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Zariski chambers on surfaces of high Picard number

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Abstract

We present an improved algorithm for the computation of Zariski chambers on algebraic surfaces. The new algorithm significantly outperforms the currently available method and therefore allows us to treat surfaces of high Picard number, where huge numbers of chambers occur. As an application, we efficiently compute the number of chambers supported by the lines on the Segre–Schur quartic.

Introduction

Zariski chambers are natural pieces into which the big cone of an algebraic surface decomposes. Their properties were first studied in [**4**]. An intriguing problem, raised in [**3**], is to determine explicitly how many Zariski chambers a given surface has. In other words, on a smooth projective surface X we want to find the quantity

$$z(X) =_{\mathrm{def}} \#\{\text{Zariski chambers on } X\} \in \mathbb{N} \cup \{\infty\}.$$

Roughly speaking, the number $z(X)$ measures how complicated the surface X is from the point of view of linear series. Specifically, it answers the following natural questions (see [**3**]).

- How many different stable base loci can occur in big linear series on X?
- How many essentially different Zariski decompositions can big divisors on X have? (Here we consider Zariski decompositions to be essentially different if their negative parts have different support.)
- How many 'pieces' does the (piecewise polynomial) volume function on the big cone of X have?

To get a more detailed picture of the geometry of X, it is also natural to consider, for integers $d \geqslant 1$, the numbers

$$z_d(X) =_{\mathrm{def}} \#\left\{\begin{matrix}\text{Zariski chambers on } X \text{ that are} \\ \text{supported by curves of degree } d \text{ or less}\end{matrix}\right\},$$

where the degree of curves is taken with respect to a fixed ample line bundle on X. One has, of course, $z(X) = \sup_d z_d(X)$. While there are surfaces X for which $z(X)$ is infinite, the numbers $z_d(X)$ are always finite, because on any smooth surface there are only finitely many irreducible negative curves of fixed degree.

In [**3**] an algorithm was presented that computes Zariski chambers from the intersection matrix of a set of negative curves, and the algorithm was applied to Del Pezzo surfaces. While this method is very efficient in those cases, further experience has shown that there do exist surfaces where the algorithm takes an inordinate amount of time, to the point of becoming impractical in such situations. This is, for instance, the case when the method is being applied to the 64 lines on the Segre–Schur quartic (see Section 2). At first glance, this phenomenon may seem somewhat surprising, as there are surfaces with many more curves to be considered

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(such as the Del Pezzo surface X_8 with its 240 negative curves) where an application of the algorithm poses no practical problems at all. It seems that it is not so much the number of negative curves that matters most but, rather, the Picard number of the surface and the number of chambers that are found. This is in accordance with the observation that the essential work done by the algorithm (and, in fact, its essential bottle-neck) is the computation of an enormous number of determinants, and the dimension of the matrices in question is bounded in terms of the Picard number of the surface. (The dimension of the determinants to be computed is bounded by $\rho(X) - 1$.)

In the present paper, we attack this problem by providing a significantly improved algorithm that is suitable also for surfaces with higher Picard number. Our focus here is on the efficient calculation of the relevant determinants. As is well known, the usual fraction-free algorithms for computing the determinant of an $n \times n$ integer matrix, such as the fraction-free Bareiss algorithm, have complexity $O(n^3)$. And, to our knowledge, even the best current fraction-free algorithms for determinants over integral domains need $O(n^{2.697263})$ ring operations (see [8]). Our main point is that within our new algorithm, each of the necessary determinant computations has a complexity of only $O(n^2)$; this is achieved by reusing information from previous computations (see the details in Section 1).

As an application, we determine the number of chambers that are supported by the lines on the Segre–Schur quartic. This remarkable surface provides an ideal application for the new algorithm: it turns out that it has an enormous number of Zariski chambers supported by lines, and it seems that the surface lies at the edge of what can be practically computed with current methods.

We prove the following result.

THEOREM. *Let $X \subset \mathbb{P}^3$ be the Segre–Schur quartic, that is, the surface given in homogeneous coordinates $(x_0 : x_1 : x_2 : x_3)$ by the equation*

$$x_0(x_0^3 - x_1^3) - x_2(x_2^3 - x_3^3) = 0.$$

(i) *We have*

$$z(X) = \infty$$

and

$$z_1(X) = 8\,260\,383\,569.$$

(ii) *The maximal number of lines that can occur in the support of a Zariski chamber is* 19 (*which is the maximal possible value, as the Picard number of X is* 20).

Note that the number $z_1(X)$ is bigger by a factor of about 10^4 than the number 1 501 681 of chambers on the Del Pezzo surface X_8 (blow-up of the plane in 8 points), even though only 64 curves are used to build chambers, as opposed to 240 curves on X_8. On the other hand, the Picard number of the Segre quartic is about twice that of X_8. One is led to wonder how the number of Zariski chambers is related to the Picard number in general, besides the fact that with higher Picard number, chambers of bigger support size become theoretically possible.

Note that since the negative curves on X of higher degrees are not known, the numbers $z_d(X)$ are at present inaccessible for $d \geqslant 2$. It would already be very interesting to know at what rate they grow as $d \to \infty$.

Concerning the organization of this paper, we start in Section 1 by presenting the improved algorithm. In Section 2 we give the proof of the theorem on the Segre–Schur quartic stated above. Finally, Section 3 compares the new algorithm with the original one, providing

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sample run-times for Del Pezzo surfaces, the Segre–Schur quartic, and matrices related to Fermat surfaces.

1. *Efficient computation of Zariski chambers*

Zariski chambers were first studied in [4], and we refer to that paper for a detailed introduction. For a very brief account, consider a smooth projective surface X over the complex numbers. To any big and nef $\mathbb{R}$-divisor P on X one associates the *Zariski chamber* Σ_P, which by definition consists of all divisor classes $[D]$ in the big cone $\mathrm{Big}(X)$ such that the irreducible curves in the negative part of the Zariski decomposition of D are precisely the curves C with $P \cdot C = 0$. The main result of [4] states that the sets Σ_P yield a locally finite decomposition of $\mathrm{Big}(X)$ into locally polyhedral subcones such that:

- on each subcone the volume function is given by a single polynomial of degree two;
- in the interior of each of the subcones the stable base loci are constant.

For the purposes of the present paper, a crucial fact is that the number of Zariski chambers can be computed from the intersection matrix of the negative curves on X, because Zariski chambers correspond to negative definite reduced divisors: by [3, Proposition 1.1] one has the following fact.

Proposition 1.1. *The set of Zariski chambers on a smooth projective surface X that are different from the nef chamber is in bijective correspondence with the set of reduced divisors on X whose intersection matrix is negative definite.*

In the statement, the *nef chamber* is the chamber Σ_H associated with an ample divisor H. Its closure is the nef cone, and its interior is the ample cone. Next, suppose that X carries only finitely many negative curves (that is, irreducible curves $C \subset X$ with negative self-intersection C^2). Then an immediate consequence of Proposition 1.1 is the following (see [3, Proposition 1.5]).

Proposition 1.2. (i) *The number $z(X)$ of Zariski chambers on X is given by*

$$z(X) = 1 + \#\left\{ \begin{array}{l} \textit{negative definite principal submatrices of the} \\ \textit{intersection matrix of the negative curves on } X \end{array} \right\}.$$

(ii) *More generally, let $C_1, \ldots, C_r$ be distinct negative curves on X, and let S be their intersection matrix. Then the number of Zariski chambers that are supported by a non-empty subset of $\{C_1, \ldots, C_r\}$ equals the number of negative definite principal submatrices of the matrix S.*

Here a *principal submatrix* of a given $n \times n$ matrix is, as usual, a submatrix that results from deleting k corresponding rows and columns of the matrix, where $0 \leqslant k < n$.

The algorithm below computes the number of positive definite principal submatrices of a given symmetric matrix. In view of Proposition 1.2, this enables us to determine the number of Zariski chambers (by considering the negative of the intersection matrix).

Algorithm 1.3. The algorithm takes as input an integer $n \geqslant 1$ and a symmetric $n \times n$ matrix A over $\mathbb{Z}$. It outputs all subsets $S \subset \{1, \ldots, n\}$ having the property that the corresponding principal submatrix A_S is positive definite.

We adopt a backtracking strategy as in [3], but instead of testing for positive definiteness by computing the determinant $\det(A_S)$ from scratch, the algorithm makes use of three procedures, *Grow*, *Shrink* and *IsPosDef*, to be discussed below.

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```
Algorithm PosDef
Input: integer n ⩾ 1, symmetric matrix A ∈ Z^{n×n}
Output: all positive definite principal submatrices of A

k ← 1; S ← ∅; B ← (); T ← ()
Grow(S, k)
while S ≠ ∅ do
    Assert((B = T · A_S) and (B is in Bareiss form))
    if IsPosDef(S) then
        output S
    else
        Shrink(S)
    end if
    if k < n then
        k ← k + 1; Grow(S, k)
    else
        if S ≠ ∅ and k = max S then
            Shrink(S)
        end if
        if S ≠ ∅ then
            k ← max S; Shrink(S); k ← k + 1; Grow(S, k)
        end if
    end if
end while
```

We now explain the procedures *Grow*, *Shrink* and *IsPosDef*. They work on matrix variables B and T that are global variables of *PosDef*. At every stage of the algorithm, B holds the Bareiss trigonalization of A_S. The auxiliary matrix variable T is the essential tool that makes is possible to do the trigonalization incrementally; it holds the triangular matrix that one obtains from the unity matrix upon applying the same transformations that have been applied to A_S to obtain B.

The procedure *Shrink(S)* removes the maximal element from S and updates B and T. The latter is achieved by simply discarding the last row and column of both B and T.

The procedure *IsPosDef(S)* determines whether A_S is positive definite. This can be done as follows. As the matrix $A_{S\setminus\max S}$ is known to be positive definite, A_S is positive definite if and only if its determinant is positive; and the latter can be read off the sign of the lower right entry of B, since in a Bareiss trigonalization this entry is always the determinant of the original matrix A_S. So we have

$$\mathit{IsPosDef}(S) = (B[\max S, \max S] > 0).$$

The procedure *Grow(S, k)* incorporates a new element $k > \max S$ into the index set S and updates B and T accordingly. It consists of the following steps.

(G1) $S \leftarrow S \cup \{k\}$.

(G2) Attach the last row and last column of A_S to the bottom and right of B. Attach the last row and column of the unit matrix to T.

(G3) Replace the last column b of B by $T \cdot b$.

(G4) Clear the last row of B by means of the following procedure:

```
s ← |S|
for i from 1 to s − 1 do
    if i = 1 then
        d ← 1
    else
```

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```
        d ← B[i − 1, i − 1]
    end if
    for j from i + 1 to s do
        B[s, j] ← (B[s, j] · B[i, i] − B[i, j] · B[s, i]) div d
    end for
    for j from 1 to s do
        T[s, j] ← (T[s, j] · B[i, i] − T[i, j] · B[s, i]) div d
    end for
    B[s, i] ← 0
end for
```

In the two loops of (G4), the division by d is the Bareiss divison, made possible by Sylvester's identity, which keeps the size of the matrix entries from exploding (see [1]). Note that *Grow* does $O(s^2)$ loop iterations and hence has complexity $O(n^2)$, while *Shrink* and *IsPosDef* need only constant time.

2. *Lines and chambers on the Segre–Schur quartic*

In this section we consider the Segre–Schur quartic (see [10, 12] and also [2, Section 2.1]). By Segre's theorem [11], this remarkable surface carries the maximal number of lines possible for a smooth quartic in $\mathbb{P}^3$. In order to find the chambers supported on lines on this particular surface, we need to determine the intersection matrix of all lines. We approach this task in a more general setting. Caporaso, Harris and Mazur [7] and Boissiere and Sarti [5] considered a class of surfaces in $\mathbb{P}^3$ known to contain many lines, which was first studied by Segre [12]. Specifically, these surfaces are given by an equation

$$\varphi(x_0, x_1) = \psi(x_2, x_3) \tag{2.0.1}$$

where φ and ψ are homogeneous polynomials of common degree d. Clearly, the surface we are interested in, the Segre–Schur quartic, is of this type with

$$\varphi(x, y) = \psi(x, y) = x(x^3 - y^3).$$

For any surface S given by an equation (2.0.1) of degree d, consider the sets of zeros $V(\varphi)$ and $V(\psi)$ in $\mathbb{P}^1$. Denote by Γ the group of automorphisms of $\mathbb{P}^1$ mapping $V(\varphi)$ onto $V(\psi)$.

Proposition 2.1 ([7, Lemma 5.1] and [5, Proposition 4.1]). *The number of lines on S equals $d^2 + d \cdot |\Gamma|$.*

In order to establish terminology for our further investigations, we briefly go through the steps of the proof given in [7].

- Consider the sets of points $P_1, \ldots, P_d$ and $P'_1, \ldots, P'_d$ of S lying on the lines $L_1 = V(x_2, x_3)$ and $L_2 = V(x_0, x_1)$, respectively. For every $i, j \in \{1, \ldots, d\}$, the line $L_{i,j}$ joining P_i and P'_j is contained in S. The d^2 lines $L_{i,j}$ are called *lines of the first type*.
- Any other line L on S is called a *line of the second type*. Any such line is disjoint from L_1 and L_2, guaranteeing that it is given by equations of the form

$$\begin{aligned} x_2 &= ax_0 + bx_1, \\ x_3 &= cx_0 + dx_1, \end{aligned}$$

  with $ad - bc \neq 0$. The invertible matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ induces an automorphism of $\mathbb{P}^1$ mapping $V(\varphi)$ onto $V(\psi)$.

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- Conversely, every automorphism in Γ given by an invertible matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ corresponds to d distinct lines of the second type on S given by equations

$$\begin{aligned} x_2 &= \lambda\eta^k(ax_0 + bx_1), \\ x_3 &= \lambda\eta^k(cx_0 + dx_1), \end{aligned} \tag{2.1.1}$$

  for some $\lambda \in \mathbb{C}$ and $1 \leqslant k \leqslant d$, where η denotes a primitive dth root of unity.

With this knowledge, we turn to determining explicitly the lines lying on the Segre–Schur quartic. The 16 lines of the first type are just the lines joining the zeros of the polynomial $\varphi(x, y) = x(x^3 - y^3)$ on L_1 and L_2, each considered as a copy of $\mathbb{P}^1$. Setting $\xi = e^{2\pi i/3}$, these points are

$$\begin{aligned} P_1 &= (0:1:0:0), \quad P_2 = (1:1:0:0), \\ P_3 &= (\xi:1:0:0), \quad P_4 = (\xi^2:1:0:0) \end{aligned}$$

and

$$\begin{aligned} P_1' &= (0:0:0:1), \quad P_2' = (0:0:1:1), \\ P_3' &= (0:0:\xi:1), \quad P_4' = (0:0:\xi^2:1). \end{aligned}$$

The lines $L_{i,j} = \overline{P_iP_j'}$ joining them can be expressed as

$$L_{i,j} : \begin{pmatrix} -1 & a_i & 0 & 0 \\ 0 & 0 & -1 & a_j \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0,$$

where a_i denotes the ith entry of the tuple $(0, 1, \xi, \xi^2)$.

For the lines of the second type we note that in the case of the Segre–Schur quartic, Γ is the tetrahedral group $\mathcal{T}$, which is isomorphic to the product $D_2 \times C_3$ of the dihedral group $D_2 \cong (\mathbb{Z}/2\mathbb{Z})^2$ and a cyclic group $C_3 \cong \mathbb{Z}/3\mathbb{Z}$. More concretely, the group D_2 operating on the points $(0:1), (1:1), (1:\xi), (1:\xi^2)$ consists of the elements

$$T_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad T_2 = \begin{pmatrix} -1 & \xi \\ 2\xi^2 & 1 \end{pmatrix},$$

$$T_3 = \begin{pmatrix} -1 & \xi^2 \\ 2\xi & 1 \end{pmatrix}, \quad T_4 = \begin{pmatrix} -1 & 1 \\ 2 & 1 \end{pmatrix},$$

and the cyclic group C_3 is generated by the element

$$Z = \begin{pmatrix} \xi & 0 \\ 0 & 1 \end{pmatrix}.$$

Hence the elements of $\mathcal{T}$ are the automorphisms Z^kT_j for $0 \leqslant k \leqslant 2$ and $1 \leqslant j \leqslant 4$. For an element $\gamma \in \Gamma$ represented by a matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, we need to find the corresponding numbers λ_m such that the lines

$$L_{\gamma,\lambda_m} : \begin{pmatrix} \lambda_m a & \lambda_m b & -1 & 0 \\ \lambda_m c & \lambda_m d & 0 & -1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

from equation (2.1.1) lie on S. If $\varphi(p_0, p_1) = 0$ for a $p \in \mathbb{P}^1$, then

$$\varphi(\lambda ap_0 + \lambda bp_1, \lambda cp_0 + \lambda dp_1) = \lambda^4\varphi(ap_0 + bp_1, cp_0 + dp_1) = 0.$$

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The line $L_{\gamma,\lambda}$ thus intersects S in the four points $(p_0 : p_1 : \lambda(ap_0 + bp_1) : \lambda(cp_0 + dp_1))$, corresponding to the zeros $(p_0 : p_1)$ of φ. Now, let $(q_0 : q_1) \in \mathbb{P}^1$ be any point on which φ does not vanish. Then the desired values for λ are given as solutions of the equation

$$\lambda^4 = \frac{\varphi(q_0, q_1)}{\varphi(aq_0 + bq_1, cq_0 + dq_1)}.$$

By way of example, let us carry out the computation for the automorphism $Z^1T_2 \in \mathcal{T}$. It is given by the matrix

$$\begin{pmatrix} -\xi & \xi^2 \\ 2\xi^2 & 1 \end{pmatrix}.$$

Choosing $(q_0 : q_1) = (1 : 0)$, we get

$$\psi(1, 0) = 1, \quad \psi(-\xi, 2\xi^2) = -\xi(-\xi^3 - 8\xi^6) = 9\xi.$$

Consequently, the corresponding values for λ_m are

$$\lambda_m = \frac{i^m \xi^2 \sqrt{3}}{3}$$

for $m = 0, \ldots, 3$.

In an analogous manner, we find the factors λ_m for the remaining automorphisms in $\mathcal{T}$, as shown in Table 1. We set $\eta = \sqrt{3}/3$ and write $i = \sqrt{-1}$ for the imaginary unit.

The lines $L_{Z^kT_j,\lambda_m}$ for $k = 0, \ldots, 2$, $j = 1, \ldots, 4$ and $m = 0, \ldots, 3$ are thus exactly the 48 lines of the second type on S.

TABLE 1. *Factors λ_m for the automorphisms in $\mathcal{T}$.*

Automorphism	Matrix	λ_m
Z^0T_1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	i^m
Z^1T_1	$\begin{pmatrix} \xi & 0 \\ 0 & 1 \end{pmatrix}$	$i^m\xi^2$
Z^2T_1	$\begin{pmatrix} \xi^2 & 0 \\ 0 & 1 \end{pmatrix}$	$i^m\xi$
Z^0T_2	$\begin{pmatrix} -1 & \xi \\ 2\xi^2 & 1 \end{pmatrix}$	$i^m\eta$
Z^1T_2	$\begin{pmatrix} -\xi & \xi^2 \\ 2\xi^2 & 1 \end{pmatrix}$	$i^m\eta\xi^2$
Z^2T_2	$\begin{pmatrix} -\xi^2 & 1 \\ 2\xi^2 & 1 \end{pmatrix}$	$i^m\eta\xi$
Z^0T_3	$\begin{pmatrix} -1 & \xi^2 \\ 2\xi & 1 \end{pmatrix}$	$i^m\eta$
Z^1T_3	$\begin{pmatrix} -\xi & 1 \\ 2\xi & 1 \end{pmatrix}$	$i^m\eta\xi^2$
Z^2T_3	$\begin{pmatrix} -\xi^2 & \xi \\ 2\xi & 1 \end{pmatrix}$	$i^m\eta\xi$
Z^0T_4	$\begin{pmatrix} -1 & 1 \\ 2 & 1 \end{pmatrix}$	$i^m\eta$
Z^1T_4	$\begin{pmatrix} -\xi & \xi \\ 2 & 1 \end{pmatrix}$	$i^m\eta\xi^2$
Z^2T_4	$\begin{pmatrix} -\xi^2 & \xi^2 \\ 2 & 1 \end{pmatrix}$	$i^m\eta\xi$

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The configuration of the lines on the Segre–Schur quartic is given by the following proposition.

PROPOSITION 2.2. *Let $\delta : \mathbb{R} \to \{0,1\}$ denote the indicator function of the set $\{0\}$, mapping any non-zero number x to 0 and the number 0 to 1. With other notation as above, we have that:*

- *every line L on S has self-intersection $L^2 = -2$;*
- *the intersection number of distinct lines of the first type is*

$$L_{i,j} \cdot L_{i',j'} = \delta(i-i') + \delta(j-j');$$

- *the intersection number of a line of the first type with a line of the second type is*

$$L_{s,t} \cdot L_{Z^k T_j, \lambda_m} = \begin{cases} \delta(-\xi^{2k+t-1} + \xi^j - \xi^{s-1} - 2\xi^{s+t+2k-j-2}) & \text{if } j \neq 1, s \neq 1, t \neq 1, \\ \delta(t+2k-j-1 \bmod 3) & \text{if } j \neq 1, t \neq 1, s = 1, \\ \delta(s-1-j \bmod 3) & \text{if } j \neq 1, s \neq 1, t = 1, \\ \delta(s+k-t \bmod 3) & \text{if } j = 1, s \neq 1, t \neq 1, \\ \delta(s-1)\cdot\delta(j-1)\cdot\delta(t-1) & \text{otherwise}; \end{cases}$$

- *the intersection number of distinct lines of the second type is*

$$L_{Z^k T_j, \lambda_m} \cdot L_{Z^{k'} T_{j'}, \lambda_{m'}} = \begin{cases} \delta((i^{m'-m} - 3)\xi^{2k} \\ \quad + i^{m'-m}(1 - 3i^{m'-m})\xi^{2k'} \\ \quad + 2i^{m'-m}(\xi^{2k-j+j'} + \xi^{2k'-j'+j})) & \text{if } j \neq 1, j' \neq 1, \\ \delta(i^{2m}\xi^{2k} - i^m i^{m'} \xi^{2k'} \eta + i^{m'} \eta i^m \xi^{2k} \\ \quad - 3(i^{2m'} \eta^2 \xi^{2k'})) & \text{if } j = 1, j' \neq 1, \\ \delta(m - m') & \text{if } j = j' = 1. \end{cases}$$

Proof. The statement about self-intersection of lines on S follows immediately from adjunction and the fact that S has trivial canonical class. To prove the statement about the intersection of distinct lines of the first type, all we need to show is that two such lines A and B cannot intersect in a point outside the (disjoint) lines L_1 and L_2 (see the sketch of the proof of Proposition 2.1): if this were the case, then since L_1 and L_2 both intersect A and B, they would have to lie in the plane spanned by A and B and would therefore intersect each other.

The remaining intersection numbers are calculated in the following way. Two lines in $\mathbb{P}^3$ given by equations

$$a_0x_0 + a_1x_1 + a_2x_2 + a_3x_3 = 0,$$
$$b_0x_0 + b_1x_1 + b_2x_2 + b_3x_3 = 0$$

and

$$c_0x_0 + c_1x_1 + c_2x_2 + c_3x_3 = 0,$$
$$d_0x_0 + d_1x_1 + d_2x_2 + d_3x_3 = 0$$

intersect each other if and only if the determinant of the matrix

$$\begin{pmatrix} a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & b_3 \\ c_0 & c_1 & c_2 & c_3 \\ d_0 & d_1 & d_2 & d_3 \end{pmatrix}$$

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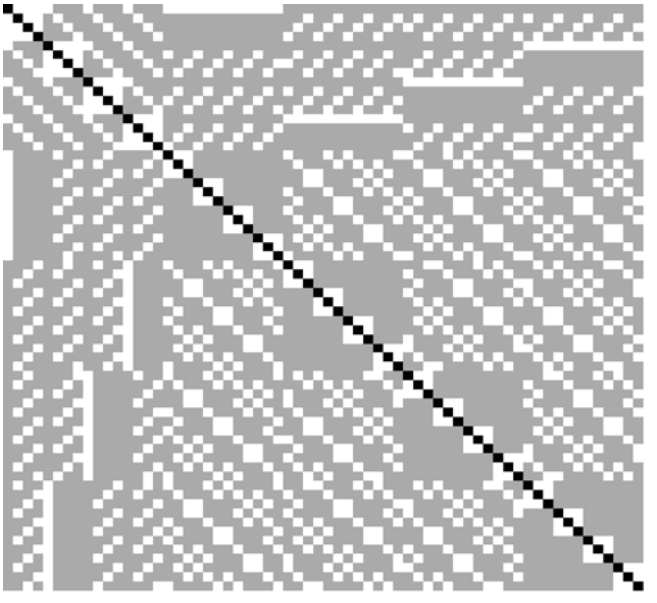

FIGURE 1. *Intersection matrix of the 64 lines on the Segre–Schur quartic. The numerical entries of* -2, 0 *and* 1 *are replaced by black, grey and white boxes, respectively.*

vanishes. Note that for $s, t \neq 1$, the line $L_{s,t}$ of the first type is given by

$$L_{s,t} : \begin{pmatrix} -1 & \xi^{s-2} & 0 & 0 \\ 0 & 0 & -1 & \xi^{t-2} \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0;$$

and if $j \neq 1$, then the line $L_{Z^k T_j, \lambda_m}$ is given by

$$L_{Z^k T_j, \lambda_m} : \begin{pmatrix} -i^m \eta & i^m \eta \xi^{j-1} & -1 & 0 \\ 2i^m \eta \xi^{2k+1-j} & i^m \eta \xi^{2k} & 0 & -1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

Computation of the determinants in question then yields the asserted formulas. By way of illustration, let us calculate the intersection number of the line $L_{1,t}$ of the first type, where $t \neq 1$, and a line $L_{Z^k T_j, \lambda_m}$ of the second type, with $j \neq 1$. We have

$$\det \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & \xi^{t-2} \\ -i^m \eta & i^m \eta \xi^{j-1} & -1 & 0 \\ 2i^m \eta \xi^{2k+1-j} & i^m \eta \xi^{2k} & 0 & -1 \end{pmatrix} = i^m \eta (\xi^{j-1} - \xi^{2k+t-2}).$$

Consequently, the lines intersect if and only if

$$\xi^{j-1} = \xi^{2k+t-2},$$

which is equivalent to

$$t + 2k - j - 1 \equiv 0 \mod 3. \qquad \square$$

Combining all the intersection numbers yields the intersection matrix of the Segre–Schur quartic, displayed in Figure 1. The order of rows and columns was chosen as follows: the first 16 rows correspond to the lines of the first type, $L_{s,t}$ with $s = 1, \ldots, 4$ and $t = 1, \ldots, 4$; the

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remaining 48 rows correspond to the lines of the second type, $L_{Z^k T_j, \lambda_m}$ with $k = 0, \ldots, 2$, $m = 0, \ldots, 3$ and $j = 1, \ldots, 4$.

Proof of the theorem stated in the introduction. The intersection matrix of the 64 lines is of rank 20; hence the classes of the lines generate the Néron–Severi group $\mathrm{NS}(X)$ over $\mathbb{Q}$, and the K3 surface is singular (in the sense that $\mathrm{NS}(X)$ is of maximal possible rank). Therefore, by a result of Shioda and Inose [**13**], its automorphism group is infinite. The latter fact implies, in turn, that there are infinitely many (-2)-curves on X (cf. [**9**, Remark 7.2]), and hence $z(X) = \infty$.

An application of Algorithm 1.3 to the negative of the intersection matrix of the 64 lines shows that it has exactly 8 260 383 568 positive definite principal submatrices. Upon taking the additional nef chamber into account, we arrive at the claimed 8 260 383 569 chambers. As $\rho(X) = 20$, there cannot be Zariski chambers supported by more than 19 curves. The results of the algorithm show that chambers with 19 supporting curves actually occur. (In fact there are exactly 1728 such chambers; see Remark 2.3 below.) □

Remark 2.3. There is more numerical data that is of interest: how many Zariski chambers are there with support of given cardinality? For $\ell \geqslant 1$, let us denote by $z_d^{(\ell)}(X)$ the number of Zariski chambers that are supported by ℓ curves of degree d or less. Our computations yield for the Segre–Schur quartic the values in the following table.

ℓ	$z_1^{(\ell)}(X)$
1	64
2	2016
3	41376
4	605856
5	6343776
6	45613512
7	217025520
8	674047818
9	1376161536
10	1900843848
11	1832006112
12	1264421472
13	635795760
14	233619648
15	61499712
16	11037702
17	1246368
18	69744
19	1728

These numbers show that the surface clearly favours chambers of 'medium size'.

Remark 2.4. The lines

$$L_{1,1}, \ldots, L_{1,4}, L_{2,1}, L_{2,2}, L_{2,3}, L_{3,1}, L_{3,2}, L_{3,3}$$

of the first type together with the lines

$$L_{Z^1 T_1 \lambda_0}, L_{Z^2 T_1 \lambda_0}, L_{Z^1 T_2 \lambda_0}, L_{Z^2 T_2 \lambda_0}, L_{Z^1 T_3 \lambda_0},$$
$$L_{Z^2 T_3 \lambda_0}, L_{Z^1 T_1 \lambda_1}, L_{Z^2 T_1 \lambda_1}, L_{Z^1 T_2 \lambda_1}, L_{Z^2 T_2 \lambda_1}$$

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of the second type generate a sublattice $\Lambda \subset \mathrm{NS}(X)$ of rank 20 and discriminant -48. So they yield a basis of $\mathrm{NS}(X) \otimes \mathbb{Q}$. It is shown in [6, Appendix B] that the discriminant of the lattice $\mathrm{NS}(X)$ is -48 as well; hence the aforementioned lines in fact generate $\mathrm{NS}(X)$.

Remark 2.5. It is interesting to note that the 16 lines of the first type that one has on any quartic surface of type

$$\varphi(x_0, x_1) = \psi(x_2, x_3)$$

give rise to 6521 Zariski chambers. As these 16 lines generate a sublattice of rank 10, one should compare the number 6521 with the number $z(X_8) = 1\,501\,681$ for the Del Pezzo surface X_8 of Picard number 9. It would be interesting to know, on a more conceptual basis, what properties of the lattice are crucial for obtaining a large number of chambers.

3. *Comparison of efficiency*

To illustrate the practicability of the new algorithm, we will now compare it with the algorithm from [3] by providing sample run-times[†].

We consider first the Del Pezzo surfaces $X_1, \ldots, X_8$ as in [3]. The following table lists their chamber numbers $z(X_r)$ and shows the run-times (all in milliseconds) for the algorithm from [3], called A_1 in the table, as well as for the new algorithm A_2. The value n is the dimension of the intersection matrix[‡] (that is, the number of negative curves on X_r).

r	1	2	3	4	5	6	7	8
n	1	3	6	10	16	27	56	240
$z(X_r)$	2	5	18	76	393	2764	33 645	1 501 681
A_1	0.55	0.56	0.70	0.85	2.3	33.5	1.48×10^3	2.88×10^4
A_2	0.80	0.85	0.82	0.91	2.3	26.8	1.09×10^3	1.92×10^4
Factor	0.69	0.66	0.85	0.93	1	1.25	1.36	1.5

For $r \leqslant 4$, the original algorithm A_1 is actually faster, presumably due to the overhead caused by the more sophisticated strategy of A_2. Starting with $r = 6$, however, A_2 gets more and more superior. This pattern will become even more evident when we consider the lines on the Segre quartic, which we do next. Specifically, we provide run-times for A_1 and A_2 when they are applied to the principal submatrices consisting of the first n rows and columns of the intersection matrix from Section 2.

n	8	16	24	32	40	48	56	64
A_1	0.90	65.55	4.16×10^3	9.46×10^4	1.43×10^6	1.57×10^7	1.3×10^8	$*$
A_2	0.96	31.54	1.70×10^3	3.61×10^4	5.40×10^5	5.66×10^6	4.6×10^7	4.9×10^8
Factor	0.94	2.08	2.45	2.62	2.65	2.77	2.83	$*$

Clearly, with growing matrix dimension, algorithm A_2 shows an advantage over algorithm A_1. (As for the asterisk in the last column, the factors in the fourth line of the table seem to

[†]The run-times were obtained using Delphi implementations of A_1 and A_2 on an Intel Core Duo CPU E8600 at 3.33 GHz.

[‡]Here and in what follows, the algorithm is actually applied to the *negative* of the intersection matrix in question.

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suggest a factor of about 3. If this assumption were true, then for $n = 64$ algorithm A_1 should have a run-time of about 15×10^8 ms, which is around three weeks. In our attempts to verify the assumption, we were, however, not able to get results within that amount of time, and we found it technically difficult to check for run-times beyond a month.)

The potential of A_2 is demonstrated even more clearly in the case of matrices with large definite principal submatrices: for example, matrices which are negative definite themselves. We use a symmetric integer matrix with negative entries -2 on the main diagonal, super- and subdiagonal entries 1, and remaining entries all 0. Note that a similar matrix can in fact be realized as an intersection matrix: for any number k, consider a surface of degree k of the type discussed in Section 2, for example the Fermat surface of degree k. Among the k^2 lines of the first type we can pick $n = 2k + 1$ lines with the desired configuration. However, the self-intersections of these lines (and therefore the diagonal entries of the matrix) are then $2 - k$.

n	15	21	23	27	33
A_1	2.73×10^2	3.84×10^4	4.26×10^5	4.50×10^6	4.74×10^8
A_2	1.10×10^2	1.17×10^4	1.18×10^5	1.23×10^6	1.06×10^8
Factor	2.48	3.28	3.61	3.66	4.48

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