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FACTORISABLE RIGHT ADEQUATE SEMIGROUPS

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On a semigroup S the relation $\mathscr{L}^*$ is defined by the rule that $(a, b) \in \mathscr{L}^*$ if and only if the elements a, b of S are related by Green's relation $\mathscr{L}$ in some oversemigroup of S. It is well known that for a monoid S, every principal right ideal is projective if and only if each $\mathscr{L}^*$-class of S contains an idempotent. Following (**6**) we say that a semigroup with or without an identity in which each $\mathscr{L}^*$-class contains an idempotent and the idempotents commute is *right adequate*. A right adequate semigroup S in which $eS \cap aS = eaS$ for any $e^2 = e$, $a \in S$ is called *right type A*. This class of semigroups is studied in (**5**).

Inverse semigroups are right adequate and in this case $\mathscr{L}^* = \mathscr{L}$. In (**2**) a class of inverse semigroups, called factorisable inverse semigroups is studied. One of the main results is that any inverse semigroup can be embedded in a fundamental factorisable inverse semigroup. In view of Proposition 1.2 in (**5**) and the results of Section 3 in (**2**) it follows that for any right type A monoid S there is a fundamental factorisable inverse semigroup F and a faithful embedding θ of S into F such that, if $a, b \in S$ and $a\,\mathscr{L}^* b$ then $a\theta \mathscr{L} b\theta$. In spite of this we find in (**3**) that factorisable right type A monoids which are not inverse semigroups play a role in the investigation of proper right type A covers. It therefore seems worthwhile to make a study of factorisable right adequate semigroups in general and this paper is devoted to such a study. The approach adopted closely follows that used in (**2**).

An inverse semigroup S is called *factorisable* if $S = GE$ for some subgroup G and set of idempotents E in S. Analogously we define a *weak factorisable right adequate semigroup* to be a right adequate semigroup S in which there is a left cancellative monoid T and a set of idempotents E in S such that $S = TE$. In this case S is necessarily a monoid and the identity of T is the identity of S. In Section 2 this notion will be specialised to define factorisable right adequate semigroups.

Sections 3 and 4 are concerned with the study of factorisable right adequate semigroups which are semilattices of left cancellative monoids. In the final section we characterise the semigroups which are direct products of a left cancellative monoid and a semilattice with greatest element. These semigroups are just the proper factorisable right adequate semigroups which are semilattices of left cancellative monoids.

We adopt the terminology and notation of (**8**).

1. Preliminaries

The following result from (**7**), (**9**) and (**10**) gives a useful alternative description of the relation $\mathscr{L}^*$.

Lemma 1.1. *Let a and b be elements in a semigroup S. Then the following are equivalent:*

(i) $(a, b) \in \mathcal{L}^*$,

(ii) *for all* $x, y \in S^1$, $ax = ay$ *if and only if* $bx = by$.

Corollary 1.2. *If e is an idempotent of a semigroup S, then for any $a \in S$ the following are equivalent:*

(i) $(e, a) \in \mathcal{L}^*$,

(ii) $ae = a$ *and* $ax = ay$ *implies* $ex = ey$ *for any* $x, y \in S^1$.

Let S be a right adequate semigroup with semilattice of idempotents E and a be an element in S. The $\mathcal{L}^*$-class in S containing a is denoted by L_a^*. It follows from the commutativity of the idempotents in S that each $\mathcal{L}^*$-class contains a unique idempotent which is denoted by a^*. For any $a, b \in S$ it is easy to see that $(ab)^* = (a^*b)^*$. From (**5**) we have that S is right type A if and only if $ea = a(ea)^*$ for any $e^2 = e, a \in S$.

In (**6**) the largest congruence μ_L contained in $\mathcal{L}^*$ on S was shown to be given by

$$\mu_L = \{(a, b) \in S \times S; \quad (ea)^* = (eb)^* \text{ for all } e \in E^1\}$$

From (**4**) and (**6**) we have:

Proposition 1.3. *Let S be a right type A monoid with semilattice of idempotents E. Then the following are equivalent:*

(i) *E is central,*

(ii) *$\mathcal{L}^*$ is μ_L,*

(iii) *S is a semilattice of left cancellative monoids.*

Let S be a right adequate semigroup with set of idempotents E. Define σ on S by:

$$a\sigma b \text{ if and only if } ae = be \text{ for some } e \in E.$$

If S is a right type A monoid, then from (**5**) σ is the minimum left cancellative congruence on S.

A right type A monoid S is called proper if $\sigma \cap \mathcal{L}^* = \iota$ in S. The proper right type A monoids have been characterised in (**5**).

2. Basic properties

Let S be a right adequate semigroup with semilattice of idempotents E. Define $\leqq$ on S by $a \leqq b$ if and only if $a = be$ for some $e \in E$. We conclude from (**1**) or easily verify that $\leqq$ is a partial order on S. If S is a right type A monoid, then $\leqq$ is compatible and the minimum left cancellative congruence on S is:

$$\sigma = \{(a, b) \in S \times S; \exists c \in S \text{ such that } c \leqq a, c \leqq b\}.$$

Let S be any right adequate semigroup and T be a subsemigroup of S. Define

$$\omega T = \{x \in S;\ x \leqq t \text{ for some } t \in T\}.$$

It is straightforward to prove the following proposition:

Proposition 2.1. *Let S be a right adequate semigroup. T be a left cancellative monoid in S and E be a set of idempotents in S. Then the following are equivalent:*

(i) $S = TE$,

(ii) *for each $s \in S$ there exists $t \in T$ such that $s \leqq t$,*

(iii) $S = \omega T$.

Now let S be a right type A monoid and T be a subsemigroup of S. Then as a consequence of the compatibility of $\leqq$ we get ωT is a subsemigroup of S and that it can be written as TE where E is a set of idempotents in S. If T is a left cancellative monoid and $E \subseteq \omega T$, then ωT is a weak factorisable right adequate semigroup and the proof of the following lemma is clear.

Lemma 2.2. *Let S be a right type A monoid and T be a left cancellative submonoid of S. Then ωT is a full subsemigroup of S which is weak factorisable.*

Lemma 2.3. *Let S be a right adequate monoid. Then L_1^* is a left cancellative submonoid.*

Proof. For any $a, b \in L_1^*$, $(ab)^* = (a^*b)^* = (1b)^* = b^* = 1$ so that L_1^* is closed and is thus a submonoid of S. It follows from Corollary 1.2 that L_1^* is left cancellative.

The inverse semigroup S is factorisable if $S = GE$ where $G = L_1$ is a subgroup of S and 1 is the identity of S (**2**). To get an analogue of this formula in right adequate semigroups S one needs to have $S = L_1^*E$. Let S be a weak factorisable right adequate semigroup, say, $S = TE$. In order to obtain an appropriate analogue of factorisable inverse semigroups in the right adequate case, it seems essential to impose the condition that $T \subseteq L_1^*$, that is, $t\mathscr{L}^*1$ for any $t \in T$.

We now define a right adequate monoid S to be *factorisable* if S is weak factorisable as $S = TE$ and $T \subseteq L_1^*$.

In the rest of this section S will denote a right adequate monoid.

If S is factorisable as $S = TE$, say, then $T \subseteq L_1^*$ so that $S = L_1^*E$.

Now suppose that $S = L_1^*E$ for some set of idempotents E in S. Then an element x of S may be written as $x = te$ for some $t \in L_1^*$, $e \in E$ and so $x^* = (te)^* = t^*e = e$. Thus e is uniquely determined by x and $x^* \in E$ for all $x \in S$, so that E is the semilattice of idempotents of S.

In the next theorem which is an analogue of Theorem 2.2 in (**2**), we give some alternative characterisations of factorisable right adequate semigroups.

Theorem 2.4. *The following are equivalent where E is the semilattice of idempotents of S;*

(i) *S is factorisable,*

(ii) $S = L_1^*E$,

(iii) $S = \omega L_1^*$,

(iv) $L_e^* = L_1^*e$ *for any $e \in E$.*

Proof. It is clear that (i), (ii) and (iii) are equivalent. If $S = L_1^*E$; then as noted above, for an element $x = te$ of S, we have $x^* = e$. Hence $x \in L_e^*$ if and only if $x = te$ where $t \in L_1^*$, that is, $L_e^* = L_1^* e$.

If $L_e^* = L_1^* e$ for any $e \in E$, then clearly

$$S = \bigcup_{e \in E} L_e^* = \bigcup_{e \in E} L_1^* e = L_1^* E.$$

Corollary 2.5. *Let S be a right type A monoid with semilattice of idempotents E. Then ωL_1^* is the largest factorisable full subsemigroup of S.*

Proof. Since L_1^* is a left cancellative monoid, we have by Lemma 2.2 that ωL_1^* is a weak factorisable full subsemigroup of S. But $\omega L_1^* = L_1^* E$ so that by Theorem 2.4, ωL_1^* is factorisable. Clearly any full factorisable subsemigroup of S has the form TE for some left cancellative monoid $T \subseteq S$. Hence 1 must be a member of T. Now if $t \in T$, $e \in E$ and t, e are $\mathscr{L}^*$-related in S, they are $\mathscr{L}^*$-related in TE and so it follows from Theorem 2.4 that $e = 1$. Hence $T \subseteq L_1^*$ and $TE \subseteq \omega L_1^*$ giving the desired result.

Corollary 2.5 can be considered as a generalisation of the main part of Theorem 2.3 in (**2**).

We are now in a position to show that when specialised to the inverse case, our definition yields the factorisable inverse semigroups of (**2**). To demonstrate this fact, let S be an inverse semigroup with semilattice of idempotents E. In particular, S is a right adequate semigroup in which $\mathscr{L} = \mathscr{L}^*$. If S is factorisable in the sense of our definition, then S is a monoid and $S = L_1 E$. In order to see that S is factorisable according to the definition of (**2**) one only needs to show that L_1 is a subgroup of S. By Lemma 2.3, L_1 is a submonoid. Let $t \in L_1$ and t^{-1} be the inverse of t in S so that $t^{-1}t = 1$. Let $tt^{-1} = e$ so that $(t^{-1})^* = e$. Since S is factorisable in our sense, $t^{-1} = t_1 f$ where $t_1 \in L_1, f \in E$. Then $(t^{-1})^* = f$ so that $e = f$ and $t^{-1} = t_1 e$. Hence $1 = t^{-1}t = t_1 et = t_1 t$; it follows that $t^{-1} = t_1$ and L_1 is a subgroup of S.

On the other hand, if S is a factorisable inverse semigroup in the sense of (**2**), then $S = GE$ where $G = L_1$, so that by Theorem 2. 4, S is factorisable in our sense.

3. Central idempotents

In this section the class of factorisable right adequate semigroups which have central idempotents will be investigated. We begin with

Lemma 3.1. *Let S be a right type A monoid with set of idempotents E and ρ be a congruence contained in $\mathscr{L}^*$ on S. Then for any $e \in E$, $e\rho$ is a left cancellative monoid. Further:*

$$e\rho \subseteq Ee\zeta = \{x \in S;\, xf = fx \text{ for any } f \in Ee\}$$

Proof. Certainly $e\rho$ is a subsemigroup of S. Since $\rho \subseteq \mathscr{L}^*$, $e\rho \subseteq L_e^*$. Let $a \in e\rho$ and $f \in Ee$. Then we have $(fa, fe) \in \rho$, that is, $(fa, f) \in \rho \subseteq \mathscr{L}^*$ so that $(fa)^* = f$. Since S is right type A, we get; $fa = a(fa)^* = af$ and $a \in Ee\zeta$. Therefore $e\rho \subseteq Ee\zeta$. In particular, $ea = ae = a$ for any $a \in e\rho$. If $a, b, c \in e\rho$ and $ca = cb$ then $ea = eb$, that is, $a = b$. Hence $e\rho$ is a left cancellative monoid with identity e.

As mentioned in (**6**), when studying right adequate semigroups, congruences contained in $\mathcal{L}^*$ replace the idempotent-separating congruences of the inverse theory. With this in mind we see that the following proposition is a generalisation of Theorem 4.3 in (**2**).

Proposition 3.2. *Let S be a right type A monoid with semilattice of idempotents E and 1 the identity of S. Let ρ be a congruence contained in $\mathcal{L}^*$ on S and $S'=(1\rho)E$. Then S' is a factorisable right type A monoid with central idempotents.*

Proof. By Lemma 3.1. 1ρ is a left cancellative monoid contained in $E\zeta$. Let $x, y \in S'$. Then $x=af$, $y=bg$ for some $a, b \in 1\rho$ and $f, g \in E$ and because $1\rho \subseteq E\zeta$, $xy = afbg = abfg$ so that $xy \in S'$. Clearly $E \subseteq S'$. Therefore S' is a full subsemigroup of S and so S' is a right type A monoid. Because $t \in 1\rho$ implies $t \in L_1^*(S')$ then S' is factorisable. Since $1\rho \subseteq E\zeta$, it is clear that the idempotents of S' are central.

Corollary 3.3. *If S is a right type A monoid, then S contains a largest factorisable full subsemigroup with central idempotents.*

Proof. Let μ_L be the maximum congruence contained in $\mathcal{L}^*$ on S as defined in Section 1. Let $T = 1\mu_L$. Put $S'=TE$ where E is the semilattice of idempotents of S. By Proposition 3.2, S' is a factorisable full subsemigroup of S with central idempotents. Suppose H is a factorisable full subsemigroup of S with central idempotents. Then by Theorem 2.4, $H=NE$ where N is the $\mathcal{L}^*$-class in H which contains 1. Then $en=ne$ for any $n \in N$, $e \in E$, so that $(en)^*=(ne)^*=1e=e1$ and $(n, 1) \in \mu_L$, that is, $n \in 1\mu_L$. Thus $N \subseteq 1\mu_L = T$. Therefore $H=NE \subseteq TE = S'$.

We observe that if, in Corollary 3.3, S has central idempotents, then by Proposition 1.3, $\mu_L = \mathcal{L}^*$ and so $S' = 1\mu_L E = L_1^* E = \omega L_1^*$ is the largest factorisable full subsemigroup of S considered in Corollary 2.5.

Now we turn to the structure of factorisable right adequate semigroups with central idempotents. We begin by recalling (see (**8**) or (**11**)) that if S is a semilattice $\mathcal{Y}$ of semigroups S_α, $\alpha \in \mathcal{Y}$, then S is a strong semilattice of the S_α's if there exist linking homomorphisms $\phi_{\alpha,\beta}: S_\alpha \to S_\beta$ whenever $\alpha \geqq \beta$ satisfying

(i) $\phi_{\alpha,\alpha}$ is the identity map on S_α,

(ii) $\phi_{\alpha,\beta}\phi_{\beta,\gamma} = \phi_{\alpha,\gamma}$, $\alpha \geqq \beta \geqq \gamma$,

(iii) if $a \in S_\alpha$ and $b \in S_\beta$, then $ab = (a\phi_{\alpha,\alpha\beta})(b\phi_{\beta,\alpha\beta})$.

If $\mathcal{Y}$ has a greatest element ι and $\beta, \gamma \in \mathcal{Y}$ with $\beta > \gamma$, then

$$\phi_{\iota,\gamma} = \phi_{\iota,\beta}\phi_{\beta,\gamma}$$

and so we have the following lemma.

Lemma 3.4. *Every linking homomorphism is onto if and only if $\phi_{\iota,\alpha}$ is onto for all $\alpha \in \mathcal{Y}$.*

From (**4**), we know that a right adequate semigroup has central idempotents if and only if it is a strong semilattice of left cancellative monoids. Specialising to the case of factorisable monoids, we have

Theorem 3.5. *Let S be right adequate monoid with central idempotents and suppose that S is a strong semilattice $\mathscr{Y}$ of left cancellative monoids $T_\alpha (\alpha \in \mathscr{Y})$. Then S is factorisable if and only if every linking homomorphism $\phi_{\alpha,\beta}$ is onto.*

Proof. Let e_α be the identity of T_α. If $a \in T_\alpha$ and $a \mathscr{L}^* e_\beta$, then $ae_\beta = a$ so that $\alpha \leqq \beta$. On the other hand $ae_\alpha = a$ so that $e_\beta e_\alpha = e_\beta$ and $\beta \leqq \alpha$. Hence $\alpha = \beta$ and it follows that $T_\alpha = L^*_{e_\alpha}$.

Clearly if ι is the greatest element of $\mathscr{Y}$, then $e_\iota = 1$. Now by Theorem 2.4, S is factorisable if and only if $L^*_{e_\alpha} = L^*_1 e_\alpha$ for each $\alpha \in \mathscr{Y}$. For $\alpha, \beta \in \mathscr{Y}$ with $\alpha \geqq \beta$ and $x \in T_\alpha$ we have $x\phi_{\alpha,\beta} = xe_\beta$, so that, in particular, $T_\iota e_\alpha = T_\iota \phi_{\iota,\alpha}$. Hence S is factorisable if and only if $T_\alpha = T_\iota \phi_{\iota,\alpha}$ for each $\alpha \in \mathscr{Y}$, that is, if and only if $\phi_{\iota,\alpha}$ is onto for each $\alpha \in \mathscr{Y}$. The result now follows by Lemma 3.4.

4. Direct products

As examples of factorisable right adequate semigroups with central idempotents we have semigroups of the form $S = T \times E$ where T is a left cancellative monoid and E is a semilattice with greatest element i. Clearly for any $(t, e) \in S$ we have $(t, e)\mathscr{L}^*(1, e)$ and $\bar{E} = \{(1, e): e \in E\}$ is the semilattice of idempotents of S. Therefore S is a right adequate semigroup with identity $(1, i)$. Certainly the idempotents are central.

Notice that T is isomorphic to $\bar{T} = L^*_{(1,i)} = \{(t, i): t \in T\}$ and $S = \bar{T}\bar{E}$. Hence by Theorem 2.4, S is a factorisable right adequate semigroup.

Further, if $(t_1, e_1), (t_2, e_2) \in S$ are such that $((t_1, e_1), (t_2, e_2)) \in \mathscr{L}^* \cap \sigma$, then $(t_1, e_1)^* = (t_2, e_2)^*$, that is $e_1 = e_2$ and there exists $(1, f) \in \bar{E}$ such that $(t_1, e_1)(1, f) = (t_2, e_2)(1, f)$, so that $t_1 = t_2$. Therefore $(t_1, e_2) = (t_2, e_2)$ and S is proper.

The direct part of the following proposition is now evident.

Proposition 4.1. *Let S be a semigroup isomorphic to a direct product $T \times E$ of a left cancellative monoid and a semilattice E with greatest element i. Then*

(i) *S is a right adequate semigroup which is factorisable as $S = \bar{T}\bar{E}$ for some left cancellative monoid $\bar{T}$ and subsemilattice $\bar{E}$ of S.*

(ii) *every element x in S is uniquely represented in the form $x = \bar{t}\bar{e}$ where $\bar{t} \in \bar{T}$ and $\bar{e} \in \bar{E}$.*

(iii) *$\bar{e}\bar{t} = \bar{t}\bar{e}$ for all $\bar{e} \in \bar{E}$ and $\bar{t} \in \bar{T}$.*

Conversely, any semigroup which satisfies (i), (ii) *and* (iii) *is isomorphic to the direct product of the left cancellative monoid $\bar{T}$ and the semilattice $\bar{E}$.*

Proof. To prove the converse part, assume S satisfies (i), (ii) and (iii). We define $\theta: \bar{T} \times \bar{E} \to S$ by $(\bar{t}, \bar{e})\theta = \bar{t}\bar{e}$. It is easily verified that θ is an isomorphism.

It is clear that Proposition 4.1 is an analogue of Theorem 4.6 in (**2**) and gives the conditions for a factorisable right adequate semigroup to be a direct product of its factors. These semigroups are proper with central idempotents. An alternative description in terms of linking homomorphisms can be given, to do this we need the following lemma.

Lemma 4.2. *Let S be a right adequate monoid which is a semilattice of left cancellative monoids $\{L_e^*: e \in E\}$ with linking homomorphisms $\phi_{e,f}$, $e, f \in E$, $e \geqq f$ where E is the semilattice of S. Then the following are equivalent:*

(i) *S is proper,*

(ii) *every linking homomorphism $\phi_{e,f}$ is one-to-one.*

Proof. If S is proper and $a, b \in L_e^*$ are such that $a\phi_{e,f} = b\phi_{e,f}$, then $af = bf$ so that $(a, b) \in \mathscr{L}^* \cap \sigma$. Hence $a = b$ and $\phi_{e,f}$ is one-to-one. On the other hand, suppose that (ii) holds and let $a, b \in S$ be such that $(a, b) \in \mathscr{L}^* \cap \sigma$. Then for some idempotents e, f, we have $a, b \in L_e^*$ and $af = bf$. Hence

$$a\phi_{e,ef} = aef = af = bf = bef = b\phi_{e,ef}$$

But $\phi_{e,ef}$ is one-to-one so that $a = b$ and S is proper.

Theorem 4.3. *Let S be a factorisable right adequate semigroup which is a semilattice of left cancellative monoids $\{L_e^*: e \in E\}$, where E is the semilattice of S. Then the following are equivalent:*

(i) *S is proper,*

(ii) *every linking homomorphism $\phi_{e,f}$ is an isomorphism,*

(iii) *S is isomorphic to $L_1^* \times E$.*

Proof. Since S is factorisable, every linking homomorphism is onto by Theorem 3.5. Thus (i) is equivalent to (ii) by Lemma 4.2.

By the remarks at the beginning of this section we known that $L_1^* \times E$ is proper so that (iii) implies (i).

Now let S be proper. Since S is factorisable, we have, by Theorem 2.4, $S = L_1^*E$. If $x \in S$ and $x = t_1e = t_2f$ where $t_1, t_2 \in L_1^*$, $e, f \in E$, then $e = (t_1e)^* = (t_2f)^* = f$. Now $t_1e = t_2e$ gives $(t_1, t_2) \in \mathscr{L}^* \cap \sigma$ so that $t_1 = t_2$. Certainly E is central in S. Hence by Proposition 4.1, S is isomorphic to $L_1^* \times E$, and (iii) holds.

Corollary 4.4. *A semigroup is isomorphic to a direct product of a left cancellative monoid and a semilattice with a greatest element if and only if it is a proper factorisable right adequate semigroup with central idempotents.*

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