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IMPROVED UPPER BOUNDS ON DIOPHANTINE TUPLES WITH THE PROPERTY $D(n)$

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Abstract

Let n be a nonzero integer. A set S of positive integers is a Diophantine tuple with the property $D(n)$ if $ab + n$ is a perfect square for each $a, b \in S$ with $a \neq b$. It is of special interest to estimate the quantity M_n, the maximum size of a Diophantine tuple with the property $D(n)$. We show the contribution of intermediate elements is $O(\log \log |n|)$, improving a result by Dujella ['Bounds for the size of sets with the property $D(n)$', *Glas. Mat. Ser. III* **39**(59)(2) (2004), 199–205]. As a consequence, we deduce that $M_n \leq (2 + o(1)) \log |n|$, improving the best known upper bound on M_n by Becker and Murty ['Diophantine m-tuples with the property $D(n)$', *Glas. Mat. Ser. III* **54**(74)(1) (2019), 65–75].

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1. Introduction

A set $\{a_1, a_2, \ldots, a_m\}$ of distinct positive integers is a *Diophantine m-tuple* if the product of any two distinct elements in the set is one less than a square. A famous example of a Diophantine quadruple is $\{1, 3, 8, 120\}$, due to Fermat. Such a construction is optimal in the sense that there is no Diophantine 5-tuple, recently confirmed by He *et al.* [8]. There are many generalisations and variants of Diophantine tuples. We refer to the recent book of Dujella [6] for a comprehensive overview.

In this paper, we focus on one natural generalisation that has been studied extensively. Let n be a nonzero integer. A set S of positive integers is a Diophantine tuple with the property $D(n)$ if $ab + n$ is a perfect square for each $a, b \in S$ with $a \neq b$. It is of special interest to estimate the quantity M_n, the maximum size of a Diophantine tuple with the property $D(n)$. We have mentioned that $M_1 = 4$ [8]. Analogously, Bliznac Trebješanin and Filipin [2] proved that $M_4 = 4$. More recently, Bonciocat *et al.* [3] proved that $M_{-1} = M_{-4} = 3$.

It is widely believed that M_n is uniformly bounded (for example, this follows from the uniformity conjecture) [1, 4]. Using elementary congruence considerations, it is easy to show that $M_n = 3$ when $n \equiv 2 \pmod 4$ (see [6, Section 5.4.1] for more discussions). In a remarkable paper [7], Dujella and Luca showed that if p is a prime, then M_p and M_{-p} are both bounded by $3 \cdot 2^{168}$. However, for a generic n, the best-known upper bound on M_n is of the form $O(\log |n|)$.

Following [1, 4, 5], for Diophantine tuples with the property $D(n)$, we separate the contribution of large, intermediate and small elements as follows:

$$A_n = \sup\{|S \cap [|n|^3, +\infty)| : S \text{ has the property } D(n)\},$$
$$B_n = \sup\{|S \cap (n^2, |n|^3)| : S \text{ has the property } D(n)\},$$
$$C_n = \sup\{|S \cap [1, n^2]| : S \text{ has the property } D(n)\}.$$

In [4], Dujella showed that $A_n \leq 31$. The best known upper bound on B_n is $B_n \leq 0.6071 \log |n| + O(1)$, due to Dujella [5]. As for C_n, the best result,

$$C_n \leq 2 \log |n| + O\left(\frac{\log |n|}{(\log \log |n|)^2}\right), \tag{1.1}$$

is due to Becker and Murty [1]. Summing the bounds on A_n, B_n and C_n yields $M_n \leq (2.6071 + o(1)) \log |n|$, the best known upper bound on M_n [1].

Our main result is the following improved upper bounds for B_n and M_n.

THEOREM 1.1. *We have*

$$B_n = O(\log \log |n|), \quad M_n \leq 2 \log |n| + O\left(\frac{\log |n|}{(\log \log |n|)^2}\right).$$

The key observation of our improvement is that the contribution of intermediate elements can be bounded more efficiently. To achieve that, we separate the contribution of large and intermediate elements differently. For each $\epsilon > 0$, let

$$A_n^{(\epsilon)} = \sup\{|S \cap (|n|^{2+\epsilon}, +\infty)| : S \text{ has the property } D(n)\},$$
$$B_n^{(\epsilon)} = \sup\{|S \cap (n^2, |n|^{2+\epsilon}]| : S \text{ has the property } D(n)\}.$$

We give the following estimates on $A_n^{(\epsilon)}$ and $B_n^{(\epsilon)}$.

THEOREM 1.2. *The following estimates hold uniformly for all $\epsilon \in (0, 1)$ and all nonzero integers n:*

$$A_n^{(\epsilon)} = O\left(\log \frac{1}{\epsilon}\right), \quad B_n^{(\epsilon)} \leq 0.631\epsilon \log |n| + O(1).$$

To estimate B_n and M_n, note that

$$B_n \leq A_n^{(\epsilon)} + B_n^{(\epsilon)}, \quad M_n \leq A_n^{(\epsilon)} + B_n^{(\epsilon)} + C_n.$$

By setting

$$\epsilon = \frac{\log \log |n|}{\log |n|},$$

we see that Theorem 1.1 follows from Theorem 1.2 and (1.1) immediately.

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2. Proofs

Our proofs are inspired by several arguments used in [4, 5, 9]. We first recall three useful lemmas from [4].

LEMMA 2.1 [4, Lemma 2]. *Let n be a nonzero integer. Let $\{a, b, c, d\}$ be a Diophantine quadruple with the property $D(n)$ and $a < b < c < d$. If $c > b^{11}|n|^{11}$, then $d \le c^{131}$.*

LEMMA 2.2 [4, Lemma 3]. *Let n be a nonzero integer. If $\{a, b, c\}$ is a Diophantine triple with the property $D(n)$ and $ab + n = r^2, ac + n = s^2, bc + n = t^2$, then there exist integers e, x, y, z such that*

$$ae + n^2 = x^2, \quad be + n^2 = y^2, \quad ce + n^2 = z^2,$$

and

$$c = a + b + \frac{e}{n} + \frac{2}{n^2}(abe + rxy).$$

LEMMA 2.3 [4, Lemma 5]. *Let n be an integer with $|n| \ge 2$. Let $\{a, b, c, d\}$ be a Diophantine quadruple with the property $D(n)$. If $n^2 < a < b < c < d$, then $c > 3.88a$ and $d > 4.89c$.*

Next, we deduce a gap principle from the above two lemmas.

COROLLARY 2.4. *Let n be an integer with $|n| \ge 2$. Let $\{a, b, c, d\}$ be a Diophantine quadruple with the property $D(n)$. If $n^2 < a < b < c < d$, then*

$$d > \frac{bc}{n^2}.$$

PROOF. We apply Lemma 2.2 to the Diophantine triple $\{b, c, d\}$. Since $b > n^2$ and $be + n^2 \ge 0$, it follows that $e \ge 0$. If $e = 0$, then Lemma 2.2 implies that

$$d = b + c + 2\sqrt{bc + n} < 2c + 2\lfloor\sqrt{c^2 + n}\rfloor \le 4c < 4.89c,$$

which is impossible in view of Lemma 2.3. Thus, $e \ge 1$ and Lemma 2.2 implies that

$$d > \frac{2bce}{n^2} > \frac{bc}{n^2}. \qquad \square$$

Now we are ready to prove Theorem 1.2.

PROOF OF THEOREM 1.2. Let $\epsilon \in (0, 1)$ and n be a nonzero integer.

We first bound $B_n^{(\epsilon)}$. Let S be a Diophantine tuple with the property $D(n)$, such that all elements in S are in $[n^2, |n|^{2+\epsilon}]$. By Lemma 2.3, the elements in S grow exponentially. More precisely,

$$|S| \le \epsilon \log_{4.89} |n| + O(1) < 0.631\epsilon \log |n| + O(1).$$

The bound on $B_n^{(\epsilon)}$ follows.

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Next we estimate $A_n^{(\epsilon)}$. Let S be a Diophantine tuple with the property $D(n)$, such that all elements in S are at least $|n|^{2+\epsilon}$. Label the elements in S in increasing order as $a_1 < a_2 < \cdots$. By Corollary 2.4,

$$a_{i+2} > \frac{a_i a_{i+1}}{n^2}$$

holds for each $i \geq 2$. For each $i \geq 2$, let $b_i = a_i/n^2$. Then we have $b_{i+2} > b_i b_{i+1}$. Note that $b_2 = a_2/n^2$ and $b_3 > b_2 = a_2/n^2$. Define the sequence $\{\beta_i\}_{i=2}^{\infty}$ recursively by

$$\beta_2 = \beta_3 = 1, \quad \beta_{i+2} = \beta_i + \beta_{i+1} \quad (i \geq 2).$$

By induction, we have $b_i > (a_2/n^2)^{\beta_i}$. It follows that

$$a_i > \frac{a_2^{\beta_i}}{|n|^{2\beta_i - 2}}. \tag{2.1}$$

Since $\beta_i \to \infty$, we can choose k sufficiently large such that

$$(\beta_k - 11)(2 + \epsilon) > 2\beta_k + 9;$$

let $k = k(\epsilon)$ be the smallest such k. If $|S| < k$, we are done. Otherwise, (2.1) implies that

$$a_k > \frac{a_2^{\beta_k}}{|n|^{2\beta_k - 2}} = a_2^{11}|n|^{11} \cdot \frac{a_2^{\beta_k - 11}}{|n|^{2\beta_k + 9}} > a_2^{11}|n|^{11}|n|^{(\beta_k - 11)(2+\epsilon) - (2\beta_k + 9)} > a_2^{11}|n|^{11}.$$

Now Lemma 2.1 implies that the largest element in S is at most a_k^{131}. By a similar argument as above, for each $i \geq 2$, we have

$$a_{k+i} > \frac{a_k^{\beta_i}}{|n|^{2\beta_i - 2}}. \tag{2.2}$$

Since $\beta_i \to \infty$, we can choose ℓ sufficiently large such that

$$(\beta_\ell - 131)(2 + \epsilon) > 2\beta_\ell - 2;$$

let $\ell = \ell(\epsilon)$ be the smallest such ℓ. Note that both k and ℓ are explicitly computable constants depending only on ϵ. Since the sequence (β_i) grows exponentially, it follows that $k(\epsilon)$ and $\ell(\epsilon)$ are of the order $\log(1/\epsilon)$.

If $|S| \geq k + \ell$, then (2.2) implies that

$$a_{k+\ell} > \frac{a_k^{\beta_\ell}}{|n|^{2\beta_\ell - 2}} = a_k^{131} \cdot \frac{a_k^{\beta_\ell - 131}}{|n|^{2\beta_\ell - 2}} > a_k^{131}|n|^{(\beta_\ell - 131)(2+\epsilon) - (2\beta_\ell - 2)} > a_k^{131},$$

which is a contradiction. Therefore, $|S| < k + \ell$. Thus, $A_n^{(\epsilon)} \leq k(\epsilon) + \ell(\epsilon) \ll \log(1/\epsilon)$, where the implicit constant is absolute. □

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