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RESEARCH ARTICLE

The Torelli theorem for ALH^* gravitational instantons

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Abstract

We give a short proof of the Torelli theorem for ALH^* gravitational instantons using the authors' previous construction of mirror special Lagrangian fibrations in del Pezzo surfaces and rational elliptic surfaces together with recent work of Sun-Zhang. In particular, this includes an identification of 10 diffeomorphism types of ALH^*_b gravitational instantons.

Contents

1. Introduction

Gravitational instantons were introduced by Hawking [11] as certain solutions to the classical Einstein equations. They are the building blocks of Euclidean quantum gravity and are analogous to self-dual Yang-Mills instantons arising from Yang-Mills theory. Mathematically, gravitational instantons are noncompact, complete hyperKähler manifolds with L^2-integrable curvature tensor. Depending on the volume growth of the geometry at infinity, there are a few known classes of gravitational instantons discovered first: ALE, ALF, ALG, ALH. Here ALE is the abbreviation for *asymptotically locally Euclidean*, ALF is for *asymptotically locally flat*, and the latter two were simply named by induction. Later, Hein [12] constructed new gravitational instantons with different curvature decay and volume growth on the complement of a fibre in a rational elliptic surface, named ALG^* (corresponding to Kodaira type I^*_b-fibre) and ALH^* (corresponding to Kodaira type I_b-fibre). The first class has the same volume growth as ALG but with different curvature decay, while the latter has a volume growth of $r^{4/3}$.

Gravitational instantons also play an important role in differential geometry, as they arise as the blow-up limits of hyperKähler metrics [14, 6]. Recently, Sun-Zhang [27] made use of the

Cheeger-Fukaya-Gromov theory of $\mathcal{N}$-structures to prove that any nonflat gravitational instanton has a unique asymptotic cone and indeed falls into one of the families in the above list. Thus, it remains to classify the gravitational instantons in each class.

The classification of the gravitational instantons has a long history. By work of Kronheimer [17], ALE gravitational instantons always have the underlying geometry of a minimal resolution of the quotient of $\mathbb{C}^2$ by a finite subgroup of $SU(2)$. Moreover, Kronheimer established a Torelli-type theorem for ALE gravitational instantons. More recently, building on work of Minerbe [21], Chen-Chen [3] studied gravitational instantons with curvature decay $|Rm| \leqslant r^{-2-\varepsilon}$ for some $\varepsilon > 0$. They proved such gravitational instantons must be of the class ALE, ALF, ALG or ALH. Moreover, Chen-Chen proved that up to hyperKähler rotation, ALH (or ALG) gravitational instantons are isomorphic to the complement of a fibre with zero (or finite) monodromy in a rational elliptic surface. Very recently, Chen-Viaclovsky [6] studied the Hodge theory of ALG^*-gravitational instantons, and then Chen-Viaclovsky-Zhang [7] proved the Torelli-type theorem for the ALG, ALG^* gravitational instantons. So the remaining case is the classification of the gravitational instantons of type ALH^*.

Examples of ALH^* gravitational instantons are constructed from del Pezzo surfaces by Tian-Yau [28] and from rational elliptic surfaces by Hein [12]. Hein observed that these two examples have the same curvature decay, injectivity radius and volume growth. The relation between the two examples was made precise by the authors [4, 5] as a by-product of their work on the Strominger-Yau-Zaslow mirror symmetry of log Calabi-Yau surfaces; in particular, it was shown that these two examples are related by a global hyperKähler rotation.

The goal is this paper is to give a short proof of a Torelli theorem for ALH^* gravitational instantons using the earlier results in [4, 5], together with the recent work of Sun-Zhang [27]. Below, we give an informal statement of the main theorem and refer the reader to Theorem 3.9 for a precise version.

Theorem 1.1. *ALH^* gravitational instantons are classified by the cohomology classes of their hyperKähler triple.*

The proof of the above theorem is similar to the Torelli theorem for K3 surfaces, which is a consequence of the results from [26, 2, 20] and the Calabi conjecture [29]. The proof goes as follows; using the exponential decay result of ALH^* gravitational instantons to the Calabi ansatz by Sun-Zhang [27], an earlier argument of the authors from [4] implies that up to hyperKähler rotation, any ALH^* gravitational instanton can be compactified to a rational elliptic surface. The complex structure of such a rational elliptic surface is determined by Gross-Hacking-Keel's [9] Torelli theorem for log Calabi-Yau surfaces. Theorem 3.9 then follows from a local model calculation in combination with the essentially optimal uniqueness theorem for solutions of the complex Monge-Ampère equations established by the authors in [5].

The paper is organised as follows. In Section 2, we review the earlier work of the authors. This includes the construction of special Lagrangian tori via the mean curvature flow in the geometry asymptotic to the Calabi ansatz and the hyperKähler rotation of the Calabi ansatz, as well as a uniqueness theorem for Ricci-flat metrics on the complement of an I_b-fibre in a rational elliptic surface. In Section 3, we first recall the result of Sun-Zhang [27] on ALH^* gravitational instantons and provide a short proof of the Torelli theorem based on the results reviewed in Section 2.

2. Previous results

2.1. Ansatz special Lagrangians and their hyperKähler rotations

We begin by reviewing the Calabi ansatz. Let $D = \mathbb{C}/(\mathbb{Z}\oplus\mathbb{Z}\tau)$ be an elliptic curve with $\mathrm{Im}\tau > 0$ and ω_D a flat metric on D. For a fixed $b \in \mathbb{N}$, let L be a degree b line bundle over D, and denote by $Y_{\mathcal{C}}$ the total space of L with projection $\pi_{\mathcal{C}} : Y_{\mathcal{C}} \to D$. Let $X_{\mathcal{C}}$ be the complement of the zero section in $Y_{\mathcal{C}}$. Choose h to be the unique hermitian metric on L with curvature given by ω_D with $\int_D \omega_D = 2\pi b$. If we let z be the coordinate on D and ξ a local trivialisation of L, we get coordinates on L via $(z, w) \mapsto (z, w\xi)$. The

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Calabi ansatz is then given by

$$\omega_{\mathcal{C}} = \sqrt{-1}\partial\bar{\partial}\frac{2}{3}\big(-\log|\xi|_h^2\big)^{\frac{2}{3}}, \qquad \Omega_{\mathcal{C}} = \frac{dw}{w}\wedge \pi_{\mathcal{C}}^* dz,$$

for a given holomorphic function $f(z)$ such that

$$\frac{i}{2}\int_D \frac{Res_D\Omega_{\mathcal{C}}}{2\pi i}\wedge\overline{\left(\frac{Res_D\Omega_{\mathcal{C}}}{2\pi i}\right)} = 2\pi b.$$

It is straightforward to check that $(\omega_{\mathcal{C}}, \Omega_{\mathcal{C}})$ is a hyperKähler triple: that is, $2\omega_{\mathcal{C}}^2 = \Omega_{\mathcal{C}}\wedge\bar{\Omega}_{\mathcal{C}}$. The induced Riemannian metric is complete but not of bounded geometry. Specifically, if r denotes the distance function to a fixed point, then as one travels towards the zero section, the curvature and injectivity radius have the following behaviour:

$$|Rm|\sim r^{-2} \qquad\qquad inj\sim r^{-\frac{1}{3}}.$$

Let $\underline{L}$ be a special Lagrangian in D with respect to (ω_D, Ω_D), where Ω_D is a holomorphic volume form on D such that $\omega_D = \frac{i}{2}\Omega_D\wedge\bar{\Omega}_D$. A straightforward calculation shows that

$$L_{\mathcal{C}} = \pi_{\mathcal{C}}^{-1}(\underline{L})\cap\{|\xi|_h^2 = \varepsilon\}$$

is a special Lagrangian submanifold of $(X_{\mathcal{C}}, \omega_{\mathcal{C}}, \Omega_{\mathcal{C}})$. We call $L_{\mathcal{C}}$ an ansatz special Lagrangian. In particular, a special Lagrangian fibration in D induces a special Lagrangian fibration in $X_{\mathcal{C}}$, and by direct calculation, the monodromy of such fibration is conjugate to $\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix}$. The middle homology $H_2(X_{\mathcal{C}}, \mathbb{Z})\cong\mathbb{Z}^2$ is generated by $[L_{\mathcal{C}}], [L'_{\mathcal{C}}]$, where $\underline{L}, \underline{L}'$ are any pair such that $[\underline{L}], [\underline{L}']$ generates $H_1(D, \mathbb{Z})$.

Since in complex dimension two, all Ricci-flat Kähler metrics are hyperKähler, one can perform a hyperKähler rotation and arrive at $\check{X}_{mod}$ with a Kähler form $\check{\omega}_{\mathcal{C}}$ and holomorphic volume form $\check{\Omega}_{\mathcal{C}}$ that has the same underlying space as $X_{\mathcal{C}}$. By choosing the hyperKähler rotation appropriately, the special Lagrangian fibration near infinity in $X_{\mathcal{C}}$ becomes an elliptic fibration $\check{X}_{mod}\to\Delta^*$ over a punctured disc Δ^*. The monodromy of the fibration implies that after a choice of section $\sigma:\Delta^*\to\check{X}_{mod}$, the space X_{mod} is biholomorphic to

$$\Delta^*\times\mathbb{C}/\Lambda(u),\ \text{where}\ \Lambda(u) = \mathbb{Z}\oplus\mathbb{Z}\frac{b}{2\pi i}\log u.$$

Here we will use u for the complex coordinate of the disc and v for the fibre. See [5, Appendix A]. There is a natural partial compactification $\check{Y}_{mod}\to\Delta$ by adding an I_b fibre over the origin of Δ.

Before we identify $\check{\omega}_{\mathcal{C}}$ and $\check{\Omega}_{\mathcal{C}}$, we recall the *standard semi-flat metric* on $\check{X}_{mod}$, written down in [10]:

$$\begin{aligned}\omega_{sf,\varepsilon} := {}&\sqrt{-1}|\kappa(u)|^2\frac{k|\log|u||}{2\pi\varepsilon}\frac{du\wedge d\bar{u}}{|u|^2}\\ &+\frac{\sqrt{-1}}{2}\frac{2\pi\varepsilon}{k|\log|u||}(dv+B(u,v)du)\wedge\overline{(dv+B(u,v)du)},\end{aligned}$$

where $B(u,v) = -\frac{\mathrm{Im}(v)}{\sqrt{-1}u|\log|u||}$. A straightforward calculation shows that

1. ε is the size of the fibre with respect to ω_{sf}.
2. ω_{sf} is flat along the fibres.
3. $(\omega_{sf}, \Omega_{sf})$ form a hyperKähler triple, where $\Omega_{sf} = \frac{\kappa(u)}{u}dv\wedge du$ is the unique volume form such that $\int_C\Omega_{sf} = 1$, and C is the 2-cycle represented by $\{|u| = const, \mathrm{Im}(v) = 0\}$.

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The cycle C is called a 'bad cycle' by Hein [12],[1] and this notion is refined by the authors in [5]. It is easy to see that $H_2(\check{X}_{mod}, \mathbb{Z})$ is freely generated by the fibre class and C; we therefore define

Definition 2.1. A cycle $C' \subset X_{mod}$ is called a quasi-bad cycle if the homology class $[C'] \in H_2(\check{X}_{mod}, \mathbb{Z})$ can be written as $m[C] + [F]$, where $[F]$ is the fibre class.

It was observed by Hein that the semi-flat metric has the same asymptotic behaviour for $|Rm|$ and inj as the Calabi ansatz [12]. This motivates the natural guess that the hyperKähler rotation of the Calabi ansatz would give the semi-flat metric. However, there is a certain subtle discrepancy, and one must first introduce a class of nonstandard semi-flat metrics, as defined in [5]. For any $b_0 \in \mathbb{R}$, we define the *nonstandard semi-flat metric* as

$$\begin{aligned}\omega_{sf,b_0,\varepsilon} &:= \sqrt{-1}\frac{|\kappa(u)|^2}{\varepsilon}W^{-1}\frac{du \wedge d\bar{u}}{|u|^2} \\ &\quad + \frac{\sqrt{-1}}{2}W\varepsilon\Big(dv + \widetilde{\Gamma}(v,u,b_0)du\Big) \wedge \overline{\Big(dv + \widetilde{\Gamma}(u,v,b_0)du\Big)},\end{aligned}$$

where $W = \frac{2\pi}{k|\log|u||}$ and $\widetilde{\Gamma}(u,v,b_0) = B(u,v) + \frac{b_0}{2\pi^2}\frac{|\log|u||}{u}$. An appealing way to think of nonstandard semi-flat metrics is that they are obtained from standard semi-flat metrics by pulling back along the fibrewise translation map defined by a multi-valued (possibly uncountably valued) section $\sigma : \Delta^* \to \check{X}_{mod}$; see [5]. A nonstandard semi-flat metric has the same curvature and injectivity radius decay as a standard semi-flat metric. However, if $\frac{2b_0}{b} \notin \mathbb{Z}$, then the cohomology class of the nonstandard semi-flat metric cannot be realised by a standard semi-flat metric. We now state the following result.

Theorem 2.2 [5, Appendix A]. *Assume that $D \cong \mathbb{C}/\mathbb{Z} \oplus \mathbb{Z}\tau$ is an elliptic curve, with τ in the upper half-plane. Let $Y_{\mathcal{C}}$ be the total space of a degree b line bundle L over D and $X_{\mathcal{C}}$ the complement of the zero section. Let $\omega_{\mathcal{C}}$ and $\Omega_{\mathcal{C}}$ be the forms arising from the Calabi ansatz on $X_{\mathcal{C}}$, as above. Consider the hyperKähler rotation of $X_{\mathcal{C}}$ with Kähler form $\check{\omega}_{\mathcal{C}}$ and holomorphic volume form $\check{\Omega}_{\mathcal{C}}$ such that the ansatz special Lagrangian corresponding to $1 \in \mathbb{Z} \oplus \mathbb{Z}\tau$ is of phase zero. Then with a suitable choice of coordinates, one has*

$$\check{\omega}_{\mathcal{C}} = \alpha\omega_{sf,b_0,\varepsilon}, \qquad \check{\Omega}_{\mathcal{C}} = \alpha\Omega_{sf},$$

where $b_0 = -\frac{1}{2}Re(\tau)b$, $\varepsilon = \frac{2\sqrt{2}\pi}{Im(\tau)}$ and $\alpha = \sqrt{b\pi Im(\tau)}$. In particular, there exists a bijection between $\tau \leftrightarrow (b_0, \varepsilon)$: that is, every (possibly nonstandard) semi-flat metric can be realised as some hyperKähler rotation of certain Calabi ansatz up to a scaling.

As a direct consequence, we get the following special Lagrangian fibrations in $\check{X}_{mod}$ via hyperKähler rotations from the Calabi ansatz:

Lemma 2.3. *Fix an m-quasi bad cycle class $[L] \in H_2(\check{X}_{mod}, \mathbb{Z})$ that is primitive. There exists a special Lagrangian fibration in $\check{X}_{mod}$ with respect to the semi-flat hyperKähler triple $(\omega_{sf,b_0,\varepsilon}, \Omega_{sf})$ if and only if $\int_{[L]} \omega_{sf,b_0,\varepsilon} = 0$.*

2.2. *A uniqueness theorem for Ricci-flat metrics on noncompact Calabi-Yau surfaces*

Recall that a rational elliptic surface is a rational surface with an elliptic fibration structure. Using the standard semi-flat metric as an asymptotic model, Hein [12] constructed many Ricci-flat metrics on the complement of a fibre in a rational elliptic surface. In the case that the removed fibre is of Kodaira type I_b, the authors established the uniqueness of these metrics as well as the existence of a parameter space. We recall the setup here.

[1]It is worth noting that the definition of the bad cycle actually implicitly depends on a choice of a section $\sigma : \Delta^* \to \check{X}_{mod}$. We refer the reader to [5] for more details on (quasi)-bad cycles.

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Let $\check{Y}$ be a rational elliptic surface and $\check{D}$ an I_b-fibre. Fix a meromorphic form $\check{\Omega}$ with a simple pole along $\check{D}$. Denote $\check{X} = \check{Y} \setminus \check{D}$, and let $\mathcal{K}_{dR,\check{X}}$ be the set of de Rham cohomology classes that can be represented by Kähler forms on $\check{X}$. Then $\mathcal{K}_{dR,\check{X}}$ is a cone in $H^2(\check{X}, \mathbb{R})$. With a slight modification of the work of Hein [12], the authors generalised the existence theorem:

Theorem 2.4 [5, Theorem 2.16]. *Given any $[\check{\omega}] \in \mathcal{K}_{dR,\check{X}}$, there exists α_0 such that for $\alpha > \alpha_0$, there exists a Ricci-flat metric $\check{\omega} \in [\check{\omega}]$ on $\check{X}$ with a suitable choice of section and a semi-flat metric $\omega_{sf,b_0,\varepsilon}$ such that*

1. *$\check{\omega}^2 = \alpha\check{\Omega} \wedge \bar{\check{\Omega}}$: that is, $\check{\omega}$ solves the Monge-Ampére equation.*
2. *The curvature $\check{\omega}$ satisfies $|\nabla^k Rm|_{\check{\omega}} \lesssim r^{-2-k}$ for every $k \in \mathbb{N}$.*
3. *$\check{\omega}$ is asymptotic to the semi-flat metric in the following sense: there exists $C > 0$ such that for every $k \in \mathbb{N}$, one has*

$$|\nabla^k(\check{\omega} - \omega_{sf,b_0,\varepsilon})|_{\check{\omega}} \sim O(e^{-Cr^{2/3}}).$$

We refer the reader to [5, Remark 2.17] for a description of some (minor) differences between Theorem 2.4 and the work of Hein. The authors then proved an essentially optimal uniqueness theorem for Ricci-flat metrics with polynomial decay to a (possibly nonstandard) semiflat metrics on $\check{X}$.

Theorem 2.5 [5, Proposition 4.8]. *Suppose $\check{\omega}_1, \check{\omega}_2$ are two complete Calabi-Yau metrics on $\check{X} = \check{Y} \setminus \check{D}$ with the following properties:*

(i) *$\check{\omega}_i^2 = \alpha^2\check{\Omega} \wedge \overline{\check{\Omega}}$, for $i = 1, 2$.*
(ii) *$[\check{\omega}_1]_{dR} = [\check{\omega}_2]_{dR} \in H^2_{dR}(\check{X}, \mathbb{R})$.*
(iii) *There are (possibly nonstandard) semi-flat metrics $\omega_{sf,\sigma_i,b_{0,i},\varepsilon_i}$ such that*

$$[\omega_{sf,\sigma_i,b_{0,i},\varepsilon_i}]_{BC} = [\check{\omega}_i]_{BC} \in H^{1,1}_{BC}(\check{X}_{\Delta^*}, \mathbb{R})$$

and

$$|\check{\omega}_i - \alpha\omega_{sf,\sigma_i,b_{0,i},\frac{\varepsilon_i}{\alpha}}| \leqslant Cr_i^{-4/3},$$

where r_i is the distance from a fixed point with respect to $\check{\omega}_i$.

Then there is a fibre preserving holomorphic map $\Phi \in \mathrm{Aut}_0(X, \mathbb{C})$ such that $\Phi^*\check{\omega}_2 = \check{\omega}_1$.

2.3. Perturbations of the model special Lagrangians

Let (X, ω) be a Kähler manifold such that the corresponding Riemannian metric is Ricci-flat. Given a Lagrangian submanifold $L \subseteq X$, we can deform L via its mean curvature $\vec{H}$, defining a family of Lagrangians L_t such that

$$\frac{\partial}{\partial t} L_t = \vec{H}.$$

It is proved by Smoczyk [25] that the Maslov zero Lagrangian condition is preserved under the flow; thus the name Lagrangian mean curvature flow (LMCF). If X admits a covariant holomorphic volume form Ω, then there exists a phase function $\theta : L \to S^1$ defined by $\Omega|_L = e^{i\theta}\mathrm{Vol}_L$. If θ is constant, then L is a special Lagrangian. Since we are working on a Calabi-Yau manifold, the mean curvature of L can be computed by $\vec{H} = \nabla\theta$. In particular, if the LMCF converges smoothly, it converges to a special Lagrangian.

Now, in general, the LMCF may develop a finite time singularity [23], which is expected to be related to the Harder-Narasimhan filtration of the Fukaya category [16]. However, using a quantitative version

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of the machinery of Li [18], the authors proved a quantitative local regularity theorem for the LMCF in the present setting; see [4, Theorem 4.23].

Theorem 2.6 (Theorem 4.23, [4]). *Let X be a noncompact Calabi-Yau surface with Ricci-flat metric ω and holomorphic volume form Ω. Fix a point in X, and let r denote the distance function to this fixed point. Assume that there exists a diffeomorphism F from the end of $\mathcal{C}$ to X such that for all $k \in \mathbb{N}$, one has*

$$\| \nabla^k_{g_{\mathcal{C}}} (F^*\omega - \omega_{\mathcal{C}}) \|_{g_{\mathcal{C}}} < C_k e^{-\delta r^{\frac{2}{3}}}, \| \nabla^k_{g_{\mathcal{C}}} (F^*\Omega - \Omega_{\mathcal{C}}) \|_{g_{\mathcal{C}}} < C_k e^{-\delta r^{\frac{2}{3}}},$$

for some constant $C_k > 0$. Then given an ansatz special Lagrangian (from Section 2.1) mapped to X via F, if it is sufficiently close to infinity along the end of X, it can be deformed to a genuine special Lagrangian with respect to (ω, Ω).

Specifically, in [4], the authors argue that an ansatz special Lagrangian can be deformed via Moser's trick to a Lagrangian with respect to the Ricci flat metric ω. After proving this deformation preserves several geometric bounds (including exponential decay of the mean curvature vector along the end of X), the authors show that the mean curvature flow converges exponentially fast to a special Lagrangian. We direct the reader to [4] for further details.

3. The Torelli theorem

First we recall the definition of ALH^* gravitational instantons following Sun-Zhang [27]:

Definition 3.1.

1. Given $b \in \mathbb{N}$, an ALH^*_b model end is the hyperKähler triple from the Gibbons-Hawking ansatz on $\mathbb{T}^2 \times [0, \infty)$ with the harmonic function $b\rho$, where T^2 is the flat two-torus and ρ is the coordinate on $[0, \infty)$
2. A gravitational instanton (X, g) is of type ALH^* if there exists a diffeomorphism F from $\mathcal{C}$ to X such that for all $k \in \mathbb{N}$, one has

$$\| \nabla^k_{g_{\mathcal{C}}} (F^*g - g_{\mathcal{C}}) \|_g = O(r^{-k-\varepsilon})$$

for some $\varepsilon > 0$.

Remark 3.2. It is explained in [14, Section 2.2] that the Calabi ansatz is actually an ALH^*_b model end for some b.

Let (X, g) be an ALH^*_b gravitational instanton, and fix a choice of hyperKähler triple (ω, Ω) such that ω is the Kähler form with respect to the complex structure determined by Ω. Sun-Zhang proved that the geometry at infinity has exponential decay to the model end.

Theorem 3.3 [27, Theorem 6.19]. *There exist $\delta > 0$ and a diffeomorphism F from the end of $\mathcal{C}$ to X such that for all $k \in \mathbb{N}$, one has $F^*\omega = \omega_{\mathcal{C}} + d\sigma$ for some 1-form σ and*

$$\| \nabla^k_{g_{\mathcal{C}}} (F^*\omega - \omega_{\mathcal{C}}) \|_{g_{\mathcal{C}}} < C_k e^{-\delta r^{\frac{2}{3}}}, \| \nabla^k_{g_{\mathcal{C}}} (F^*\Omega - \Omega_{\mathcal{C}}) \|_{g_{\mathcal{C}}} < C_k e^{-\delta r^{\frac{2}{3}}},$$

for some constant $C_k > 0$.

Consider $L_{\mathcal{C}} \in X_{\mathcal{C}}$ for any primitive class $[L_{\mathcal{C}}] \in H_2(X_{\mathcal{C}}, \mathbb{Z})$ with ε small enough. Then as above, one can use Moser's trick to modify $F(L_{\mathcal{C}})$ to a Lagrangian $L \subseteq X$. The LMCF starting at L will then converge smoothly to a special Lagrangian tori by Theorem 3.3 and Theorem 2.6. Notice that from [4, Proposition 5.24], the LMCF flows the ansatz special Lagrangian fibration near infinity to a genuine special Lagrangian fibration on $X \setminus K$ for some compact set K.

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Now consider the hyperKähler rotation $\check{X}$ equipped with Kähler form $\check{\omega}$ and holomorphic volume form $\check{\Omega}$ such that

$$\check{\omega} = \mathrm{Re}\Omega, \qquad \check{\Omega} = \omega - i\mathrm{Im}\Omega. \tag{3.1}$$

Then $\check{X} \setminus K$ admits an elliptic fibration to a noncompact Riemann surface $\check{B}$, which is diffeomorphic to an annuli. From the uniformisation theorem, $\check{B}$ is either biholomorphic to a punctured disc or a holomorphic annulus. Notice that the j-invariants of the elliptic fibres converge to infinity at the end from Theorem 2.2 and Theorem 3.3. Since the j-invariant is a holomorphic function on $\check{B}$, one has $\check{B}$ must be biholomorphic to a punctured disc.

Again from [4, Proposition 5.24], the monodromy of the fibration $\check{X} \setminus K \to \check{B}$ near infinity is the same as the explicit model special Lagrangian fibration. There are two consequences. Firstly, there is no sequence of multiple fibres converging to infinity. Secondly, the monodromy is conjugate to $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ from direct calculation. Then one can compactify $\check{X}$ to a compact complex surface $\check{X}$ by adding an I_b-fibre $\check{D}$ at infinity by [4, Corollary 6.3]. Now we can use to the classification of surfaces to deduce the following:

Proposition 3.4. *$\check{Y}$ is a rational elliptic surface.* [2]

Proof. From Appendix [5, Appendix A], the form $\check{\Omega}$ is meromorphic with a simple pole along $\check{D}$. Therefore, we have $K_{\check{Y}} = \mathcal{O}_{\check{Y}}(-\check{D})$. From the elliptic fibration on $\check{Y} \setminus K$, we have $c_1(\check{Y})^2 = 0$. There are no (-1) curves in the fibre by the adjunction formula. Since $b_1(\check{X}) = 0$ by [27, Corollary 7.6],[3] we also have $b_1(Y) = 0$ from the Mayer-Vietoris sequence. Assume that $\check{Y}$ is minimal. Since $c_1(\check{Y})^2 = 0$ and $b_1(\check{Y}) = 0$, by the Enriques-Kodaira classification (see, for example, [1, Chapter VI, Table 10]), it follows that $\check{Y}$ can only be an Enriques surface, a K3 surface or a minimal properly elliptic surface. $K_{\check{Y}} = \mathcal{O}_{\check{Y}}(-\check{D})$ obviously excludes the first two possibilities. Furthermore, recall that a properly elliptic surface has Kodaira dimension 1. This is again impossible because $K_{\check{Y}} = \mathcal{O}_{\check{Y}}(-\check{D})$. To sum up, it must be the case that $\check{Y}$ is not minimal.

Now, any (-1) curve E in $\check{Y}$ has intersection one with $\check{D}$, so $(\check{D}+E)^2 > 0$. Therefore, $\check{Y}$ is projective by [1, Chapter IV, Theorem 5.2]. Then $h^1(\check{Y}, \mathcal{O}_{\check{Y}}) = 0$ from Hodge theory and $h^0(\check{Y}, K^2_{\check{Y}}) = 0$ since $-K_{\check{Y}}$ is effective. Finally, Castelnuovo's rationality criterion implies that $\check{Y}$ is rational. Thus the local elliptic fibration near $\check{D}$ in $\check{Y}$ actually extends to an elliptic fibration. Indeed, one has $\mathrm{Pic}(\check{Y}) \cong H^2(\check{Y}, \mathbb{Z})$ since $H^1(\check{Y}, \mathcal{O}_{\check{Y}}) = H^2(\check{Y}, \mathcal{O}_{\check{Y}}) = 0$. Thus, $\check{Y}$ is a rational elliptic surface. □

To sum up, we proved the following uniformisation theorem:

Theorem 3.5. *Any ALH^*_b gravitational instanton (up to hyperKähler rotation) can be compactified to a rational elliptic surface.*

Remark 3.6. An analogue result of Hein-Sun-Viaclovsky-Zhang [15] proves that up to hyperKähler rotation, any ALH^*_b gravitational instanton can be compactified to a weak del Pezzo surface.

The possible singular fibres of a rational elliptic surface are classified by Persson [24]. The rational elliptic surface $\check{Y}$ can only admit an I_b-fibre for $b \leqslant 9$, which gives a constraint on b. From the work of Persson [24] (see also [13, Section 3.3], or Proposition 9.15, Proposition 9.16 of [8]), there exists a single deformation family of pairs of rational elliptic surfaces with an I_b fibre for $b \neq 8$, and there are two deformation families for $b = 8$. Different families have different Betti numbers. In particular, there exist ALH^*_b gravitational instantons for every $1 \leqslant b \leqslant 9$ from the work of Hein [12]. Thus, we have the following consequence:

Corollary 3.7.

1. *There are only ALH^*_b gravitational instantons for $b \leqslant 9$.*
2. *There are only 10 diffeomorphism types of ALH^*_b gravitation instantons.*

[2] This is a slight modification of [4, Theorem 1.6] taking advantage of Theorem 2.2.
[3] One may also see that from [15, Theorem 1.1].

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Before we prove the Torelli theorem of ALH_b^*-gravitational instantons, we first recall the Torelli theorem of log Calabi-Yau surfaces [9]. Let (Y, D) be a Looijenga pair: that is, Y is a rational surface, and $D \in |-K_Y|$ is an anti-canonical cycle. Consider the homology long exact sequence of pairs (Y, D) with coefficients in $\mathbb{Z}$:

$$0 = H_3(Y) \to H_3(Y, Y \setminus D) \xrightarrow{\partial_*} H_2(Y \setminus D) \xrightarrow{\iota} H_2(Y) \to H_2(Y, Y \setminus D). \tag{3.2}$$

Here we identify $H_k(Y, Y \setminus D)$ with $H^{4-k}(D)$ by Poincare duality. Let $\varepsilon \in H^1(D)$ denote a generator, which determines its orientation. There exists a unique meromorphic volume form Ω_Y with a simple pole along D and normalisation $\int_{\partial_*(\varepsilon)} \Omega_Y = 1$. Denote by C_Y^{++} the subcone of $\mathrm{Pic}(Y)$ that consists of element β satisfying

1. $\beta^2 > 0$: that is, β is in the positive cone.
2. $\beta.[E] \geqslant 0$ for any (-1)-curve E in Y.

By [9, Lemma 2.13], C_Y^{++} is invariant under parallel transport. We denote by Δ_Y the set of nodal classes of Y: that is,

$$\Delta_Y = \{\alpha \in \mathrm{Pic}(Y) | \alpha \text{ can be represented by a } (-2)\text{-curve in } Y \setminus D\}.$$

For each element $\alpha \in \Delta_Y$, there is an associate reflection as an automorphism on $\mathrm{Pic}(Y)$ given by

$$s_\alpha : \beta \mapsto \beta + \langle \alpha, \beta \rangle.$$

The Weyl group W_Y is then the group generated by $s_\alpha, \alpha \in \Delta_Y$.

With the above notations, the Gross-Hacking-Keel weak Torelli theorem for Looijenga pairs is stated as follows:

Theorem 3.8 (Theorem 1.8, [9]). *Let $(Y_1, D), (Y_2, D)$ be two Looijenga pairs and $\mu : Pic(Y_1) \to Pic(Y_2)$ be an isomorphism of lattices. Assume that*

1. $\mu([D_i]) = ([D_i])$ *for all i.*
2. $\mu(C_{Y_1}^{++}) = C_{Y_2}^{++}$.
3. $\mu([\Omega_{Y_1}]) = [\Omega_{Y_2}]$, *where Ω_i is the meromorphic form on Y_i with a simple pole along D_i and the normalisation described above.* [4]

Then there exists a unique $g \in W_{Y_1}$ such that $\mu \circ g = f^*$ for an isomorphism of pairs $f : (Y_2, D) \to (Y_1, D)$.

We are now ready to prove our Torelli theorem.

Theorem 3.9. *Let $(X_i, \omega_i, \Omega_i)$ be ALH_b^* gravitational instantons such that there exists a diffeomorphism $F : X_2 \cong X_1$ with*

$$F^*[\omega_1] = [\omega_2] \in H^2(X_2, \mathbb{R}), F^*[\Omega_1] = [\Omega_2] \in H^2(X_2, \mathbb{C}).$$

Then there exists a diffeomorphism $f : X_2 \to X_1$ such that $f^*\omega_1 = \omega_2$ and $f^*\Omega_1 = \Omega_2$.

Proof. Assume that $(\check{Y}_2, \check{D}_2)$ are the pair of a rational elliptic surface and an I_b fibre such that $\check{X}_2 = \check{Y}_2 \setminus \check{D}_2$ is a hyperKähler rotation of $(X_2, \omega_2, \Omega_2)$ with elliptic fibration and fibre class $[L] \in H_2(X_2, \mathbb{Z})$. Thanks to the assumption $F^*[\Omega_1] = [\Omega_2]$, there exists a special Lagrangian fibration on $(X_1, \omega_1, \Omega_1)$ with fibre class $F_*[L]$. Let $(\check{Y}_1, \check{D}_1)$ be the pair of rational elliptic surface and I_b fibre such that $\check{X}_1 = \check{Y}_1 \setminus \check{D}_1$ is a hyperKähler rotation of X_1 with elliptic fibration with fibre class $F_*[L]$. Denote $(\check{\omega}_i, \check{\Omega}_i)$ for the hyperKähler triple on $\check{X}_i$. From Theorem 2.2 and Theorem 3.3, the resulting holomorphic volume form $\check{\Omega}_i$ on $\check{X}_i$ is meromorphic on $\check{Y}_i$ and has a simple pole along $\check{D}_i$.

[4] Here we use a different period interpretation, which is stronger. See [8, Proposition 3.12].

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We will first use the weak Torelli theorem of Looijenga pairs (Theorem 3.8) to show that there exists a biholomorphism $\check{Y}_2 \to \check{Y}_1$ such that the induced map on $H^2(\check{X}_2, \mathbb{Z})$ is the same as F^*. To achieve that, we will construct an isomorphism of lattices $\tilde{F}^* : H^2(\check{Y}_1, \mathbb{Z}) \to H^2(\check{Y}_2, \mathbb{Z})$ from the diffeomorphism F such that $\tilde{F}^*([\check{D}_{1,i}]) = [\check{D}_{2,i}]$.

Lemma 3.10. *There exists a diffeomorphism $F' : X_2 \to X_1$ such that*

1. *F' is homotopic to F and*
2. *if $C \subseteq \check{Y}_2$ is a 2-cycle that is a local section of the fibration $\check{Y}_2 \to \mathbb{P}^1$ near infinity and intersects $\check{D}_{i,2}$ transversally for some i, then the closure of $F'(C \cap X_2)$ intersects $\check{D}_{i,1}$ transversally and is again a local section of $\check{Y}_1 \to \mathbb{P}^1$ near infinity.*

Proof. There exist compact sets $K_i \subset X_i$ such that $g_i : X_i \backslash K_i \cong X_{\mathcal{C}}$. Recall that F sends a neighbourhood of infinity of X_2 to a neighbourhood of infinity of X_1; and for each $i = 1, 2$, there exists a special Lagrangian fibration on $X_i \setminus K_i \to \Delta^*$, where Δ^* is the punctured disc. We may choose K_1, K_2 such that $F(X_2 \setminus K_2) \subseteq X_1 \setminus K_1$ and ∂K_i is the preimage of a loop in Δ^* under the special Lagrangian fibration; that is, there exists $\nu_i : \partial K_i \to S^1$. Since both $\partial K_1, F(\partial K_2)$ are the boundary of a neighbourhood of infinity of X_1 and $X_1 \setminus K_1 \cong X_{\mathcal{C}} \cong \check{X}_{mod} \cong \partial K_1 \times (0, 1)$, there exists a vector field on $X_1 \setminus K_1$ such that the induced flow takes ∂K_1 to $F(\partial K_2)$. We will denote such a diffeomorphism by $\nu : \partial K_1 \cong F(\partial K_2)$.

Since S^1 is the Eilenberg-MacLane space $K(\mathbb{Z}, 1)$, we have $[\partial K_1, S^1] = H^1(\partial K_1, \mathbb{Z}) \cong \mathbb{Z}^2$. Restricting the model special Lagrangian fibrations in $g_1^{-1}(X_{\mathcal{C}})$ and possibly composing with multiple cover $S^1 \to S^1$ gives $\mathbb{Z}^2$ nonhomotopic maps from ∂K_1 to S^1. Notice that they all have different fibre homology classes. Therefore, two maps from ∂K_1 to S^1 are homotopic if and only if the corresponding fibre classes are homologous. Therefore, we have $\nu_1 \sim \nu_2 \circ F^{-1} \circ \nu$, and we can modify ν such that ν sends fibres of ν_1 to fibres of $\nu_2 \circ F^{-1}$, which are 2-tori. Let T^2 be a fibre of $\nu_1(\partial K_1)$; then $\varphi = \nu_2 \circ F^{-1} \circ \nu \circ \nu_1^{-1}$ induces an element in the mapping class group $MCG(T^2) \cong SL(2, \mathbb{Z})$. The monodromy M of $\partial K \to S^1$ is conjugate to $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ and commutes with φ. Thus, φ is also of the form $\pm\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}$ for some $m \in \mathbb{Z}$. Therefore, we may modify F such that fibrewise, it is given by φ on $X_2 \setminus K_2'$ for large enough compact set K_2'. In terms of the coordinates in Section 2.1, F' (after the identification $X_{\mathcal{C}} \cong \check{X}_{mod}$) is given by

$$u \mapsto u, v \mapsto \pm v + m\frac{\operatorname{Im}(v)}{\operatorname{Im}(\tau(u))}. \tag{3.3}$$

Now every continuous section of $\check{X}_{mod}$ that extends to $\check{Y}_{mod}$ is of the form

$$h(u) + \frac{a}{2\pi i}\log u,$$

where $h(u)$ is a continuous function over Δ and $a \in \mathbb{Z}$. A straightforward calculation shows that equation (3.3) maps sections of $\check{Y}_{mod}$ to sections $\check{Y}_{mod}$; this finishes the proof of the lemma. □

From now on, we will replace F by F' in Lemma 3.10 and still denote it by F. Recall that the second homology group of a rational elliptic surface is generated by the components of fibres and sections. The lemma implies that there exists a map $\tilde{F}^* : H^2(\check{Y}_1, \mathbb{Z}) \to H^2(\check{Y}_2, \mathbb{Z})$ such that the following diagram commutes

$$\begin{tikzcd} H^2(\check{Y}_1, \mathbb{Z}) \arrow[r, "\tilde{F}^*"] \arrow[d] & H^2(\check{Y}_2, \mathbb{Z}) \arrow[d] \\ H^2(X_1, \mathbb{Z}) \arrow[r, "F^*"] & H^2(X_2, \mathbb{Z}) \end{tikzcd}$$

and the intersection pairing is preserved. Here the vertical maps are the natural ones induced from the restriction. From Poincare duality, $\tilde{F}^*$ must be an isometry of lattices.

From [9, Construction 5.7], there exists a universal family $(\mathcal{Y}, \mathcal{D})$ over $\mathrm{Hom}(\mathrm{Pic}(\check{Y}_1), \mathbb{C}^*)$ such that $(\check{Y}_1, \check{D}_1) = (\mathcal{Y}_1, \mathcal{D}_1)$ is the reference fibre and there exists an isomorphism of pairs $\rho : (\check{Y}_2, \check{D}_2) \cong (\mathcal{Y}_2, \mathcal{D}_2)$ with some fibre $(\mathcal{Y}_2, \mathcal{D}_2)$. Now $\tilde{F}^*$ can be decomposed as

$$\tilde{F}^* : H^2(\check{Y}_1, \mathbb{Z}) = H^2(\mathcal{Y}_1, \mathbb{Z}) \overset{\mathrm{Par}}{\cong} H^2(\mathcal{Y}_2, \mathbb{Z}) \overset{\rho^*}{\cong} H^2(\check{Y}_2, \mathbb{Z}),$$

where Par denotes a choice of the parallel transport via the universal family. Since $\rho : \check{Y}_2 \cong \mathcal{Y}_2$ is a biholomorphism, it preserves the set of exceptional curves and positive cones. Together with the fact that C^{++} is preserved under the parallel transport, we have $\tilde{F}^*(C^{++}_{\check{Y}_1}) = C^{++}_{\check{Y}_2}$. Now, from Theorem 3.8, there exists an isomorphism of pairs $h : (\check{Y}_2, \check{D}) \to (\check{Y}_1, \check{D})$ such that $\tilde{F}^* \circ g = h^*$ for some $g \in W_{\check{Y}_1}$.

Next, we will show that g is the identity. From [9, Theorem 3.2], the hyperplanes $\alpha^\perp, \alpha \in W_{\check{Y}_1} \cdot \Delta_{\check{Y}_1}$ divide $C^{++}_{\check{D}}$ into chambers, and the Weyl group $W_{\check{Y}_1}$ simply acts transitively on the chambers. Moreover, there exists a unique chamber containing the nef cone and thus the ample cone. Chambers divided by $\alpha^\perp$ in $H^2(\check{Y}_1)$ have disjoint image under the restriction map $\iota^* : H^2(\check{Y}_1) \to H^2(\check{X}_1)$. Indeed, if $\delta_1, \delta_2 \in H^2(\check{Y}_1)$ and $\iota^*\delta_1 = \iota^*\delta_2$, then from the dual of the long exact sequence of equation (3.2), we have

$$\delta_2 = \delta_1 + \sum_i a_i [D_i].$$

Thus δ_1, δ_2 fall in the same chamber because $\alpha \cdot [D_i] = 0$ for all $\alpha \in W_{\check{Y}_1} \cdot \Delta_{\check{Y}_1}$. Again from the long exact sequence in equation (3.2), the image of ι^* is a hyperplane in $H^2(\check{X}_1)$. For each $\alpha \in \Delta_{\check{Y}_1}$, there is a corresponding (-2)-curve C_α of $\check{Y}_1$ that completely falls in $\check{X}_1$. Given a compact 2-cycle C of $\check{X}$, we can associate a hyperplane $[C]^{\perp_{\check{X}_1}}$ of $H^2(\check{X}_1)$ given by

$$[C]^{\perp_{\check{X}_1}} = \{[\omega] \in H^2(\check{X}_1) | \int_C [\omega] = 0\}.$$

Then $\iota^*(\alpha^\perp)$ is the intersection of the hyperplanes $[C_\alpha]^{\perp_{\check{X}_1}}$ and $[\partial_*(\varepsilon)]^{\perp_{\check{X}_1}}$. Again, the hyperplanes $[C_\alpha]^{\perp_{\check{X}_1}}, \alpha \in W_{\check{Y}_1} \cdot \Delta_{\check{Y}_1}$ divide $H^2(\check{X}_1)$ into chambers. There exists a unique one that contains the image of the Kähler cone of $\check{Y}_1$, which consists of 2-forms integrating positively on C_α for all $\alpha \in \Delta_{\check{Y}_1}$. Since F^* sends $[\omega_1]$ to $[\omega_2]$ and h^* preserves the Kähler classes of $\check{Y}_1$, one must have that g is the identity and $\tilde{F}^* = h^*$.

When restricting to $\check{X}_1$, we have $h^* = \tilde{F}^* = F^*$. Since $F^*[\check{\Omega}_1] = [\check{\Omega}_2]$ from the assumption and $h^*\check{\Omega}_1 = c\check{\Omega}_2$ for some constant $c \in \mathbb{C}^*$, we then have $h^*\check{\Omega}_1 = \check{\Omega}_2$. From Theorem 2.2 and Theorem 3.3, the resulting Kähler form $\check{\omega}_i$ is exponentially decaying to a possibly nonstandard semi-flat metric. Then Theorem 2.5 implies that $T^*_\sigma h^* \check{\omega}_1 = \check{\omega}_2$, where T_σ is a translation by a global section of $\check{X}_2$, which doesn't alter the $(2, 0)$-forms. We may take $f = h \circ T_\sigma$, and this finishes the proof of the Torelli theorem. □

Remark 3.11. Ongoing work by Mazzeo and Zhu [22] studies the Fredholm mapping properties of the Laplace operator on ALH* space with applications to Hodge theory and perturbation theory.

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