Bull. Aust. Math. Soc. **107** (2023), 10–21
doi:10.1017/S0004972722000545

A SHARP UPPER BOUND FOR THE SUM OF RECIPROCALS OF LEAST COMMON MULTIPLES II

SIAO HONG ✉, MEILING HUANG and YUANLIN LI

(Received 27 March 2022; accepted 26 April 2022; first published online 9 June 2022)

Abstract

Let n and k be positive integers with $n \geq k+1$ and let $\{a_i\}_{i=1}^n$ be a strictly increasing sequence of positive integers. Let $S_{n,k} := \sum_{i=1}^{n-k} 1/\mathrm{lcm}(a_i, a_{i+k})$. In 1978, Borwein ['A sum of reciprocals of least common multiples', *Canad. Math. Bull.* **20** (1978), 117–118] confirmed a conjecture of Erdős by showing that $S_{n,1} \leq 1 - 1/2^{n-1}$. Hong ['A sharp upper bound for the sum of reciprocals of least common multiples', *Acta Math. Hungar.* **160** (2020), 360–375] improved Borwein's upper bound to $S_{n,1} \leq {a_1}^{-1}(1 - 1/2^{n-1})$ and derived optimal upper bounds for $S_{n,2}$ and $S_{n,3}$. In this paper, we present a sharp upper bound for $S_{n,4}$ and characterise the sequences $\{a_i\}_{i=1}^n$ for which the upper bound is attained.

2020 *Mathematics subject classification*: primary 11A05; secondary 11B65.

Keywords and phrases: least common multiple, reciprocal, upper bound, prime number theorem.

1. Introduction

Chebyshev [4] investigated the least common multiple of consecutive positive integers when he made the first important attempt to prove the prime number theorem stating that $\log \mathrm{lcm}(1, 2, \ldots, n) \sim n$ as n goes to infinity (see, for example, [13]). Hanson [8] and Nair [14] gave upper and lower bounds for $\mathrm{lcm}(1, 2, \ldots, n)$ and Nair's lower bound was extended in [6, 11]. Goutziers [7] studied the asymptotic behaviour of the least common multiple of a set of integers not exceeding N. Bateman *et al.* [1] obtained an asymptotic estimate for the least common multiple of arithmetic progressions that is generalised in [12] to products of linear polynomials. In another direction, Behrend [2] strengthened an inequality of Heilbronn [9] and Rohrbach [15]. Erdős and Selfridge [5] proved a remarkable old conjecture that predicts that the product of any two or more consecutive positive integers is never a perfect power.

Erdős observed another interesting phenomena related to least common multiples. Let n and k be positive integers with $n \geq k+1$ and let $\{a_i\}_{i=1}^n$ be a strictly increasing

This work was supported in part by research grants from the Natural Sciences and Engineering Research Council of Canada (Grant No. DDG-2019-04206 and RGPIN 2017-03903).

sequence of positive integers. Let

$$S_{n,k} := \sum_{i=1}^{n-k} \frac{1}{\operatorname{lcm}(a_i, a_{i+k})}.$$

In 1978, Borwein [3] confirmed a conjecture of Erdős by showing that $S_{n,1} \le 1 - 1/2^{n-1}$ with equality if and only if $a_i = 2^{i-1}$ for $1 \le i \le n$. Recently, Hong [10] improved this upper bound and used the new result to get sharp upper bounds for $S_{n,2}$ and $S_{n,3}$. He also characterised the sequences $\{a_i\}_{i=1}^{\infty}$ for which these upper bounds are attained. In this paper, we concentrate on $S_{n,4}$. We will present an optimal upper bound for $S_{n,4}$ and characterise the sequences $\{a_i\}_{i=1}^{n}$ for which this upper bound is attained.

As usual, for any real number x, we denote by $\lfloor x \rfloor$ and $\lceil x \rceil$ respectively the largest integer no more than x and the smallest integer no less than x. For brevity, we write $S_n := S_{n,4}$.

The main result of this paper can be stated as follows.

THEOREM 1.1. *Let n be an integer with $n \ge 5$ and let $\{a_i\}_{i=1}^{n}$ be a strictly increasing sequence of positive integers. Then:*

(i) $S_5 \le 1/5$ *with equality if and only if* $a_i = i$ *for all* $i \in \{1, 2, 3, 4, 5\}$*;*
(ii) $S_6 \le 11/30$ *with equality if and only if* $a_i = i$ *for all* $i \in \{1, 2, 3, 4, 5, 6\}$*;*
(iii) $S_7 \le 43/90$ *with equality if and only if* $a_i = i$ *for all* $i \in \{1, 2, 3, 4, 5, 6\}$ *and* $a_7 = 9$*;*
(vi) $S_8 \le 101/180$ *with equality if and only if* $a_i = i$ *for all* $i \in \{1, 2, 3, 4, 5, 6\}$, $a_7 = 9$ *and* $a_8 = 12$*;*
(v) *if* $n \ge 9$, *then*

$$S_n \le \frac{493}{420} - \frac{533}{105} \cdot \frac{1}{2^{\lfloor n/4 \rfloor + 1}} + \frac{\epsilon_n}{2^{\lfloor n/4 \rfloor}}, \tag{1.1}$$

where

$$\epsilon_n := \begin{cases} 0 & \text{if } n \equiv 0 \pmod 4, \\ \frac{2}{5} & \text{if } n \equiv 1 \pmod 4, \\ \frac{11}{15} & \text{if } n \equiv 2 \pmod 4, \\ \frac{107}{105} & \text{if } n \equiv 3 \pmod 4, \end{cases}$$

and equality in (1.1) occurs if and only if $a_i = i$ *for all* $i \in \{1, 2, 3, 4\}$ *and* $a_{4i+1} = 5 \times 2^{i-1}$ $(1 \le i \le \lfloor (n-1)/4 \rfloor)$, $a_{4i+2} = 3 \times 2^{i}$ $(1 \le i \le \lfloor (n-2)/4 \rfloor)$, $a_{4i+3} = 7 \times 2^{i-1}$ $(1 \le i \le \lfloor (n-3)/4 \rfloor)$ *and* $a_{4i+4} = 2^{i+2}$ $(1 \le i \le \lfloor n/4 \rfloor - 1)$.

The rest of the paper is organised as follows. In Section 2, we prove several preliminary lemmas. In Section 3, we provide a proof for our main result.

https://doi.org/10.1017/S0004972722000545 Published online by Cambridge University Press

2. Auxiliary lemmas

In this section, we supply several auxiliary lemmas that are needed in the proof of Theorem 1.1. The first is Hong's upper bound [10, Theorem 1.2] which improves Borwein's upper bound [3].

LEMMA 2.1 [10, Theorem 1.2]. *Let n be an integer with $n \geq 2$ and let $\{a_i\}_{i=1}^n$ be a strictly increasing sequence of positive integers. Then*

$$\sum_{i=1}^{n-1} \frac{1}{\mathrm{lcm}(a_i, a_{i+1})} \leq \frac{1}{a_1}\Big(1 - \frac{1}{2^{n-1}}\Big) \tag{2.1}$$

with equality in (2.1) if and only if $a_i = 2^{i-1}a_1$ for all integers i with $1 \leq i \leq n$.

LEMMA 2.2. *Let m be an integer with $m \geq 3$. Then*

$$\frac{1}{7} + \frac{1}{9} + \frac{1}{9}\Big(1 - \frac{1}{2^{m-2}}\Big) < \frac{1}{5} + \frac{1}{21} + \frac{1}{5}\Big(1 - \frac{1}{2^{m-2}}\Big)$$

and

$$\frac{1}{9} + \frac{1}{12} + \Big(\frac{1}{9} + \frac{1}{10}\Big)\Big(1 - \frac{1}{2^{m-2}}\Big) < \frac{1}{8} + \frac{1}{21} + \Big(\frac{1}{7} + \frac{1}{8}\Big)\Big(1 - \frac{1}{2^{m-2}}\Big).$$

PROOF. Since $m \geq 3$, a direct computation gives the desired inequalities. □

LEMMA 2.3. *Let S_n be given as above. Then:*

(i) $S_5 \leq 1/5$ *with equality if and only if $a_i = i$ for all $i \in \{1, 2, 3, 4, 5\}$;*
(ii) $S_6 \leq 11/30$ *with equality if and only if $a_i = i$ for all $i \in \{1, 2, 3, 4, 5, 6\}$;*
(iii) $S_7 \leq 43/90$ *with equality if and only if $a_i = i$ for all $i \in \{1, 2, 3, 4, 5, 6\}$ and $a_7 = 9$.*

PROOF. We first deal with S_5. Since $\mathrm{lcm}(a_1, a_4) \geq a_5 \geq 5$,

$$S_5 = \frac{1}{\mathrm{lcm}(a_1, a_5)} \leq \frac{1}{5}. \tag{2.2}$$

The equality in (2.2) holds if and only if $\mathrm{lcm}(a_1, a_5) = 5$, which is true if and only if $a_1 = 1$ and $a_5 = 5$. However, $a_1 < a_2 < a_3 < a_4 < a_5$. So the equality in (2.2) holds if and only if $a_i = i$ for all $i \in \{1, 2, 3, 4, 5\}$.

Now consider S_6. Since $a_2 \geq 2$, $a_2 \mid \mathrm{lcm}(a_2, a_6)$ and $\mathrm{lcm}(a_2, a_6) \geq a_6 \geq 6$, we deduce that $\mathrm{lcm}(a_2, a_6) \geq 6$ with equality if and only if $a_2 = 2$ and $a_6 = 6$. So

$$S_6 = \frac{1}{\mathrm{lcm}(a_1, a_5)} + \frac{1}{\mathrm{lcm}(a_2, a_6)} \leq \frac{1}{5} + \frac{1}{6} = \frac{11}{30}, \tag{2.3}$$

with equality in (2.3) if and only if $\mathrm{lcm}(a_1, a_5) = 5$ and $\mathrm{lcm}(a_2, a_6) = 6$, which is true if and only if $a_1 = 1, a_2 = 2, a_5 = 5$ and $a_6 = 6$, which is true if and only if $a_i = i$ for all $i \in \{1, 2, 3, 4, 5, 6\}$.

Finally, we consider S_7. Since $a_3 \geq 3$, $a_3 \mid \mathrm{lcm}(a_3, a_7)$ and $\mathrm{lcm}(a_3, a_7) \geq a_7 \geq 7$, we deduce that either $\mathrm{lcm}(a_3, a_7) = 8$ which is true if and only if $a_3 = 4$ and $a_7 = 8$, or

https://doi.org/10.1017/S0004972722000545 Published online by Cambridge University Press

$\text{lcm}(a_3, a_7) = 9$ which is true if and only if $a_3 = 3$ and $a_7 = 9$, or $\text{lcm}(a_3, a_7) \geq 10$. We divide the rest of the proof into three cases.

If $\text{lcm}(a_3, a_7) \geq 10$, then

$$S_7 = \frac{1}{\text{lcm}(a_1, a_5)} + \frac{1}{\text{lcm}(a_2, a_6)} + \frac{1}{\text{lcm}(a_3, a_7)} \leq \frac{11}{30} + \frac{1}{10} < \frac{43}{90}$$

as desired.

If $\text{lcm}(a_3, a_7) = 8$, then $a_3 = 4$ and $a_7 = 8$. This implies that $a_4 = 5, a_5 = 6$ and $a_6 = 7$. Since $(a_1, a_2) \in \{(1, 2), (1, 3), (2, 3)\}$, we have $\text{lcm}(a_1, a_5) = 6$ and $\text{lcm}(a_2, a_6) \in \{14, 21\}$. It then follows that

$$S_7 \leq \frac{1}{6} + \frac{1}{14} + \frac{1}{8} < \frac{43}{90}.$$

If $\text{lcm}(a_3, a_7) = 9$, then we must have $a_3 = 3$ and $a_7 = 9$. So $\text{lcm}(a_3, a_7) = 9$. It then follows that

$$S_7 = S_6 + \frac{1}{\text{lcm}(a_3, a_7)} \leq \frac{11}{30} + \frac{1}{9} = \frac{43}{90}, \tag{2.4}$$

with equality in (2.4) if and only if $a_i = i$ for all $i \in \{1, 2, 3, 4, 5, 6\}$ and $\text{lcm}(a_3, a_7) = 9$, if and only if $a_i = i$ for all $i \in \{1, 2, 3, 4, 5, 6\}$ and $a_7 = 9$ as required.

This completes the proof of Lemma 2.3. □

LEMMA 2.4. *Let m be a positive integer with $m \geq 2$ and $\mathcal{A} = \{a_i\}_{i=1}^{8}$ a strictly increasing sequence of eight positive integers. Let*

$$\square_m = \square_m(\mathcal{A}) := \sum_{i=1}^{4} \bigg(\frac{1}{\text{lcm}(a_i, a_{i+4})} + \frac{1}{a_{i+4}} \Big(1 - \frac{1}{2^{m-2}} \Big) \bigg). \tag{2.5}$$

Then both of the following statements are true.

(i) *Either $\square_2 = 101/180$ which is true if and only if $a_i = i$ for all $i \in \{1, 2, 3, 4, 5, 6\}$, $a_7 = 9$ and $a_8 = 12$, or $\square_2 = 389/720$ which holds if and only if $a_i = i$ for all integers $i \in \{1, 2, 3, 4, 5, 6\}$, $a_7 = 9$ and $a_8 = 16$, or $\square_2 = 453/840$ which is true if and only if $a_i = i$ for all integers $i \in \{1, 2, 3, 4, 5, 6, 7, 8\}$, or $\square_2 < 453/840$.*

(ii) *If $m \geq 3$, then*

$$\square_m \leq \frac{493}{420} - \frac{533}{105} \cdot \frac{1}{2^{m+1}}, \tag{2.6}$$

with equality in (2.6) if and only if $a_i = i$ for all integers i with $1 \leq i \leq 8$.

PROOF

(i). Evidently, $\square_2 = \sum_{i=1}^{4} 1/\text{lcm}(a_i, a_{i+4})$. We consider the following cases.

Case 1: $a_5 \geq 6$. Then $a_8 \geq 9$. If $a_8 \geq 10$, then by the fact $\text{lcm}(a_i, a_{i+4}) \geq a_{i+4}$ for all $i \in \{1, 2, 3, 4\}$, we derive

$$\square_2 \leq \frac{1}{a_5} + \frac{1}{a_6} + \frac{1}{a_7} + \frac{1}{a_8} \leq \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{10} < \frac{1}{5} + \frac{1}{6} + \frac{1}{8} + \frac{1}{21} = \frac{453}{840}.$$

https://doi.org/10.1017/S0004972722000545 Published online by Cambridge University Press

If $a_8 = 9$, then $a_5 = 6$, $a_6 = 7$ and $a_7 = 8$. This implies that $a_1 \in \{1, 2\}$, $a_2 \in \{2, 3\}, a_3 \in \{3, 4\}$ and $a_4 \in \{4, 5\}$. It follows that $\mathrm{lcm}(a_1, a_5) = 6$, $\mathrm{lcm}(a_2, a_6) \in \{14, 21\}, \mathrm{lcm}(a_3, a_7) = \mathrm{lcm}(a_3, 8) \in \{8, 24\}$ and $\mathrm{lcm}(a_4, a_8) = \mathrm{lcm}(a_4, 9) \in \{36, 45\}$. So

$$\square_2 \le \frac{1}{6} + \frac{1}{14} + \frac{1}{8} + \frac{1}{36} < \frac{1}{5} + \frac{1}{6} + \frac{1}{8} + \frac{1}{21} = \frac{453}{840}.$$

Case 2: $a_5 = 5$. Then $a_i = i$ for all integers i with $1 \le i \le 4$. If $a_6 \ge 7$, then $a_7 \ge 8$ and $a_8 \ge 9$. So $\mathrm{lcm}(a_1, a_5) = 5, \mathrm{lcm}(a_2, a_6) \ge 8, \mathrm{lcm}(a_3, a_7) \ge 9$ and $\mathrm{lcm}(a_4, a_8) \ge 12$. However, $1/9 + 1/12 < 1/6 + 1/21$. Thus

$$\square_2 \le \frac{1}{5} + \frac{1}{8} + \frac{1}{9} + \frac{1}{12} < \frac{1}{5} + \frac{1}{6} + \frac{1}{8} + \frac{1}{21} = \frac{453}{840}.$$

In what follows, we let $a_6 = 6$. If $a_7 \ge 10$, then $a_8 \ge 11$. It follows that $\mathrm{lcm}(a_3, a_7) \ge 12$ with equality holding if and only if $a_7 = 12$, and $\mathrm{lcm}(a_4, a_8) \ge 12$ with equality occurring if and only if $a_8 = 12$. Since $a_7 < a_8$,

$$\square_2 < \frac{1}{5} + \frac{1}{6} + \frac{1}{12} + \frac{1}{12} < \frac{1}{5} + \frac{1}{6} + \frac{1}{8} + \frac{1}{21} = \frac{453}{840}.$$

It remains to consider the case $a_7 \in \{7, 8, 9\}$. We consider three subcases.

Subcase 2.1: $a_7 = 7$. Then $\mathrm{lcm}(a_3, a_7) = 21$ and $\mathrm{lcm}(a_4, a_8) = \mathrm{lcm}(4, a_8) \ge 8$ with equality if and only if $a_8 = 8$. So

$$\square_2 \le \frac{1}{5} + \frac{1}{6} + \frac{1}{21} + \frac{1}{8} = \frac{453}{840}$$

with equality if and only if $a_i = i$ for all integers i with $1 \le i \le 8$.

Subcase 2.2: $a_7 = 8$. Then $a_8 \ge 9$. Hence

$$\square_2 \le \frac{1}{5} + \frac{1}{6} + \frac{1}{24} + \frac{1}{12} < \frac{1}{5} + \frac{1}{6} + \frac{1}{8} + \frac{1}{21} = \frac{453}{840}.$$

Subcase 2.3: $a_7 = 9$. Then $a_8 \ge 10$. It follows that either $\mathrm{lcm}(a_4, a_8) = 12$ which is true if and only if $a_8 = 12$, or $\mathrm{lcm}(a_4, a_8) = 16$ which is true if and only if $a_8 = 16$, or $\mathrm{lcm}(a_4, a_8) \ge 20$. We then deduce that either

$$\square_2 = \frac{1}{5} + \frac{1}{6} + \frac{1}{9} + \frac{1}{12} = \frac{101}{180}$$

which is true if and only if $a_i = i$ for all $i \in \{1, 2, 3, 4, 5, 6\}$, $a_7 = 9$ and $a_8 = 12$, or

$$\square_2 = \frac{1}{5} + \frac{1}{6} + \frac{1}{9} + \frac{1}{16} = \frac{389}{720}$$

https://doi.org/10.1017/S0004972722000545 Published online by Cambridge University Press

which holds if and only if $a_i = i$ for all $i \in \{1,2,3,4,5,6\}$, $a_7 = 9$ and $a_8 = 16$, or

$$\square_2 \le \frac{1}{5}+\frac{1}{6}+\frac{1}{9}+\frac{1}{20} < \frac{1}{5}+\frac{1}{6}+\frac{1}{8}+\frac{1}{21} = \frac{453}{840}$$

as expected. This completes the proof of part (i).

(ii). Let $m \ge 3$. Since $\mathrm{lcm}(a_i, a_{i+4}) \ge a_{i+4}$ for all integers i with $1 \le i \le 4$,

$$\square_m \le \sum_{i=1}^{4}\Big(\frac{1}{a_{i+4}} + \frac{1}{a_{i+4}}\Big(1-\frac{1}{2^{m-2}}\Big)\Big) = \Big(2-\frac{1}{2^{m-2}}\Big)\sum_{i=5}^{8}\frac{1}{a_i} \tag{2.7}$$

with equality in (2.7) if and only if $a_i \mid a_{i+4}$ for all integers $i \in \{1,2,3,4\}$. Let $S_0 := 493/420 - 533/105 \cdot 1/(2^{m+1})$. We divide the rest of the proof into two cases.

Case 1: $a_5 \ge 6$. Then $a_6 \ge 7, a_7 \ge 8$ and $a_8 \ge 9$. So by (2.7) and Lemma 2.2,

$$\begin{aligned}\square_m &\le \Big(2-\frac{1}{2^{m-2}}\Big)\sum_{i=5}^{8}\frac{1}{a_i} \le \Big(\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}\Big)\Big(2-\frac{1}{2^{m-2}}\Big)\\ &< \frac{1}{5}+\frac{1}{6}+\frac{1}{8}+\frac{1}{21}+\Big(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\Big)\Big(1-\frac{1}{2^{m-2}}\Big) = S_0\end{aligned}$$

since $m \ge 3$. This gives the desired result for Case 1.

Case 2: $a_5 = 5$. Then $a_i = i$ for all $i \in \{1,2,3,4\}$. We consider three subcases.

Subcase 2.1: $a_6 = 6$. Then $a_7 \ge 7$ and $\mathrm{lcm}(a_3, a_7) = \mathrm{lcm}(3, a_7) \ge 9$. So

$$\begin{aligned}\square_m &= \frac{1}{\mathrm{lcm}(1,5)}+\frac{1}{\mathrm{lcm}(2,6)}+\frac{1}{\mathrm{lcm}(3,a_7)}+\frac{1}{\mathrm{lcm}(4,a_8)}\\ &\quad+\Big(\frac{1}{5}+\frac{1}{6}+\frac{1}{a_7}+\frac{1}{a_8}\Big)\Big(1-\frac{1}{2^{m-2}}\Big).\end{aligned} \tag{2.8}$$

If $a_7 = 7$, then it follows from $a_8 \ge 8$ that $\mathrm{lcm}(4, a_8) \ge 8$ with equality if and only if $a_8 = 8$. Therefore,

$$\begin{aligned}\square_m &= \frac{1}{5}+\frac{1}{6}+\frac{1}{21}+\frac{1}{\mathrm{lcm}(4,a_8)}+\Big(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{a_8}\Big)\Big(1-\frac{1}{2^{m-2}}\Big)\\ &\le \frac{1}{5}+\frac{1}{6}+\frac{1}{21}+\frac{1}{8}+\Big(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\Big)\Big(1-\frac{1}{2^{m-2}}\Big) = S_0\end{aligned}$$

with equality if and only if $a_i = i$ for all integers i with $1 \le i \le 8$.

If $a_7 = 8$, then $a_8 \ge 9$ and so $\mathrm{lcm}(4, a_8) \ge 12$. Thus by (2.8),

$$\square_m < \frac{1}{5}+\frac{1}{6}+\frac{1}{24}+\frac{1}{12}+\Big(\frac{1}{5}+\frac{1}{6}+\frac{1}{8}+\frac{1}{9}\Big)\Big(1-\frac{1}{2^{m-2}}\Big) < S_0.$$

https://doi.org/10.1017/S0004972722000545 Published online by Cambridge University Press

If $a_7 = 9$, then $\operatorname{lcm}(3, a_7) = 9$, $a_8 \geq 10$ and so $\operatorname{lcm}(4, a_8) \geq 12$. Since $m \geq 3$ and by Lemma 2.2,

$$\square_m < \frac{1}{5} + \frac{1}{6} + \frac{1}{9} + \frac{1}{12} + \Big(\frac{1}{5} + \frac{1}{6} + \frac{1}{9} + \frac{1}{10}\Big)\Big(1 - \frac{1}{2^{m-2}}\Big)$$
$$< \frac{1}{5} + \frac{1}{6} + \frac{1}{8} + \frac{1}{21} + \Big(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\Big)\Big(1 - \frac{1}{2^{m-2}}\Big) = S_0.$$

If $a_7 \geq 10$, then $a_8 \geq 11$. Hence $\operatorname{lcm}(3, a_7) \geq 12$ with equality holding if and only if $a_7 = 12$, and $\operatorname{lcm}(4, a_8) \geq 12$ with equality occurring if and only if $a_8 = 12$. Since $a_7 < a_8$ and $m \geq 3$,

$$\square_m < \frac{1}{5} + \frac{1}{6} + \frac{1}{12} + \frac{1}{12} + \Big(\frac{1}{5} + \frac{1}{6} + \frac{1}{10} + \frac{1}{11}\Big)\Big(1 - \frac{1}{2^{m-2}}\Big)$$
$$< \frac{1}{5} + \frac{1}{6} + \frac{1}{21} + \frac{1}{8} + \Big(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\Big)\Big(1 - \frac{1}{2^{m-2}}\Big) = S_0.$$

Subcase 2.2: $a_6 = 7$. Then $a_7 \geq 8$ and $a_8 \geq 9$. So $\operatorname{lcm}(3, a_7) \geq 9$ with equality if and only if $a_7 = 9$, and $\operatorname{lcm}(4, a_8) \geq 12$ with equality if and only if $a_8 = 12$. Since $1/14 + 1/9 + 1/12 < 1/6 + 1/8 + 1/21$, it then follows immediately that

$$\square_m = \frac{1}{\operatorname{lcm}(1,5)} + \frac{1}{\operatorname{lcm}(2,7)} + \frac{1}{\operatorname{lcm}(3,a_7)} + \frac{1}{\operatorname{lcm}(4,a_8)}$$
$$+ \Big(\frac{1}{5} + \frac{1}{7} + \frac{1}{a_7} + \frac{1}{a_8}\Big)\Big(1 - \frac{1}{2^{m-2}}\Big)$$
$$< \frac{1}{5} + \frac{1}{14} + \frac{1}{9} + \frac{1}{12} + \Big(\frac{1}{5} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9}\Big)\Big(1 - \frac{1}{2^{m-2}}\Big)$$
$$< \frac{1}{5} + \frac{1}{6} + \frac{1}{21} + \frac{1}{8} + \Big(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\Big)\Big(1 - \frac{1}{2^{m-2}}\Big) = S_0.$$

Subcase 2.3: $a_6 \geq 8$. Then $a_7 \geq 9$ and $a_8 \geq 10$. Thus $\operatorname{lcm}(a_2, a_6) = \operatorname{lcm}(2, a_6) \geq 8$, $\operatorname{lcm}(a_3, a_7) = \operatorname{lcm}(3, a_7) \geq 9$ and $\operatorname{lcm}(a_4, a_8) = \operatorname{lcm}(4, a_8) \geq a_8 \geq 10$ which implies that $\operatorname{lcm}(a_4, a_8) \geq 12$ since $4 \mid \operatorname{lcm}(a_4, a_8)$. It then follows from the inequality $1/9 + 1/12 < 1/6 + 1/21$ that

$$\square_m = \frac{1}{\operatorname{lcm}(1,5)} + \frac{1}{\operatorname{lcm}(2,a_6)} + \frac{1}{\operatorname{lcm}(3,a_7)} + \frac{1}{\operatorname{lcm}(4,a_8)}$$
$$+ \Big(\frac{1}{5} + \frac{1}{a_6} + \frac{1}{a_7} + \frac{1}{a_8}\Big)\Big(1 - \frac{1}{2^{m-2}}\Big)$$

https://doi.org/10.1017/S0004972722000545 Published online by Cambridge University Press

$$\leq \frac{1}{5}+\frac{1}{8}+\frac{1}{9}+\frac{1}{12}+\Big(\frac{1}{5}+\frac{1}{8}+\frac{1}{9}+\frac{1}{10}\Big)\Big(1-\frac{1}{2^{m-2}}\Big)$$
$$< \frac{1}{5}+\frac{1}{6}+\frac{1}{21}+\frac{1}{8}+\Big(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\Big)\Big(1-\frac{1}{2^{m-2}}\Big) = S_0.$$

This completes the proof of part (ii). □

3. Proof of Theorem 1.1

Let $m \geq 2$ be an integer and let $\square_m$ be defined as in (2.5). Then $\square_2 = S_8$, so the results for parts (i) to (iv) follow from Lemmas 2.3 and 2.4. It remains to prove (v).

We first deal with the upper bounds for S_9, S_{10} and S_{11}. For $r \in \{1, 2, 3\}$,

$$S_{8+r} = \square_2 + \sum_{i=1}^{r} \frac{1}{\mathrm{lcm}(a_{4+i}, a_{8+i})}.$$

By Lemma 2.4, either $\square_2 = 101/180$ which is true if and only if $a_i = i$ for all $i \in \{1, 2, 3, 4, 5, 6\}$, $a_7 = 9$ and $a_8 = 12$, or $\square_2 = 389/720$ which holds if and only if $a_i = i$ for all integers $i \in \{1, 2, 3, 4, 5, 6\}$, $a_7 = 9$ and $a_8 = 16$, or $\square_2 = 453/840$ which is true if and only if $a_i = i$ for all integers $i \in \{1, 2, 3, 4, 5, 6, 7, 8\}$, or $\square_2 < 453/840$.

If $\square_2 < 453/840$, then it follows from $\mathrm{lcm}(a_5, a_9) \geq 10, \mathrm{lcm}(a_6, a_{10}) \geq 12$ and $\mathrm{lcm}(a_7, a_{11}) \geq 14$ that

$$S_9 < \frac{453}{840} + \frac{1}{\mathrm{lcm}(a_5, a_9)} \leq \frac{453}{840} + \frac{1}{10} = \frac{537}{840},$$
$$S_{10} < \frac{453}{840} + \sum_{i=1}^{2} \frac{1}{\mathrm{lcm}(a_{4+i}, a_{8+i})} \leq \frac{453}{840} + \frac{1}{10} + \frac{1}{12} = \frac{607}{840},$$
$$S_{11} < \frac{453}{840} + \sum_{i=1}^{3} \frac{1}{\mathrm{lcm}(a_{4+i}, a_{8+i})} \leq \frac{453}{840} + \frac{1}{10} + \frac{1}{12} + \frac{1}{14} = \frac{667}{840}.$$

If $\square_2 = 101/180$, then by Lemma 2.4, we must have $a_i = i$ for all integers i with $1 \leq i \leq 6$, $a_7 = 9$ and $a_8 = 12$. So $a_9 \geq 13$, $a_{10} \geq 14$ and $a_{11} \geq 15$. This implies that $\mathrm{lcm}(a_5, a_9) = \mathrm{lcm}(5, a_9) \geq 15$ with equality if and only if $a_9 = 15$, $\mathrm{lcm}(a_6, a_{10}) = \mathrm{lcm}(6, a_{10}) \geq 18$ with equality if and only if $a_{10} = 18$, and $\mathrm{lcm}(a_7, a_{11}) = \mathrm{lcm}(9, a_{11}) \geq 18$ with equality if and only if $a_{11} = 18$. Hence

$$S_9 = \frac{101}{180} + \frac{1}{\mathrm{lcm}(a_5, a_9)} \leq \frac{101}{180} + \frac{1}{15} = \frac{113}{180} < \frac{537}{840},$$
$$S_{10} = \frac{101}{180} + \sum_{i=1}^{2} \frac{1}{\mathrm{lcm}(a_{4+i}, a_{8+i})} \leq \frac{101}{180} + \frac{1}{15} + \frac{1}{18} = \frac{123}{180} < \frac{607}{840},$$

https://doi.org/10.1017/S0004972722000545 Published online by Cambridge University Press

$$S_{11} = \frac{101}{180} + \sum_{i=1}^{3} \frac{1}{\mathrm{lcm}(a_{4+i}, a_{8+i})} < \frac{101}{180} + \frac{1}{15} + \frac{1}{18} + \frac{1}{18} = \frac{133}{180} < \frac{667}{840}$$

as desired.

If $\square_2 = 389/720$, then by Lemma 2.4, we must have $a_i = i$ for all integers i with $1 \le i \le 6$, $a_7 = 9$ and $a_8 = 16$. So $a_9 \ge 17$, $a_{10} \ge 18$ and $a_{11} \ge 19$ which implies that $\mathrm{lcm}(a_5, a_9) = \mathrm{lcm}(5, a_9) \ge 20$ with equality if and only if $a_9 = 20$, $\mathrm{lcm}(a_6, a_{10}) = \mathrm{lcm}(6, a_{10}) \ge 18$ with equality if and only if $a_{10} = 18$ and $\mathrm{lcm}(a_7, a_{11}) = \mathrm{lcm}(9, a_{11}) \ge 27$ with equality if and only if $a_{11} = 27$. One then deduces that

$$S_9 = \frac{389}{720} + \frac{1}{\mathrm{lcm}(a_5, a_9)} \le \frac{389}{720} + \frac{1}{20} = \frac{425}{720} < \frac{537}{840},$$

$$S_{10} = \frac{389}{720} + \sum_{i=1}^{2} \frac{1}{\mathrm{lcm}(a_{4+i}, a_{8+i})} < \frac{389}{720} + \frac{1}{20} + \frac{1}{18} = \frac{465}{720} < \frac{607}{840},$$

$$S_{11} = \frac{389}{720} + \sum_{i=1}^{3} \frac{1}{\mathrm{lcm}(a_{4+i}, a_{8+i})} \le \frac{389}{720} + \frac{1}{20} + \frac{1}{18} + \frac{1}{27} = \frac{465}{720} + \frac{1}{27} < \frac{667}{840}$$

as desired.

If $\square_2 = 453/840$, then by Lemma 2.4, we must have $a_i = i$ for all integers i with $1 \le i \le 8$. So $a_9 \ge 9$ which implies that $\mathrm{lcm}(a_5, a_9) \ge 10$ with equality if and only if $a_9 = 10$. Furthermore, $\mathrm{lcm}(a_6, a_{10}) \ge 12$ with equality if and only if $a_{10} = 12$ and $\mathrm{lcm}(a_7, a_{11}) \ge 14$ with equality if and only if $a_{11} = 14$. Thus

$$S_9 = \frac{453}{840} + \frac{1}{\mathrm{lcm}(a_5, a_9)} \le \frac{453}{840} + \frac{1}{10} = \frac{537}{840}, \tag{3.1}$$

$$S_{10} = \frac{453}{840} + \sum_{i=1}^{2} \frac{1}{\mathrm{lcm}(a_{4+i}, a_{8+i})} \le \frac{453}{840} + \frac{1}{10} + \frac{1}{12} = \frac{607}{840}, \tag{3.2}$$

$$S_{11} = \frac{453}{840} + \sum_{i=1}^{3} \frac{1}{\mathrm{lcm}(a_{4+i}, a_{8+i})} \le \frac{453}{840} + \frac{1}{10} + \frac{1}{12} + \frac{1}{14} = \frac{667}{840}, \tag{3.3}$$

where each equality in (3.1) to (3.3) holds if and only if $a_i = i$ for all integers i with $1 \le i \le 8$, $a_9 = 10$, $a_{10} = 12$ and $a_{11} = 14$. So part (v) is true when $9 \le n \le 11$.

In what follows, we always assume that $n \ge 12$. Then we can write $n = 4m$ or $n = 4m + r$ for some integers m and r with $m \ge 3$ and $1 \le r \le 3$. For any integer i with $1 \le i \le 4$, we define

$$S_m^{(i)} := \sum_{j=1}^{m-2} \frac{1}{\mathrm{lcm}(a_{4j+i}, a_{4j+4+i})}.$$

https://doi.org/10.1017/S0004972722000545 Published online by Cambridge University Press

Then

$$S_{4m} = \sum_{i=1}^{4} \Big(\frac{1}{\operatorname{lcm}(a_i, a_{i+4})} + S_m^{(i)} \Big) \tag{3.4}$$

and

$$S_{4m+r} = S_{4m} + \sum_{i=1}^{r} \frac{1}{\operatorname{lcm}(a_{4m-4+i}, a_{4m+i})}. \tag{3.5}$$

For any integer i with $1 \le i \le 4$, applying Lemma 2.1 to the subsequence $\{a_{i+4}, a_{i+8}, \ldots, a_{i+4(m-1)}\}$ yields

$$S_m^{(i)} = \sum_{j=1}^{m-2} \frac{1}{\operatorname{lcm}(a_{i+4j}, a_{i+4j+4})} \le \frac{1}{a_{i+4}} \Big(1 - \frac{1}{2^{m-2}}\Big) \tag{3.6}$$

with equality in (3.6) if and only if $a_{i+4j} = a_{i+4} \times 2^{j-1}$ for all integers j with $1 \le j \le m-1$. Further, for any integer i with $1 \le i \le r$, applying Lemma 2.1 to the subsequence $\{a_{4+i}, a_{8+i}, \ldots, a_{4m+i}\}$ gives

$$S_m^{(i)} + \frac{1}{\operatorname{lcm}(a_{4m-4+i}, a_{4m+i})} = \sum_{j=1}^{m-1} \frac{1}{\operatorname{lcm}(a_{4j+i}, a_{4j+4+i})} \le \frac{1}{a_{4+i}} \Big(1 - \frac{1}{2^{m-1}}\Big) \tag{3.7}$$

with equality in (3.7) if and only if $a_{4j+i} = a_{4+i} \times 2^{j-1}$ for all integers j with $1 \le j \le m$. Then by (3.4) and (3.6),

$$S_{4m} \le \sum_{i=1}^{4} \Big(\frac{1}{\operatorname{lcm}(a_i, a_{i+4})} + \frac{1}{a_{i+4}} \Big(1 - \frac{1}{2^{m-2}}\Big)\Big) = \square_m \tag{3.8}$$

with equality in (3.8) if and only if $a_{4j+i} = a_{4+i} \times 2^{j-1}$ for all integers i and j with $1 \le j \le m-1$ and $1 \le i \le 4$. By (3.5), (3.6) and (3.7),

$$\begin{aligned} S_{4m+r} &= \sum_{i=1}^{4} \frac{1}{\operatorname{lcm}(a_i, a_{i+4})} + \sum_{i=1}^{r} \Big(S_m^{(i)} + \frac{1}{\operatorname{lcm}(a_{4m-4+i}, a_{4m+i})} \Big) + \sum_{i=r+1}^{4} S_m^{(i)} \\ &\le \sum_{i=1}^{4} \frac{1}{\operatorname{lcm}(a_i, a_{i+4})} + \sum_{i=1}^{r} \frac{1}{a_{4+i}} \Big(1 - \frac{1}{2^{m-1}}\Big) + \sum_{i=r+1}^{4} \frac{1}{a_{4+i}} \Big(1 - \frac{1}{2^{m-2}}\Big) \\ &= \sum_{i=1}^{4} \Big(\frac{1}{\operatorname{lcm}(a_i, a_{i+4})} + \frac{1}{a_{i+4}} \Big(1 - \frac{1}{2^{m-2}}\Big)\Big) + \frac{1}{2^{m-1}} \sum_{i=1}^{r} \frac{1}{a_{4+i}} \\ &= \square_m + \frac{1}{2^{m-1}} \sum_{i=1}^{r} \frac{1}{a_{4+i}}, \end{aligned} \tag{3.9}$$

and equality in (3.9) holds if and only if $a_{4j+i} = a_{4+i} \times 2^{j-1}$ for all integers i and j with $1 \le j \le m-1$ and $1 \le i \le 4$ and $a_{4m+i} = a_{4+i} \times 2^{m-1}$ for all integers i with $1 \le i \le r$.

https://doi.org/10.1017/S0004972722000545 Published online by Cambridge University Press

Now by Lemma 2.4, if $m \geq 3$, then

$$\begin{aligned}\square_m \leq & \frac{1}{5}+\frac{1}{6}+\frac{1}{8}+\frac{1}{21}+\Big(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\Big)\Big(1-\frac{1}{2^{m-2}}\Big)\\ = & \frac{493}{420}-\frac{533}{105}\cdot\frac{1}{2^{m+1}} := S_0,\end{aligned} \tag{3.10}$$

with equality in (3.10) if and only if $a_i = i$ for all integers i with $1 \leq i \leq 8$. Notice that

$$\sum_{i=1}^{r}\frac{1}{a_{4+i}} \leq \sum_{i=1}^{r}\frac{1}{4+i} \tag{3.11}$$

with equality in (3.11) if and only if $a_{4+i} = 4+i$ for all $1 \leq i \leq r$. Therefore, by (3.8) and (3.10), $S_{4m} \leq S_0$ with equality if and only if $a_i = i$ for all $i \in \{1,2,3,4\}$ and $a_{4j+i} = (4+i)\times 2^{j-1}$ for all integers i and j with $1 \leq j \leq m-1$ and $1 \leq i \leq 4$. It follows from (3.9) and (3.11) that

$$S_{4m+r} \leq S_0 + \frac{1}{2^{m-1}}\sum_{i=1}^{r}\frac{1}{4+i} = \frac{493}{420}-\frac{533}{105}\cdot\frac{1}{2^{m+1}}+\frac{1}{2^{m-1}}\sum_{i=1}^{r}\frac{1}{4+i}, \tag{3.12}$$

with equality in (3.12) if and only if $a_i = i$ for all $i \in \{1,2,3,4\}$, $a_{4j+i} = (4+i)\times 2^{j-1}$ for all integers i and j with $1 \leq j \leq m-1$ and $1 \leq i \leq 4$ and $a_{4m+i} = (4+i)\times 2^{m-1}$ for $1 \leq i \leq r$. So part (v) is proved when $n \geq 12$.

This completes the proof of Theorem 1.1.

Acknowledgement

This work was carried out during a visit by the first author to Brock University as a postdoctoral fellow. He would like to sincerely thank the host institution for its hospitality and for providing an excellent atmosphere for research.

References

[1] P. Bateman, J. Kalb and A. Stenger, 'A limit involving least common multiples', *Amer. Math. Monthly* **109** (2002), 393–394.

[2] F. A. Behrend, 'Generalization of an inequality of Heilbronn and Rohrbach,' *Bull. Amer. Math. Soc. (N.S.)* **54** (1948), 681–684.

[3] D. Borwein, 'A sum of reciprocals of least common multiples', *Canad. Math. Bull.* **20** (1978), 117–118.

[4] P. L. Chebyshev, 'Memoire sur les nombres premiers', *J. Math. Pures Appl. (9)* **17** (1852), 366–390.

[5] P. Erdős and J. Selfridge, 'The product of consecutive integers is never a power', *Illinois J. Math.* **19** (1975), 292–301.

[6] B. Farhi, 'Nontrivial lower bounds for the least common multiple of some finite sequences of integers', *J. Number Theory* **125** (2007), 393–411.

[7] C. J. Goutziers, 'On the least common multiple of a set of integers not exceeding *N*', *Indag. Math. (N.S.)* **42** (1980), 163–169.

[8] D. Hanson, 'On the product of the primes', *Canad. Math. Bull.* **15** (1972), 33–37.

[9] H. A. Heilbronn, 'On an inequality in the elementary theory of numbers', *Math. Proc. Cambridge Philos. Soc.* **33** (1937), 207–209.

[10] S. A. Hong, 'A sharp upper bound for the sum of reciprocals of least common multiples', *Acta Math. Hungar.* **160** (2020), 360–375.
[11] S. F. Hong, Y. Y. Luo, G. Y. Qian and C. L. Wang, 'Uniform lower bound for the least common multiple of a polynomial sequence', *C. R. Math. Acad. Sci. Paris Ser. I* **351** (2013), 781–785.
[12] S. F. Hong, G. Y. Qian and Q. R. Tan, 'The least common multiple of a sequence of products of linear polynomials', *Acta Math. Hungar.* **135** (2012), 160–167.
[13] K. Ireland and M. Rosen, *A Classical Introduction to Modern Number Theory*, Graduate Texts in Mathematics, 84 (Springer-Verlag, New York, 1990).
[14] M. Nair, 'On Chebyshev-type inequalities for primes', *Amer. Math. Monthly* **89** (1982), 126–129.
[15] H. Rohrbach, 'Beweis einer zahlentheoretischen Ungleichung', *J. reine angew. Math.* **177** (1937), 193–196.

SIAO HONG, Department of Mathematics and Statistics,
Brock University, St. Catharines, ON L2S 3A1, Canada
e-mail: sahongnk@163.com

MEILING HUANG, Department of Mathematics and Statistics,
Brock University, St. Catharines, ON L2S 3A1, Canada
e-mail: mhuang@brocku.ca

YUANLIN LI, Department of Mathematics and Statistics,
Brock University, St. Catharines, ON L2S 3A1, Canada
e-mail: yli@brocku.ca