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# FRAMES IN HILBERT C*-MODULES AND MORITA EQUIVALENT C*-ALGEBRAS

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**Abstract.** We show that the property of a C*-algebra that all its Hilbert modules have a frame, in the case of $\sigma$-unital C*-algebras, is preserved under Rieffel–Morita equivalence. In particular, we show that a $\sigma$-unital continuous-trace C*-algebra with trivial Dixmier–Douady class, all of whose Hilbert modules admit a frame, has discrete spectrum. We also show this for the tensor product of any commutative C*-algebra with the C*-algebra of compact operators on any Hilbert space.

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**1. Introduction.** In 1952, Duffin and Schaeffer [**3**] introduced the concept of a Hilbert space frame to deal with certain problems in non-harmonic Fourier analysis. Frank and Larsen generalized the notion of frame in a Hilbert space to the setting of a Hilbert C*-module [**4**]. Unlike the case of Hilbert spaces it is not known exactly for which C*-algebras $A$, every Hilbert $A$-module has a frame. Using the Kasparov stabilization theorem [**5**], Frank and Larson [**4**] showed that for any C*-algebra $A$, every countably generated Hilbert $A$-module has a frame. The problem of finding those C*-algebras $A$ for which all Hilbert $A$-modules have a frame is open [**4**]. In

2010, Li solved the problem in the case of a commutative unital C*-algebra as follows [**7**, Theorem 1.1].

THEOREM 1.1. *A commutative unital C*-algebra A is finite-dimensional if, and only if, every Hilbert A-module has a frame.*

QUESTION 1.2. Let $A$ be a unital C*-algebra such that every Hilbert $A$-module has a frame. Must $A$ be finite-dimensional?

Close inspection of the proof of the main result of Li [**7**] shows that this holds also for a locally compact Hausdorff space. In particular, just as in Proposition 2.4 and Lemma 3.2 of Li [**7**], we have the following construction.

PROPOSITION 1.3. *Let $Z$ be an infinite locally compact Hausdorff space having a countable subset $W \subseteq Z$ with a point $z_\infty \in \overline{W} \setminus W$. Then, there exists a continuous field of Hilbert spaces $((H_z)_{z\in Z}, \Gamma)$ over $Z$, such that $H_z$ is separable for every $z \in W$ while $H_{z_\infty}$ is non-separable. Any such $\Gamma$, as a Hilbert $C_0(Z)$-module, has no frame.*

It is clear (as a consequence of the case of a compact space, dealt with in Proposition 2.4 of Li [**7**]), that every infinite locally compact Hausdorff space $Z$ satisfies the condition stated in the above proposition, except when $Z$ is discrete. Also, it is well known that a commutative C*-algebra $A = C_0(Z)$ is a C*-algebra of compact operators exactly when $Z$ is discrete [**2**, 4.7.20].

On the other hand, Bakic and Guljas showed [**1**] that if $A$ is a C*-algebra of compact operators, then every Hilbert $A$-module has a basis (in the sense of Bakic and Guljas [**1**]). (The converse is also true [**9**].)

It follows that a non-unital version of Li's theorem [**7**] can be stated as follows.

THEOREM 1.4. *Let $A$ be a commutative C*-algebra. Then, $A$ is a C*-algebra of compact operators (equivalently, has discrete spectrum) if, and only if, every Hilbert $A$-module has a frame.*

The following question naturally arises.

QUESTION 1.5. Let $A$ be a C*-algebra such that every Hilbert $A$-module has a frame. Is it true that $A$ must be a C*-algebra of compact operators?

It is clear that an affirmative answer to Question 1.5 also gives an affirmative answer to Question 1.2, since unital C*-algebras of compact operators are finite-dimensional.

Recall that a C*-algebra $A$ is a C*-algebra of compact operators if and only if the spectrum of every maximal commutative C*-subalgebra is discrete [**2**, 4.7.20]. Therefore, Theorem 1.4 implies the following result.

THEOREM 1.6. *For every C*-algebra $A$, the following conditions are equivalent:*

(*i*) *$A$ is a C*-algebra of compact operators;*

(*ii*) *every Hilbert C*-module over every maximal commutative C*-subalgebra $B$ of $A$ has a frame.*

Consequently, an affirmative answer to the following question would give an affirmative answer to Question 1.5 (and so to Question 1.2).

QUESTION 1.7. Let $A$ be a C*-algebra. Are the following conditions equivalent:

(i) every Hilbert C*-module over $A$ has a frame;

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(ii) every Hilbert C*-module over every (maximal commutative) C*-subalgebra $B$ of $A$ has a frame?

In the next section, we shall show that if two C*-algebras $A$ and $B$ are Rieffel–Morita equivalent and $A$ is unital (or $\sigma$-unital), then if every Hilbert C*-module over $A$ has a frame, the same is true for every Hilbert C*-module over $B$. This provides an example of a non-commutative unital C*-algebra $A$ which has a Hilbert $A$-module with no frame.

**2. Frames in Hilbert C*-modules.** Let $A$ be a C*-algebra and $E$ be a Hilbert $A$-module. A family $\{x_i\}_{i\in I}$ of elements in $E$ is called a frame if there are real constants $C, D > 0$ such that $\sum_{i\in I}\langle x, x_i\rangle_A\langle x_i, x\rangle_A$ converges in the ultraweak topology to some element in the universal enveloping von Neumann algebra $A^{**}$ of $A$ and

$$C\langle x, x\rangle_A \leq \sum_{i\in I}\langle x, x_i\rangle_A\langle x_i, x\rangle_A \leq D\langle x, x\rangle_A,$$

for every $x \in E$. (The ultraweak convergence in fact follows from the second inequality.) A frame is said to be standard if the sum in the middle of the above inequality converges in norm for every $x \in E$, and to be normalized if $C = D = 1$. (Normalized implies standard, by Dini's theorem.)

Definition 2.1. Two C*-algebras $A$ and $B$ are Rieffel–Morita equivalent (called strongly Morita equivalent in Rieffel [**11**]) if there is an $A$-$B$-imprimitivity bimodule, i.e., an $A$-$B$-bimodule, $X$, such that

(i) $X$ is a full left Hilbert $A$-module and a full right Hilbert $B$-module, and
(ii) for all $x, y, z \in X$, $a \in A$, and $b \in B$,

$$\langle ax, y\rangle_B = \langle x, a^*y\rangle_B \ , \ \ \langle xb, y\rangle_A = \langle x, yb^*\rangle_A \ \text{ and } \ \langle x, y\rangle_A z = x\langle y, z\rangle_B.$$

For every $x, y$ in a right Hilbert $A$-module $E$, we define an operator $\theta_{x,y} : E \to E$ by $\theta_{x,y}(z) = x\langle y, z\rangle_A$. The closed linear span of $\{\theta_{x,y} | x, y \in E\}$ is a C*-algebra, $K(E)$, and is referred to as (informally) as the algebra of compact operators on $E$ (these are not necessarily compact on $E$ as a Banach space). If $E$ is a full right Hilbert $A$-module, then $E$ is a $K(E)$-$A$-imprimitivity bimodule, and $K(E)$ and $A$ are Rieffel–Morita equivalent.

The main result of this paper, Theorem 2.4, below, asserts that the existence of frames in all Hilbert $A$-modules is preserved under Rieffel–Morita equivalence, when $A$ is unital or $\sigma$-unital.

The $\sigma$-unital case is more subtle than the unital case, and a discussion of this case, and an even somewhat more general case, is given in Lemma 2.3, which is based on the assumption of a certain kind of approximate unit, including the countable case. To motivate the discussion, let $A$ be a C*-algebra and let $X$ be a full left Hilbert $A$-module. Let $X^+$ denote the set of all finite sums $\sum_{i=1}^{n}\langle x_i, x_i\rangle$ with $n \in \mathbb{N}$ and $x_i \in X$, for $1 \leq i \leq n$. Then, it is easy to see that $A$ has an increasing approximate identity $\{u_\lambda\}_{\lambda\in\Lambda}$ in $X^+$. Indeed, denote by $\Lambda$ the set of all finite subsets of $X$, ordered by inclusion. For $\lambda = \{x_1, \ldots, x_n\} \in \Lambda$, set $v_\lambda = \sum_{i=1}^{n}\langle x_i, x_i\rangle$ and $u_\lambda = v_\lambda(\frac{1}{n} + v_\lambda)^{-1} = (\frac{1}{n} + v_\lambda)^{-\frac{1}{2}}v_\lambda(\frac{1}{n} + v_\lambda)^{-\frac{1}{2}}$. Then, $u_\lambda = \sum_{i=1}^{n}\langle y_i^\lambda, y_i^\lambda\rangle$, where $y_i^\lambda = (\frac{1}{n} + v_\lambda)^{-\frac{1}{2}}x_i$. As shown in Dixmier [**2**, 1.7.2 ], $\{u_\lambda\}_{\lambda\in\Lambda}$ is an increasing approximate identity for $A$.

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Next, observe that this construction does not guarantee the existence of an approximate identity for $A$ of the form $\{\sum_{i\in F}\langle x_i, x_i\rangle\}_{F\in\mathcal{F}}$, where $\{x_i\}_{i\in I}$ is in $X$ and $\mathcal{F}$ is the set of all finite subsets of $I$, i.e., with $\sum_{i\in I}\langle x_i, x_i\rangle = 1_{M(A)}$, in the strict topology in the multiplier algebra $M(A)$ of $A$ (equivalently, by Dini's theorem, in the ultraweak topology in $A^{**}$).

In fact, if there is such a family, then $\{x_i\}_{i\in I}$ is a normalized frame for $X$ as a Hilbert $K(X)$-module. Conversely, if $\{x_i\}_{i\in I}$ is a normalized frame for $X$ as a Hilbert $A$-module then $\{\sum_{i\in F}\theta_{x_i,x_i}\}_{F\in\mathcal{F}}$ is an increasing approximate identity for $K(X)$.

For example, for $Z = [0, 1]$, there exists a right Hilbert $C(Z)$-module $E$ with no frame [**7**]. Hence, for $A = K(E)$, there is no approximate identity for $A$ of the above form (not even one consisting of the finite sums of just some family of positive elements, as each of these would be a sum of (possibly infinitely many) rank one positive elements of the form $\theta_{x,x}$). (Using the increasing approximate unit consisting of finite sums of rank one operators $\theta_{x,x}$, one obtains (multiplying on both sides by the square root) that an arbitrary positive element of $K(E)$ can be approximated from below by such an element, and subtracting and repeating one obtains an expression of the element as an infinite sum of (finite sums of) elements $\theta_{x,x}$.)

Consider a full left Hilbert $A$-module $X$. When $A$ is unital, there is a finite family $\{x_i\}_{i=1}^n$ in $X$ such that $\sum_{i=1}^n {}_A\langle x_i, x_i\rangle = 1_A$ [**8**, Lemma 2.4.3]. Hence, the family $\{x_i\}_{i=1}^n$ is a finite frame for the Hilbert $K(X)$-module $X$. When $A$ is $\sigma$-unital, there is a sequence $\{x_i\}_{i=1}^\infty$ in $X$ such that $\sum_{i=1}^\infty {}_A\langle x_i, x_i\rangle = 1_{M(A)}$ in the strict topology [**6**, Lemma 7.3].

As a non-unital example, suppose that $H$ is an infinite dimensional Hilbert space with orthonormal basis $\{e_i\}_{i\in I}$ and consider the corresponding rank-one projections $p_i = e_i \otimes e_i$. If $E$ is a full Hilbert $K(H)$-module, then for each $i \in I$, there is $x_i \in E$ such that $\langle x_i, x_i\rangle = p_i$ [**1**], and $\{\sum_{i\in F}\langle x_i, x_i\rangle\}_{F\in\mathcal{F}}$ is an increasing approximate identity for $K(H)$.

To prove the main result of this section, we need some technical lemmas. Note that $A^k = A \oplus \cdots \oplus A$ ($k$ times) is a Hilbert $A$-module with the inner product

$$\langle a, b\rangle = \sum_{r=1}^{k} a_r^* b_r, \quad a = [a_1, \ldots, a_k]^{tr}, b = [b_1, \ldots, b_k]^{tr} \in A^k.$$

In particular, $\mathbb{M}_k(A) \cong K(A^k)$.

Also, observe that if $c$ is a self-adjoint element of $A^{**}$ such that $aca^* \geq 0$, for all $a \in A$, then $c \geq 0$, since $e_\alpha c e_\alpha \to c$, in the strict operator topology, for any bounded approximate identity $\{e_\alpha\}$ of $A$.

LEMMA 2.2. *Let $M$ be a right Hilbert $A$-module and $\{m_j\}_{j\in J}$ be a frame for $M$ with frame bounds $C$ and $D$. Then, for each $k \in \mathbb{N}$ and $n_1, \ldots, n_k$ in $M$*

$$C[\langle n_r, n_s\rangle_A]_{r,s} \leq \left[\sum_{j\in J}\langle n_r, m_j\rangle_A\langle m_j, n_s\rangle_A\right]_{r,s} \leq D[\langle n_r, n_s\rangle_A]_{r,s},$$

*in $\mathbb{M}_k(A^{**})$.*

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*Proof.* We identify $\mathbb{M}_k(A^{**})$ with $K((A^{**})^k)$. For each $a = [a_1, \ldots, a_k]^{tr}$ in $A^k$,

$$\langle a, [\langle n_r, n_s\rangle_A]_{r,s} a\rangle = \sum_{r,s=1}^{k} a_r^* \langle n_r, n_s\rangle_A a_s = \left\langle \sum_{r=1}^{k} n_r a_r, \sum_{s=1}^{k} n_s a_s \right\rangle_A ,$$

and

$$\left\langle a, \left[\sum_{j\in J} \langle n_r, m_j\rangle_A \langle m_j, n_s\rangle_A\right]_{r,s} a\right\rangle = \sum_{j\in J}\sum_{r,s=1}^{k} a_r^* \langle n_r, m_j\rangle_A \langle m_j, n_s\rangle_A a_s$$
$$= \sum_{j\in J} \left\langle \sum_{r=1}^{k} n_r a_r, m_j\right\rangle_A \left\langle m_j, \sum_{s=1}^{k} n_s a_s\right\rangle_A ,$$

and we get the desired frame inequalities, namely,

$$C\langle a, [\langle n_r, n_s\rangle_A]_{r,s} a\rangle \leq \left\langle a, \left[\sum_{j\in J} \langle n_r, m_j\rangle_A \langle m_j, n_s\rangle_A\right]_{r,s} a\right\rangle$$
$$\leq D\langle a, [\langle n_r, n_s\rangle_A]_{r,s} a\rangle,$$

for every $a = [a_1, \ldots, a_k]^{tr} \in A^k$, and hence by continuity for every $a \in (A^{**})^k$. It follows that

$$C[\langle n_r, n_s\rangle_A]_{r,s} \leq \left[\sum_{j\in J} \langle n_r, m_j\rangle_A \langle m_j, n_s\rangle_A\right]_{r,s} \leq D[\langle n_r, n_s\rangle_A]_{r,s}.$$

□

LEMMA 2.3. *Suppose that $X$ is a full left Hilbert $A$-module and there is a family $\{x_i\}_{i\in I}$ in $X$ such that $\{\sum_{i\in F} {}_A\langle x_i, x_i\rangle\}_{F\in\mathcal{F}}$ is an approximate identity for $A$, where $\mathcal{F}$ is the set of all finite subsets of $I$.*

*If $M$ is a right Hilbert $A$-module with a frame, then the right Hilbert $K(X)$-module $M \otimes_A X$ has a frame.*

*Proof.* Let $\{m_j\}_{j\in J}$ be a frame for $M$ with frame bounds $C$ and $D$. For every $n \in M$, we have

$$C\langle n, n\rangle_A \leq \sum_{j\in J} \langle n, m_j\rangle_A \langle m_j, n\rangle_A \leq D\langle n, n\rangle_A .$$

We know that $M \otimes_A X$ is a right Hilbert $K(X)$-module with $K(X)$-valued inner product $\langle m_1 \otimes x_1, m_2 \otimes x_2\rangle := \langle x_1, \langle m_1, m_2\rangle_A x_2\rangle_{K(X)}$. Also, by assumption, $\sum_{i\in I} {}_A\langle x_i, x_i\rangle a = a$, for all $a \in A$. We assert that $\{m_j \otimes x_i : j \in J, i \in I\}$ is a frame for $M \otimes_A X$.

If $f$ is a positive functional on $K(X)$, then for every $y \in X$, we define a positive functional $\varphi$ on $A$ by $\varphi(a) = f(\langle y, ay\rangle_{K(X)})$, for $a \in A$. For every $n \in M$, we have

$$C\varphi(\langle n, n\rangle_A) \leq \sum_{j\in J} \varphi(\langle n, m_j\rangle_A \langle m_j, n\rangle_A) \leq D\varphi(\langle n, n\rangle_A). \tag{1}$$

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On the other hand,

$$\varphi(\langle n,n\rangle_A)=f(\langle y,\langle n,n\rangle_A y\rangle_{K(X)})=f(\langle n\otimes y,n\otimes y\rangle_{K(X)}),$$

and also

$$\begin{aligned}\sum_{j\in J}\varphi(\langle n,m_j\rangle_A\langle m_j,n\rangle_A)&=\sum_{j\in J}f(\langle y,\langle n,m_j\rangle_A\langle m_j,n\rangle_A y\rangle_{K(X)})\\&=\sum_{j\in J}f(\langle y,\langle n,m_j\rangle_A\left(\sum_{i\in I}{}_A\langle x_i,x_i\rangle\langle m_j,n\rangle_A\right)y\rangle_{K(X)})\\&=\sum_{j\in J}\sum_{i\in I}f(\langle y,\langle n,m_j\rangle_{A\,A}\langle x_i,x_i\rangle\langle m_j,n\rangle_A y\rangle_{K(X)})\\&=\sum_{j\in J}\sum_{i\in I}f(\langle y,\langle n,m_j\rangle_A x_i\langle x_i,\langle m_j,n\rangle_A y\rangle_{K(X)}\rangle_{K(X)})\\&=\sum_{j\in J}\sum_{i\in I}f(\langle y,\langle n,m_j\rangle_A x_i\rangle_{K(X)}\langle x_i,\langle m_j,n\rangle_A y\rangle_{K(X)})\\&=\sum_{j\in J}\sum_{i\in I}f(\langle n\otimes y,m_j\otimes x_i\rangle\langle m_j\otimes x_i,n\otimes y\rangle)\\&=\sum_{(i,j)\in I\times J}f(\langle n\otimes y,m_j\otimes x_i\rangle\langle m_j\otimes x_i,n\otimes y\rangle).\end{aligned}$$

(To see the third equality, note that $\sum_{i\in I}{}_A\langle x_i,x_i\rangle\langle m_j,n\rangle_A$ is convergent in norm.) Hence, by (2.1), for every state $f$ of $K(X)$ and every $n\in M$ and $y\in X$, we have

$$\begin{aligned}Cf(\langle n\otimes y,n\otimes y\rangle)&\le\sum_{(i,j)\in I\times J}f(\langle n\otimes y,m_j\otimes x_i\rangle\langle m_j\otimes x_i,n\otimes y\rangle)\\&\le Df(\langle n\otimes y,n\otimes y\rangle).\end{aligned}$$

Now, as in the proof of Proposition 3.1 in Li [7], we conclude that

$$\sum_{(i,j)\in I\times J}\langle n\otimes y,m_j\otimes x_i\rangle\langle m_j\otimes x_i,n\otimes y\rangle,$$

is convergent in $K(X)^{**}$ in the ultra-weak topology, and

$$\begin{aligned}C\langle n\otimes y,n\otimes y\rangle&\le\sum_{(i,j)\in I\times J}\langle n\otimes y,m_j\otimes x_i\rangle\langle m_j\otimes x_i,n\otimes y\rangle\\&\le D\langle n\otimes y,n\otimes y\rangle.\end{aligned}$$

These inequalities pass to the finite sums $\sum_{r=1}^k n_r\otimes y_r$ by the previous lemma and the following equalities

$$\begin{aligned}\left\langle\sum_{r=1}^k n_r\otimes y_r,\sum_{s=1}^k n_s\otimes y_s\right\rangle&=\sum_{r,s=1}^k\langle y_r,\langle n_r,n_s\rangle_A y_s\rangle\\&=\langle[y_1,\ldots,y_k]^{tr},[\langle n_r,n_s\rangle_A]_{r,s}[y_1,\ldots,y_k]^{tr}\rangle.\end{aligned}$$

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Finally, the frame inequalities (2.2) hold for $n \otimes y$ replaced by an arbitrary element in $M \otimes_A X$, by continuity of the inner product. □

If the C*-algebras $A$ and $B$ are Rieffel–Morita equivalent and $X$ is an $A$-$B$-imprimitivity bimodule (giving rise to the Morita equivalence of $A$ and $B$), then $B \cong K({}_A X)$. Now, if $A$ and $X$ satisfy the conditions of the above lemma (for instance, if $A$ is unital or $\sigma$-unital—see above), then the right Hilbert $B$-module $M \otimes_A X$ has a frame whenever the right Hilbert $A$-module $M$ has a frame. On the other hand, if $E$ is any right Hilbert $B$-module then $E \cong E \otimes_B \overline{X} \otimes_A X$ and so if the right Hilbert $A$-module $M = E \otimes_B \overline{X}$ has a frame then so does $E$. In particular, we obtain the following result.

THEOREM 2.4. *Suppose that the C*-algebras $A$ and $B$ are Rieffel–Morita equivalent and $A$ is $\sigma$-unital. If every Hilbert $A$-module has a frame, then every Hilbert $B$-module has a frame.*

Note that in Lemma 2.3 and Theorem 2.4, "frame" can be replaced by "standard frame". One just needs to observe that convergence in the ultraweak operator topology could be replaced by norm convergence, everywhere. Therefore, the existence of standard frames for Hilbert modules over unital (or $\sigma$-unital) C*-algebras is also preserved under Rieffel–Morita equivalence of the base C*-algebras.

The above theorem implies certain well known results. For instance, since every finite-dimensional C*-algebra $A$ is Rieffel–Morita equivalent to $\mathbb{C}^k$ for some $k$, by Theorems 1.1 and 2.4, every Hilbert $A$-module has a frame. Also, if $H$ is a Hilbert space, then every Hilbert $\mathcal{K}(H)$-module has a frame, since $\mathbb{C}$ and $\mathcal{K}(H)$ are Rieffel–Morita equivalent. For more general commutative algebras, Theorem 1.1 yields:

COROLLARY 2.5. *If $Z$ is a locally compact, $\sigma$-compact, Hausdorff space and $n \in \mathbb{N}$, then every Hilbert $C_0(Z) \otimes M_n(\mathbb{C})$-module has a frame if and only if $Z$ is discrete.*

The above corollary provides an example of a non-commutative C*-algebra $A$ which has a Hilbert $A$-module with no frame.

COROLLARY 2.6. *Let $A$ be a nuclear C*-algebra which is also a von Neumann algebra. If every Hilbert $A$-module has a frame, then $A$ is finite-dimensional.*

*Proof.* As a nuclear von Neumann algebra, $A$ must be of the form $\bigoplus_{i=1}^m A_i \otimes M_{n_i}(\mathbb{C})$, where $m \in \mathbb{N}$ and $A_i$ is an abelian von Neumann algebra, for $i \in \{1, \ldots, m\}$. If every Hilbert $A$-module has a frame, then so does every Hilbert $A_i \otimes M_{n_i}(\mathbb{C})$-module, for each $i \in \{1, \ldots, m\}$. Hence, by Corollary 2.5, each $A_i$ is finite-dimensional, and so is $A$. □

Recall that if $A$ is a unital continuous-trace C*-algebra with finite spectrum, then $A$ is finite-dimensional.

COROLLARY 2.7. *Let a unital C*-algebra $A$ be Rieffel–Morita equivalent to a unital commutative C*-algebra. Then, every Hilbert $A$-module has a frame if and only if $A$ is finite-dimensional.*

*Proof.* Since $A$ is Rieffel–Morita equivalent to a commutative C*-algebra, $A$ is a continuous-trace C*-algebra such that the Dixmier–Douady class $\delta(A)$ vanishes in $H^3(T; \mathbb{Z})$ (see [**10**]). Therefore, $A$ and $C(\hat{A})$ are Rieffel–Morita equivalent, where $\hat{A}$ is the spectrum of $A$. If every Hilbert $A$-module has a frame, then Theorem 2.4

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and Li's theorem (Theorem 1.1 above) imply that $\hat{A}$ is finite. Therefore, $A$ is finite-dimensional. $\square$

Finally, let us consider the question when every Hilbert module over a C*-algebra of the form $C_0(Z) \otimes A$ has a frame, where $Z$ is locally compact and Hausdorff and $A$ is an arbitrary C*-algebra. We have the following result (compare with Proposition 1.3).

THEOREM 2.8. *Let $A$ be a non-zero C*-algebra (in Proposition 1.3, $A$ is just $\mathbb{C}$). Let $((H_z)_{z\in Z}, \Gamma)$ be a continuous field of Hilbert spaces over a locally compact Hausdorff space $Z$. Suppose that there is a countable subset $W \subseteq Z$ with a point $z_\infty \in \overline{W} \setminus W$ such that $H_z$ is separable for every $z \in W$, while $H_{z_\infty}$ is non-separable. (As stated in Proposition 1.3, such a field exists whenever $Z$ is not discrete.) Then, $\Gamma \otimes A$ as a left Hilbert $C_0(Z) \otimes A$-module has no frame.*

*Proof.* First, we note that the $C_0(Z) \otimes A$-valued inner product on $\Gamma \otimes A$ is given by $\langle x \otimes a, y \otimes b\rangle = \langle x, y\rangle \otimes a^*b$, for every $x, y \in \Gamma$ and $a, b \in A$.

Let $\{u_j\}_{j\in J}$ be a frame for $\Gamma \otimes A$ with frame bounds $C$ and $D$. We may suppose that for every $j \in J$, $u_j = \lim_n u_{jn}$, where $u_{jn} = \sum_{i=1}^{\infty} x^i_{jn} \otimes a^i_{jn}$, $x^i_{jn} \in \Gamma$, $a^i_{jn} \in A$, and for each $(j, n) \in J \times \mathbb{N}$, $x^i_{jn} = 0$ and $a^i_{jn} = 0$, for all but finitely many $i$. Therefore,

$$C\varphi(\langle u, u\rangle) \leq \sum_{j\in J} \varphi(\langle u, u_j\rangle\langle u_j, u\rangle) \leq D\varphi(\langle u, u\rangle), \tag{2}$$

for every $u \in \Gamma \otimes A$ and every state $\varphi$ of $C_0(Z) \otimes A$.

Fix an element of norm one $b \in A$ and a state $\psi_b$ of $A$ such that $\psi_b(b^*b) = 1$. For each $z \in Z$ denote by $\varphi_z$ the state of $C_0(Z)$ given by evaluation at $z$. Then, $\varphi_z \otimes \psi_a$ is a state of $C_0(Z) \otimes A$. Taking $u = x \otimes b$ and $\varphi = \varphi_z \otimes \psi_b$ in inequality (2.3), we get

$$C\|x(z)\|^2 \leq \sum_{j\in J} \varphi_z \otimes \psi_b(\langle u, u_j\rangle\langle u_j, u\rangle) \leq D\|x(z)\|^2.$$

For any $z \in Z$ and any vector $w \in H_z$, by Dixmier [**2**, Proposition 10.1.10] we can find $x \in \Gamma$ with $x(z) = w$. Thus, we have

$$C\|w\|^2 \leq \sum_{j\in J} \Phi_j(z, w) \leq D\|w\|^2, \tag{3}$$

where

$$\begin{aligned}\Phi_j(z, x(z)) &:= \varphi_z \otimes \psi_b(\langle u, u_j\rangle\langle u_j, u\rangle)\\ &= \lim_n \sum_{i,k=1}^{\infty} \langle x^i_{jn}(z), x(z)\rangle\langle x(z), x^k_{jn}(z)\rangle\psi_b(b^* a^i_{jn}(a^k_{jn})^* b).\end{aligned}$$

Using the identification $C_0(Z) \otimes A = C_0(Z, A)$, we see that the map $(z, u) \mapsto \varphi_z \otimes \psi_b(\langle u, u_j\rangle\langle u_j, u\rangle)$ is continuous and hence $\Phi_j(z, x(z))$ is continuous with respect to $z$ and $x$, for every $j \in J$.

For each $z \in Z$, choose an orthonormal basis $S_z$ of $H_z$. The second inequality in (2.4) implies that for each $w \in S_z$ the set $F_w = \{j \in J : \Phi_j(z, w) \neq 0\}$ is countable. If $F_z = \{j \in J : \exists w \in S_z, \Phi_j(z, w) \neq 0\}$, then we have $F_z = \bigcup_{w\in S_z} F_w$ and so $F_z$ is countable, for every $z \in W$. Hence, the set $F := \bigcup_{z\in W} F_z$ is countable.

We note that the zero set of $\Phi_j(z,\cdot)$ is a linear subspace of $H_z$. In fact, if $\varphi(\langle u,u_j\rangle\langle u_j,u\rangle)=0$, then $\varphi(\langle u,u_j\rangle\langle u_j,u'\rangle)=0$ for all $u'\in C_0(Z)\otimes A$. Consequently, if also $\varphi(\langle u',u_j\rangle\langle u_j,u'\rangle)=0$, then $\varphi(\langle u+u',u_j\rangle\langle u_j,u+u'\rangle)=0$.

Therefore, for each $j\in J\setminus F$, $z\in W$ and $w_1,\ldots,w_n\in S_z$ and $\lambda_1,\ldots,\lambda_n\in\mathbb{C}$, we have $\Phi_j(z,\sum_{i=1}^n\lambda_i w_i)=0$.

Now, by continuity of $\Phi_j(z,x(z))$ with respect to $x$, we can conclude that if $j\in J\setminus F$, then $\Phi_j(z,x(z))=0$ for all $z\in W$ and $x\in\Gamma$.

Since $F\times\mathbb{N}\times\mathbb{N}$ is countable and $H_{z_\infty}$ is non-separable, there is a unit vector $w\in H_{z_\infty}$ orthogonal to $x^i_{jn}(z_\infty)$ for all $(j,n,i)\in F\times\mathbb{N}\times\mathbb{N}$, and so $\Phi_j(z_\infty,w)=0$ for all $j\in F$. On the other hand, if $j\in J\setminus F$, then $\Phi_j(z,x(z))=0$ for all $z\in W$ and $x\in\Gamma$. This implies that $\Phi_j(z_\infty,w)=0$, since the map $z\mapsto\Phi_j(z,x(z))$ is continuous. Therefore, $\Phi_j(z_\infty,w)=0$ for all $j\in J$, which contradicts (2.4). This shows that $\Gamma\otimes A$ does not have a frame. □

LEMMA 2.9. *Let $A$ be a $C^*$-algebra and $I$ an arbitrary index set, and set $A^\infty=\bigoplus_{i\in I}A_i$ ($C^*$-algebra direct sum), where $A_i=A$, for all $i\in I$. Then, every Hilbert $A^\infty$-module has a frame if and only if every Hilbert $A$-module has a frame.*

*Proof.* We only need to prove the sufficiency. Let $E$ be a Hilbert $A^\infty$-module. Then $E=\bigoplus_{i\in I}E_i$ (Hilbert $A^\infty$-module direct sum), where $E_i:=E\cdot\tilde{A}_i$ and

$$\tilde{A}_i:=\{(a_j)_{j\in I}:a_j\in A_j,a_j=0\ \ (j\neq i)\}.$$

Note that each $E_i$ could be considered as a Hilbert $A$-module and

$$\langle(x_i)_{i\in I},(y_i)_{i\in I}\rangle_{A^\infty}=(\langle x_i,y_i\rangle_A)_{i\in I},$$

for all $(x_i)_{i\in I},(y_i)_{i\in I}\in E$.

Consider the Hilbert $A$-module direct sum

$$H:=\{(x_i)_{i\in I}\in\bigoplus_{i\in I}E_i|\ \sum_{i\in I}\langle x_i,x_i\rangle_A \text{ is convergent in } A\},$$

which has a frame $F=\{(m^j_i)_{i\in I}\}_{j\in J}$, say with bounds $C$ and $D$. Clearly, for every $i\in I$, $F_i=\{m^j_i\}_{j\in J}$ is a frame for the Hilbert $A$-module $E_i$ with the same bounds $C$ and $D$.

We show that $F$ is a frame for $E$, too. Indeed, for every $(x_i)_{i\in I}\in E$ we have

$$\langle(x_i)_{i\in I},(x_i)_{i\in I}\rangle_{A^\infty}=(\langle x_i,x_i\rangle_A)_{i\in I},$$

and

$$\begin{aligned}\sum_{j\in J}\langle(x_i)_{i\in I},(m^j_i)_{i\in I}\rangle_{A^\infty}\langle(m^j_i)_{i\in I},(x_i)_{i\in I}\rangle_{A^\infty}&=\sum_{j\in J}(\langle x_i,m^j_i\rangle_A\langle m^j_i,x_i\rangle_A)_{i\in I}\\&=\left(\sum_{j\in J}\langle x_i,m^j_i\rangle_A\langle m^j_i,x_i\rangle_A\right)_{i\in I},\end{aligned}$$

therefore,

$$\begin{aligned}C\langle(x_i)_{i\in I},(x_i)_{i\in I}\rangle_{A^\infty}&\le\sum_{j\in J}\langle(x_i)_{i\in I},(m^j_i)_{i\in I}\rangle_{A^\infty}\langle(m^j_i)_{i\in I},(x_i)_{i\in I}\rangle_{A^\infty}\\&\le D\langle(x_i)_{i\in I},(x_i)_{i\in I}\rangle_{A^\infty}.\end{aligned}$$

□

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The following results follow from Lemma 2.9 and Theorem 2.8.

COROLLARY 2.10. *Let $A$ be a $C^*$-algebra and $Z$ be a locally compact Hausdorff space. Every Hilbert $C_0(Z) \otimes A$-module has a frame if, and only if, $Z$ is discrete and every Hilbert $A$-module has a frame.*

COROLLARY 2.11. *Let $Z$ be a locally compact Hausdorff space and $H$ be a Hilbert space. The following statements are equivalent:*
(*i*) *every Hilbert $C_0(Z) \otimes \mathcal{K}(H)$-module has a frame;*
(*ii*) *$Z$ is discrete;*
(*iii*) *$C_0(Z) \otimes \mathcal{K}(H)$ is a $C^*$-algebra of compact operators.*

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