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FREE l-GROUPS AND VECTOR LATTICES

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The purpose of this paper is to present three somewhat disparate results on free objects in three different classes of ℓ-groups. The first is that no proper ideal of a finitely generated free vector lattice can itself be a free vector lattice. Second, each free abelian ℓ-group is characteristically simple. The third result is that each disjoint subset of a free (non-abelian) ℓ-group is countable.

The reader is referred to Conrad (1970) for the general algebraic theory of ℓ-groups and (real) vector lattices. We review here only the standard definitions of freedom. A vector lattice V is *free* if $V \neq 0$ and V possesses a generating subset S such that each function $\bar{\alpha}: S \to W$, where W is a vector lattice, extends to a vector lattice homomorphism $\alpha: V \to W$. We say V is *free on* S. Free l-groups and free abelian ℓ-groups are similarly defined.

Given any non-empty set S there exists a (unique up to isomorphism) free vector lattice on S. A very useful model for this free vector lattice was given in Baker (1968), and earlier in a generalized form in Henriksen and Isbell (1962). We confine our attention here to the case that S is finite of cardinality n. The set W of all functions $f: \mathbb{R}^n \to \mathbb{R}$ with pointwise operations is a vector lattice. The sublattice of W generated by the linear functions is (isomorphically) the free vector lattice on S. It will be denoted by FVLn. S is to be identified with the set of coordinate projection maps $\pi_i: \mathbb{R}^n \to \mathbb{R}$ $(i = 1, \cdots, n)$.

We have $FVLn = \{f: \mathbb{R}^n \to \mathbb{R} \,|\, f = \vee_I \wedge_J f_{ij}$ where I and J are finite and the f_{ij} are linear$\}$. In particular, each $f \in FVLn$ is continuous and *positively homogeneous* ($f(rx) = rf(x)$ for all $x \in \mathbb{R}^n$ and $0 \leqq r \in \mathbb{R}$). The set $T(f) = \{x \in \mathbb{R}^n \,|\, f(x) \neq 0\}$ is open since f is continuous.

Let S^{n-1} denote the unit sphere in $\mathbb{R}^n$. The map restricting $f \in FVLn$ to S^{n-1} is an isomorphism (by positive homogeneity). It is often convenient to identify $FVLn$ with this isomorphic copy. In this case $T(f) = \{x \in S^{n-1} \,|\, f(x) \neq 0\}$ is an open subset of the sphere. K. Baker (1968; Lemma 3.2) showed that the sets $T(f)$, $f \in FVLn$, form a base for the topology of S^{n-1}.

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1. Finitely generated free vector lattices

We intend to prove that no proper ideal of $FVLn$ can be a free vector lattice. The free vector lattice on one element is isomorphic to the cardinal sum of two copies of $\mathbb{R}$. Thus the only proper ideals of $FVL1$ are $\mathbb{R}$ and 0, neither of which is free. In the remainder of this section m and n will denote positive integers with $n \geqq 2$.

Let A be a proper non-zero ideal of $FVLn$. If, on the assumption that A is free, one could produce an idempotent endomorphism of $FVLn$ with image A, then a contradiction would result since it is known (Bleier (1973), corollary to Theorem 2.3) that $F\,VLn$ has no proper cardinal summands. Howeber, the author sees no direct method of producing such an endomorphism, and hence a different line of proof will be pursued.

LEMMA 1.1. (Baker (1968), *Lemma* 3.3). *If $f, g \in FVLn$ and $T(f) \subseteq T(g)$, then there exists an integer k such that $|f| \leqq k|g|$; in other words, $f \in \langle g \rangle$, where $\langle g \rangle$ is the principal ideal of $FVLn$ generated by g.*

LEMMA 1.2. (Bleier(1973), Lemma 3.6). *If $x \in S^{n-1}$ then there exists $f \in FVLn$ such that $T(f) = S^{n-1} \backslash \{x\}$.*

LEMMA 1.3. *Let V be a vector sublattice of $FVLn$, and suppose $\alpha: V \to FVLm$ is a homomorphism. Let $0 \leqq f, g \in V$. If $T(f) \subseteq T(g)$, then $T(\alpha f) \subseteq T(\alpha g)$. In particular, if $T(f) = T(g)$ then $T(\alpha f) = T(\alpha g)$.*

PROOF. $T(f) \subseteq T(g)$ implies by Lemma 1.1 that $f \leqq kg$ for some positive integer k. Hence $0 \leqq \alpha f \leqq \alpha k g$, and thus $T(\alpha f) \subseteq T(\alpha k g) = T(k \alpha g) = T(\alpha g)$.

LEMMA 1.4. *Let $h \in FVLn$. Let $\mathscr{S}_1 = \{T(f) \,|\, 0 \leqq f \in FVLn$ and $T(f) \subseteq T(h)\}$ and let $\mathscr{S}_2 = \{T(g) \,|\, 0 \leqq g \in FVLm)$. If $\langle h \rangle$ is isomorphic to $FVLm$, then there is a one-to-one inclusion-preserving correspondence between $\mathscr{S}_1$ and $\mathscr{S}_2$ with inclusion-preserving inverse.*

PROOF. Suppose α is an isomorphism of $\langle h \rangle$ onto $FVLm$ with inverse β. We define $\bar{\alpha}: \mathscr{S}_1 \to \mathscr{S}_2$ by $\bar{\alpha}(T(f)) = T(\alpha f)$. By Lemma 1.1 αf is defined for all f for which $T(f) \in \mathscr{S}_1$, and by Lemma 1.3 $\bar{\alpha}$ is well-defined and inclusion-preserving.

Similarly, define $\bar{\beta}: \mathscr{S}_2 \to \mathscr{S}_1$ by $\bar{\beta}(T(g)) = T(\beta g)$. Again by Lemma 1.3, this time applied to β, we have that $\bar{\beta}$ is well-defined and inclusion-preserving.

$\bar{\alpha}$ and $\bar{\beta}$ are inverses since α and β are.

LEMMA 1.5. Let the notation be as in the statement of the preceding lemma. If $\langle h \rangle$ is a proper ideal of $FVLn$, then there is no one-to-one inclusion-preserving correspondence between $\mathscr{S}_1$ and $\mathscr{S}_2$ with inclusion-preserving inverse.

PROOF. Suppose λ is a one-to-one inclusion preserving correspondence between $\mathscr{S}_1$ and $\mathscr{S}_2$ with (inclusion preserving) inverse μ. If $T(h) = S^{n-1}$, then by Lemma 1.1 we have $\langle h \rangle = FVLn$, a contradiction. $T(h) = \varnothing$ is clearly impossible. Thus $T(h)$ is a non-empty proper open subset of S^{n-1}, and hence is not compact.

Thus there is a covering of $T(h)$ by a countable number of open sets $T(f_1), T(f_2), \cdots$, where $0 \leqq f_i \in FVLn$ and $T(f_i) \subseteq T(h)$, such that no finite number of the $T(f_i)$ cover $T(h)$. Let $D_i = T(f_1 \vee \cdots \vee f_i) = T(f_1) \cup \cdots \cup T(f_i)$. Then $D_i \in \mathscr{S}_1$, and $D_1 \subseteq D_2 \subseteq \cdots$ is an ascending sequence of open sets each properly contained in $T(h)$. Moreover, $T(h) = \cup D_i$. Note that S^{m-1} corresponds to $T(h)$ under λ, and thus each λD_i is properly contained in S^{m-1}.

Suppose (by way of contradiction) there exists $x \in S^{m-1} \setminus \cup \lambda D_i$. By Lemma 1.2 there exists $g \in FVLm$ such that $T(g) = S^{m-1} \setminus \{x\}$. $T(g)$ contains each λD_i, and hence $\mu T(g)$ contains each D_i. Thus $\mu T(g) = T(h)$, contradicting the fact that S^{m-1} corresponds to $T(h)$.

Hence $\lambda D_1 \subseteq \lambda D_2 \subseteq \cdots$ is an ascending sequence of proper open subsets of S^{m-1} whose union is S^{m-1}. This contradicts the fact that S^{m-1} is compact, and completes the proof.

THEOREM 1.6. *If A is a proper ideal of $FVLn$, then A is not a free vector lattice.*

PROOF. Suppose (by way of contradiction) that A is free with I as a free set of generators. Then (Weinberg (1963), Thm. 2.13) I is finite and so A has a strong order unit h. Thus $A = \rho h \sigma$ is a principal ideal of $FVLn$. Let $|I| = m$. Then there exists an isomorphism α of A onto $FVLm$. By Lemma 1.4 α induces a one-to-one correspondence of the type prohibited by Lemma 1.5. This contradiction completes the proof.

2. Free abelian l-groups

It is known that the free abelian ℓ-group $FALn$ on n elements is the l-sub group of $FVLn$ generated by the coordinate projection maps $\pi_i \colon \mathbb{R}^n \to \mathbb{R}$. (See Baker (1968) or Conrad (1971).) Moreover, $\{\pi_i,\ i = 1, \cdots, n\}$ is a set of free generators for $FALGn$.

Our goal is to prove that $FALGn$ is characteristically simple; that is, the only ideals of $FALGn$ which are invariant under all ℓ-automorphisms of $FALGn$ are 0 and $FALGn$ itself. We imitate the method used in Bleier (1973) for the free vector lattices. However, there are several technical difficulties to be overcome. (We remark in passing that these difficulties can be stubborn an$Y$ sometimes critical, as noted in the appendix to Bleier (1973). The proof in Section 1 of this paper does not go through for free abelian ℓ-groups because Lemma 1.2 fails. Whether or not the abelian ℓ-group analogue to Theorem 1.6 holds is an open question.)

We denote the integers by Z, and view Z^n as a subset of $\mathbb{R}^n$ in the natural way. Z^n consists of those points in $\mathbb{R}^n$ having only integer coordinates.

LEMMA 2.1. Suppose $\alpha: R^n \to R^n$ is a vector space isomorphism such that $\alpha(Z^n) = Z^n$. If $f \in FALGn$, then $f \circ \alpha^{-1} \in FALGn$ and $T(f \circ \alpha^{-1}) = \alpha(T(f))$.

PROOF. Assume f is in the subgroup $G(n)$ of $FALGn$ generated by the $\pi_i: \mathbb{R}^n \to R$. Then $f = m_1\pi_1 + \cdots + m_n\pi_n$, for some $m_1, \cdots, m_n \in Z$. Let e_i be the unit vector in R^n with 1 in the i-th place and 0 elsewhere. Write $\alpha^{-1}(e_i) = (a_{1i}, \cdots, a_{ni}) \in Z^n$. Then $f \circ \alpha^{-1}(e_i) = m_1 a_{1i} + \cdots + m_n a_{ni}$. Thus $f \circ \alpha^{-1} = (m_1 a_{11} + \cdots + m_n a_{n1})\pi_1 + \cdots + (m_1 a_{1n} + \cdots + m_n a_{nn})\pi_n$. Hence $f \circ \alpha^{-1} \in G(n)$.

Now suppose $f \in FALGn$. Then $f = \bigvee_I \bigwedge_J f_{ij}$ where I and J are finite and $f_{ij} \in G(n)$. Thus $f \circ \alpha^{-1} = (\bigvee_I \bigwedge_J f_{ij}) \circ \alpha^{-1} = \bigvee_I \bigwedge_J (f_{ij} \circ \alpha^{-1})$. By the preceding paragraph $f_{ij} \circ \alpha^{-1} \in G(n)$. Thus $f \circ \alpha^{-1} \in FALGn$.

LEMMA 2.2. *Suppose* $\alpha: R^n \to R^n$ *is a vector space isomorphism such that* $\alpha(Z^n) = Z^n$. *Then* $\alpha^*: FALGn \to FALGn$ by $\alpha^*(f) = f \circ \alpha^{-1}$ *is an* ℓ*-group automorphism.*

PROOF. By Lemma 2.1 $f \circ \alpha^{-1} \in FALGn$ whenever $f \in FALGn$. The rest of the proof is completely routine.

LEMMA 2.3. *Let* $0 \neq f \in FALGn$. *There exists a finite number of* ℓ*-group automorphisms* $\sigma_1, \cdots, \sigma_k$ *of* $FALGn$ *and* $m \in Z$ *such that* $|\pi_1| \leqq m(|\sigma_1 f| + \cdots + |\sigma_k f|)$.

PROOF. There exists some point $z = (z_1, \cdots, z_n)$ such that $f(z) \neq 0$ and $z_1, \cdots, z_n$ are relatively prime integers. Indeed, the rays from the origin through such points form a dense subset or R^n, and $T(f) = \{x \in R^n \mid f(x) \neq 0\}$ is a nonempty open cone in R^n.

By Fuchs (1970; Lemma 15.3, page 78) there exist $w_2, \cdots, w_n \in Z^n$ such that the n elements $z, w_2, \cdots, w_n$ generate Z^n as an abelian group, It is known then that $z, w_2, \cdots, w_n$ form a basis for R^n as a vector space. Hence there is a vector space isomorphism $\beta: R^n \to R^n$ such that $\beta(Z^n) = Z^n$ and $\beta(z) = (1, 0, \cdots, 0)$; simply let $\beta(w_i) = e_i$, for $i = 2, \cdots, n$, where e_i is the unit vector in R^n with 1 as the ith coordinate and 0 elsewhere.

Let $U = \beta(T(f))$. Then U is an open cone in R^n and $e_1 \in U$. Let $\bar{k} = (k, 0, \cdots, 0)$ where $k \in Z$, and let $a_1 = e_1$ and $a_i = e_i + \bar{k}$ for $i = 2, \cdots, n$. Let k be sufficiently large that $a_1, a_2, \cdots, a_n \in U$. Note that $a_1, \cdots, a_n$ generate Z^n as a group. For each fixed choice C of signs $\pm e_1, \cdots, \pm e_n$, there is a vector space isomorphism $\gamma_C: R^n \to R^n$ such that $\gamma_C(a_i) = \pm e_i$. Let $\alpha_C = \gamma_C \circ \beta$, and let $\alpha_C^*: FALGn \to FALGn$ be given by $\alpha_C^*(g) = g \circ \alpha_C^{-1}$. By Lemma 2.2 α_C^* is an ℓ-group automorphism of $FALGn$. By Lemma 2.1 $T(\alpha_C^* f) = T(f \circ \alpha_C^{-1}) = \alpha_C(T(f)) = \gamma_C(U)$. Each $\gamma_C(U)$ contains an orthant of $\mathbb{R}^n \backslash \{0\}$, and the union of all the $\gamma_C(U)$, over all choices of signs C, is $\mathbb{R}^n \backslash \{0\}$.

Let $\sigma_1, \cdots, \sigma_k$ be a listing of all the α_C^*. Then

$$T(|\sigma_1 f| + \cdots + |\sigma_k f|) = \bigcup_{i=1}^{k} T(\sigma_i f) = R^n \backslash \{0\}.$$

By Lemma 1.1 (or alternatively by the compactness of S^{n-1}) there is a positive integer m such that $|\pi_1| \leqq m(|\sigma_1 f| + \cdots + |\sigma_k f|)$.

THEOREM 2.4. *Each free abelian ℓ-group is characteristically simple.*

PROOF. The free abelian ℓ-group on S is that ℓ-subgroup of the free vector lattice on S generated by S (Conrad (1971)). We have developed the machinery in the lemmas so that the proof of Theorem 2.7 in Bleier (1973) can be imitated directly. We omit the details.

3. Free l-groups

Credit for the argument in this section belongs to I. Amemiya (1966). The present author's sole contribution lies in noting that Amemiya's argument applies to free (non-abelian) ℓ-groups and in writing out the details in the careful form given below.

Let $F(S)$ denote the free ℓ-group with S as a free generating subset. If $T \subseteq S$, then the ℓ-subgroup of $F(S)$ generated by T is $F(T)$. This is readily verified from the definition of freedom. Let $x \in F(S)$. Then $x \in F(T) \subseteq F(S)$ for some finite subset T of S. This is immediate since S generates $F(S)$ and the algebraic operations are finitary. Also, if S is finite, then $F(S)$ is countable.

THEOREM 3.1. *No free ℓ-group contains an uncountable disjoint subset.*

PROOF. Suppose (by way of contradiction) that $\{x_\alpha\}_{\alpha \in \mathscr{A}}$ is an uncountable subset of disjoint elements of the free ℓ-group $F(S)$. For each x_α choose a finite subset T_α of S such that $x_\alpha \in F(T_\alpha) \subseteq F(S)$. Then there exists some integer n such that $|T_\alpha| = n$ for uncountably many $\alpha \in \mathscr{A}$.

Let $A = \{T_\alpha \mid |T_\alpha| = n\}$. A is uncountable since otherwise some $F(T_\alpha)$ would contain uncountably many of the x_α, which is impossible since each $F(T_\alpha)$ is countable. Let k be the largest integer such that there exists some subset X of S with $|X| = k$ and $X \subseteq T_\alpha$ for uncountably many $T_\alpha \in A$. Note that $0 \leqq k < n$. Fix X with $|X| = k$ and $X \subseteq T_\alpha$ for uncountably many $T_\alpha \in A$. (If $k = 0$, take $X = \varnothing$.)

Let $B = \{T_\alpha \in A \mid X \subseteq T_\alpha\}$, and let $Y_\alpha = T_\alpha \backslash X$. Then $C = \{Y_\alpha \mid T_\alpha \in B\}$ is uncountable. Let D be a subset of C which is maximal with respect to $Y_\alpha \cap Y_\beta = \varnothing$. Suppose D is countable. Then $W = \{y \in Y_\alpha \mid Y_\alpha \in D\}$ is countable since each Y_α is finite. Because of the maximality of D, each member of C has non-empty intersection with W. Hence some $w \in W$ belongs to uncountably many members of C. But now $X \cup \{w\}$ is contained in uncountably many members of A, contradicting the maximality of k.

Thus D is uncountable. Let $\mathscr{A}^* = \{\alpha \in \mathscr{A} \mid Y_\alpha \in D\}$, and let Y be a set of cardinality $n-k$. For $\alpha \in \mathscr{A}^*$ let $h_\alpha: X \cup Y_\alpha \to X \cup Y$ be a bijection such that h_α restricted to X is the identity on X. h_α extends uniquely to an isomorphism $\bar{h}_\alpha: F(T_\alpha) \to F(X \cup Y)$.

We prove $\{\bar{h}_\alpha(x_\alpha) \mid \alpha \in \mathscr{A}^*\}$ is a pairwise disjoint subset of $F(X \cup Y)$. Let $\alpha, \beta \in \mathscr{A}^*$. Define $f: X \cup Y_\alpha \cup Y_\beta \to F(X \cup Y)$ by

$$f(z) = \begin{cases} z & \text{if } z \in X \\ h_\alpha(z) & \text{if } z \in Y_\alpha \\ h_\beta(z) & \text{if } z \in Y_\beta \end{cases}$$

X, Y_α, and Y_β are pairwise disjoint subsets of A, and hence f is indeed single-valued. f extends uniquely to a homomorphism $\bar{f}: F(T_\alpha \cup T_\beta) \to F(X \cup Y)$. Note that $\bar{f}$ extends both $\bar{h}_\alpha$ and $\bar{h}_\beta$. Since $x_\alpha \wedge x_\beta = 0$, we have $\bar{f}(x_\alpha) \wedge \bar{f}(x_\beta) = 0$, and hence $\bar{h}_\alpha(x_\alpha) \wedge \bar{h}_\beta(x_\beta) = 0$.

Thus we have an uncountable disjoint subset of $F(X \cup Y)$. This contradicts the fact that $F(X \cup Y)$ is countable.

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