \mathbb{Z})^*(n)[2n].$$

(b) If $W - Y' = \coprod_\alpha W_\alpha$ is the decomposition of $W - Y'$ into connected components, then the *motive with compact support of Y and with coefficients in $i_Y^* j_* \mathbb{Z}$* is defined as

$$M_{\mathrm{gm}}^{\mathrm{c}}(Y, i_Y^* j_* \mathbb{Z}) := \bigoplus_\alpha M_{\mathrm{gm}}^{\mathrm{c}}(Y, i_Y^* j_{\alpha*} \mathbb{Z}).$$

Here, the j_α denote the open immersions of the W_α into W.

Using [VSF00, Corollary V.4.3.6] and the definition of $M_{\mathrm{gm}}(Y, i_Y^! j_! \mathbb{Z})$ (Definition 3.1), one sees that $M_{\mathrm{gm}}^{\mathrm{c}}(Y, i_Y^* j_* \mathbb{Z})$ belongs to $DM_{\mathrm{gm}}^{\mathrm{eff}}(k)$.

Remark 7.2. The object on the right-hand side in Definition 7.1 is defined without the hypothesis of smoothness on $W - Y'$. In general, the object $M_{\mathrm{gm}}(Y, i_Y^! j_! \mathbb{Z})$ is dual to what should be considered as the motive with compact support of Y and with coefficients in $i_Y^* j_* \mathbb{D}(\mathbb{Z})$, where $\mathbb{D}(\mathbb{Z})$ is the coefficient system on $W - Y'$ which is Verdier-dual to $\mathbb{Z}$.

Now assume a given filtration

$$\emptyset = \mathfrak{F}_{-1} Y \subset \mathfrak{F}_0 Y \subset \cdots \subset \mathfrak{F}_d Y = Y$$

of Y by closed sub-schemes. It induces a stratification of Y by locally closed sub-schemes $Y_{\mathrm{m}} := \mathfrak{F}_{\mathrm{m}} Y - \mathfrak{F}_{\mathrm{m}-1} Y$, for $m = 0, \ldots, d$. Define W^m as the complement of $\mathfrak{F}_{\mathrm{m}-1} Y$ in W. Write $i_{Y_{\mathrm{m}}}$ for the closed immersion of Y_{m} into W^m. By abuse of notation, we use the letter j to also denote the open immersions of $W - Y'$ into W^m.

THEOREM 7.3 (Localization). *Assume that $W - Y' \in Sm/k$. Then there is a canonical chain of morphisms*

$$M_0 = M_{\mathrm{gm}}^{\mathrm{c}}(Y, i_Y^* j_* \mathbb{Z}) \overset{\gamma_0}{\longrightarrow} M_1 \overset{\gamma_1}{\longrightarrow} M_2 \overset{\gamma_2}{\longrightarrow} \cdots \overset{\gamma_{d-1}}{\longrightarrow} M_d \overset{\gamma_d}{\longrightarrow} M_{d+1} = 0$$

in $DM_{\mathrm{gm}}(k)$. For each $m \in \{0, \ldots, d\}$, there is a canonical exact triangle

$$M_{\mathrm{gm}}^{\mathrm{c}}(Y_{\mathrm{m}}, i_{Y_{\mathrm{m}}}^* j_* \mathbb{Z}) \longrightarrow M_{\mathrm{m}} \overset{\gamma_{\mathrm{m}}}{\longrightarrow} M_{\mathrm{m}+1} \longrightarrow M_{\mathrm{gm}}^{\mathrm{c}}(Y_{\mathrm{m}}, i_{Y_{\mathrm{m}}}^* j_* \mathbb{Z})[1]$$

in $DM_{\mathrm{gm}}(k)$. In particular, all the M_{m} are in $DM_{\mathrm{gm}}^{\mathrm{eff}}(k)$.

Proof. Dualize co-localization (Theorem 3.4). $\qquad\square$

COROLLARY 7.4. *In the above situation, assume that $Y = \partial \overline{X} := \overline{X} - X$, with $X \in Sm/k$, and $\overline{X}$ a compactification of X (which may be non-smooth). Write $(\partial \overline{X})_{\mathrm{m}} := Y_{\mathrm{m}}$. Then there is a canonical chain of morphisms*

$$M_0 = \partial M_{\mathrm{gm}}(X) \overset{\gamma_0}{\longrightarrow} M_1 \overset{\gamma_1}{\longrightarrow} M_2 \overset{\gamma_2}{\longrightarrow} \cdots \overset{\gamma_{d-1}}{\longrightarrow} M_d \overset{\gamma_d}{\longrightarrow} M_{d+1} = 0$$

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in $DM_{\mathrm{gm}}(k)$. For each $m \in \{0, \ldots, d\}$, there is a canonical exact triangle

$$M_{\mathrm{gm}}^{\mathrm{c}}((\partial\overline{X})_{\mathrm{m}}, i_{(\partial\overline{X})_{\mathrm{m}}}^* j_* \mathbb{Z}) \longrightarrow M_{\mathrm{m}} \xrightarrow{\gamma_{\mathrm{m}}} M_{\mathrm{m}+1} \longrightarrow M_{\mathrm{gm}}^{\mathrm{c}}((\partial\overline{X})_{\mathrm{m}}, i_{(\partial\overline{X})_{\mathrm{m}}}^* j_* \mathbb{Z})[1]$$

in $DM_{\mathrm{gm}}(k)$. In particular, all the M_{m} are in $DM_{\mathrm{gm}}^{\mathrm{eff}}(k)$.

Proof. This follows from Corollary 6.16 and Theorem 7.3. $\qquad\square$

Remark 7.5. Given our definition, it is easy to deduce from duality and Theorems 4.1 and 5.1 that $M_{\mathrm{gm}}^{\mathrm{c}}(Y, i_Y^* j_* \mathbb{Z})$ is invariant under abstract blow-up and under analytical isomorphism. In concrete situations, this observation helps to control the 'graded pieces' $M_{\mathrm{gm}}^{\mathrm{c}}((\partial\overline{X})_{\mathrm{m}}, i_{(\partial\overline{X})_{\mathrm{m}}}^* j_* \mathbb{Z})$ of $\partial M_{\mathrm{gm}}(X)$.

8. The case of normal crossings

Throughout this section, we consider the following situation: X lies in Sm/k and is irreducible and $\overline{X} \in Sm/k$ is a smooth compactification of X, such that $\partial\overline{X} := \overline{X} - X$ is a normal crossing divisor with smooth irreducible components. We stratify $\partial\overline{X}$ by defining $(\partial\overline{X})^m$ as the geometric locus of points lying on exactly m irreducible components, $m = 1, \ldots, \dim X$. Note that the $(\partial\overline{X})^m$ are all smooth. Denote by j the open immersion of X, and by i_{m} the immersion of $(\partial\overline{X})^m$ into $\overline{X}$. We re-state co-localization and localization (Corollaries 3.5 and 7.4) as follows.

COROLLARY 8.1. *Denote by n the dimension of X, and assume that k admits resolution of singularities.*

(a) *There is a canonical chain of morphisms*

$$M^n = 0 \xrightarrow{\gamma^{n-1}} M^{n-1} \xrightarrow{\gamma^{n-2}} M^{n-2} \xrightarrow{\gamma^{n-3}} \cdots \xrightarrow{\gamma^0} M^0 = \partial M_{\mathrm{gm}}(X)[1]$$

in $DM_{\mathrm{gm}}^{\mathrm{eff}}(k)$. For each $m \in \{1, \ldots, n\}$, there is a canonical exact triangle

$$M_{\mathrm{gm}}((\partial\overline{X})^m, i_{\mathrm{m}}^! j_! \mathbb{Z})[-1] \longrightarrow M^{n-m+1} \xrightarrow{\gamma^{n-m}} M^{n-m} \longrightarrow M_{\mathrm{gm}}((\partial\overline{X})^m, i_{\mathrm{m}}^! j_! \mathbb{Z})$$

in $DM_{\mathrm{gm}}^{\mathrm{eff}}(k)$.

(b) *There is a canonical chain of morphisms*

$$M_0 = \partial M_{\mathrm{gm}}(X) \xrightarrow{\gamma_0} M_1 \xrightarrow{\gamma_1} M_2 \xrightarrow{\gamma_2} \cdots \xrightarrow{\gamma_{n-2}} M_{n-1} \xrightarrow{\gamma_{n-1}} M_n = 0$$

in $DM_{\mathrm{gm}}^{\mathrm{eff}}(k)$. For each $m \in \{1, \ldots, n\}$, there is a canonical exact triangle

$$M_{\mathrm{gm}}^{\mathrm{c}}((\partial\overline{X})^m, i_{\mathrm{m}}^* j_* \mathbb{Z}) \longrightarrow M_{n-m} \xrightarrow{\gamma_{n-m}} M_{n-m+1} \longrightarrow M_{\mathrm{gm}}^{\mathrm{c}}((\partial\overline{X})^m, i_{\mathrm{m}}^* j_* \mathbb{Z})[1]$$

in $DM_{\mathrm{gm}}^{\mathrm{eff}}(k)$.

The aim of this section is to give a description of the motives with coefficients $M_{\mathrm{gm}}((\partial\overline{X})^m, i_{\mathrm{m}}^! j_! \mathbb{Z})$ and $M_{\mathrm{gm}}^{\mathrm{c}}((\partial\overline{X})^m, i_{\mathrm{m}}^* j_* \mathbb{Z})$ occurring in the above statement. Denote by N_{m} the normal bundle of $(\partial\overline{X})^m$ in $\overline{X}$. For any component D of the divisor $\partial\overline{X}$, consider the normal bundle of $(\partial\overline{X})^m \cap D$ in D. Using all possible D, this gives m sub-bundles of N_{m} of codimension one. Their intersection is the zero bundle over $(\partial\overline{X})^m$. Define N_{m}^* as the complement in N_{m} of the union of these sub-bundles. Fix an order $\prec$ of the index set of components of $\partial\overline{X}$, i.e. write

$$\partial\overline{X} = \bigcup_{i=1}^{r} D_i,$$

where $D_1, \ldots, D_r$ are the r distinct components of $\partial\overline{X}$. Note that N_{m}^* is a torsor under a torus of dimension m, and that the choice of $\prec$ gives an identification of this torus with $\mathbb{G}_{\mathrm{m}}^m$.

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THEOREM 8.2.

(a) *The order $\prec$ induces an isomorphism*

$$M_{\mathrm{gm}}((\partial\overline{X})^m, i_{\mathrm{m}}^! j_! \mathbb{Z}) \xrightarrow{\sim} M_{\mathrm{gm}}(N_{\mathrm{m}}^*)[m]$$

in $DM_{\mathrm{gm}}^{\mathrm{eff}}(k)$.

(a) *Assume that k admits resolution of singularities. Then the order $\prec$ induces an isomorphism*

$$M_{\mathrm{gm}}^{\mathrm{c}}((\partial\overline{X})^m, i_{\mathrm{m}}^* j_* \mathbb{Z}) \xrightarrow{\sim} M_{\mathrm{gm}}^{\mathrm{c}}(N_{\mathrm{m}}^*)[-m]$$

in $DM_{\mathrm{gm}}^{\mathrm{eff}}(k)$.

Our main computational tool in the proof of Theorem 8.2 will be the following.

LEMMA 8.3. *Let $S \in Sch/k$ and let V be a vector bundle of rank m over S. Assume given subbundles V_i, $i = 1, \ldots, \ell$ of codimension one of V, with $1 \leqslant \ell \leqslant m$. Assume also that Zariski-locally over S, there exist trivializations $V \cong \mathbb{A}_S^m$ identifying V_i with the hyperplane given by the vanishing of the ith coordinate, $i = 1, \ldots, \ell$. Define V^* as the complement in V of the union of the V_i. Then for any proper subset I of $\{1, \ldots, \ell\}$, the immersion $V - V^* - \cup_{i \in I} V_i \hookrightarrow V - \cup_{i \in I} V_i$ induces an isomorphism*

$$M_{\mathrm{gm}}(V - V^* - \cup_{i \in I} V_i) \xrightarrow{\sim} M_{\mathrm{gm}}(V - \cup_{i \in I} V_i)$$

in $DM_{\mathrm{gm}}^{\mathrm{eff}}(k)$.

Proof. The Mayer–Vietoris property [VSF00, Proposition V.3.1.3] for the functor L shows that we may assume that a trivialization as in the hypotheses exists globally. Then the immersion $V^* \hookrightarrow V - \bigcup_{i \in I} V_i$ is isomorphic to

$$\mathbb{G}_{m,S}^\ell \times_S \mathbb{A}_S^{m-\ell} \hookrightarrow \mathbb{G}_{m,S}^p \times_S \mathbb{A}_S^{m-p},$$

for some $p < \ell$. Hence $V - V^* - \cup_{i \in I} V_i \hookrightarrow V - \cup_{i \in I} V_i$ is a homotopy equivalence (in fact, both sides are homotopic to $\mathbb{G}_{m,S}^p$). $\qquad\square$

Proof of Theorem 8.2. (a) Recall that $M_{\mathrm{gm}}((\partial\overline{X})^m, i_{\mathrm{m}}^! j_! \mathbb{Z})$ is defined as the motive $M_{\mathrm{gm}}(\mathfrak{Y})$, where $\mathfrak{Y}$ is the diagram

$$
\begin{array}{ccc}
\partial\overline{X}' - (\partial\overline{X})^m & \longrightarrow & \overline{X}' - (\partial\overline{X})^m \\
\downarrow & & \downarrow \\
\partial\overline{X}' & \longrightarrow & \overline{X}'
\end{array}
$$

and $\overline{X}'$ and $\partial\overline{X}'$ denote the complements of the strata $(\partial\overline{X})^p$, for $p > m$. Now because of our assumptions on $\overline{X}$ and $\partial\overline{X}$, this diagram is analytically isomorphic to the following diagram $\mathfrak{N}$, in the sense that the assumptions of Theorem 5.1 are satisfied.

$$
\begin{array}{ccc}
N_{\mathrm{m}} - N_{\mathrm{m}}^* - 0(\partial\overline{X})^m & \longrightarrow & N_{\mathrm{m}} - 0(\partial\overline{X})^m \\
\downarrow & & \downarrow \\
N_{\mathrm{m}} - N_{\mathrm{m}}^* & \longrightarrow & N_{\mathrm{m}}
\end{array}
$$

Here, $0(\partial\overline{X})^m$ denotes the zero section of the bundle N_{m} over $(\partial\overline{X})^m$. Analytical invariance thus allows us to compute $M_{\mathrm{gm}}(\mathfrak{N})$ instead of $M_{\mathrm{gm}}(\mathfrak{Y})$. In order to do so, consider the upper line

$$N_{\mathrm{m}} - N_{\mathrm{m}}^* - 0(\partial\overline{X})^m \longrightarrow N_{\mathrm{m}} - 0(\partial\overline{X})^m$$

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of $\mathfrak{N}$, and the finite covering

$$N_{\mathrm{m}} - 0(\partial\overline{X})^m = \bigcup_{i=1}^{m} N_{\mathrm{m}} - V_i.$$

Here, the V_i are the sub-bundles of codimension one constructed from the components of $\partial\overline{X}$. For $1 \leqslant i_1 < \cdots < i_p \leqslant m$, we have

$$\bigcap_{q=1}^{p} N_{\mathrm{m}} - V_{i_q} = N_{\mathrm{m}} - \bigcup_{q=1}^{p} V_{i_q}.$$

Define the double complex $L(\mathfrak{N}')$ of Nisnevich sheaves as follows. First consider the Čech complex associated to the above covering, and to the induced covering of $N_{\mathrm{m}} - N_{\mathrm{m}}^* - 0(\partial\overline{X})^m$, then add the lower line of $L(\mathfrak{N})$ as follows.

$$
\begin{array}{ccc}
0 = L(\emptyset) & \longrightarrow & L(N_{\mathrm{m}}^*) \\
\downarrow & & \downarrow \\
\vdots & & \vdots \\
\downarrow & & \downarrow \\
\prod_{i_1 < i_2} L(N_{\mathrm{m}} - N_{\mathrm{m}}^* - (V_{i_1} \cup V_{i_2})) & \longrightarrow & \prod_{i_1 < i_2} L(N_{\mathrm{m}} - (V_{i_1} \cup V_{i_2})) \\
\downarrow & & \downarrow \\
\prod_i L(N_{\mathrm{m}} - N_{\mathrm{m}}^* - V_i) & \longrightarrow & \prod_i L(N_{\mathrm{m}} - V_i) \\
\downarrow & & \downarrow \\
L(N_{\mathrm{m}} - N_{\mathrm{m}}^*) & \longrightarrow & L(N_{\mathrm{m}})
\end{array}
$$

Here, we have total degree zero for $L(N_{\mathrm{m}})$, hence degree $-m$ for $L(N_{\mathrm{m}}^*)$. We thus get a canonical morphism of complexes of Nisnevich sheaves

$$L(\mathfrak{N}') \longrightarrow L(\mathfrak{N}).$$

The Mayer–Vietoris property [VSF00, Proposition V.3.1.3] shows that it induces an isomorphism in $DM_-^{\mathrm{eff}}(k)$. By Lemma 8.3, the projection from $L(\mathfrak{N}')$ to its upper line

$$L(\mathfrak{N}') \longrightarrow L(N_{\mathrm{m}}^*)[m]$$

also induces an isomorphism in $DM_-^{\mathrm{eff}}(k)$

(b) results from (a) by dualizing. $\qquad\square$

As the proof of Theorem 8.2 shows, the dependence of the isomorphism on the order $\prec$ can be made explicit. A canonical isomorphism between $M_{\mathrm{gm}}((\partial\overline{X})^m, i_{\mathrm{m}}^! j_! \mathbb{Z})$ and a twisted form of $M_{\mathrm{gm}}(N_{\mathrm{m}}^*)[m]$ can be constructed by proceeding as in [Del71, (3.1.4)]. We leave the details to the reader.

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