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Conjugacy invariants for Brouwer mapping classes

JULIETTE BAVARD

Sorbonne Universités, UPMC Université Paris 06, Institut de Mathématiques de Jussieu-Paris Rive Gauche, UMR 7586, CNRS, Université Paris Diderot, Sorbonne Paris Cité, F-75005, Paris, France
(e-mail: juliette.bavard@imj-prg.fr)

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Abstract. We give new tools for homotopy Brouwer theory. In particular, we describe a canonical reducing set called the set of *walls*, which splits the plane into *maximal translation areas* and *irreducible areas*. We then focus on Brouwer mapping classes relative to four orbits and describe them explicitly by adding a *tangle* to Handel's diagram and to the set of walls. This is essentially an isotopy class of simple closed curves in the cylinder minus two points.

Contents

1. *Introduction*

1.1. *Homotopy Brouwer theory.* Homotopy Brouwer theory was introduced by Handel in [**Han99**] to prove his famous fixed point theorem for planar homeomorphism, which has many applications to surface homeomorphisms (for examples, see the introduction of [**LC06**]). This theory has been used by John Franks and Michael Handel, for example to study Hamiltonian surface diffeomorphisms, in [**FH03b**] and prove the Zimmer conjecture for area preserving diffeomorphisms of surfaces in [**FH03a**].

Homotopy Brouwer theory can be seen as the study of the elements of the mapping class group of the plane minus $\mathbb{Z}$ which are classes of Brouwer homeomorphisms relative to a finite number of orbits. More precisely, consider a Brouwer homeomorphism h: that is, a fixed point-free homeomorphism of the plane preserving the orientation. Choose a finite number of disjoint orbits of this homeomorphism and denote their union by $\mathcal{O}$. Classical Brouwer theory tells us that each orbit of a Brouwer homeomorphism is properly embedded in the plane: that is, intersects every compact set of the plane in only a finite number of points (see, for example, [**Gui94**]). Hence $\mathcal{O}$ is homeomorphic to $\mathbb{Z}$ in the plane. Denote by $MCG(\mathbb{R}^2, \mathcal{O})$ the mapping class group of the plane relative to $\mathcal{O}$: that is, the quotient of the group of orientation-preserving homeomorphisms of the plane which

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globally fix $\mathcal{O}$ by its connected component of the identity for the compact-open topology. Now we can look at the class of h in $MCG(\mathbb{R}^2, \mathcal{O})$: since h is a Brouwer homeomorphism, this class is said to be a *Brouwer mapping class*. We denote it by $[h; \mathcal{O}]$. Two Brouwer mapping classes $[h; \mathcal{O}]$ and $[h'; \mathcal{O}']$ are said to be conjugate if there exists an orientation-preserving homeomorphism ϕ of the plane such that $\phi(\mathcal{O}) = \mathcal{O}'$ and $[\phi h \phi^{-1}; \phi(\mathcal{O})]$ is equal to $[h'; \mathcal{O}']$ in $MCG(\mathbb{R}^2; \mathcal{O}')$. One aim of homotopy Brouwer theory is to describe (up to conjugacy) every Brouwer mapping class relative to a finite number of given orbits.

1.2. *Brouwer mapping classes relative to one, two and three orbits.* In [**Han99**], Michael Handel gives a complete description of Brouwer mapping classes relative to one and two orbits. He shows that, relative to one orbit, there exists only one Brouwer mapping class up to conjugacy: the class of the translation relative to one of its orbits. Relative to two orbits, he proves that they are exactly three Brouwer mapping classes (up to conjugacy): the class of the translation, the class of the time-one map R of the Reeb flow and the class of R^{-1}.

In [**LR13**], Le Roux gives a complete description of Brouwer mapping classes relative to three orbits and uses this description to define an index for Brouwer homeomorphisms. In particular, he shows that there are only a finite number of Brouwer mapping classes relative to three orbits, and that each of them contains the time-one map of a flow (see [**LR13**] for more details and the complete description of this classes).

The situation changes if we look at the Brouwer mapping classes relative to more than three orbits: indeed, if $r \geq 4$, there are a infinite number of Brouwer mapping classes relative to r orbits, and only a finite number of them contain the time-one map of a flow. One aim of this paper is to give a complete description of Brouwer mapping classes relative to four orbits.

1.3. *Walls.* In [**Han99**], Michael Handel defines reducing lines of a Brouwer mapping class $[h; \mathcal{O}]$: such a line is homotopic to its image by h and splits the set of orbits into two smaller sets. He proves that every Brouwer mapping class relative to more than one orbit has at least one reducing line [**Han99**, Theorem 2.7].

We propose to call the isotopy class of a reducing line Δ a *wall* if every other reducing line is homotopically disjoint from Δ. The set of walls is clearly a conjugacy invariant for Brouwer mapping classes. We prove the following theorem.

THEOREM 3.5. *Let $[h; \mathcal{O}]$ be a Brouwer mapping class. Let $\mathcal{W}$ be a family of pairwise disjoint reducing lines containing exactly one representative of each wall for $[h; \mathcal{O}]$. If Z is a connected component of $\mathbb{R}^2 - \mathcal{W}$, then exactly one of the following holds:*

- *Z is an irreducible area;*
- *Z is a maximal translation area; or*
- *Z does not intersect $\mathcal{O}$.*

Precise definitions of irreducible and maximal translation areas will be given in §3. A translation area Z is, in particular, an area which is invariant under h and on which h has very simple dynamics: indeed, up to conjugacy, h is conjugated to a homeomorphism

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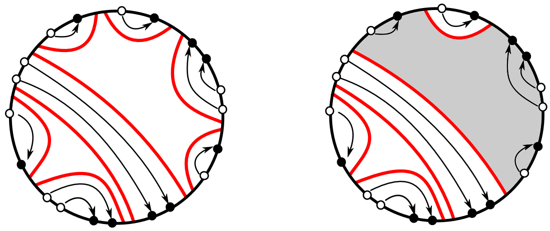

FIGURE 1. Examples of diagrams with walls: the one on the left is determinant, and the one on the right is non-determinant.

whose restriction to Z is a translation. Contrariwise, an irreducible area has more complex dynamics and cannot be reduced into more simple areas: it does not contain any reducing line. It will follow that if the complement of the walls of a Brouwer mapping class does not contain an irreducible area, then we will easily be able to understand the Brouwer mapping class, which will indeed be a time-one map of a flow (see §3). Moreover, we will prove that if the complement of the walls of a Brouwer mapping class has an irreducible area, then it also has at least two maximal translation areas.

1.4. *Diagrams.* Following, essentially [**Han99**, **LR13**], we can associate to every conjugacy class of Brouwer mapping class a unique diagram, for which a precise definition will be given in §3. This diagram is a disk with r arrows, where r is the number of orbits that we consider; each arrow represents an orbit. The cyclic order of the endpoints of the arrows is determined by the existence of a *nice family of homotopy translation arcs* (see §3). A diagram of Brouwer mapping class is said to be *determinant* if there exists only one conjugacy class of Brouwer mapping class associated to it. Every diagram for Brouwer mapping class relative to one, two or three orbits is determinant. For four orbits or more, there exist diagrams which are not determinant.

We can add the set of walls on this diagram: we obtain a *diagram with walls*, which is again a conjugacy invariant for Brouwer mapping classes (see Figure 1 for examples). This invariant is more precise than the diagram without walls, but it is still not total for Brouwer mapping classes relative to more than three orbits. Again, we can define the notion of *deteminant* diagram with walls (which corresponds to only one conjugacy class). We give an elementary combinatoric condition to identify the determinant diagrams among diagrams with walls without crossing arrows.

PROPOSITION 3.8. *A diagram with walls and without crossing arrows is determinant if and only if there is no irreducible area in the wall complement.*

1.5. *A total conjugacy invariant for Brouwer mapping class relative to four orbits.* For Brouwer mapping classes relative to four orbits, we add a new invariant to the non-determinant diagrams with walls: the *tangle*. This invariant is an isotopy class of curves on the cylinder with two marked points (up to horizontal twists). See Figure 2 for an example. Using, in particular, the set of walls and the description of determinant diagrams with four orbits, we get a total conjugacy invariant.

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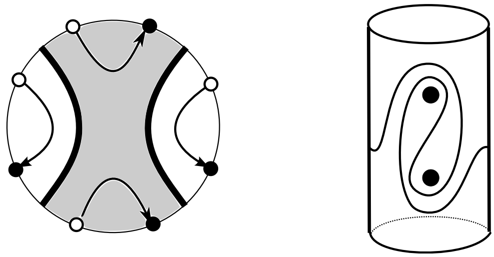

FIGURE 2. Example of a couple (diagram with walls, representative of the tangle).

THEOREM 3.13. *Two Brouwer mapping classes relative to four orbits are conjugated if and only if they have the same couple (diagram with walls, tangle).*

We finally get a complete description of Brouwer mapping classes relative to four orbits.

In the first section, we recall useful tools for homotopy Brouwer theory from [**Han99**, **LR13**]. A precise description of the results will be given in §3. The remainder of the text is devoted to proofs.

2. *First tools of homotopy Brouwer theory*

We recall the following definitions and properties (see [**Han99**, **LR12**, **LR13**]). Let $[h; \mathcal{O}]$ be a Brouwer mapping class: that is, the isotopy class of a Brouwer homeomorphism h relative to a finite set of orbits $\mathcal{O}$. Denote by r the number of orbits of $\mathcal{O}$. We choose a complete hyperbolic metric of the first kind on $\mathbb{R}^2 - \mathcal{O}$. Even if it is not explicitly specified, we will always consider complete hyperbolic metrics *of the first kind* on surfaces: that is, such that the surface is isomorphic to $\mathbb{H}^2/\Gamma$, where Γ is of the first kind (see [**Mat00**] for details).

2.1. *Examples: flows and product with a free half-twist.*

Flows. For abbreviation, we say that a homeomorphism f is a flow if it is the time-one map of a flow. If a Brouwer homeomorphism f is isotopic to a flow without fixed points relative to $\mathcal{O}$, then we say that $[f; \mathcal{O}]$ is a *flow class*.

Example A. The first example is the flow class of Figure 3. In this example, we choose 5 streamlines of a flow f and get a Brouwer mapping class relative to five orbits: $\mathcal{O}_1$, $\mathcal{O}_2$, $\mathcal{O}_3$, $\mathcal{O}_4$ and $\mathcal{O}_5$. We denote by $\mathcal{O}$ their union.

Free half-twist. We call a *half-twist* any homeomorphism that is:

- supported in a topological disk D of $\mathbb{R}^2$ which contains exactly two points of $\mathcal{O}$, denoted by x and y; and
- isotopic to a homeomorphism supported in D which is a rotation of a half turn on a disk included in D, which exchanges x and y.

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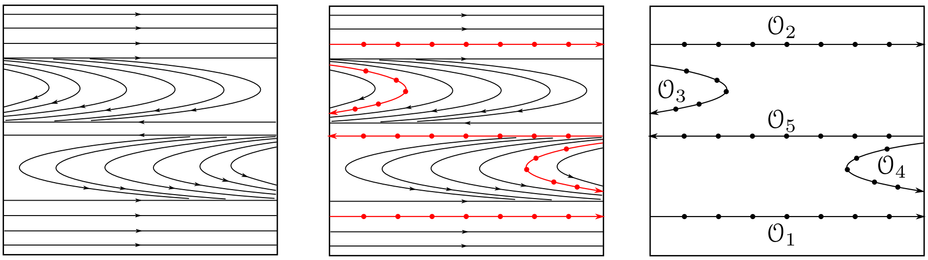

FIGURE 3. Example of a Brouwer mapping class $[f; \mathcal{O}]$ relative to five orbits.

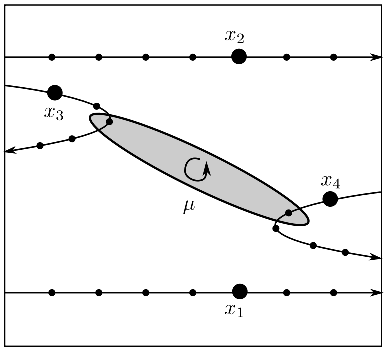

FIGURE 4. Example B: the Brouwer mapping class $[g; \mathcal{O}']$.

If h is a Brouwer homeomorphism, we call an *h-free half-twist* every half-twist μ supported in an h-free disk: that is, in a disk D such that $h^n(D) \cap D = \emptyset$ for every non-zero $n \in \mathbb{Z}$. Note that μh is a Brouwer homeomorphism.

Example B. Our second example is the product of f with the free half-twist μ of Figure 4, which exchanges the two points of the disk which are in $\mathcal{O}$. We denote the product μf by g. For $i = 1, 2, 3, 4$ we choose x_i on $\mathcal{O}_i$, as in Figure 4, and denote the g-orbit of x_i by $\mathcal{O}'_i$: that is, $\{g^n(x_i)\}_{n\in\mathbb{Z}}$. In particular, $\mathcal{O}_1 = \mathcal{O}'_1$ and $\mathcal{O}_2 = \mathcal{O}'_2$. We denote the union of the $\mathcal{O}'_i$ by $\mathcal{O}'$. Note that $\mathcal{O}'$ coincides with $\mathcal{O}_1 \cup \mathcal{O}_2 \cup \mathcal{O}_3 \cup \mathcal{O}_4$. We consider the Brouwer mapping class $[g; \mathcal{O}']$.

2.2. *Arcs and topological lines in* $\mathbb{R}^2 - \mathcal{O}$. We call an *arc* an embedding α of $]0, 1[$ in $\mathbb{R}^2 - \mathcal{O}$ such that it can be continuously extending to zero and one with $\alpha(0), \alpha(1) \in \mathcal{O}$. By abuse of notation, we will denote the image of the embedding $\alpha(]0, 1[)$ by α and call it an arc again. The extensions $\alpha(0)$ and $\alpha(1)$ are said to be the *endpoints* of α. A *topological line* is an embedding α of $\mathbb{R}$ in $\mathbb{R}^2 - \mathcal{O}$ which is proper in $\mathbb{R}^2$: that is, an embedding α such that for every compact K of the plane, there exists $t_0 \in \mathbb{R}$ such that if $|t| > |t_0|$, then $\alpha(t) \notin K$. Again, by abuse of notation, we denote by α the image of $\mathbb{R}$ by α in $\mathbb{R}^2 - \mathcal{O}$ and call it a (topological) line.

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2.3. *Isotopy classes of arcs and lines.* We say that two arcs (respectively, two lines) α and β are isotopic relative to $\mathcal{O}$ if there exists a continuous and proper application H : $]0, 1[\times[0, 1] \to \mathbb{R}^2 - \mathcal{O}$ such that:

- $H(\cdot, 0) = \alpha(\cdot)$ and $H(\cdot, 1) = \beta(\cdot)$; and
- if α and β are arcs, H can be continuously extended to $[0, 1] \times [0, 1]$ in such a way that the endpoints coincide: that is, for every $t \in]0, 1[$, $\alpha(0) = H(0, t) = \beta(0)$ and $\alpha(1) = H(1, t) = \beta(1)$.

If α is an arc or a line of $\mathbb{R}^2 - \mathcal{O}$, we denote the geodesic representative in the isotopy class of α relative to $\mathcal{O}$ by $\alpha_\#$. It is known that this geodesic representative is unique and that if α and β are two arcs or lines, then $\alpha_\#$ and $\beta_\#$ are in minimal position. In particular, if α and β are homotopically disjoint (that is, they have disjoint representatives in their isotopy classes), then $\alpha_\#$ and $\beta_\#$ are disjoint.

2.4. *Straightening principle.* We will need the following lemma, which is from [**Han99**, Lemma 3.5] (see also [**LR13**, Lemmas 1.4, 3.2 and Corollary 1.5]).

LEMMA 2.1. (Straightening principle) *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two locally finite families of lines and arcs of $\mathbb{R}^2 - \mathcal{O}$ such that:*

- *the elements of $\mathcal{F}_1$ (respectively, $\mathcal{F}_2$) are mutually homotopically disjoint; and*
- *if $\alpha \in \mathcal{F}_1$ and $\beta \in \mathcal{F}_2$, then α and β are in minimal position.*

Then the following statements hold.

(1) *There exists a homeomorphism u isotopic to* Id *relative to $\mathcal{O}$ such that, for every element γ in $\mathcal{F}_1 \cup \mathcal{F}_2$, $u(\gamma) = \gamma_\#$, where $\gamma_\#$ is the geodesic representative of the isotopy class of γ.*
(2) *If h is an orientation preserving homeomorphism of the plane such that $h(\mathcal{O}) = \mathcal{O}$, then there exists $h' \in [h; \mathcal{O}]$ such that, for every α in $\mathcal{F}_1 \cup \mathcal{F}_2$, $h'(\alpha_\#) = h(\alpha)_\#$.*

Remark 2.1.

- We get (2) by applying (1) to $h(\mathcal{F}_1 \cup \mathcal{F}_2)$.
- Note that the h' of (2) may not be a Brouwer homeomorphism.

2.5. *Homotopy translation arc.* A *homotopy translation arc* for $[h; \mathcal{O}]$ is an arc α such that:

- there exists $x \in \mathcal{O}$ such that $\alpha(0) = x$ and $\alpha(1) = h(x)$; and
- for every $n \in \mathbb{Z}$, $h^n(\alpha)$ is homotopically disjoint from α.

In particular, every translation arc for h with endpoints in $\mathcal{O}$ is a homotopy translation arc for $[h; \mathcal{O}]$. In general, there exist homeomorphisms h and arcs which are not homotopic to translation arcs for h, but which are homotopy translation arcs for $[h; \mathcal{O}]$.

Example A. Figure 5 (left) shows different homotopy translation arcs for example A: the arcs β and γ are homotopy translation arcs for $[f; \mathcal{O}]$. The arc α is not a homotopy translation arc for $[f; \mathcal{O}]$. Note that, however, if we forget the orbit $\mathcal{O}_5$, α is a homotopy translation arc for $[h; \mathcal{O}_1 \cup \mathcal{O}_2 \cup \mathcal{O}_3 \cup \mathcal{O}_4]$.

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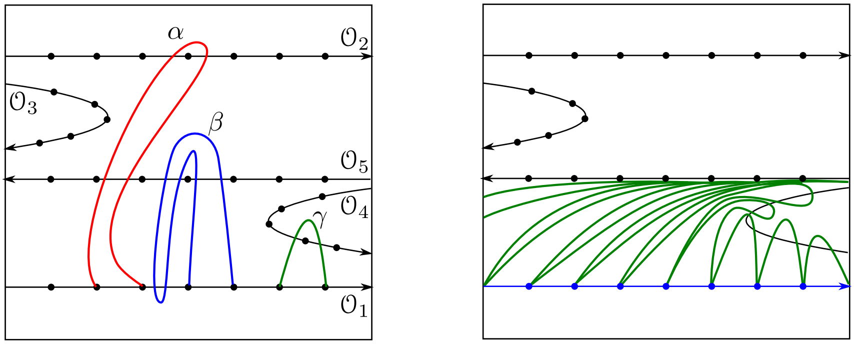

FIGURE 5. Examples of homotopy translation arcs and homotopy streamlines.

2.6. *Half homotopy streamlines.* If α is a homotopy translation arc for $[h; \mathcal{O}]$, we define the *homotopy streamline* by

$$T(\alpha, h, \mathcal{O}) := \bigcup_{n\in\mathbb{Z}} (h^n(\alpha([0, 1[)))_\#.$$

Since α is a homotopy translation arc, the geodesic iterates are mutually disjoint, and hence $T(\alpha, h, \mathcal{O})$ is an embedding of $\mathbb{R}$, which can eventually be non-proper.

We also define the *backward (respectively, forward) homotopy streamline* $T^-(\alpha, h, \mathcal{O})$ (respectively, $T^+(\alpha, h, \mathcal{O})$) by

$$T^-(\alpha, h, \mathcal{O}) := \bigcup_{n\leq 0} (h^n(\alpha([0, 1[)))_\#,$$

$$T^+(\alpha, h, \mathcal{O}) := \bigcup_{n\geq 0} (h^n(\alpha([0, 1[)))_\#.$$

Example A. The streamline $T(\beta, f, \mathcal{O})$ of example A is proper and coincides with the horizontal streamline which contains $\mathcal{O}_1$. The streamline $T(\gamma, h, \mathcal{O})$ is drawn on Figure 5. It is not proper, but $T^+(\gamma, h, \mathcal{O})$ is proper.

2.7. *Backward proper and forward proper arcs.* Let α be a homotopy translation arc. If the backward homotopy streamline $T^-(\alpha, h, \mathcal{O})$ is the image of $\mathbb{R}^-$ under a proper embedding, that is, if for every compact K of the plane there exists $n_0 \leq 0$ such that for every $n \leq n_0$, $(h^n(\alpha))_\#$ does not intersect K, then α is said to be a *backward proper arc* for $[h; \mathcal{O}]$. Similarly, if the forward homotopy streamline $T^+(\alpha, h, \mathcal{O})$ is proper, then α is said to be a *forward proper arc* for $[h; \mathcal{O}]$.

Example A. In the example of Figure 5, β is a backward proper and forward proper arc, and γ is a forward proper arc but it is not a backward proper arc. Note that if h is a flow, then every homotopy translation arc which lies on a flow streamline is backward proper and forward proper.

Example B. In Figure 6, we draw iterates of an arc lying on a streamline of f which intersects the support of the free half-twist μ. We see that this arc is backward proper but not forward proper (the iterates are 'stuck' by the orbit $\mathcal{O}'_1$).

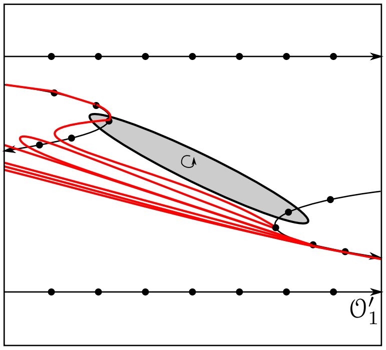

FIGURE 6. Example of a homotopy streamline for $[g; \mathcal{O}']$.

Translation. If a Brouwer homeomorphism f is conjugate to a translation relative to $\mathcal{O}$, then we say that $[f; \mathcal{O}]$ is a *translation class*. In this case, every homotopy translation arc for $[f; \mathcal{O}]$ is backward proper and forward proper.

2.8. *Nice family* $(\alpha_i^{\pm})_{1\leq i\leq r}$. A *nice family* $(\alpha_i^{\pm})_{1\leq i\leq r}$ associated to $[h; \mathcal{O}]$ is a family of homotopy translation arcs for $[h; \mathcal{O}]$ such that:

- for every $1 \leq i \leq r$:
  - α_i^- is a backward proper arc;
  - α_i^+ is a forward proper arc; and
  - α_i^- and α_i^+ have the same endpoints, lying in the orbit $\mathcal{O}_i$;
- the backward proper half-streamlines $T^-(\alpha_i^-, h, \mathcal{O})$ are mutually disjoint; and
- the forward proper half-streamlines $T^+(\alpha_i^+, h, \mathcal{O})$ are mutually disjoint.

Note that if $(\alpha_i^{\pm})_i$ is a nice family for a Brouwer mapping class $[h; \mathcal{O}]$, then the previous proper half-streamlines $T^{\pm}(\alpha_i^{\pm}, h, \mathcal{O})$ are mutually disjoint outside a topological disk of the plane.

Examples. Figure 7 give an example of a nice family for $[f; \mathcal{O}]$ with some arcs not homotopic to arcs included in streamlines, and an example of a nice family for $[g; \mathcal{O}']$. In particular, α_3^+ (respectively, α_4^-) is constructed with an iteration by g^{-1} (respectively, g) of an arc lying on the f-streamline of $\mathcal{O}_4$ after the support of μ (respectively, of an arc lying on the f-streamline of $\mathcal{O}_3$ before the support of μ).

The following theorem of Handel [**Han99**] allows us to consider a nice family for every Brouwer mapping class in the following sections.

THEOREM 2.2. [**Han99**] *For every* $[h; \mathcal{O}]$*, there exists a nice family associated to* $[h; \mathcal{O}]$.

Remark 2.2. For a statement closer to this one, see [**LR13**], proposition 3.1. Here we describe a way to deduce our statement from [**LR13**, Proposition 3.1], where the statement is given for *generalized homotopy half-streamlines*. As seen in Figure 8, in any open neighborhood of a generalized homotopy half-streamline there exist disjoint homotopy streamlines whose union contains every point of $\mathcal{O}$ included in the generalized homotopy half-streamline. It follows that the result is still true with the statement given here (that

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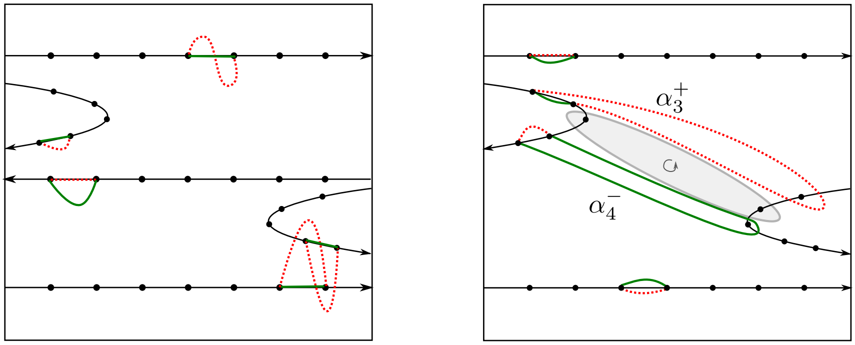

FIGURE 7. Example of nice families for $[f; \mathcal{O}]$ and $[g; \mathcal{O}']$.

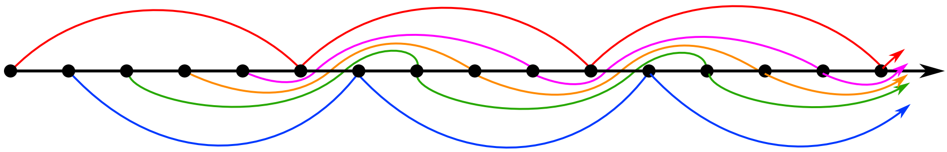

FIGURE 8. Finding disjoint homotopy half-streamlines in a neighborhood of a generalized homotopy half-streamline.

is, when we replace disjoint generalized homotopy half-streamlines by (non-generalized) disjoint homotopy half-streamlines).

2.9. *Reducing line.* We say that a line of $\mathbb{R}^2$ *splits* a given set of points X included in $\mathbb{R}^2 - \Delta$ if both connected components of $\mathbb{R}^2 - \Delta$ intersect X.

A *reducing line* Δ for $[h; \mathcal{O}]$ is a line in $\mathbb{R}^2 - \mathcal{O}$ such that $h(\Delta)$ is properly isotopic to Δ relative to $\mathcal{O}$ and such that Δ splits $\mathcal{O}$. Note that all the elements of a same orbit of $\mathcal{O}$ are included in the same connected component of $\mathbb{R}^2 - \Delta$. Indeed, according to the straightening principle 2.1, there exists $h' \in [h; \mathcal{O}]$ such that $h'(\Delta) = \Delta$. Figure 9 gives examples.

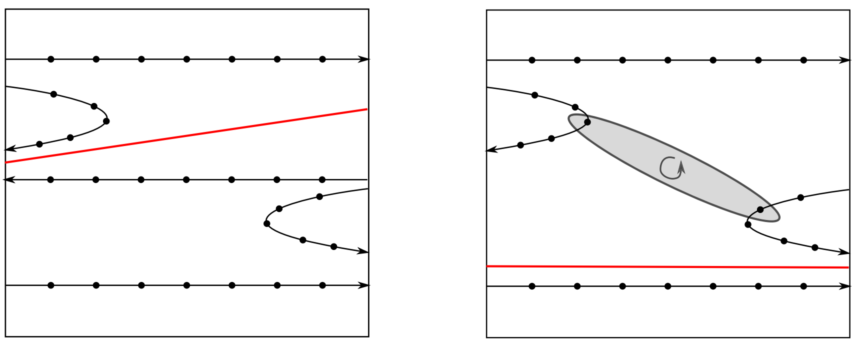

FIGURE 9. Examples of reducing lines for $[f; \mathcal{O}]$ and $[g; \mathcal{O}']$.

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We will need the following (Handel's) theorem. See [**LR13**, Proposition 3.3], for this formulation.

THEOREM 2.3. [**Han99**] *Let* $[h; \mathcal{O}]$ *be a Brouwer mapping class relative to more than one orbit. Let* $(\alpha_i^{\pm})_i$ *be a nice family for* $[h; \mathcal{O}]$. *There exists a reducing line for* $[h; \mathcal{O}]$ *which is disjoint from every backward proper half-streamline* $T^-(\alpha_i^-, h, \mathcal{O})$.

2.10. *Homotopy Brouwer line.* A *homotopy Brouwer line* L for $[h; \mathcal{O}]$ is a topological line in $\mathbb{R}^2 - \mathcal{O}$ such that:

- L is homotopically disjoint from $h(L)$;
- L is not isotopic to $h(L)$; and
- if we denote by V the connected components of $\mathbb{R}^2 - L_\#$ containing $h(L)_\#$, then we have $h(V)_\# \subset V$, where $h(V)_\#$ is the connected component of $\mathbb{R}^2 - h(L)_\#$ which does not contain $L_\#$.

This definition does not depend on the chosen metric on $\mathbb{R}^2 - \mathcal{O}$. Figure 10 gives examples of Brouwer lines for $[f; \mathcal{O}]$ and $[g; \mathcal{O}']$.

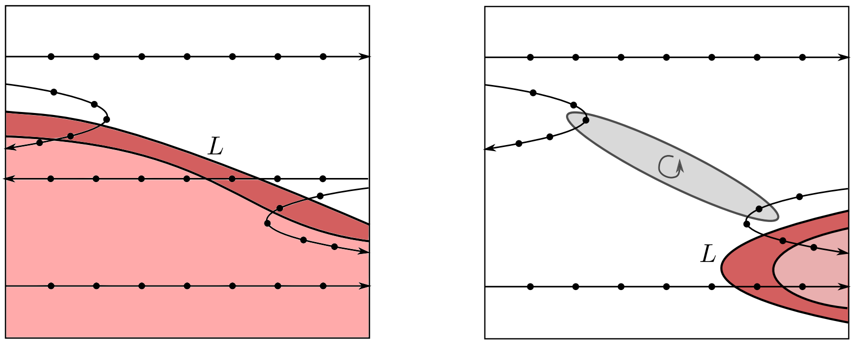

FIGURE 10. Examples of Brouwer lines for $[f; \mathcal{O}]$ and $[g; \mathcal{O}']$.

3. *Description of the results*

Here we give the main definitions and statements of the paper. Proofs will be given in the following sections.

3.1. *Adjacency areas, diagrams and special nice families (§4).*

3.1.1. *Cyclic order of a nice family.* Let $[h; \mathcal{O}]$ be a Brouwer mapping class and let $(\alpha_i^{\pm})_i$ be a nice family for $[h; \mathcal{O}]$. There is a natural cyclic order on the elements of the nice family $(\alpha_i^{\pm})_i$ given by the order of the homotopy half-streamlines generated by the $\alpha_i^{\pm}$ at infinity: if we choose a big enough topological circle which intersects each half-streamline only once, with transverse intersections, the order on the half-streamlines is given by the order of these intersections (which is independent of the choice of the circle). In the following, we will call this cyclic order the *cyclic order of the nice family*.

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3.1.2. *Adjacency.* If several forward proper arcs (respectively, several backward proper arcs), are consecutive for the cyclic order of the nice family, then they are said to be *adjacent*. A subfamily of the nice family consisting only of consecutive arcs of the same type (all backwards or all forwards) is said to be a *subfamily of adjacency*. If two orbits have forward proper arcs (respectively, backward proper arcs) in the same nice family which are adjacent, they are said to be *forward adjacent* (respectively, *backward adjacent*). The following proposition is essentially due to Handel [**Han99**] (a proof will be given in §4).

PROPOSITION 3.1. *Let* $[h; \mathcal{O}]$ *be a Brouwer mapping class. If* $(\alpha_i^{\pm})_i$ *and* $(\beta_i^{\pm})_i$ *are two nice families for this class, then they have the same cyclic order up to permutation of arcs of* $(\beta_i^{\pm})_i$ *inside the same subfamilies of adjacency.*

3.1.3. *Diagram associated to a Brouwer mapping class.* Using Proposition 3.1, we can associate a diagram to each Brouwer mapping class (see Figures 11 and 12 for examples).

(1) Let $[h; \mathcal{O}]$ be a Brouwer mapping class relative to r orbits. Choose a nice family $(\alpha_i^{\pm})_{1\leq i\leq r}$.
(2) On the boundary component of a disk, choose one point for each arc of $(\alpha_i^{\pm})_i$ in such a way that the $2r$ chosen points respect the cyclic order of $(\alpha_i^{\pm})_i$.
(3) For every i, draw an arrow from the point representing α_i^- to the point representing α_i^+. Label this arrow with i.
(4) Exchange the points in a same subfamily of adjacency, if necessary, to eliminate as many crossings as possible between the arrows.

We identify two diagrams if they have the same combinatorics.

PROPOSITION 3.2. *The diagram associated to a Brouwer mapping class is a conjugacy invariant: if two Brouwer mapping classes are conjugated, then they have the same diagram.*

We say that a diagram $\mathcal{D}$ is *determinant* if, up to conjugacy, there exists only one Brouwer mapping class whose associated diagram is $\mathcal{D}$. It is a natural question to ask which diagrams are determinant.

For Brouwer mapping classes relative to one, two and three orbits, the diagram is a total invariant: every diagram with one, two or three arrows is determinant (see [**Han99**] for one and two orbits and [**LR13**] for three orbits).

However, for Brouwer mapping classes relative to more than three orbits, the diagram is not a total invariant. For example, consider the flow f and the f-free half-twist μ of examples A and B. The Brouwer mapping classes $[\mu^2 f; \mathcal{O}']$ and $[f; \mathcal{O}']$ have the same diagram but are not conjugated (we will prove later that they are not conjugated). In §6, we will give combinatorial conditions on diagrams to prove that some of them are determinant. In §8 we will describe all the determinant diagrams for Brouwer mapping classes relative to four orbits.

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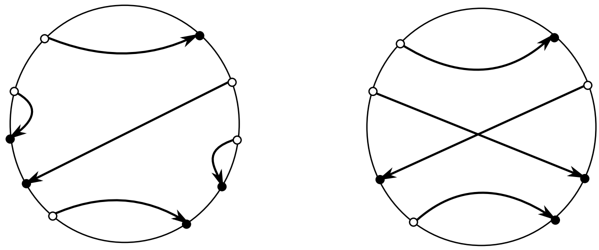

FIGURE 11. Diagrams associated to $[f;\mathcal{O}]$ and $[g;\mathcal{O}']$ (examples A and B of §2).

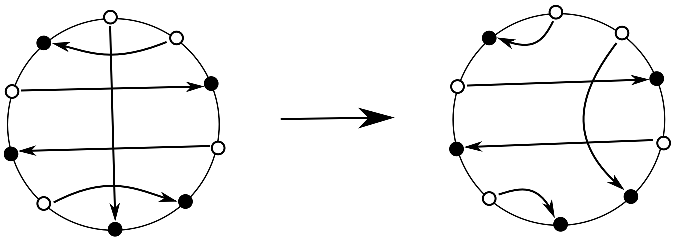

FIGURE 12. An example of step 4 (this is Example 2.9 of Handel [**Han99**]).

3.1.4. *Special nice families.* A *reducing set* is a union of mutually disjoint and non-isotopic reducing lines. Any connected component of the complement of a reducing set is said to be a *stable area*. In particular, every stable area for $[h;\mathcal{O}]$ is isotopic to its image by h relative to $\mathcal{O}$. If the reducing lines of a given reducing set $\mathcal{R}$ are geodesic, then, according to the straightening principle 2.1, there exists $h' \in [h;\mathcal{O}]$ such that for every stable area Z of the complement of $\mathcal{R}$ $h'(Z) = Z$. The following proposition will be proved in §4.3.

PROPOSITION 3.3. *Let $[h;\mathcal{O}]$ be a Brouwer mapping class. Let $(\Delta^k)_k$ be a reducing set. There exists a nice family $(\alpha_i^{\pm})_i$ for $[h;\mathcal{O}]$ such that, for every k, α_i^- and α_i^+ are homotopically disjoint from Δ^k for every i.*

3.2. *Walls for a Brouwer mapping class (§5).* We define a canonical reducing set (the walls) and prove that this set splits the plane into three types of stable area: stable areas disjoint from $\mathcal{O}$, translation areas and irreducible areas. In the next section, we will give combinatorial conditions for the existence of irreducible areas.

3.2.1. *Translation areas.* Henceforth, if Z is a stable area of $[h;\mathcal{O}]$, whenever we say 'orbits of Z', it means 'orbits of $\mathcal{O}$ contained in Z'.

Definition 3.1. (Translation area) Let $[h;\mathcal{O}]$ be a Brouwer mapping class. We say that a stable area Z of $[h;\mathcal{O}]$ is a *translation area* if all the orbits of Z are backward adjacent and forward adjacent for $[h;\mathcal{O}]$.

Moreover, a translation area Z is said to be *maximal* if there exists no translation area Z' non-isotopic to Z and is such that $Z \subset Z'$.

Note that every translation area is included in a maximal translation area.

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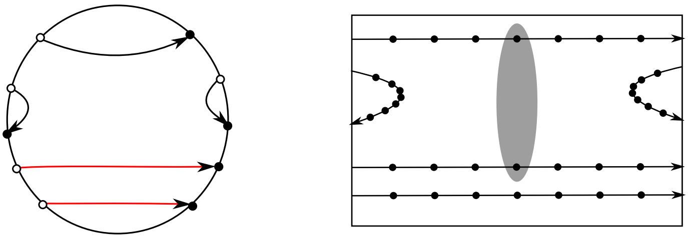

FIGURE 13. Example of a Brouwer class whose diagram has two backward and forward adjacent arrows which are not in the same translation area.

Remark 3.1. The orbits of a same translation area for a Brouwer mapping class are represented by arrows which are backward and forward adjacent in the diagram associated to the Brouwer class. However, some arrows which are backward and forward adjacent do not represent orbits of a translation area. For an example, see Figure 13: the Brouwer class that we consider is the product of a flow with a double free half-twist between the two arrows that intersect the gray disk of the figure. The two arrows on the bottom of the diagram are backward and forward adjacent but not in the same translation area.

The following proposition implies that every Brouwer class has flow streamlines on its translation areas. It will be proved in §5.1.

PROPOSITION 3.4. *If Z is a translation area, every backward (respectively, forward) proper arc of a nice family which is included in Z is also forward (respectively, backward) proper.*

3.2.2. *Irreducible areas.*

Definition 3.2. (Irreducible area) Let $[h; \mathcal{O}]$ be a Brouwer mapping class. We say that a stable area Z of $[h; \mathcal{O}]$ is an *irreducible area* if:

- Z contains at least two orbits of $\mathcal{O}$; and
- there is no reducing line of $[h; \mathcal{O}]$ strictly included in Z (that is, homotopically disjoint from every boundary component of Z and non-isotopic to any of those).

3.2.3. *Walls.*

Definition 3.3. (Wall) Let $[h; \mathcal{O}]$ be a Brouwer mapping class. An isotopy class of a reducing line Δ for $[h; \mathcal{O}]$ is called a *wall for* $[h; \mathcal{O}]$ if every reducing line for $[h; \mathcal{O}]$ is homotopically disjoint from Δ.

The proof of the following theorem is the aim of §5.

THEOREM 3.5. *Let $[h; \mathcal{O}]$ be a Brouwer mapping class. Let $\mathcal{W}$ be a family of pairwise disjoint reducing lines containing exactly one representative of each wall for $[h; \mathcal{O}]$. If Z is a connected component of $\mathbb{R}^2 - \mathcal{W}$, then exactly one of the following holds:*

- *Z is an irreducible area;*
- *Z is a maximal translation area; or*
- *Z does not intersect $\mathcal{O}$.*

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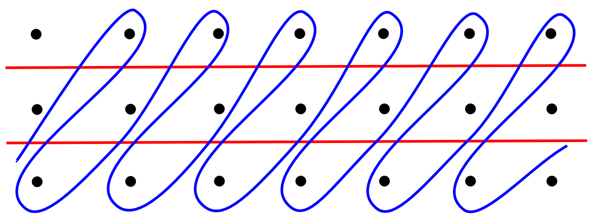

FIGURE 14. Examples of reducing lines for a translation relative to three orbits.

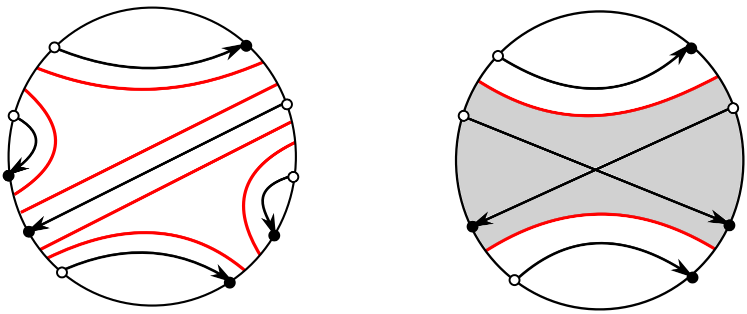

FIGURE 15. Diagrams with walls associated to $[f;\mathcal{O}]$ and $[g;\mathcal{O}']$ (examples A and B of §2).

Remark 3.2. A maximal translation area has either one or two boundary components.

Remark 3.3. The set of walls is empty if and only if $[h;\mathcal{O}]$ is a translation class.

Proof. Assume that $[h;\mathcal{O}]$ is a translation class relative to $r \geq 2$ orbits, then it is conjugated to the unitary horizontal translation relative to $\mathbb{Z} \times \{1, \ldots, r\}$. Hence there are $r - 1$ horizontal non-homotopic reducing lines (see Figure 14). Every non-horizontal reducing line intersects at least one horizontal line and, for every horizontal line Δ, there exists at least one reducing line which intersects Δ (see Figure 14). Thus the set of walls of $[h;\mathcal{O}]$ is empty.

Conversely, assume that the set of walls of $[h;\mathcal{O}]$ is empty; then, according to Theorem 3.5, R^2 is either an irreducible area or a maximal translation area. But, according to Theorem 2.3, every Brouwer mapping class relative to more than one orbit admits at least one reducing line, and hence $\mathbb{R}^2$ is never an irreducible area. Thus $[h;\mathcal{O}]$ is a translation class. □

Since there exists a nice family disjoint from the walls (according to Proposition 3.3), it makes sense to add the set of walls on the diagram defined in §4 (see Figure 15 for two examples). We can see the maximal translation areas on this diagram: the backward ends of their arrows are adjacent in the diagram and the forward ends of their arrows are adjacent in the diagram. Consequently, according to Theorem 3.5, we can also see the irreducible areas. To help the reader, we have colored the irreducible areas gray. The resulting *diagram with walls* is a conjugacy invariant of the Brouwer mapping class which is more precise than the diagram (without walls), but still not total. In the next section we will give conditions to determine which diagrams with walls are determinant.

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3.3. *Determinant diagrams and irreducible areas (§6).*

3.3.1. *Determinant diagrams.* The following propositions motivate the search for necessary combinatoric conditions on diagrams (without walls) for the existence of irreducible areas. They will be proved in §6.1.

PROPOSITION 3.6. *A Brouwer mapping class* $[h; \mathcal{O}]$ *is a flow class if and only if no connected component of the complement of the set of walls for* $[h; \mathcal{O}]$ *is an irreducible area.*

PROPOSITION 3.7. *If two flow classes have the same diagram, then they are conjugated.*

PROPOSITION 3.8. *A diagram with walls and without crossing arrows is determinant if and only if there is no irreducible area in the wall complement.*

3.3.2. *Combinatorics of irreducible areas.* If $[h; \mathcal{O}]$ is a Brouwer class relative to r orbits, we denote by $2r'$ the number of adjacency subfamilies of $[h; \mathcal{O}]$. If $r' = r$, then we say that the orbits of $[h; \mathcal{O}]$ alternate (in this situation, every adjacency subfamily has only one element). We will prove Proposition 3.9 in §6.2.

PROPOSITION 3.9. (Combinatorics of irreducible areas) *Let* $[h; \mathcal{O}]$ *be a Brouwer mapping class and let* Z *be an irreducible area for* $[h; \mathcal{O}]$*. Then:*

(1) *the orbits of* Z *are not all backward adjacent, nor all forward adjacent for* $[h; \mathcal{O}]$*;*
(2) Z *has at least two boundary components; and*
(3) *the orbits of* $[h; \mathcal{O} \cap Z]$ *do not alternate.*

This proposition gives tools for knowing which diagrams are determinant. In particular, we have the following corollary, which will be proved in §6.3.

COROLLARY 3.10. *Let* $[h; \mathcal{O}]$ *be a Brouwer mapping class relative to* r *orbits. Denote by* $2r'$ *the number of adjacency subfamilies of* $[h; \mathcal{O}]$*. If* $r' = 1$, 2 *or* r*, then* $[h; \mathcal{O}]$ *is a flow class.*

Remark 3.4. It was proved in [**Han99**] (for $r' = 1$) and [**LR13**] (for $r' = r$) that if $r' = 1$ or $r' = r$, then $[h; \mathcal{O}]$ is a flow class (see [**LR13**, Proposition 3.1 and Lemma 3.6]).

COROLLARY 3.11. *Let* $[h; \mathcal{O}]$ *be a Brouwer mapping class relative to* r *orbits.*

(1) *If* $r \geq 3$*, there exist at least two disjoint and non-isotopic reducing lines for* $[h; \mathcal{O}]$*.*
(2) *If* $r \geq 2$*, there exist at least two translation areas for* $[h; \mathcal{O}]$ *which have exactly one boundary component.*
(3) *There exists a nice family* $(\alpha_i^{\pm})_i$ *and* $j \neq k$ *such that:*
   - *relative to* $\mathcal{O}$, α_j^- *is isotopic to* α_j^+ *and* α_k^- *is isotopic to* α_k^+*; and*
   - *in the cyclic order,* α_j^- *and* α_j^+ *(respectively,* α_k^- *and* α_k^+*) are neighbors.*

The third point of Corollary 3.11 can be reformulated as follows.

COROLLARY 3.12. *Let* $[h; \mathcal{O}]$ *be a Brouwer class relative to* $r \geq 2$ *orbits. There exist at least two disjoint backward and forward proper streamlines* T *and* S *for* $[h; \mathcal{O}]$*.*

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Moreover, the orbits which are neither in T nor S are included in the same connected component of the complement of $S \cup T$.

3.4. *Classification relative to four orbits (§8).* The aim of §8 is to give a complete description of Brouwer mapping classes relative to four orbits. We first find every diagram with walls which are not determinant. For the Brouwer mapping classes with a non-determinant diagram, we define a new conjugacy invariant, *the tangle* (see §8.2.2). This tangle is an isotopy class of curves on the cylinder with two marked points, up to horizontal twists (see Figure 16 for an example). We set that the tangle of Brouwer mapping classes without irreducible area is the empty set. We claim that the couple (diagram with walls, tangle) is a total conjugacy invariant for Brouwer mapping classes relative to four orbits.

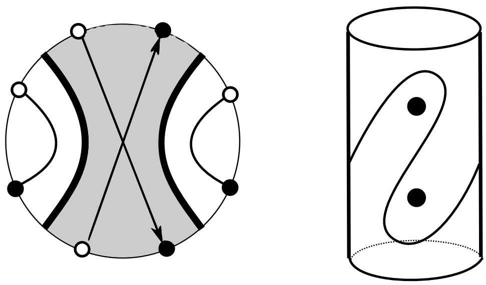

FIGURE 16. Example of a couple (diagram with walls, representative of the tangle).

THEOREM 3.13. *Two Brouwer mapping classes relative to four orbits are conjugated if and only if they admit the same couple (diagram with walls, tangle).*

We describe which tangles are realized by Brouwer mapping classes and call them *adapted tangles*.

Finally, every couple (diagram with walls, adapted tangle) is realized by:

- a flow if the diagram is determinant or if the tangle is trivial; and
- a product of a flow and a finite number of free half-twists if the diagram is not determinant and the tangle is not trivial.

This gives a complete description of the Brouwer mapping classes relative to four orbits.

4. *Adjacency areas, diagrams and special nice families*

4.1. *Adjacency areas.* Let $[h; \mathcal{O}]$ be a Brouwer mapping class. Let $(\alpha_i^{\pm})_{1 \leq i \leq r}$ be a nice family. Let $\{\alpha_{i_1}^{+}, \ldots, \alpha_{i_n}^{+}\}$ be a subfamily of adjacency for $[h; \mathcal{O}]$. For simplicity of notation, we assume that $i_k = k$ for every $1 \leq k \leq n$. Choose a complete hyperbolic metric on $\mathbb{R}^2 - \mathcal{O}$. Let L be a geodesic topological line in $\mathbb{R}^2 - \mathcal{O}$ such that (see Figure 17):

- one connected component of $\mathbb{R}^2 - L$, denoted by A, contains an infinite component of each T_i^{+} for $1 \leq i \leq n$;
- for every $1 \leq i \leq n$, L intersects $T_i^{+} := T^{+}(\alpha_i^{+}, h, \mathcal{O})$ at exactly one point; and
- A does not contain any point of $\mathcal{O}$ outside those n half-streamlines.

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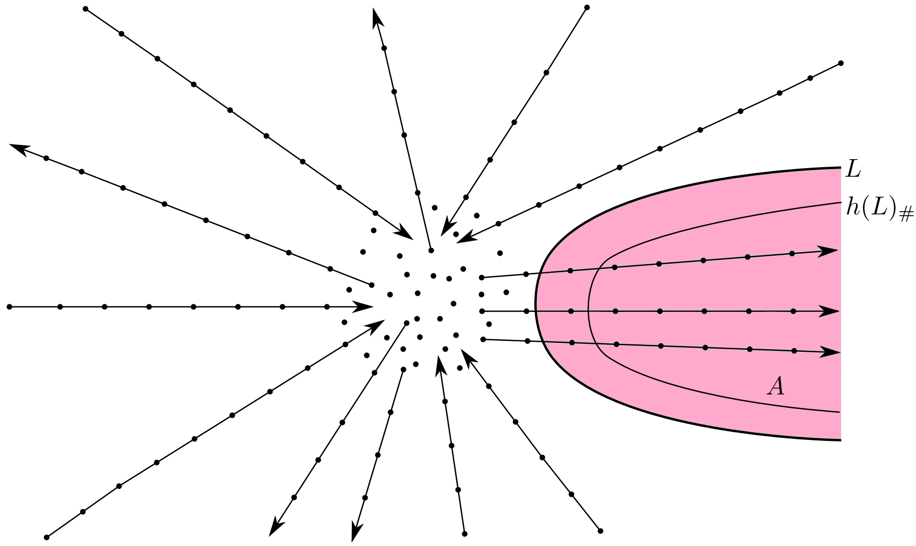

FIGURE 17. Example of a forward adjacency area.

Definition 4.1. (Adjacency area) With the previous notation, we say that A is a *forward adjacency area* for $[h; \mathcal{O}]$. A forward adjacency area for $[h^{-1}; \mathcal{O}]$ is said to be a *backward adjacency area* for $[h; \mathcal{O}]$.

Note that every adjacency area can be obtained with the following construction.

Construction of an adjacency area. With the previous notation, suppose that the α_i^+ for $i = 1, \ldots, n$ are ordered from α_1^+ to α_n^+ in the cyclic order of the nice family. Choose one point $x_i \in \mathcal{O}_i$ in every T_i^+. Denote by L_i the unbounded component of $T_i^+ - \{x_i\}$.

- If $n = 1$, let $\mathcal{U}$ be an open neighborhood of L_1 which is homotopic to L_1 relative to $\mathcal{O}$.
- If $n > 1$, denote by γ_i a geodesic arc of $\mathbb{R}^2 - \mathcal{O}$ which admits x_i and x_{i+1} as endpoints for every $i \leq n - 1$, and such that one connected component of $L_i \cup \gamma_i \cup L_{i+1}$ does not intersect $\mathcal{O}$. In particular, note that $\{h^n(\gamma_i)_\#\}_{n \geq 0}$ is locally finite. Now consider the line $\tilde{L} := L_1 \cup \gamma_1 \cup \cdots \cup \gamma_n \cup L_n$. Let $\mathcal{V}$ be the connected component of $\mathbb{R}^2 - \tilde{L}$ which contains L_2 if $n \geq 3$ and which does not intersect $\mathcal{O}$ if $n = 2$. Let $\mathcal{U}$ be an open neighborhood of $\mathcal{V}$, isotopic to $\mathcal{V}$ relative to $\mathcal{O}$.

Assume that the boundary component L of the closure of $\mathcal{U}$ is geodesic: then $\mathcal{U}$ is an adjacency area. Since there exist pairwise disjoint homotopy half-streamlines (Theorem 2.2), there exist pairwise disjoint adjacency areas whose union contains an infinite component of every half-orbit (see Figure 18).

Definition 4.2. Choose an adjacency area for every subfamily of adjacency such that the chosen areas are mutually disjoint. Such a family is said to be a *complete family of adjacency areas for* $[h; \mathcal{O}]$.

In particular, the union of the elements of a complete family of adjacency areas contains every point of $\mathcal{O}$ (but a finite number). Moreover, if we consider two complete families of

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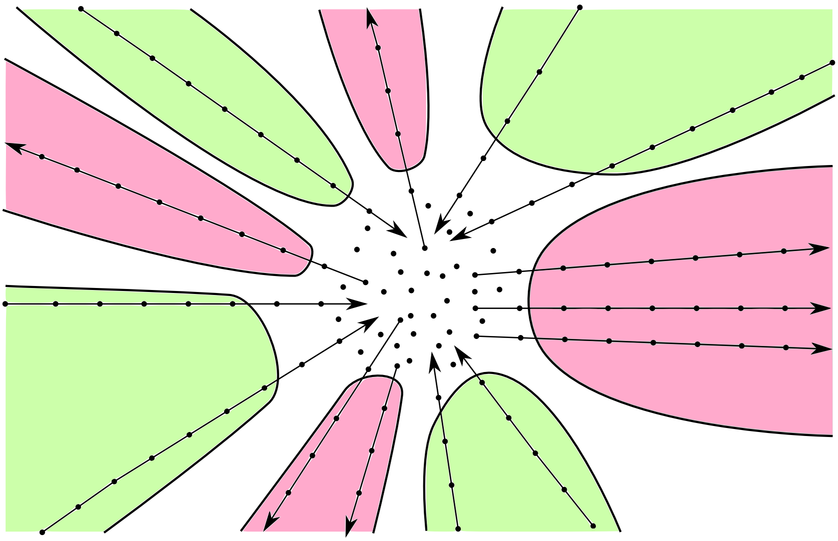

FIGURE 18. Example of a complete family of backward and forward adjacency areas.

adjacency areas, then there exists a compact K such that these two families are isotopic relative to $\mathcal{O}$ in $\mathbb{R}^2 - K$.

The following theorem is essentially due to Handel [**Han99**]. This statement is a reformulation of [**LR13**, Proposition 3.1].

THEOREM 4.1. (Handel) *Let* $[h; \mathcal{O}]$ *be a Brouwer class which is not a translation class. Choose a complete family of adjacency. There exists a reducing line disjoint from all the backward adjacency areas of the complete family.*

Proof. There exists a family of generalized homotopy half-streamlines such that, for every backward adjacency area A of the complete family, $\mathcal{O} \cap A$ is included in one of the backward generalized homotopy half-streamlines. Proposition 3.1 of [**LR13**] gives a reducing line disjoint from all the backward generalized homotopy half-streamlines of the family. The result follows. □

PROPOSITION 4.2. *Let* A *be an adjacency area for* $[h; \mathcal{O}]$ *and let* L *be its boundary component.*

(1) *L is a homotopy Brouwer line.*
(2) *The family* $(h^k(L)_\#)_{k\in\mathbb{N}}$ *is locally finite.*
(3) *(Handel) Let* β^+ *be any forward proper arc with endpoints in an orbit of* $\mathcal{O}$ *which intersects* A*. There exists* k_0 *such that for every* $k > k_0$, $h^k(\beta^+)_\#$ *is included in* A.

Proof. Every adjacency area can be seen as in the previous construction. Thus (1) and (2) follow, because $\{h^n(\gamma_i)_\#\}_{n\geq 0}$ is locally finite for every i, as well as $\{h^n(L_i)_\#\}_{n\geq 0}$. The constructed $\mathcal{U}$ is a neighborhood of a 'generalized homotopy streamline' (see [**Han99**] and Figure 19), and hence property (3) holds, according to [**Han99**, Lemma 4.6]. □

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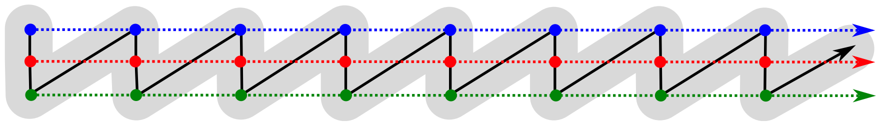

FIGURE 19. Neighborhoods of generalized homotopy half-streamlines are homotopic to adjacency areas.

COROLLARY 4.3. *Let* $[h; \mathcal{O}]$ *be a Brouwer mapping class, and let* A *be an adjacency area for* $[h; \mathcal{O}]$. *Then there exists a homeomorphism* χ *of the plane, preserving the orientation, such that* $\chi h \chi^{-1}$ *is isotopic relatively to* $\chi(\mathcal{O})$ *to a homeomorphism which coincides with a translation on* $\chi(A)$.

Proof. This follows from (1) and (2) of Proposition 4.2. □

4.2. *Diagrams.*

Proofs of Propositions 3.1 and 3.2. Let $(\alpha_i^{\pm})_i$ and $(\beta_i^{\pm})_i$ be two nice families for a Brouwer class $[h; \mathcal{O}]$. According to (3) of Proposition 4.2, for every i there exists an adjacency area A and there exists $k_0 \in \mathbb{N}$ such that for every $k > k_0$, $h^k(\alpha_i^+)$ and $h^k(\beta_i^+)$ are included in A. We have a similar result for backward arcs. It follows that $(\alpha_i^{\pm})_i$ and $(\beta_i^{\pm})_i$ have the same cyclic order up to permutation of arcs of $(\beta_i^{\pm})_i$ inside the same subfamilies of adjacency, which is Proposition 3.1.

As a corollary, we get Proposition 3.2: if two Brouwer mapping classes are conjugated, then they have the same diagram. □

4.3. *Special nice families.*

The aim of this section is to prove Proposition 3.3, that is, that for every reducing set $\mathcal{R}$, there exists a nice family which is disjoint from $\mathcal{R}$.

4.3.1. *Intersections between reducing lines and adjacency areas.*

LEMMA 4.4. *Let* $[h; \mathcal{O}]$ *be a Brouwer mapping class, with a complete family of adjacency areas. Let* Δ *be a reducing line. Then* Δ *is isotopic relative to* $\mathcal{O}$ *to a topological line* Δ' *which intersects at most two adjacency areas of the family.*

Moreover, for any complete hyperbolic metric on $\mathbb{R}^2 - \mathcal{O}$*, the intersection between the geodesic representative of* Δ *for this metric and any adjacency area has at most a finite number of connected components.*

Proof. Let $(\alpha_i^{\pm})_i$ be a nice family for $[h; \mathcal{O}]$. Choose a complete hyperbolic metric on $\mathbb{R}^2 - \mathcal{O}$. For every i, we denote by T_i^+ (respectively, T_i^-), the homotopy half-streamline $T^+(\alpha_i^+, h, \mathcal{O})$ (respectively, $T^-(\alpha_i^+, h, \mathcal{O})$). According to the straightening principle 2.1, there exists $h' \in [h; \mathcal{O}]$ such that $h'(T_i^+) \subset T_i^+$, $T_i^- \subset h'(T_i^-)$ and $h'(\Delta_\#) = \Delta_\#$.

CLAIM 1. *Let* A *be an adjacency area. We denote by* ∂A *its boundary component. If* $\Delta_\# \cap \partial A$ *is non-empty, then* $\Delta_\# \cap h'^n(\partial A)_\#$ *is non-empty for every* $n \in \mathbb{Z}$.

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Proof of the Claim 1. Since $\Delta_\# \cap \partial A$ is non-empty, $h'^n(\Delta_\#) \cap h'^n(\partial A)$ is non-empty for every $n \in \mathbb{Z}$. Since $\Delta_\#$ and ∂A are geodesic, they are in minimal position. Hence for every n, $h'^n(\Delta_\#)$ and $h'^n(\partial A)$ are also in minimal position. It follows that $h'^n(\Delta_\#) \cap h'^n(\partial A)_\#$ is non-empty. □

CLAIM 2. *Let A be an adjacency area. If $\Delta_\# \cap A$ is non-empty, then for every compact subset K of the plane, $(\Delta_\# \cap A) - K$ is non-empty.*

Proof of the Claim 2. Assume A is a forward adjacency area (if not, consider h'^{-1} instead of h). Let K be any compact subset of the plane. Assume that $\Delta_\# \cap A$ is non-empty. Since $(h'^n(\partial A)_\#)_{n\in\mathbb{N}}$ is locally finite (according to (2) of Proposition 4.2), there exists $k \in \mathbb{N}$ such that $h'^k(\partial A)_\#$ does not intersect K. Since ∂A is a homotopy Brouwer line (according to (1) of Proposition 4.2), $h'^k(\partial A)_\#$ is included in A. According to Claim 1, $h'^k(\partial A)_\#$ intersects $\Delta_\#$. Claim 2 follows. □

Denote by $(A_i)_{1\le i\le l}$ the adjacency areas of the chosen complete family. According to Claim 1, if we prove that for some $(n_i)_i \in \mathbb{Z}^l$, $\Delta_\#$ intersects at most two of the $h'^{n_i}(\partial A_i)_\#$, then $\Delta_\#$ intersects at most two of the ∂A_i. Hence, up to replacing $(A_i)_{1\le i\le l}$ by $(h'^{n_i}(\partial A_i)_\#)_{1\le i\le l}$ such that $\Delta_\#$ intersects at most two of the $h'^{n_i}(\partial A_i)_\#$, we can assume that for every i, j, A_i is disjoint from α_j^-.

CLAIM 3. *There exists a topological disk K of the plane such that every connected component of $\Delta_\# - K$ intersects at most one adjacency area.*

Proof of the Claim 3. Assume, by contradiction, that for every K, one connected component of $\Delta_\# - K$ intersects two adjacency areas. Then there exist two adjacency areas, say A_i^- and A_j^+, such that $\Delta_\# \cap A_i^-$ and $\Delta_\# \cap A_j^+$ have an infinite number of connected components. Moreover, taking K which intersects every adjacency area of the complete family, we can suppose that A_i^- follows A_j^+ in the cyclic order at infinity of the adjacency areas. Hence we can suppose that A_i^- is a backward adjacency area, and A_j^+ is a forward adjacency area.

It follows that there exists a subsegment γ of $\Delta_\#$ such that γ is the concatenation of three smaller subsegments $\gamma_1 \star \gamma_2 \star \gamma_3$ such that (see Figure 20):

- $\gamma_1 \subset A_i^-$ and its endpoints are included in ∂A_i^-;
- $\gamma_3 \subset A_j^+$ and its endpoints are included in ∂A_j^+; and
- γ_2 does not intersect $\mathring{A}_i^-$ not $\mathring{A}_j^+$, where $\mathring{A}$ denotes the interior $A - \partial A$ of A.

Moreover, we can choose γ outside any chosen topological disk of the plane: in particular, we choose it disjoint from the α_i^-. Since $\Delta_\#$ and the boundary components of the adjacency areas are in minimal position, it follows that:

- γ_1 intersects a backward homotopy half-streamline T_i^- of A_i^-;
- γ_2 does not intersect any homotopy half-streamline of the family $(T_k^\pm)_k$; and
- γ_3 intersects a forward homotopy half-streamline T_j^+ of A_j^+.

Hence there exists a subsegment δ of γ, which contains γ_2, such that its endpoints are in T_i^- and T_j^+ but its interior does not intersect any $T_k^\pm$. Since h' acts as a translation on the $T_k^\pm$, it follows that $h'(\delta)$ intersects δ. This gives a contradiction because $\Delta_\#$ is invariant by h' and without self-intersection (see Figure 21). □

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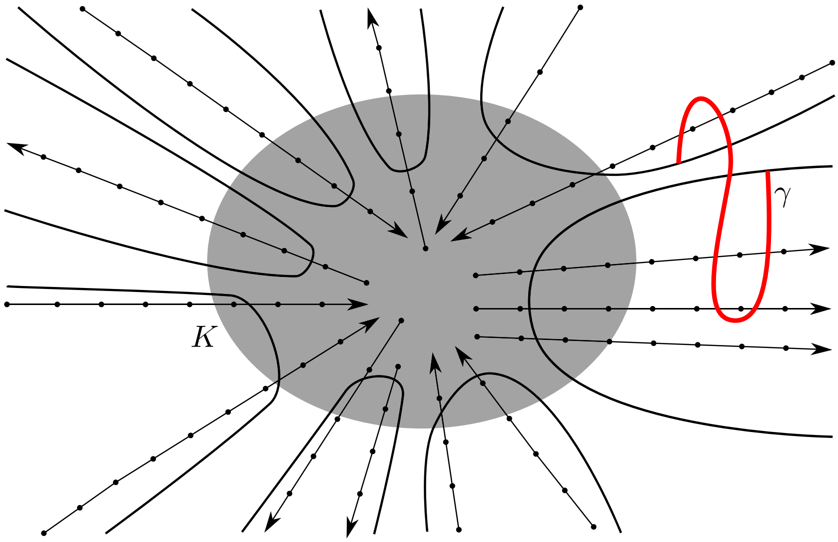

FIGURE 20. Example of a configuration with some K and some γ.

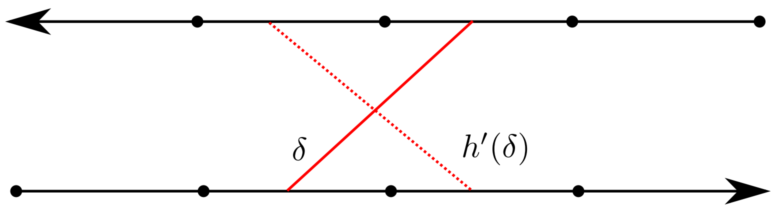

FIGURE 21. Image of δ under h'.

Let K be a topological disk given by Claim 3. Since $\Delta_\#$ is proper, there are only two unbounded connected components in $\Delta_\# - K$. According to Claim 2, as $\Delta_\#$ and the boundary components of the adjacency areas are in minimal position, every connected component of $\Delta_\# - K$ which intersects an adjacency area is unbounded. Hence $\Delta_\#$ intersects at most two adjacency areas. Moreover, since $\Delta_\#$ and ∂A_k are geodesics, the second part of the lemma follows. □

Remark. If a reducing line intersects two adjacency areas, it does not necessarily intersect one backward adjacency area and one forward adjacency area: some reducing lines intersect two adjacency areas of the same type (see Figure 22 for an example).

The following lemma will be used in the proof of (2) of Proposition 3.9.

LEMMA 4.5. *Let $[h; \mathcal{O}]$ be a Brouwer class which is not a translation class. Choose a complete family of adjacency areas. There exists a reducing line Δ such that the intersection of Δ with the complement of the forward adjacency areas is bounded.*

Proof. Let $(\alpha_i^{\pm})_i$ be a nice family such that each α_i^- intersects the boundary component of a backward adjacency area of the chosen family (such a family exists: to find it, we can

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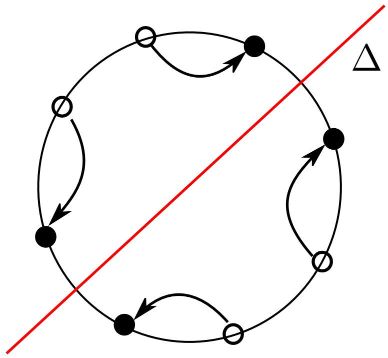

FIGURE 22. Example. Let $[f;\mathcal{O}]$ be a flow class with this diagram: the reducing line Δ intersects two forward adjacency areas.

take well-chosen iterates of the arcs of any nice family). Since $[h;\mathcal{O}]$ is not a translation class, according to Theorem 4.1, there exists a reducing line Δ which is disjoint from every backward adjacency area. Choose a complete hyperbolic metric on $\mathbb{R}^2-\mathcal{O}$. Suppose Δ is geodesic. Let K be a topological compact disk of the plane such that:

- the boundary component ∂K of K is geodesic;
- K intersects every adjacency areas of the family; and
- the union $\mathcal{A}$ of K with all the adjacency areas of the family contains $\mathcal{O}$.

Since Δ is proper, it intersects ∂K only a finite number of times. Denote the two unbounded components of $\Delta-K$ by Δ^+ and Δ^-. Since Δ^+ and Δ^- are disjoint from the backward adjacency areas, we can isotopy each of them, if necessary, to include Δ in a forward adjacency area. □

4.3.2. *Proof of Proposition 3.3.*

PROPOSITION 3.3. *Let $[h;\mathcal{O}]$ be a Brouwer mapping class. Let $(\Delta^k)_k$ be a reducing set. There exists a nice family $(\alpha_i^{\pm})_i$ for $[h;\mathcal{O}]$ such that for every k, α_i^- and α_i^+ are homotopically disjoint from Δ^k for every i.*

Proof. Choose a complete hyperbolic metric on $\mathbb{R}^2-\mathcal{O}$. As usual, we denote by $L_\#$ the geodesic representative of any line or arc L. Choose a complete family of adjacency areas for $[h;\mathcal{O}]$. In every adjacency area, we will construct pairwise disjoint backward or forward proper arcs for each orbit which intersects the area, such that every constructed arc is disjoint from the reducing set. By iterating those arcs so that the backward and forward arcs of a given orbit have the same endpoints, we get the required nice family. Let A be an adjacency area of the complete family of adjacency. Suppose A is a forward adjacency area (if not, consider h^{-1}). Note that ∂A is geodesic (by definition). For simplicity of notation, we assume that $\Delta^1,\ldots,\Delta^N$ are the reducing lines of $(\Delta^k)_k$ intersecting A. We assume that these reducing lines are geodesic. According to Lemma 4.4, each of them has an unbounded connected component included in A. Denote this unbounded component for Δ^k by Δ^k_+. Suppose that $\mathcal{O}_1,\ldots,\mathcal{O}_m$ are the orbits of $\mathcal{O}$ which intersect A. We will

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find m mutually disjoint forward proper arcs $(\beta_i^+)_{1\le i\le m}$ for $[h; \mathcal{O}]$, included in A and homotopically disjoint from Δ^k for every k.

Applying the straightening principle 2.1 to the families $(\Delta^k)_k$ and $(h^n(\partial A))_{n\ge 0}$, we can find $h' \in [h; \mathcal{O}]$ such that $h'(\Delta^k) = \Delta^k$ for every k and $(h')^n(\partial A) = h^n(\partial A)_\#$ for every n. Note that h' is conjugate to the translation on A (according to Corollary 4.3).

Let $\mathcal{C}$ be the quotient of A by h'. Denote by π the quotient map. In particular:

- $\mathcal{C}$ is a topological cylinder;
- $\pi(\mathcal{O} \cap A)$ is a set of m points $\hat{x}_1, \dots, \hat{x}_m$ on $\mathcal{C}$;
- for every k, $\pi(\Delta^k \cap A) = \pi(\Delta_+^k)$ is a separating topological circle of $\mathcal{C} - \{\hat{x}_1, \dots, \hat{x}_m\}$; and
- the $\pi(\Delta^k \cap A)$ are mutually disjoint.

For simplicity of notation, we see $\mathcal{C}$ as a vertical cylinder. There exists a homeomorphism ϕ of $\mathcal{C}$ sending each $\pi(\Delta^k \cap A)$ on a horizontal circle γ_k and the family $(\hat{x}_i)_i$ on points of $\mathcal{C}$ with mutually distinct heights. Now, for every $1 \le i \le m$, consider the horizontal circle γ_i' containing $\phi(\hat{x}_i)$. Every γ_i' is disjoint from $\phi\pi(\Delta^k \cap A)$ for every k. Hence $(\phi^{-1}(\gamma_i'))_i$ is a family of mutually disjoint curves disjoint from $\pi(\Delta^k \cap A)$ for every k. For every i, choose a lift β_i^+ of $\phi^{-1}(\gamma_i')$ included in A: that is, an arc included in A such that $\pi(\beta_i^+) = \phi^{-1}(\gamma_i')$. Such a β_i^+ is a translation arc. Since $\{h'^n(\partial A)_\#\}_{n\ge 0}$ is locally finite (Proposition 4.2), the β_i^+ are forward proper . They are disjoint from the Δ^k, as required. □

5. *Walls for a Brouwer mapping class*

The main aim of this section is to prove that the set of walls splits $\mathbb{R}^2$ into translation areas, irreducible areas and stable areas that do not intersect $\mathcal{O}$ (Theorem 3.5).

5.1. *Translation areas.*

LEMMA 5.1. *Let Z be a stable area of a Brouwer class such that all the orbits of Z are forward adjacent. Then every backward proper arc included in Z is forward proper.*

Proof. Let $[h; \mathcal{O}]$ be a Brouwer class with a complete family of adjacency areas. Let Z be a stable area for $[h; \mathcal{O}]$ which intersects only one forward adjacency area. Denote by A this adjacency area. Up to replacing h by $h' \in [h; \mathcal{O}]$, according to the straightening principle 2.1, we can assume that $h(Z) = Z$. Let $(\alpha_i^\pm)_i$ be a nice family for $[h; \mathcal{O}]$ disjoint from the boundary components of Z (such a family exists, according to Proposition 3.3). We prove the following claim, which is a consequence of Theorem 5.5 of Handel [**Han99**].

Claim. For every i such that α_i^- is in Z, there exists n such that $h^n(\alpha_i^-)_\#$ is in A.

We use the definitions and notations of Handel [**Han99**, §5] ('fitted family'). We denote by W the Brouwer subsurface $\mathbb{R}^2 - \bigcup_k A_k^+$, where $\bigcup_k A_k^+$ is the union of forward adjacency areas. If $\alpha_i^- \in Z$ is such that for every $n \ge 0$, $h^n(\alpha_i^-)_\# \cap W \ne \emptyset$, then there exists a fitted family $T \subset RH(W, \delta_+ W)$ such that:

- every $s \in T$ is included in Z (because the elements of T are subsegments of iterates of α_i^-, which is included in Z, and $h(Z) = Z$); and

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- there exists $t \in T$ whose endpoints lie on distinct components of $\delta_+ W$ (this is [**Han99**, Theorem (5.5.c)]).

Since $\delta_+ W \cap Z$ has only one component (the boundary component of A), the last point does not hold, and thus every $\alpha_i^- \in Z$ is such that, for every sufficiently big n, $h^n(\alpha_i^-)$ is homotopically included in A. It follows that every $\alpha_i^- \in Z$ is forward proper. □

PROPOSITION 3.4. *If Z is a translation area, every backward (respectively, forward) proper arc of a nice family which is included in Z is also forward (respectively, backward) proper.*

Proof. By definition, all the orbits of a translation area are backward adjacent and forward adjacent. The result is a consequence of Lemma 5.1 applied to the Brouwer class, and respectively, to its inverse. □

5.2. *Intersections between reducing lines.*

LEMMA 5.2. (Intersection of two reducing lines) *Let $[h; \mathcal{O}]$ be a Brouwer mapping class and let Δ and Δ' be two reducing lines for $[h; \mathcal{O}]$. We assume that Δ and Δ' are in minimal position. Then one of the following situations holds.*

(1) $\Delta \cap \Delta' = \emptyset$.

(2) *$\Delta \cap \Delta'$ contains exactly one point.*

(3) *$\Delta \cap \Delta'$ is an infinite set.*

Proof. Choose a complete hyperbolic metric on $\mathbb{R}^2 - \mathcal{O}$. Taking their images by isotopies if necessary, we can suppose that Δ and Δ' are geodesic. We use the straightening principle 2.1 to find a homeomorphism $h' \in [h; \mathcal{O}]$ such that h' preserves Δ and Δ'.

If $\Delta \cap \Delta'$ contains more than one point, then there exists a bounded connected component of $\mathbb{R}^2 - (\Delta \cup \Delta')$ which contains one point $x \in \mathcal{O}$. Denote this component by C_x. Then $h'(C_x)$ is a bounded component of $\mathbb{R}^2 - (\Delta \cup \Delta')$ that is different from C_x. Indeed, if $h'(C_x)$ coincides with C_x, then $h'^n(C_x) = C_x$ for every $n \geq 0$. Hence $\{h'^n(x)\}_{n \geq 0}$ is included in C_x. This is not possible because $h'^n(x) = h^n(x)$ for every n: since h is a Brouwer homeomorphism, $\{h^n(x)\}_{n \geq 0}$ is not bounded [**Gui94**, Proposition 3.5].

For the same reasons, for every $k < n \in \mathbb{N}$, $h^n(C_x)$ is disjoint from $h^k(C_x)$. Thus there exists an infinite number of bounded connected component of $\mathbb{R}^2 - (\Delta \cup \Delta')$. Hence Δ and Δ' have an infinite number of intersection points. □

LEMMA 5.3. (Intersection between a reducing line and a translation area) *Let $[h; \mathcal{O}]$ be a Brouwer mapping class and let Z be a translation area for $[h; \mathcal{O}]$. Let Δ be a reducing line. If there exists a boundary component L of Z such that L and Δ are not homotopically disjoint, then $\Delta \cap L$ is an infinite set.*

Proof. The line L is a reducing line,and hence it is isotopic to its image by h. Choose a complete hyperbolic metric on $\mathbb{R}^2 - \mathcal{O}$. Suppose that L and Δ are geodesic. For every orbit $\mathcal{O}_i$ of $\mathcal{O}$ included in Z, we choose a homotopic proper translation arc α_i included in Z such that the α_i are mutually disjoint (given by Proposition 3.3). If α is one of these

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homotopy translation arcs, we denote the proper streamline $\bigcup_{n\in\mathbb{Z}} h^n(\alpha)_\#$ by T_α. Since L and Δ are not homotopically disjoint, there exists α_i such that $T_{\alpha_i} \cap \Delta \neq \emptyset$. Since T_{α_j} and L are disjoint for every j, the straightening principle 2.1 gives us a homeomorphism $h' \in [h; \mathcal{O}]$ which preserves L, Δ and T_{α_j} for every i.

Suppose that $\Delta \cap L$ is not infinite. According to Lemma 5.2, since this intersection is not empty, it contains only one point, say x. In particular, we have $h'(x) = x$. Choose an orientation on Δ. Let y be the first intersection point between Δ and T_{α_i} after x on Δ. Denote the segment of Δ between x and y by $[xy]$. We have $h'(y) \in T_{\alpha_i}$ and $h'(]xy[) \cap (L \cup T_{\alpha_i}) = \emptyset$, and hence $y = h'(y)$. This gives a contradiction because y is contained in a proper translation arc for h'. □

LEMMA 5.4. (Intersection between reducing lines: infinite set case) *Let* $[h; \mathcal{O}]$ *be a Brouwer mapping class and let* Δ *and* Δ' *be two reducing lines for* $[h; \mathcal{O}]$. *We assume that* Δ *and* Δ' *are in minimal position.*

If $\Delta \cap \Delta'$ *is an infinite set, then* $\Delta \cup \Delta'$ *is included in a translation area.*

Proof. Choose a complete hyperbolic metric on $\mathbb{R}^2 - \mathcal{O}$. Isotopying if necessary, we can assume that Δ and Δ' are geodesic. The straightening principle 2.1 gives us $h' \in [h; \mathcal{O}]$ which preserves Δ and Δ'. Choose a complete family of mutually disjoint adjacency areas for $[h; \mathcal{O}]$. Choose a bounded connected component C_x of the complement of $\Delta \cup \Delta'$ which contains a point x of an orbit $\mathcal{O}_i$ of $\mathcal{O}$. Denote by A^- and A^+ the backward and forward adjacency areas of the chosen complete family which are intersected by $\mathcal{O}_i$. As shown in the proof of Lemma 5.2, $h'(C_x)$ is a bounded connected component of $R^2 - (\Delta \cup \Delta')$ that is different from C_x. Hence every path from x to $h'(x)$ intersects $\Delta \cup \Delta'$.

According to Proposition 3.3, there exists a forward proper arc α^+ for $\mathcal{O}_i$ which joints x to $h'(x)$ and which is disjoint from Δ'. Denote the forward half-streamline $\bigcup_{n\geq 0} h^n(\alpha^+)_\#$ by $T^+(\alpha^+)$. Note that $T^+(\alpha^+)$ is disjoint from Δ'. According to Proposition 4.2, there exists an unbounded component of $T^+(\alpha^+)$ which is included in A^+. Since $T^+(\alpha^+)$ is proper and disjoint from Δ', the straightening principle 2.1 gives us $h_1 \in [h; \mathcal{O}]$ which preserves $T^+(\alpha^+)$, Δ and Δ'. The arc α^+ intersects Δ, and hence $h_1^n(\alpha^+)$ also intersects Δ for every $n \in \mathbb{N}$. It follows that Δ intersects A^+.

The same argument with a backward proper arc α^- disjoint from Δ' shows that Δ also intersects A^-. According to Lemma 4.4, every geodesic reducing line intersects at most two adjacency areas: for Δ, these adjacency areas are A^- and A^+. Interchanging Δ and Δ', we get, by the same argument, that Δ' also intersects A^- and A^+.

We choose an orientation on Δ and Δ' such that they are oriented from A^- to A^+. There exists an unbounded connected component of the complement of $\Delta \cup \Delta'$ which is on the left of Δ and Δ'. We denote its boundary component by L_l. Likewise, we denote by L_r the boundary component of the unbounded connected component of the complement of $\Delta \cup \Delta'$ which is on the right of Δ and Δ'. The two lines L_l and L_r are proper, because they are unions of segments of two topological lines. Moreover, they are preserved by h'.

Now we have the following cases, depending on the positions of the orbits.

- If L_r and L_l split the set of orbits, then their geodesic representatives $(L_r)_\#$ and $(L_g)_\#$ are disjoint reducing lines which intersect the same adjacency areas. Denote by Z the

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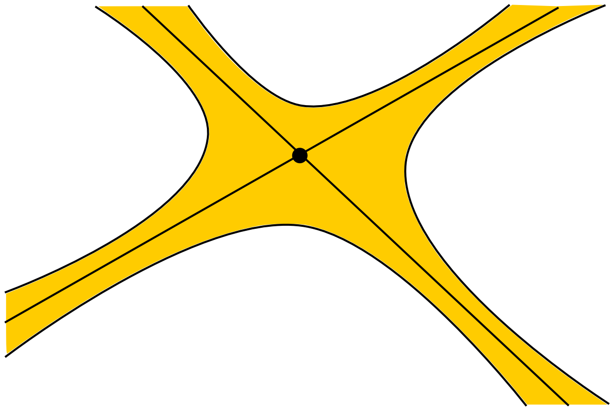

FIGURE 23. Neighborhood of $\Delta \cup \Delta'$.

stable area bounded by $(L_r)_\#$ and $(L_g)_\#$. Thus Z intersects only two adjacency areas, A^- and A^+, and hence Z is a translation area, which contains Δ and Δ'.

- If neither L_r nor L_l split the set of orbits, then there exist only two adjacency areas. Hence $[h; \mathcal{O}]$ is a translation, and the whole plane is a translation area.
- If only one of L_r and L_l splits the set of orbits, L_r, for example, then L_r is a reducing line for $[h; \mathcal{O}]$. The connected component of $\mathbb{R}^2 - (L_r)_\#$ which contains Δ and Δ' is a translation area. □

LEMMA 5.5. (Intersection between reducing lines: case with exactly one point) *Let* $[h; \mathcal{O}]$ *be a Brouwer mapping class and let* Δ *and* Δ' *be two reducing lines for* $[h; \mathcal{O}]$. *We assume that* Δ *and* Δ' *are in minimal position.*

If $\Delta \cap \Delta'$ *contains exactly one point, then there exist four reducing lines that are mutually non-isotopic and homotopically disjoint and disjoint from* Δ *and* Δ'.

Moreover, if we denote the intersection point by p *and the two half-lines of* $\Delta - \{p\}$ *(respectively, of* $\Delta' - \{p\}$*) by* Δ_1 *and* Δ_2 *(respectively,* Δ'_1 *and* Δ'_2*), these four reducing lines are isotopic relative to* $\mathcal{O}$ *to* $\Delta_1 \cup \Delta_2$, $\Delta'_1 \cup \Delta_2$, $\Delta_1 \cup \Delta'_2$ *and* $\Delta'_1 \cup \Delta'_2$.

Proof. Choose a complete hyperbolic metric on $\mathbb{R}^2 - \mathcal{O}$. Isotopying if necessary, we can assume that Δ and Δ' are geodesic. The straightening principle 2.1 gives us $h' \in [h; \mathcal{O}]$ which preserves Δ and Δ'.

Consider a proper open neighborhood $\mathcal{U}$ of $\Delta \cup \Delta'$ which does not intersect $\mathcal{O}$ and which is isotopic to $\Delta \cup \Delta'$ relative to $\mathcal{O}$ (as in Figure 23). The complement of $\mathcal{U}$ has four connected components. Each of them contains at least one orbit, because $\Delta \cup \Delta'$ are in minimal position. Hence the boundary component of the closure of $\mathcal{U}$ in $\mathbb{R}^2$ is a union of four reducing lines which are mutually non-isotopic, mutually disjoint, and each of them is disjoint from $\Delta \cup \Delta'$. □

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5.3. *Study of the set of walls.*

5.3.1. *Maximal translation areas.* We show that there exist a finite number of maximal translation areas (Proposition 5.6), and that the boundary components of these areas are walls (Proposition 5.7).

PROPOSITION 5.6. *Let $[h; \mathcal{O}]$ be a Brouwer mapping class. Up to isotopy, there exist a finite number of maximal translation areas. Moreover, they are mutually homotopically disjoint.*

Remark 5.1. The statement 5.6 is generally false if we replace *maximal translation area* by *translation area*. Indeed, if a translation area Z of a Brouwer class $[h; \mathcal{O}]$ contains at least two orbits, then there are an infinite number of non-isotopic subtranslation areas included in Z. See Figure 24 for examples of reducing lines for the translation.

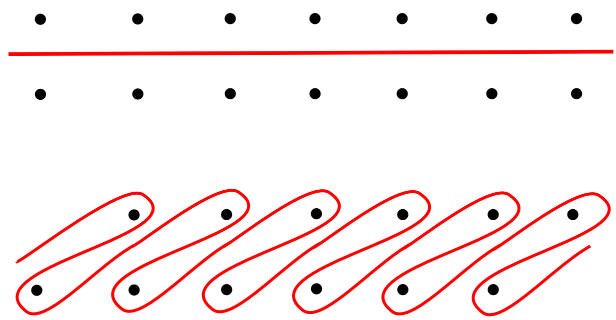

FIGURE 24. Example of two reducing lines for the translation: the complements are translation areas.

Proof of Proposition 5.6. We choose a complete hyperbolic metric on $\mathbb{R}^2 - \mathcal{O}$ and a complete family of adjacency areas for $[h; \mathcal{O}]$. Let Z and Z' be two non-homotopic maximal translation areas. Denote by A^-, A^+ (respectively, B^-, B^+), the adjacency areas intersected by Z (respectively, Z'). We claim that:

(1) the boundary components of Z and Z' are homotopically disjoint; and
(2) no boundary component of Z is included in Z'.

Proof of (1). If a boundary component L of Z intersects a boundary component L' of Z', then L and L' have an infinite number of intersection points (according to Lemma 5.3). Hence L and L' are included in the same translation area (according to Lemma 5.4). Denote this translation area by Z'', and the two adjacency areas intersected by Z'' by C^- and C^+. Since L and L' are reducing lines, they intersect at most two adjacency areas: C^- and C^+. The cyclic order of the adjacency areas at infinity is such that backward areas and forward areas alternate (by definition of adjacency areas). It follows that $A^- = B^- = C^-$ and $A^+ = B^+ = C^+$.

According to Lemma 2.1, there exists $h' \in [h; \mathcal{O}]$ such that $h'(Z) = Z$ and $h'(Z') = Z'$. It follows that the boundary components of $Z \cup Z'$ are reducing lines, and hence $Z \cup Z'$

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is a stable area. Since $Z \cup Z'$ intersects only two adjacency areas (C^- and C^+), it is a translation area. This gives a contradiction with the maximality of Z and Z' as translation areas.

Proof of (2). If a boundary component L of Z is included in Z', then there exists an orbit $\mathcal{O}_i$ included in $Z \cap Z'$. Hence, again, $A^- = B^-$ and $A^+ = B^+$, which gives a contradiction.

We complete the proof of Proposition 5.6. Every maximal translation area contains at least one orbit. Since there are a finite number of orbits and since these areas are mutually disjoint, there are a finite number of maximal translation areas. □

PROPOSITION 5.7. *Let $[h; \mathcal{O}]$ be a Brouwer mapping class and let Z be a maximal translation area. Each isotopy class of a boundary component of Z is a wall of $[h; \mathcal{O}]$.*

Proof. Let L be a boundary component of a maximal translation area Z. We need to show that if Δ is a reducing line which is non-isotopic to L, then $\Delta \cap L = \emptyset$. According to Lemma 5.3, if a reducing line Δ intersects L, then $\Delta \cap L$ is an infinite set. According to Lemma 5.4, it follows that $L \cup \Delta$ is included in a translation area. Since maximal translation areas are mutually disjoint (according to Proposition 5.6), Δ is included in Z (which contains L): this is impossible, hence every reducing line Δ is homotopically disjoint from L. □

5.3.2. *Outside the translation areas.* This subsection completes the picture: there are a finite number of maximal translation area which are mutually homotopically disjoint, and outside those areas there are only a finite number of geodesic reducing lines which intersect mutually in zero or one point.

LEMMA 5.8. *Let $[h; \mathcal{O}]$ be a Brouwer mapping class. Let Δ and Δ' be two disjoint reducing lines. If Δ and Δ' split the orbits into the same two subfamilies, then Δ and Δ' are isotopic.*

Proof. The set $\Delta \cup \Delta'$ splits the plane into three connected components. One of them (the one in the middle) is disjoint from $\mathcal{O}$. □

PROPOSITION 5.9. *Let $[h; \mathcal{O}]$ be a Brouwer mapping class. Outside the maximal translation areas, there exists only a finite number of non-isotopic reducing lines.*

Proof. We prove that there exists only a finite number of non-isotopic reducing lines in every connected components of the complement of the union of the translation areas. If two such reducing lines are not homotopically disjoint, then they have only one intersection point (according to Lemmas 5.2 and 5.4). Hence they do not split the orbits into the same subfamilies. This remark, together with Lemma 5.8, imply that if we choose a partition of the orbits in the chosen component into two subfamilies, then there exists at most one reducing line included in the complement which splits the orbits into the same partition. Since there exist only a finite number of different partitions of the orbits into two subfamilies, there are only a finite number of isotopy classes of reducing lines. □

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5.3.3. *Proof of Theorem 3.5.*

THEOREM 3.5. *Let $[h; \mathcal{O}]$ be a Brouwer mapping class. Let $\mathcal{W}$ be a family of pairwise disjoint reducing lines containing exactly one representative of each wall for $[h; \mathcal{O}]$. If Z is a connected component of $\mathbb{R}^2 - \mathcal{W}$, then exactly one of the following holds:*

- *Z is an irreducible area;*
- *Z is a maximal translation area; or*
- *Z does not intersect $\mathcal{O}$.*

Proof. Choose a complete hyperbolic metric on $\mathbb{R}^2 - \mathcal{O}$. Up to isotopy relative to $\mathcal{O}$, we can assume that the elements of $\mathcal{W}$ are geodesic. According to Proposition 5.7, the isotopy classes of boundary components of maximal translation areas are walls. Let Z be a connected component of $\mathbb{R}^2 - \mathcal{W}$ which is not a translation area. Suppose that Z is not irreducible. Then there exists a reducing line included in Z which is not homotopic to any boundary component of Z. According to Proposition 5.9, Z contains a finite number of mutually non-isotopic reducing lines. Since they are not walls, each of them intersects another one. In particular, there are at least two reducing lines included in Z which are not homotopic to any boundary component of Z.

Denote by $\mathcal{U}$ the finite union of geodesic reducing lines included in Z which are not homotopic to any boundary component of Z. Denote by $\tilde{\mathcal{U}}$ a tubular neighborhood of $\mathcal{U}$ which does not intersect $\mathcal{O}$ (see Figure 25). Choose $\tilde{\mathcal{U}}$ such that the boundary components of its closure are geodesic. Denote by L one of them. We make the following claim.

CLAIM. *The line L is a reducing line for $[h; \mathcal{O}]$.*

Proof of the claim. The line L is homotopic to a union L' of a finite number of segments included in distinct reducing lines (see Figure 26). The number of segments is finite because of the following properties.

(1) The area Z is homotopically disjoint from the translation areas.
(2) Up to isotopy there are only a finite number of reducing lines outside the translation areas (according to Proposition 5.9).
(3) If two reducing lines outside the translation areas intersect, then their intersection is exactly one point: according to Lemma 5.2, this intersection is either one point or infinite and, according to Lemma 5.4, if the intersection is infinite then the reducing lines are included in a translation area.

Denote this number by n, and the segments whose union is L' by $\delta_1 \cup \cdots \cup \delta_n$. We assume that the δ_i are in this order on L' (as in Figure 25). For every i, denote by Δ_i a reducing line of $\mathcal{U}$ which contains δ_i. Denote by L_1 the line obtained as the union of δ_1 and the half-line of Δ_2 whose endpoint is the intersection point between δ_1 and δ_2 and which contains δ_2. According to Lemma 5.5, L_1 is a reducing line for $[h; \mathcal{O}]$. For every $2 \leq i \leq n-1$, denote inductively by L_i the line obtained as the union of the half-line L_{i-1} which contains δ_1 and the half-line Δ_{i+1} which contains δ_{i+1} (both half-lines have the intersection point between endpoints δ_i and δ_{i+1}). Applying Lemma 5.5 inductively, we see that L_i is a reducing line for every i. Hence $L' = L_{n-1}$ is a reducing line for $[h; \mathcal{O}]$. □

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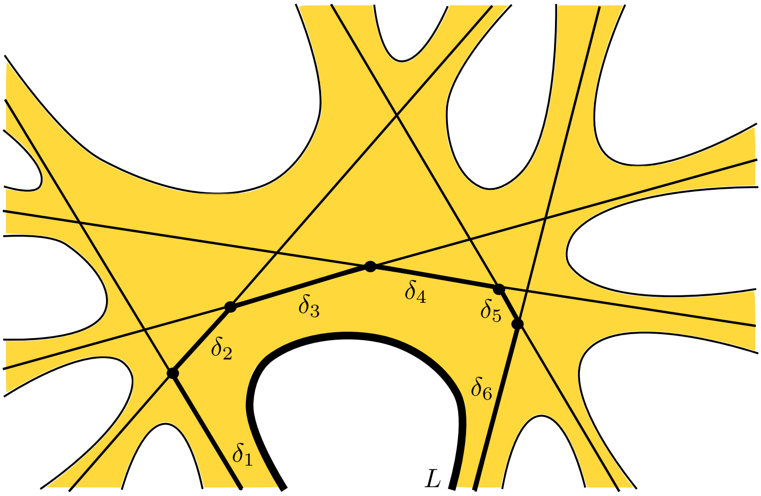

FIGURE 25. Example of $\mathcal{U}$, $\tilde{\mathcal{U}}$ and a boundary component L isotopic to $\delta_1 \cup \cdots \cup \delta_6$.

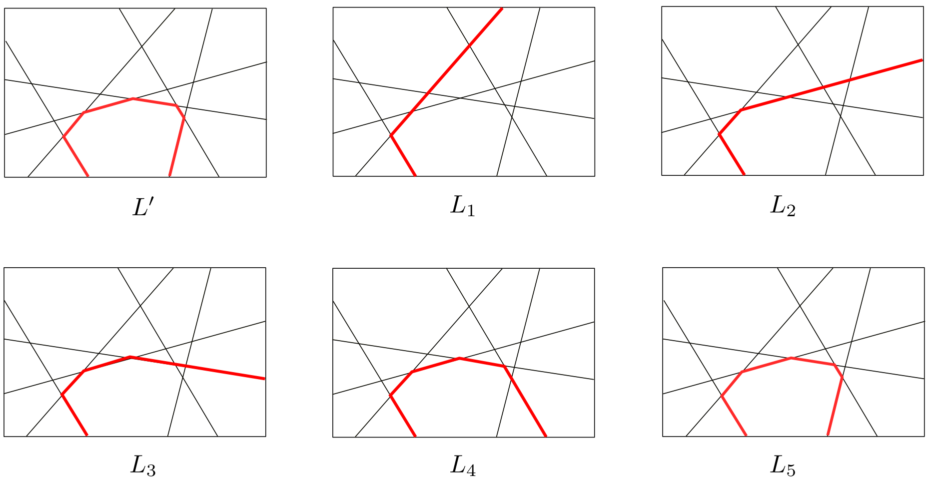

FIGURE 26. Proof of the claim in the case of the example.

End of the proof of Theorem 3.5. Since L is in Z but not in $\mathcal{U}$, it is homotopic to a boundary component of Z (by definition of $\mathcal{U}$). Hence $\partial\tilde{\mathcal{U}}$ is included in ∂Z,and thus Z is included in $\tilde{\mathcal{U}}$ (because they both are intersections of topological half-planes). It follows that Z is isotopic to $\tilde{\mathcal{U}}$ relative to $\mathcal{O}$, and hence Z does not intersect $\mathcal{O}$. □

Remark 5.2. Note that a stable area is irreducible if and only if it is an irreducible area of the complement of the set of walls. In particular, the isotopy classes of the boundary components of any irreducible area are walls.

6. *Determinant diagrams and irreducible areas*

6.1. *Determinant diagrams.* We prove here Propositions 3.6, 3.7 and 3.8. We use the following two lemmas of [**LR13**] in the proofs.

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LEMMA 6.1. [**LR13**, Lemma 1.8] *Let $\mathcal{F}$ be a finite family of pairwise disjoint topological lines in the plane. Let h_0 be an orientation preserving homeomorphism of the plane. Let H_{h_0} be the space of orientation preserving homeomorphisms that coincide with h_0 on the union of the elements of $\mathcal{F}$. Then H_{h_0} is arcwise connected.*

LEMMA 6.2. [**LR13**, Lemma 1.6] *The Brouwer mapping class $[h; \mathcal{O}]$ is a fixed point free flow class if and only if it admits a family of pairwise disjoint proper geodesic homotopy streamlines whose union contains $\mathcal{O}$.*

PROPOSITION 3.6. *A Brouwer mapping class $[h; \mathcal{O}]$ is a flow class if and only if no connected component of the complement of the set of walls for $[h; \mathcal{O}]$ is an irreducible area.*

Proof. If $[f; \mathcal{O}]$ is a flow class, then, according to Lemma 6.2, we can choose a family of pairwise disjoint proper geodesic homotopy streamlines whose union contains $\mathcal{O}$. We find reducing lines in the neighborhood of each streamline, and hence there is no irreducible area.

We now prove that if the set of walls $\mathcal{W}$ of a Brouwer mapping class $[h; \mathcal{O}]$ is such that no component of the complement of $\mathcal{W}$ is irreducible, then it is a flow class. According to Proposition 3.3, there exists a nice family $(\alpha_i^{\pm})_i$ for $[h; \mathcal{O}]$, disjoint from the walls. According to Theorem 3.5, every connected component of the complement of $\mathcal{W}$ which contains orbits is a translation area. According to Proposition 3.4, every backward proper streamline which is included in a translation area is also forward proper: it follows that every $T(\alpha_i^-, h, \mathcal{O})$ is a proper streamline. Lemma 6.2 gives us the conclusion. □

PROPOSITION 3.7. *If two flow classes have the same diagram, then they are conjugated.*

Proof. Let $[f; \mathcal{O}]$ and $[g; \mathcal{O}']$ be two flow classes with the same diagram. According to Lemma 6.2, there exists a nice family $(\alpha_i^{\pm})_i$ for $[f; \mathcal{O}]$ and a nice family $(\beta_i^{\pm})_i$ for $[g; \mathcal{O}']$ such that, for every i, α_i^- is isotopic to α_i^+ and β_i^- is isotopic to β_i^+. We set $\alpha_i := \alpha_i^- = \alpha_i^+$ and $\beta_i := \beta_i^- = \beta_i^+$. Since $[f; \mathcal{O}]$ and $[g; \mathcal{O}']$ have the same diagram, $(\alpha_i^{\pm})_i$ and $(\beta_i^{\pm})_i$ have the same cyclic order at infinity (we permute the numbering of the orbits of $\mathcal{O}'$, if necessary). Thus the Schoenflies theorem provides a homeomorphism of the plane which send $T(\alpha_i, h, \mathcal{O})$ to $T(\beta_i, h, \mathcal{O})$ for every i. Lemma 6.1 gives the conclusion. □

PROPOSITION 3.8. *A diagram with walls and without crossing arrows is determinant if and only if there is no irreducible area in the wall complement.*

Proof. Let D be a diagram with walls without crossing arrows. Suppose D is determinant. Since D is without crossing arrows, there exists a flow class $[f; \mathcal{O}]$ whose associated diagram is D (this is [**LR13**, Lemma 1.7]). Since $[f; \mathcal{O}]$ is a flow class, every orbit of $\mathcal{O}$ is included in a translation area, hence in a maximal translation area. In this situation, Theorem 3.5 imply that every connected component of the complement of walls which contains orbit is a maximal translation area. The result follows.

If a diagram with walls D has no irreducible area in the wall complement, then, according to Theorem 3.5, it has only translation areas and areas without orbits. According to Proposition 3.6, $[h; \mathcal{O}]$ is a flow class. If $[h'; \mathcal{O}']$ is another Brouwer class whose associated diagram with walls is D, then $[h'; \mathcal{O}']$ is also a flow class. It follows from Proposition 3.7 that $[h; \mathcal{O}]$ and $[h'; \mathcal{O}']$ are conjugated. Hence the diagram with walls D is determinant. □

6.2. *Combinatorics of irreducible areas.* We first prove a criterion for reducing lines and then use it to prove Proposition 3.9.

6.2.1. *A criterion for reducing lines.*

LEMMA 6.3. *Let $[h; \mathcal{O}]$ be a Brouwer mapping class and let $(\alpha_i^{\pm})_i$ be a nice family for $[h; \mathcal{O}]$. If Δ is a topological line of $\mathbb{R}^2 - \mathcal{O}$ such that:*

(1) *Δ is a topological line;*

(2) *both components of $\mathbb{R}^2 - \Delta$ contain points of $\mathcal{O}$; and*

(3) *for every i, Δ is homotopically disjoint from $h^k(\alpha_i^-)$ relative to $\mathcal{O}$ for every $k \in \mathbb{Z}$.*

Then Δ is a reducing line for $[h; \mathcal{O}]$.

Proof. We need to show that Δ and $h(\Delta)$ are isotopic relative to $\mathcal{O}$. Choose a hyperbolic metric on $\mathbb{R}^2 - \mathcal{O}$. Taking its image by an isotopy relative to $\mathcal{O}$, if necessary, we can assume that Δ is geodesic. We denote by f a representative of $[h; \mathcal{O}]$ mapping Δ onto $h(\Delta)_\#$. Such an f exists, again according to the straightening principle 2.1. Hence Δ and $f(\Delta)$ are geodesic. We need to show that $\Delta = f(\Delta)$. Suppose that $\Delta \neq f(\Delta)$. We know that these two streamlines are in minimal intersection position (because they are geodesic), and we study the three possible cases separately: either Δ and $f(\Delta)$ have several intersection points, either they have only one intersection point or they do not intersect. These three cases lead us to contradictions.

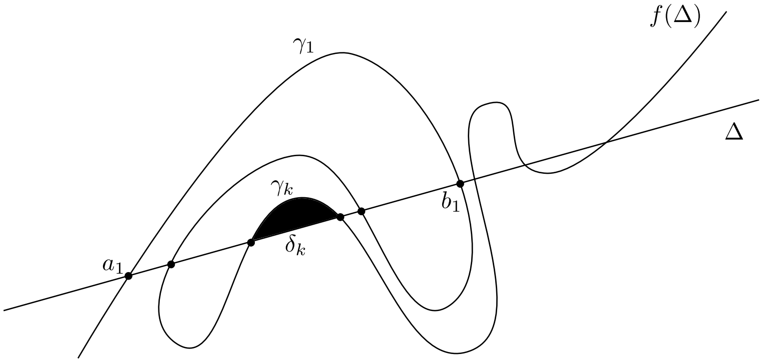

FIGURE 27. Example where Δ and $f(\Delta)$ have several intersection points.

If Δ and $f(\Delta)$ have several intersection points. We consider a subsegment γ_1 of $f(\Delta)$, whose endpoints are intersection points between Δ and $f(\Delta)$, denoted by a_1 and b_1, such that the open segment $\mathring{\gamma}_1$ is disjoint from Δ (see Figure 27). Denote by δ_1 the subsegment of Δ between a_1 and b_1. Since δ_1 is compact, it contains a finite number of intersection points between Δ and $f(\Delta)$. Denote by n the number of intersections between Δ and $f(\Delta)$. Let us show that we can assume that $n = 0$. If $n > 0$, choose an intersection point a_2 between $\mathring{\delta}_1$

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and $f(\Delta)$. Consider the half-line $f(\Delta)^+$ defined as the connected component of $f(\Delta) - a_2$ which has a subsegment with endpoint a_2 and which is included in the topological disk bounded by $\gamma_1 \cup \delta_1$. Because $f(\Delta)^+$ is proper, it goes out to the topological disk bounded by $\gamma_1 \cup \delta_1$. Hence it intersects δ_1 again , because $f(\Delta)$ is without self-intersection. Denote by b_2 the first intersection point between $f(\Delta)^+$ and δ_1. The subsegments δ_2 and γ_2 of Δ and $f(\Delta)$ with endpoints a_2 and b_2 have the same properties as δ_1 and γ_1, but the number of intersection points between Δ and $f(\Delta)$ on δ_2 is less than on δ_1. Hence, by recurrence, there exists k and two subsegments $\delta_k \subset \Delta$ and $\gamma_k \subset f(\Delta)$ with endpoints a_k and b_k, such that $\delta_k \cap f(\Delta) = \gamma_k \cap \Delta = a_k \cup b_k$.

Denote by D the topological disk bounded by $\delta_k \cup \gamma_k$. We claim that D does not intersect $\mathcal{O}$: if it contains a point of an orbit $\mathcal{O}_i$, denote by x_i this point. Let $n \in \mathbb{Z}$ be such that x_i is an endpoint of $h^n(\alpha_i^-)$. The family $(h^k(\alpha_i^-))_{k\leq n}$ is proper, because α_i^- is backward proper. Applying again the straightening principle 2.1, we find a homeomorphism g, isotopic to h, which maps Δ on $h(\Delta)_\# = f(\Delta)$ and $h^k(\alpha_i^-)$ on $(h^{k+1}(\alpha_i^-))_\#$ for every $k \leq n$. Moreover, the family $(h^k(\alpha_i^-))_{k\leq n}$ is homotopically disjoint from Δ, according to the third hypothesis of the lemma. Hence $g^k(\alpha_i)$ is disjoint from Δ and $f(\Delta)$ for every $k \leq n$. It follows that $\{h^k(x_i)\}_{k\leq n}$ is included in D. Since the orbits of a Brouwer homeomorphism are proper, this gives a contradiction: D should be disjoint from $\mathcal{O}$, and hence it is a bigone, which is also not possible because Δ and $f(\Delta)$ are in minimal position.

If Δ and $f(\Delta)$ have exactly one intersection point. The set $\Delta \cup f(\Delta)$ splits $\mathbb{R}^2$ into four connected components. Each of those connected components contains at least one orbit of $\mathcal{O}$: if not, we can find an isotopy relative to $\mathcal{O}$ which eliminates the intersection point, and hence Δ and $f(\Delta)$ are not in minimal position. Choose an orientation on Δ and consider the induced orientation by f on $f(\Delta)$. Then there exists at least one orbit of $\mathcal{O}$ which is on the left of Δ and on the right of $f(\Delta)$. We claim that this is not possible.

- Δ splits $\mathbb{R}^2$ into two topological half-plane, denoted by $\mathcal{P}$ and $\mathcal{Q}$.
- Δ splits the orbits of $\mathcal{O}$ into two families: the family P is the orbits included in $\mathcal{P}$ and the family Q is the orbits included in $\mathcal{Q}$.
- For every orbit $\mathcal{O}_i$ of $\mathcal{O}$, $f(\mathcal{O}_i) = \mathcal{O}_i$.

Then $f(\Delta)$ splits $\mathbb{R}^2$ into $f(\mathcal{P})$ and $f(\mathcal{Q})$, and hence the orbits into $f(P) = P$ and $f(Q) = Q$.

If Δ and $f(\Delta)$ have no intersection point. The set $\Delta \cup f(\Delta)$ splits $\mathbb{R}^2$ into three connected components. One of them is between Δ and $f(\Delta)$. This component contains at least one orbit, because Δ and $f(\Delta)$ are not isotopic. Now the same argument as in the previous case leads us to a contradiction: choose an orientation on Δ and consider the induced orientation by f on $f(\Delta)$. There exists at least one orbit of $\mathcal{O}$ which is on the left of Δ and on the right of $f(\Delta)$. This is not possible. □

COROLLARY 6.4. *Let $[h; \mathcal{O}]$ be a Brouwer mapping class. Let Z be a stable area for $[h; \mathcal{O}]$ which contains at least two orbits of $\mathcal{O}$. Choose a complete family of adjacency areas for $[h; \mathcal{O} \cap Z]$. Let Δ be a reducing line for $[h; \mathcal{O} \cap Z]$ which is included in Z and disjoint from every chosen backward adjacency area. Then Δ is a reducing line for $[h; \mathcal{O}]$.*

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Proof. Choose a complete hyperbolic metric on $\mathbb{R}^2 - \mathcal{O}$. We suppose that the boundary components of Z and Δ are geodesic. The straightening principle 2.1 gives us $h' \in [h; \mathcal{O}]$ which preserves Z. Let $(\alpha_i^{\pm})_i$ be a nice family, disjoint from the boundary component of Z, such that every α_i^- of Z is included in an adjacency area of the chosen family.

Since Δ is disjoint from every chosen backward adjacency area, it is disjoint from α_i^- for every i such that α_i^- is in Z. Hence, for every $k \in \mathbb{Z}$, Δ is homotopically disjoint from $h'^k(\alpha_i^-)$ relative to $\mathcal{O} \cap Z$ (because Δ is isotopic to its image by h' relative to $\mathcal{O} \cap Z$). Since $h'^k(\alpha_i^-)$ is compact and included in Z, which is preserved by h', we get that Δ and $h'^k(\alpha_i^-)$ are homotopically disjoint relative to $\mathcal{O}$ and not only relative to $\mathcal{O} \cap Z$.

It follows from Lemma 6.3 that Δ is a reducing line for $[h; \mathcal{O}]$. □

6.2.2. *Proof of Proposition 3.9.*

PROPOSITION 3.9. (Combinatorics of irreducible areas) *Let* $[h; \mathcal{O}]$ *be a Brouwer mapping class and let* Z *be an irreducible area for* $[h; \mathcal{O}]$. *Then:*

(1) *the orbits of* Z *are not all backward adjacent, nor all forward adjacent for* $[h; \mathcal{O}]$*;*
(2) Z *has at least two boundary components; and*
(3) *the orbits of* $[h; \mathcal{O} \cap Z]$ *do not alternate.*

Proof of (1). Assume that Z intersects only one forward adjacency area. According to Lemma 5.1, every $\alpha_i^- \in Z$ is forward proper. Hence Z is not irreducible.

Proof of (2). Let $[h; \mathcal{O}]$ be a Brouwer mapping class. Let Z be a stable area for $[h; \mathcal{O}]$ which has only one boundary component and at least two orbits. Denote this boundary component by L. Suppose that Z is not a translation area. We will find a reducing line for $[h; \mathcal{O}]$, which is included in Z and non-isotopic to L. Let $(\alpha_i^{\pm})_i$ be a nice family for $[h; \mathcal{O}]$ that is disjoint from L. There is a subfamily of $(\alpha_i^{\pm})_i$ which is a nice family for $[h; \mathcal{O} \cap Z]$. Denote this subfamily by $(\beta_i^{\pm})_i$. We consider the cyclic order of $(\beta_i^{\pm})_i$, and look at where is L in this cyclic order: the position of L in the cyclic order is the position of L in the plane relative to the homotopy half-streamlines generated by the $\beta_i^{\pm}$. There are two different cases:

(a) if L is between two backward proper arcs or between two forward proper arcs in the cyclic order of $(\beta_i^{\pm})_i$; or
(b) if L is between one backward proper arc and one forward proper arc in the cyclic order of $(\beta_i^{\pm})_i$.

Case (a). If L is between two backward proper arcs or between two forward proper arcs in the cyclic order of $(\beta_i^{\pm})_i$, we claim that there exists an adjacency area for $[h; \mathcal{O} \cap Z]$ which contains L. Indeed, assume that L is between two backward proper arcs (the same proof holds with two forward proper arcs, replacing h by h^{-1} when it is necessary). Denote by β_i^- and β_j^- these two backward proper arcs, and suppose that their endpoints are $x_i, h(x_i)$ and $x_j, h(x_j)$, respectively. Then there exists an arc γ, disjoint from L, whose endpoints are $h(x_i)$ and $h(x_j)$ and such that one connected component of the complement of $T^-(\beta_i^-, h, \mathcal{O}) \cup \gamma \cup T^-(\beta_j^-, h, \mathcal{O})$ does not intersect $\mathcal{O} \cap Z$. This shows what we claimed. Now Handel's Theorem 4.1 implies that there exists a reducing line Δ

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for $[h; \mathcal{O} \cap Z]$ which is disjoint from every backward adjacency area. Hence Δ is included in Z and, according to Corollary 6.4, it is a reducing line for $[h; \mathcal{O}]$.

Case (b). Assume L is between one backward proper arc and one forward proper arc in the cyclic order of $(\beta_i^{\pm})_i$. Denote these two arcs by β_i^- and β_j^+. Following the construction 4.1, we get a complete family of adjacency areas for $[h; \mathcal{O} \cap Z]$ disjoint from L. Now let Δ be a reducing line for $[h; \mathcal{O} \cap Z]$ given by Lemma 4.5, that is, such that the intersection between Δ and the complement of the forward adjacency areas of $[h; \mathcal{O} \cap Z]$ is bounded. It follows that Δ intersects L at most in a finite set (because L is disjoint from the adjacency areas). Isotopying Δ if this set is not empty, we can suppose that $\Delta \cap L = \emptyset$. Since Δ is also disjoint from every backward adjacency area of $[h; \mathcal{O} \cap Z]$, according to Corollary 6.4, it is a reducing line for $[h; \mathcal{O}]$.

Proof of (3). According to Proposition 3.3, there exists a nice family $(\alpha_i^{\pm})_i$ for $[h; \mathcal{O}]$ which is disjoint from the boundary components of Z. We consider the subfamily $(\alpha_j^{\pm})_{j\in J}$, where J is the set of indices j such that $\alpha_j^{\pm}$ is in Z: $(\alpha_j^{\pm})_{j\in J}$ is the subfamily of arcs corresponding to the orbits of $\mathcal{O} \cap Z$. A backward or forward proper arc relative to $\mathcal{O}$ is again backward or forward proper relative to $\mathcal{O} \cap Z$, and hence $(\alpha_j^{\pm})_{j\in J}$ is a nice family for $[h; \mathcal{O} \cap Z]$.

By contradiction, assume that the orbits of $[h; \mathcal{O} \cap Z]$ alternate. According to [**LR13**, Lemma 3.6], every family of alternating orbits satisfies the uniqueness of homotopy translation arcs. It follows that, for every $j \in J$, α_j^- and α_j^+ are isotopic relative to $\mathcal{O} \cap Z$. Since they are included in Z, they are also isotopic relative to $\mathcal{O}$. Hence, for such a j, $T(\alpha_j^-, h, \mathcal{O})$ is a proper streamline. At least one of the boundary components of a tubular neighborhood of this streamline is a reducing line included in Z. Thus Z is not irreducible.

6.3. *Corollaries of Proposition 3.9.*

6.3.1. *Proof of Corollary 3.10.*

COROLLARY 3.10. *Let $[h; \mathcal{O}]$ be a Brouwer mapping class relative to r orbits. Denote the number of adjacency subfamilies of $[h; \mathcal{O}]$ by $2r'$. If $r' = 1, 2$ or r, then $[h; \mathcal{O}]$ is a flow class.*

Proof. If $[h; \mathcal{O}]$ is not a flow, then there is an irreducible area in its wall complement (according to Proposition 3.6). This irreducible area has at least two boundary components (according to Proposition 3.9), which are reducing lines. Denote by Δ_1 and Δ_2 these two boundary components. The complement of $\Delta_1 \cup \Delta_2$ has three components, denoted by Z_1, Z_2 and Z. Assume that Z is the area in the middle, which contains the irreducible area. Choose a nice family $(\alpha_i^{\pm})_i$ for $[h; \mathcal{O}]$ which is disjoint from Δ_1 and Δ_2 (use Proposition 3.3). Since Z contains an irreducible area, according to Proposition 3.9, it intersects at least two different backward adjacency areas of $[h; \mathcal{O}]$ and at least two different forward adjacency areas of $[h; \mathcal{O}]$, and the orbits of $[h; \mathcal{O} \cap Z]$ do not alternate. Hence the situation is the one of Figure 28: there exists a subfamily $(\alpha_{i_1}^-, \alpha_{i_2}^+, \alpha_{i_3}^-, \alpha_{i_4}^+)$ of $(\alpha_i^{\pm})_i$ containing only arcs included in Z and such that the cyclic order of this subfamily

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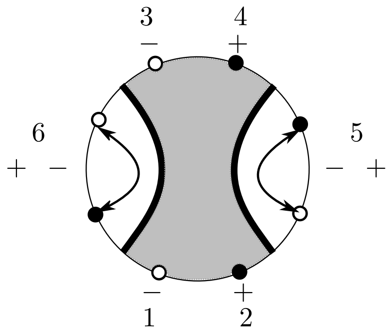

FIGURE 28. Combinatorics of an irreducible area.

is $(\alpha_{i_1}^-, \alpha_{i_2}^+, \alpha_{i_4}^+, \alpha_{i_3}^-)$, with Δ_1 between $\alpha_{i_2}^+$ and $\alpha_{i_4}^+$ and Δ_2 between $\alpha_{i_3}^-$ and $\alpha_{i_1}^-$. Since Z_1 and Z_2 contain at least one orbit (because Δ_1 and Δ_2 are reducing lines), there exists a backward proper arc of $(\alpha_i^\pm)_i$ in Z_1, denoted by $\alpha_{i_5}^-$, and a forward proper arc of $(\alpha_i^\pm)_i$ in Z_2, denoted by $\alpha_{i_6}^+$. It follows that $(\alpha_i^\pm)_i$ has a subfamily of arcs whose cyclic order at infinity is $(\alpha_{i_1}^-, \alpha_{i_2}^+, \alpha_{i_5}^-, \alpha_{i_4}^+, \alpha_{i_3}^-, \alpha_{i_6}^+)$. Hence $r' \geq 3$. It was shown in [**LR13**, Lemma 6.6], that $r' < r$. □

6.3.2. *Proof of Corollary 3.11.*

COROLLARY 3.11. *Let* $[h; \mathcal{O}]$ *be a Brouwer mapping class relative to r orbits.*

(1) *If* $r \geq 3$*, there exist at least two disjoint and non-isotopic reducing lines for* $[h; \mathcal{O}]$.
(2) *If* $r \geq 2$*, there exist at least two translation areas for* $[h; \mathcal{O}]$ *which have exactly one boundary component.*
(3) *There exists a nice family* $(\alpha_i^\pm)_i$ *and* $j \neq k$ *such that:*
   - *relative to* $\mathcal{O}$, α_j^- *is isotopic to* α_j^+ *and* α_k^- *is isotopic to* α_k^+*; and*
   - *in the cyclic order,* α_j^- *and* α_j^+ *(respectively,* α_k^- *and* α_k^+*) are neighbors.*

Proof of (1). Let r be greater than two. Theorem 2.3 gives us a first reducing line. This line splits the plane into two stable areas. One of them contains at least two orbits. According to (2) of Proposition 3.9, every stable area with one boundary component which contains at least two orbits contains at least one reducing line non-isotopic to the boundary component: this gives a second reducing line. □

Proof of (2) and (3). If $r = 2$ then any reducing line of $[h; \mathcal{O}]$ splits the plane into two translation areas. If $r \geq 3$ we find two reducing lines as seen in the proof of (1). In the complement of these two reducing lines we have, in particular, two stable areas with one boundary component. In each of these areas, by finding again a reducing line in the area as in (1), if necessary, we find, inductively, a stable area with one orbit and one boundary component: this gives (2) and (3). □

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7. *Deflectors*

This section is independent of §§4–6. The main result is Proposition 7.1, which we will need to prove Theorem 3.13.

PROPOSITION 7.1. *Let $\tau : \mathbb{R}^2 \to \mathbb{R}^2$ be the horizontal translation $(x, y) \mapsto (x + 1, y)$. Let $n \in \mathbb{N}$. Let $(\alpha_i)_{1\le i\le n}$ and $(\beta_i)_{1\le i\le n}$ be two families of translation arcs for τ such that:*
- *for every i, α_i and β_i join $(0, i)$ to $(1, i)$; and*
- *the α_i (respectively, the β_i) are mutually disjoint.*

Then there exists a homeomorphism μ of $\mathbb{R}^2$ with a compact support such that:
(1) *$\mu(\mathbb{Z} \times \{1, \ldots, n\}) = \mathbb{Z} \times \{1, \ldots, n\}$;*
(2) *for every sufficiently large $k \in \mathbb{N}$, $(\mu\tau)^k(\alpha_i)$ is isotopic relative to $\mathbb{Z} \times \{1, \ldots, n\}$ to $\tau^k(\beta_i)$ for every i; and*
(3) *$\mu\tau$ is a Brouwer homeomorphism. More precisely, μ is a finite product of τ-free half-twists disjointly supported.*

Definition 7.1. Such a homeomorphism is called a *deflector* associated to $(\alpha_i, \beta_i)_{1\le i\le n}$.

Let $\mathcal{C}_n$ be the open vertical cylinder with n marked points at distinct heights. Recall that $MCG(\mathcal{C}_n)$ is defined as the quotient of the group of homeomorphisms of the cylinder, fixing each boundary puncture and fixing the set of marked points (not necessarily pointwise) by its connected component of the identity (for the compact-open topology). In particular, it is the subgroup of the braid group of the $(n + 2)$-punctured sphere $B_{n+1}(\mathbb{S}^2)$ which fixes two punctures.

We use the following lemma in the proof of Proposition 7.1.

LEMMA 7.2. *Let $(\gamma_i)_{1\le i\le n}$ be a family of mutually disjoint simple closed curves on $\mathcal{C}_n$, such that each curve contains exactly one marked point, and such that each curve is isotopic in the cylinder without marked point to the separating circle. Let $(\gamma'_i)_{1\le i\le n}$ be the family of disjoint horizontal circles on the cylinder, such that each γ'_i contains a marked point.*

Then there exists $\phi \in MCG(\mathcal{C}_n)$ such that:
- *for every i, there exists j such that $\phi(\gamma_i)$ is isotopic to γ'_j in $\mathcal{C}_n$; and*
- *ϕ is a finite product of half-twists.*

Proof of Lemma 7.2. We again denote the isotopy classes of γ_i, γ'_i by γ_i, γ'_i when there is no confusion. Let $\varphi \in MCG(\mathcal{C}_n)$ such that $(\varphi(\gamma_i))_i = (\gamma'_i)_i$. Then, if T is a product of horizontal Dehn twists (around one of the γ'_i or around a boundary component), then $T\varphi$ coincides with φ on the γ_i. Choose $\varphi \in MCG(\mathcal{C}_n)$, which send the γ_i on the γ'_i.

Given a braid b on the cylinder, choose b' supported on a disk such that bb' is a pure braid. Then bb' can be made a 'parallel braid' b'' by several cross changes, where parallel braid means a braid whose strand is contained in a constant level of the cylinder. A cross change corresponds to a half-twist. Therefore $b''^{-1}bb'$, and hence $b''^{-1}b$ is a product of half-twists.

For $b = \varphi$, it follows that there exists $b''^{-1} = T$ a parallel braid (which is also a product of horizontal Dehn twists) such that $\phi = T\varphi$ is a finite product of half-twists, which sends the γ_i on the γ'_i. □

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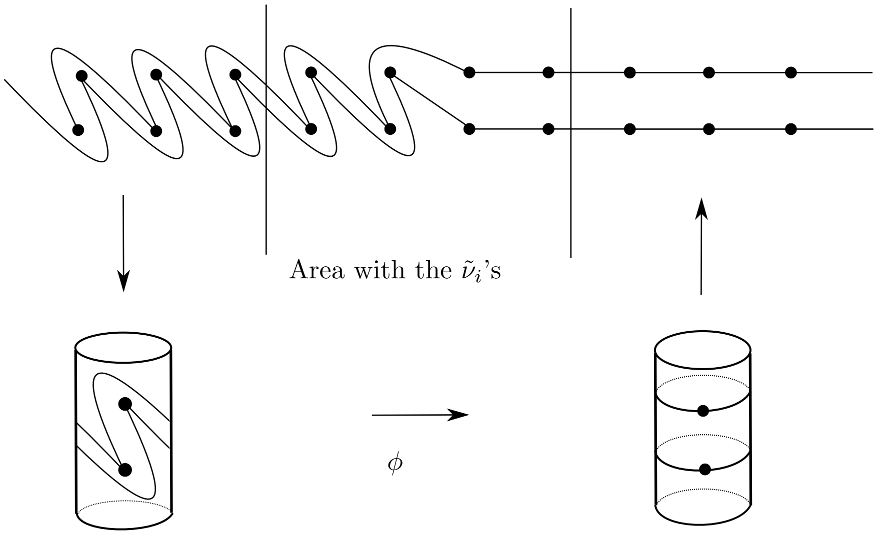

FIGURE 29. Half streamlines $T^+(\alpha_i, \mu\tau, \mathbb{Z} \times \{1, \ldots, n\})$.

Proof of Proposition 7.1. Denote by $\mathcal{C}$ the vertical cylinder (quotient of $\mathbb{R}^2$ by τ) and by $\mathcal{C}_n$ the cylinder $\mathcal{C}$ with n marked points (quotient of $\mathbb{R}^2 - \mathbb{Z} \times \{1, \ldots, n\}$ by τ). Let π be the quotient map. For every i, we denote the isotopy class of $\pi(\alpha_i)$ (respectively, $\pi(\beta_i)$), in $\mathcal{C}_n$ by $\hat{\alpha}_i$ (respectively, $\hat{\beta}_i$). There exists $\psi \in MCG(\mathcal{C}_n)$ such that $\psi(\hat{\alpha}_i)$ is isotopic in $\mathcal{C}_n$ to the horizontal circle containing the marked point $x_i = \pi(\mathbb{Z} \times \{i\})$. There exists $\chi \in MCG(\mathcal{C}_n)$ such that $\chi(\hat{\beta}_i)$ is isotopic in $\mathcal{C}_n$ to the horizontal circle containing the marked point $x_i = \pi(\mathbb{Z} \times \{i\})$. We set $\phi := \chi^{-1}\psi$. Hence $\phi(\hat{\alpha}_i)$ is isotopic in $\mathcal{C}_n$ to $\hat{\beta}_i$. According to Lemma 7.2, we can assume that $\phi := \nu_1 \cdots \nu_k$ is a finite product of half-twists supported in topological disks of $\mathcal{C}$. We want to use ϕ to construct the desired homeomorphism μ (Figure 29 gives the main idea of the proof). □

Local lift of a half-twist. We first construct a homeomorphism of $\mathbb{R}^2$ which lift the action of *one* half-twist of $\mathcal{C}_n$ to the lift of the curves. Let ν be a half-twist of $\mathcal{C}_n$. Choose a lift D_ν of the support of ν in $\mathbb{R}^2$. Denote by $\tilde{\nu}$ the homeomorphism of $\mathbb{R}^2$ such that $\pi\tilde{\nu}_{|D_\nu} := \nu\pi_{|D_\nu}$, and such that $\tilde{\nu}$ coincides with Id outside D_ν. Note that $\mathbb{Z} \times \{1, \ldots, k\}$ is preserved by $\tilde{\nu}\tau$. We say that $\tilde{\nu}$ is a *local lift of* ν.

Choice of disks. To lift the action of $\phi := \nu_1 \cdots \nu_k$, we choose a local lift $\tilde{\nu}_k$ of ν_k supported in a disk D_k on the right of the α_i and, for every $1 \leq j < k$, we choose a local lift $\tilde{\nu}_j$ of ν_j supported in a disk D_j on the right of D_{j+1} (and disjoint from D_{j+1}).

Conclusion. Let $\tilde{\mu}$ be $\tilde{\nu}_1 \cdots \tilde{\nu}_k$. For every x which is on the left of the D_j, for every n sufficiently large, there exists $(n_j)_j \in \mathbb{N}^{k+1}$ such that

$$(\tilde{\mu}\tau)^n = \tau^{n_{k+1}}\tilde{\nu}_1 \cdots \tilde{\nu}_{k-1}\tau^{n_2}\tilde{\nu}_k\tau^{n_1}(x).$$

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Moreover,

$$\pi\tilde{\nu} = \nu\pi,$$
$$\pi\tau = \pi.$$

Hence we get:

$$\pi(\tilde{\mu}\tau)^n\alpha_i = \pi\beta_i.$$

8. *Classification relative to four orbits*

8.1. *Identification of the determinant diagrams.* Here we want to identify which diagrams are determinant diagrams. In the next section, we will study the diagrams which are not determinant. Note that all the diagrams with four orbits are represented in Appendix A.

For every Brouwer mapping class relative to four orbits, we denote the number of sub-families of adjacency by $2r'$.

PROPOSITION 8.1. *A diagram for a Brouwer mapping class relative to four orbits is non-determinant if and only if it is one of the seven in Figure 30.*

Proof. According to Proposition 3.9, every irreducible area for Brouwer mapping classes relative to four orbits is as in Figure 31. □

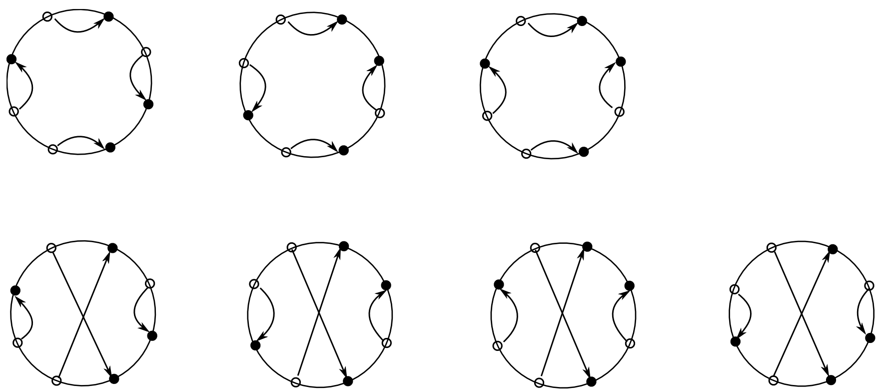

FIGURE 30. Non determinant diagrams for $r = 4$.

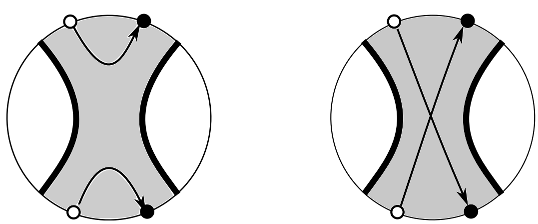

FIGURE 31. Possible irreducible areas for $r = 4$.

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8.2. *Study of the non-determinant diagrams.*

8.2.1. *Brouwer mapping classes which realize non-determinant diagrams.* If h is a homeomorphism of the plane, recall that an h-free disk is a topological disk D which is disjoint from every $h^n(D)$, with $n \neq 0$. If $[h; \mathcal{O}]$ is a Brouwer mapping class and if D is an h-free disk containing exactly two points of $\mathcal{O}$, then we call any half-twist supported in D that permutes the two points of $D \cap \mathcal{O}$ a *free half-twist.*

Remark 8.1. Each non-determinant diagram can be realized by a Brouwer mapping class.

- For each non-determinant diagram without crossing, there exists a flow having this diagram relative to some of its orbits (see [**LR13**, Lemma 1.7]).
- For each non-determinant diagram with a crossing, we can obtain it by composing a flow by a free half-twist (as in example B of §2).

8.2.2. *Tangle of the irreducible area.* Let $[h; \mathcal{O}]$ be a Brouwer mapping class relative to four orbits which is not a flow class: the diagram with walls of $[h; \mathcal{O}]$ is as in Figure 32. To simplify the notation, suppose that the two orbits of the irreducible areas are $\mathcal{O}_1$ and $\mathcal{O}_2$. To define the tangle, we will forget about $\mathcal{O}_3$ and $\mathcal{O}_4$ for a while.

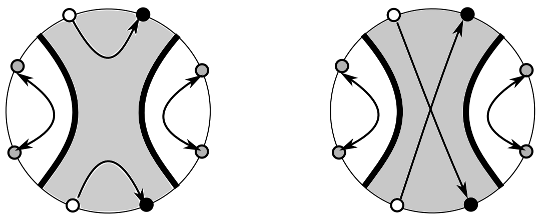

FIGURE 32. Diagrams with an irreducible area for $r = 4$.

Choose a nice family $(\alpha_i^{\pm})_i$ for $[h; \mathcal{O}]$. Choose a complete hyperbolic metric on $\mathbb{R}^2 - (\mathcal{O}_1 \cup \mathcal{O}_2)$. According to Corollary 3.10 and Proposition 3.7 (see, also, Handel [**Han99**]), the diagram relative to two orbits is a total conjugacy invariant, and hence $[h; \mathcal{O}_1 \cup \mathcal{O}_2]$ is a translation class. It follows that the four homotopic trajectories relative to $\mathcal{O}_1 \cup \mathcal{O}_2$ given by

$$T_1^+ := \bigcup_{k\in\mathbb{Z}} h^k(\alpha_1^+)_\#,$$
$$T_1^- := \bigcup_{k\in\mathbb{Z}} h^k(\alpha_1^-)_\#,$$
$$T_2^+ := \bigcup_{k\in\mathbb{Z}} h^k(\alpha_2^+)_\#,$$
$$T_2^- := \bigcup_{k\in\mathbb{Z}} h^k(\alpha_2^-)_\#$$

are proper homotopic lines. Moreover, the T_i^+ (respectively, the T_i^-) are mutually disjoint. Let ϕ be a homeomorphism of the plane that preserves the orientation and sends, for $i = 1, 2$:

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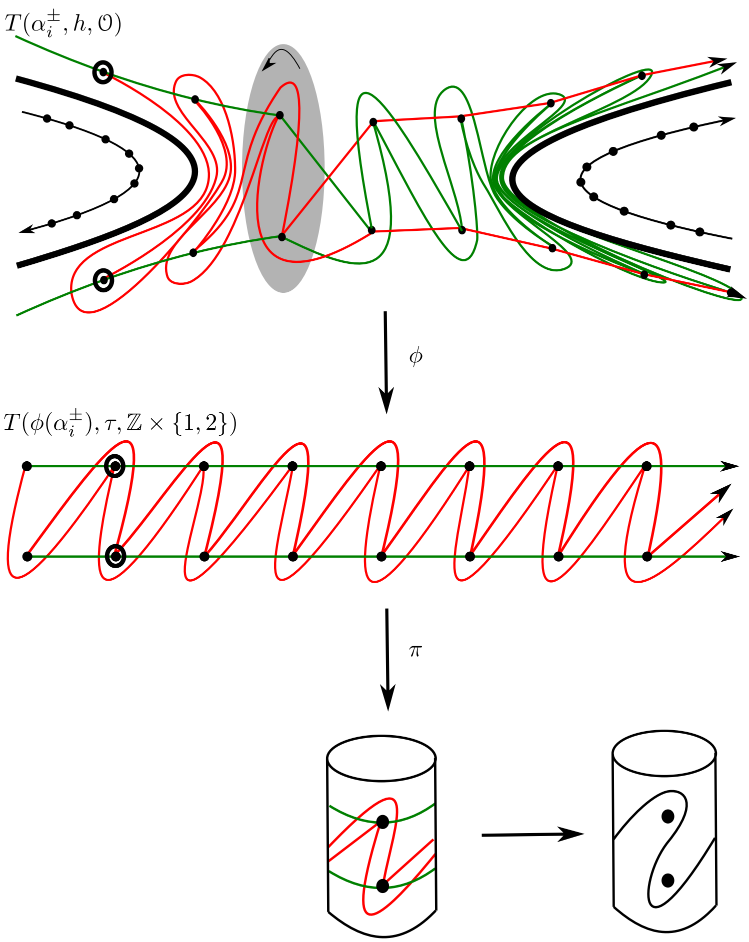

FIGURE 33. Definition of the relative tangle for the example B of §2 (Brouwer class of the product of a free half-twist with a flow).

- T_i^- onto $\mathbb{R} \times \{i\}$;
- $\{x_i\}$ onto $(0, i)$ and $\{h(x_i)\}$ onto $(1, i)$, where x_i and $h(x_i)$ are the endpoints of α_i^-; and
- $\mathcal{O}_i$ onto $\mathbb{Z} \times \{i\}$.

Let τ be the horizontal translation of the plane which maps $(x, y) \in \mathbb{R}^2$ to $(x + 1, y)$.

Let π be the quotient map which quotients $\mathbb{R}^2 - (\mathbb{Z} \times \{1, 2\})$ by τ and let $\mathcal{C}_2$ denote the quotient pointed cylinder. Note that, if we consider $\mathcal{C}_2$ as a vertical cylinder, then $\pi(\phi(\alpha_i^-))$ is homotopic in $\mathcal{C}_2$ to a horizontal circle for $i = 1, 2$ (see Figure 33 for an example).

LEMMA 8.2. *With the previous notation, the homotopy classes of the arcs $\pi(\phi(\alpha_i^+))$ in $\mathcal{C}_2$ are independent of ϕ.*

Proof. If ψ is another homeomorphism with the same properties, then ϕ and ψ coincide on the two topological lines T_1^- and T_2^-. According to Lemma 6.1, ϕ and ψ are isotopic relative to $\mathcal{O}_1 \cup \mathcal{O}_2$, and hence $\phi(\alpha_i^+)$ is isotopic to $\psi(\alpha_i^+)$ for $i = 1, 2$. □

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Denote by γ a curve which is disjoint from $\pi(\phi(\alpha_1^+))$ and $\pi(\phi(\alpha_2^+))$ and which separates $\mathcal{C}_2$ into two cylinders with a puncture, each of them containing one of the $\pi(\phi(\alpha_i^+))$. Note that γ is unique up to isotopy in $\mathcal{C}_2$.

Definition 8.1. We say that the isotopy class of $\gamma \in \mathcal{C}_2$ is the *tangle of the irreducible area of* $[h;\, \mathcal{O}]$ *relative to* $(\alpha_i^{\pm})_i$.

Remark 8.2. Note that γ is never a horizontal circle. Indeed, we could get a horizontal curve only if we had α_i^- isotopic to α_i^+ relative to $\mathcal{O}_1 \cup \mathcal{O}_2$ for $i = 1, 2$. In this situation, we get proper streamlines for every orbit, and hence $[h;\, \mathcal{O}]$ is a flow class: this gives a contradiction because we assumed that there exists an irreducible area for $[h;\, \mathcal{O}]$.

This relative tangle depends on the choice of the nice family $(\alpha_i^{\pm})_i$, and hence it is not a conjugacy invariant. However, we have the following lemma.

LEMMA 8.3. *Let* $[h;\, \mathcal{O}]$ *be a Brouwer mapping class relative to four orbits. Suppose that* $[h;\, \mathcal{O}]$ *is not a flow class. Then if* $(\alpha_i^{\pm})_i$ *and* $(\beta_i^{\pm})_i$ *are two nice families for* $[h;\, \mathcal{O}]$ *disjoint from the walls, then for every* i *there exists* n_i *such that* α_i^- *(respectively,* α_i^+*), is isotopic to* $h^{n_i}(\beta_i^-)$ *(respectively,* $h^{n_i}(\beta_i^+)$*) relative to* $\mathcal{O}$.

Proof. This will follow from the description of the adjacency areas of $[h;\, \mathcal{O}]$. Choose a complete family of adjacency areas for $[h;\, \mathcal{O}]$ and a representative $\{\Delta_1, \Delta_2\}$ of the set of walls. For every nice family $(\alpha_i^{\pm})_i$ disjoint from $\Delta_1 \cup \Delta_2$, there exists $(m_i, n_i)_i \in (\mathbb{Z}^2)^4$ such that $h^{m_i}(\alpha_i^-)$ and $h^{n_i}(\alpha_i^+)$ are included in adjacency areas. If we fix the endpoints in the adjacency area, there is only one isotopy class of homotopic translation arc included in the chosen adjacency area and disjoint from Δ_1 and Δ_2: indeed there is only one isotopy class of translation arcs for Brouwer class relative to one orbit (according to Corollary 6.3 of Handel [**Han99**]). □

We denote by T the left Dehn twist around a separating horizontal circle between the two punctures in $\mathcal{C}_2$.

LEMMA 8.4. *With the previous notation, if* $\gamma \in \mathcal{C}_2$ *(respectively,* $\gamma' \in \mathcal{C}_2$*) is the tangle of the irreducible area of* $[h;\, \mathcal{O}]$ *relative to* $(\alpha_i^{\pm})_i$ *(respectively,* $(\beta_i^{\pm})_i$*), then there exists* $n \in \mathbb{Z}$ *such that* $\gamma = T^n \gamma'$.

Proof. This is a consequence of Lemma 8.3: this lemma implies that if ϕ is as in the previous notation, for $i = 1, 2$,

$$T(\phi(\alpha_i^{\pm}), \tau, \mathbb{Z} \times \{1, 2\}) = T(\phi(h^{n_i}(\beta_i^{\pm})), \tau, \mathbb{Z} \times \{1, 2\}) = T(\phi(\beta_i^{\pm}), \tau, \mathbb{Z} \times \{1, 2\}).$$

Moreover, $\phi(h^{n_i}(x_i)) = \tau^{n_i}(\phi(x_i))$.

Since $\phi(x_i) = (0, i)$, it follows that $\phi(h^{n_i}(x_i)) = (n_i, i)$, and hence

$$\pi(\phi(\beta_i^{\pm})) = T^{n_1 - n_2}(\pi(\phi(\alpha_i^{\pm}))).$$ □

Definition 8.2. With the previous notation, we define the *tangle of* $[h;\, \mathcal{O}]$ to be the isotopy class $\gamma \in \mathcal{C}_2$ up to composition by T.

By convention, we set that every flow mapping class has trivial tangle.

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COROLLARY 8.5. *The couple (diagram with walls, tangle) is a conjugacy invariant for Brouwer mapping classes relative to four orbits.*

We will need the following result in the §8.3.

LEMMA 8.6. *With the previous notation, let γ be the tangle of the irreducible area of $[h; \mathcal{O}]$ relative to $(\alpha_i^{\pm})_i$, and let γ' be $T^n(\gamma)$ for some $n \in \mathbb{Z}$, where T is the left Dehn twist, as above. Then there exists a nice family $(\beta_i^{\pm})_i$ for $[h; \mathcal{O}]$ such that the tangle of the irreducible area of $[h; \mathcal{O}]$ relative to $(\beta_i^{\pm})_i$ is the isotopy class of γ'.*

Proof. Define $(\beta_i^{\pm})_i$ as $\beta_1^{\pm} := h^n(\alpha_1^{\pm})$ and $\beta_i^{\pm} := \alpha_i^{\pm}$ for $i = 2, 3, 4$. □

8.2.3. *Realized couples (diagram with walls, tangle).* In §8.3, we will show that the couple (diagram with walls, tangle) is a total conjugacy invariant. Here we find which couples are realized by Brouwer mapping classes relative to four orbits.

Necessary condition to be realized. Not every couple (diagram with walls, tangle) can be realized by a Brouwer mapping class. Indeed, some tangles are associated to non-determinant diagrams with crossing arrows, and some other tangles are associated to non-determinant diagrams without crossing arrows. To be more precise, denote by p and q the two marked points of $\mathcal{C}_2$, and suppose that p is above q. If γ is a curve of $\mathcal{C}_2$ representing the tangle, the marked point of $\mathcal{C}_2$ which is *above* γ (that is, in the connected component of the complement of γ which contains the top of the cylinder $\mathcal{C}_2$) can be p or q, depending on the tangles.

Moreover, this point represents the orbit whose forward half-streamline is above on the picture, and hence whose arrow ends above the other on the diagram. It follows that if this point is p, then the diagram is without crossing, and if this point is q, the diagram has a crossing. We say that such a tangle is *adapted* to the diagram. See Figure 34 for these two examples.

- On the tangle of the left, p is above γ, and hence it is the tangle of a diagram without crossing.
- On the tangle of the right, q is above γ, and hence it is the tangle of a diagram with crossing.

Note that there are an infinite number of tangles adapted to each diagram.

Realizing the adapted tangles. Given a couple (diagram with walls, tangle) such that the tangle is adapted to the diagram, we can produce a Brouwer homeomorphism which realizes this couple as follows. Denote by (D, τ) the given couple.

(1) (a) If the diagram D does not have crossing arrows, then we define $D' = D$.
(b) If the diagram D has crossing arrows, we consider the diagram D' obtained by exchanging the ends of the two crossing arrows. This is a diagram without crossing arrows.

(2) We choose a flow f which realizes the diagram D' without walls and such that there exists an f-free disk which contains one point of each orbit $\mathcal{O}_1$ and $\mathcal{O}_2$ (as in example A of §2), where $\mathcal{O}_1$ and $\mathcal{O}_2$ are the two orbits of the irreducible area of D.

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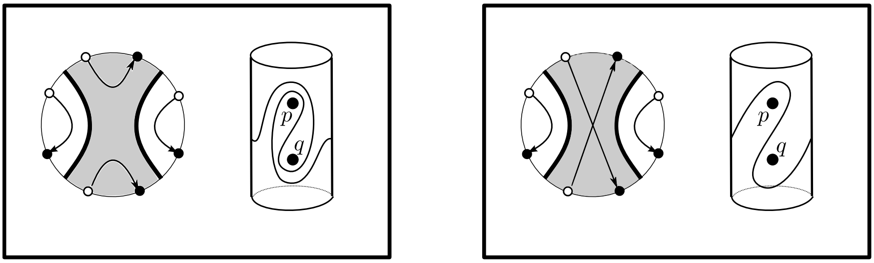

FIGURE 34. Examples of two diagrams with adapted tangles.

(3) By reversing the process of the definition of the tangle given in §8.2.2, we get two families of translation arcs. Proposition 7.1 provides us with a finite product μ of mutually disjointly supported f-free half-twists such that the tangle of $[\mu f;\, \mathcal{O}]$ is τ. Note also that the diagram associated to $[\mu f;\, \mathcal{O}]$ is D.

8.2.4. *Infinite number of Brouwer mapping classes relative to four orbits.*

PROPOSITION 8.7. *Up to conjugacy, there are an infinite (countable) number of Brouwer mapping classes relative to four orbits.*

Proof. There are an infinite number of tangles adapted to each non-determinant diagram (see Figure 34 for examples). Each of them is realized by a product of a flow with a finite number of free half-twists disjointly supported (using Proposition 7.1, see the second paragraph of §8.2.3). It follows from Corollary 8.5 that there are an infinite number of Brouwer mapping classes relative to four orbits. □

8.3. *A total conjugacy invariant.* In this section we want to show Theorem 3.13, namely that two Brouwer mapping classes relative to four orbits are conjugated if and only if they have the same invariant couple (diagram with walls, tangle).

The following Lemma 8.8 together with Lemma 8.6 give the proof of Theorem 3.13: indeed if $[h;\, \mathcal{O}]$ and $[h';\, \mathcal{O}']$ have the same invariant couple, then Lemma 8.6 provides us with a nice family $(\alpha_i^{\pm})_i$ for $[h;\, \mathcal{O}]$ and a nice family $(\beta_i^{\pm})_i$ for $[h';\, \mathcal{O}']$ such that $[h;\, \mathcal{O}]$ and $[h';\, \mathcal{O}']$ have the same tangle relative to their nice family. Hence they satisfy the hypothesis of Lemma 8.8, which says that they are conjugated.

LEMMA 8.8. *Let $[h;\, \mathcal{O}]$ and $[h';\, \mathcal{O}']$ be two Brouwer mapping classes relative to four orbits such that:*

- *they have the same diagram with walls; and*
- *there exist two nice families $(\alpha_i^{\pm})_i$ for $[h;\, \mathcal{O}]$ and $(\beta_i^{\pm})_i$ for $[h';\, \mathcal{O}']$ such that the two Brouwer mapping classes have the same relative tangle relative to their nice family.*

Then $[h;\, \mathcal{O}]$ and $[h';\, \mathcal{O}']$ are conjugated.

Proof. Note that if the diagram with walls of $[h;\, \mathcal{O}]$ has crossing arrows, then these two crossing arrows are in an irreducible area: indeed they are in the same connected component of the walls, which is not a translation area, and hence it is an irreducible area

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(according to Theorem 3.5). If the diagram with walls is without any irreducible area, then $[h; \mathcal{O}]$ and $[h'; \mathcal{O}']$ are conjugated (this is Proposition 3.8).

Let us consider the case when there exists an irreducible area. We suppose that the orbits which intersect this area are indexed by one and two. Denote by Z and Z' the irreducible areas of $[h; \mathcal{O}]$ and $[h'; \mathcal{O}']$, respectively. We assume that h preserves Z and h' preserves Z'. Denote by τ the translation of the plane which maps every $(x, y) \in \mathbb{R}^2$ onto $(x + 1, y)$. Denote by ϕ a homeomorphism of the plane as needed to define the tangle for $[h; \mathcal{O}]$: that is, which maps:

- $T_1^- \cup T_2^-$ onto $\mathbb{R} \times \{1, 2\}$;
- $\{x_1, x_2\}$ onto $\{0\} \times \{1, 2\}$, where x_i and $h(x_i)$ are the endpoints of α_i^-; and
- $\mathcal{O}_1 \cup \mathcal{O}_2$ onto $\mathbb{Z} \times \{1, 2\}$;

where $T_i^\pm := \bigcup_{k\in\mathbb{Z}} h^k(\alpha_i^\pm)_\#$.

Likewise, denote by ψ a homeomorphism which maps:

- $T_1^{'-} \cup T_2^{'-}$ on $\mathbb{R} \times \{1, 2\}$;
- $\{x'_1, x'_2\}$ on $\{0\} \times \{1, 2\}$, where x'_i and $h(x'_i)$ are the endpoints of β_i^-; and
- $\mathcal{O}'_1 \cup \mathcal{O}'_2$ on $\mathbb{Z} \times \{1, 2\}$;

where $T_i^{'\pm} := \bigcup_{k\in\mathbb{Z}} h'^k(\alpha_i^{'\pm})_\#$.

Since the two classes have the same tangle relative to their nice families, by definition of ϕ and ψ, $(\phi\alpha_i^\pm)_\# = (\psi\beta_i^\pm)_\#$ for $i = 1, 2$. Denote by $\gamma_i^\pm$ these arcs.

CLAIM 1. *There exists $\phi' \in [\phi; \mathcal{O}_1 \cup \mathcal{O}_2]$ and $\psi' \in [\psi'; \mathcal{O}'_1 \cup \mathcal{O}'_2]$ such that $\phi' Z = \psi' Z'$.*

Proof of the claim. Denote by Δ_1 (respectively Δ_2), the boundary component of Z which is disjoint from $T_1^- \cup T_2^-$ (respectively, disjoint from $T_1^+ \cup T_2^+$). Since they are disjoint, we may assume that $\phi(\Delta_1)$ is included in the left half-plane and $\phi(\Delta_2)$ is included in the right half-plane. Similarly, we denote by Δ'_1 and Δ'_2 the boundary components of Z', disjoint from $T_1^{'-} \cup T_2^{'-}$ and $T_1^{'+} \cup T_2^{'+}$, respectively, and we assume that $\psi(\Delta'_1)$ is included in the left half-plane and $\psi(\Delta'_2)$ is included in the right half-plane. Now $\phi(\Delta_1)$ and $\psi(\Delta'_1)$ are lines included in the left half-strip between $\mathbb{R} \times \{1\}$ and $\mathbb{R} \times \{2\}$: there exists a homeomorphism λ_1 supported in this half-strip which sends $\phi(\Delta_1)$ on $\psi(\Delta'_1)$. Similarly, there exists a homeomorphism λ_2 supported in the right half-strip between $\bigcup_{n\geq 0} \tau^n(\gamma_1)$ and $\bigcup_{n\geq 0} \tau^n(\gamma_2)$ which sends $\phi(\Delta_2)$ onto $\psi(\Delta'_2)$. It follows that $\lambda_1\lambda_2\phi(Z) = \psi(Z')$. Since $\lambda_1\lambda_2$ is isotopic to the identity relative to $\mathbb{R} \times \{1, 2\}$, $\lambda_1\lambda_2\phi$ is isotopic to ϕ. □

Back to the proof of Lemma 8.8. Up to isotopying ϕ relative to $\mathcal{O}_1 \cup \mathcal{O}_2$ and ψ relative to $\mathcal{O}'_1 \cup \mathcal{O}'_2$ as in Claim 1, we may assume that $\phi Z = \psi Z'$.

According to Proposition 7.1, there exists a finite product of τ-free half-twists disjointly supported and such that for every sufficiently large $k \in \mathbb{N}$, $(\mu\tau)^k(\gamma_i^-)$ is isotopic relative to $\mathbb{Z} \times \{1, 2\}$ to $\tau^k(\gamma_i^+)$ for $i = 1, 2$. Since μ is compactly supported, we can suppose that this support is included in $\phi(Z) = \psi(Z')$.

CLAIM 2. *With the previous notation, we claim that* $[\phi^{-1}\mu\phi h; \mathcal{O}]$ *and* $[\psi^{-1}\mu\psi h'; \mathcal{O}']$ *are flow classes.*

Proof of the claim. We carry out the proof for $[\phi^{-1}\mu\phi h; \mathcal{O}]$: relative to $\mathcal{O}_1 \cup \mathcal{O}_2$, for every sufficiently large $k \in \mathbb{N}$, $(\phi^{-1}\mu\phi h)^k(\alpha_i^-)$ is isotopic to $h^k(\alpha_i^+)$ for $i = 1, 2$. Because μ is supported in $\phi(Z)$, $\phi^{-1}\mu\phi$ is supported in Z and, since h preserves Z, it follows that $\phi^{-1}\mu\phi h$ also preserves Z. Since α_i^- is included in Z, and since $\mathcal{O}_3$ and $\mathcal{O}_4$ do not intersect Z, $(\phi^{-1}\mu\phi h)^k(\alpha_i^-)$ is isotopic to $h^k(\alpha_i^+)$ relative to $\mathcal{O}$ (and not only relative to $\mathcal{O}_1 \cup \mathcal{O}_2$). Since α_i^+ is a forward proper arc for $[h; \mathcal{O}]$, it follows that $T(\alpha_i^-, \phi^{-1}\mu\phi h, \mathcal{O})$ is a proper streamline. Since $\phi^{-1}\mu\phi$ is supported in Z, h is equal to $\phi^{-1}\mu\phi h$ outside Z, and hence for $j = 3, 4$, $T(\alpha_j^-, \phi^{-1}\mu\phi h, \mathcal{O}) = T(\alpha_j^-, h, \mathcal{O})$ and thus is also a proper streamline. By Lemma 6.2, it follows that $[\phi^{-1}\mu\phi h; \mathcal{O}]$ is a flow class. □

Back to the proof of Lemma 8.8. Denote by f and g two flows such that

$$[f; \mathcal{O}] = [\phi^{-1}\mu\phi h; \mathcal{O}] \quad \text{and} \quad [g; \mathcal{O}'] = [\psi^{-1}\mu\psi h'; \mathcal{O}'].$$

For every i, denote by T_i (respectively, by T_i'), the proper streamline $T(\alpha_i^-, f, \mathcal{O})$ (respectively, $T(\beta_i^-, g, \mathcal{O}')$). Changing ψ in the complement of Z', if necessary, we assume that $\phi^{-1}\psi$ maps T_3' onto T_3, $\mathcal{O}_3'$ onto $\mathcal{O}_3$, T_4' onto T_4 and $\mathcal{O}_4'$ onto $\mathcal{O}_4$. This is possible because the diagrams of $[h; \mathcal{O}]$ and $[h'; \mathcal{O}']$ are the same. Note that, for every $k \in \mathbb{Z}$, $\phi^{-1}\psi(\psi^{-1}\mu\psi h')^k(\beta)$ is isotopic to $(\phi^{-1}\mu\phi h)^k(\alpha_i^-)$ relative to $\mathcal{O}$ for $i = 1, 2$. Hence $T_i = \phi^{-1}\psi(T_i')$ for every i. According to Lemma 6.1,

$$[\phi^{-1}\psi g\psi^{-1}\phi; \mathcal{O}] = [f; \mathcal{O}].$$

Composing both parts by $\phi^{-1}\mu^{-1}\phi$, we can check that

$$[(\phi^{-1}\psi)h'(\phi^{-1}\psi)^{-1}; \mathcal{O}] = [h; \mathcal{O}].$$

Hence $[h; \mathcal{O}]$ and $[h'; \mathcal{O}']$ are conjugated. □

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A. *Appendix. Diagrams with four orbits*

Here we represent all the diagrams with four orbits (Figures A.1–A.4). If a diagram can be obtained with a Brouwer mapping class, then we also draw the possible sets of walls, and color the eventual irreducible areas in gray. We put together in the same dashed box the diagrams which are the same without walls but which have different possible sets of walls. We get three different types of diagrams.

(1) The full-framed diagrams are the ones with a Handel's cycle: according to Handel's fixed point theorem [**Han99**, Theorem 2.3], they cannot be obtained with Brouwer homeomorphisms. Also, we forget them when describing Brouwer mapping classes relative to four orbits.

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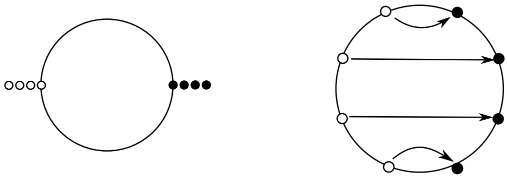

FIGURE A.1. Diagram for $r = 4$ and $r' = 1$.

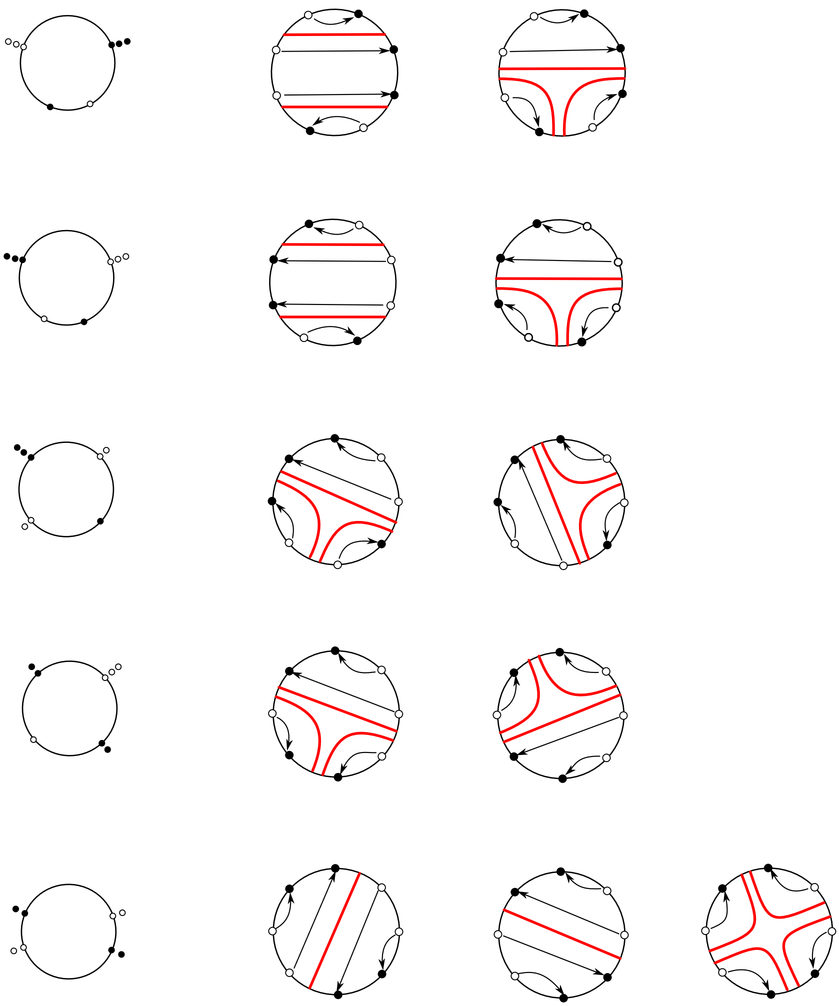

FIGURE A.2. Diagrams for $r = 4$ and $r' = 2$.

(2) The diagrams without irreducible areas are the determinant ones. Every of them can be realized by a Brouwer mapping class (according to [**LR13**, Lemma 1.7]). Moreover, up to conjugation, this Brouwer mapping class is unique and it is a flow class (Propositions 3.6 and 3.7).

(3) The diagrams with an irreducible area (in gray) are the eight non-determinant ones.

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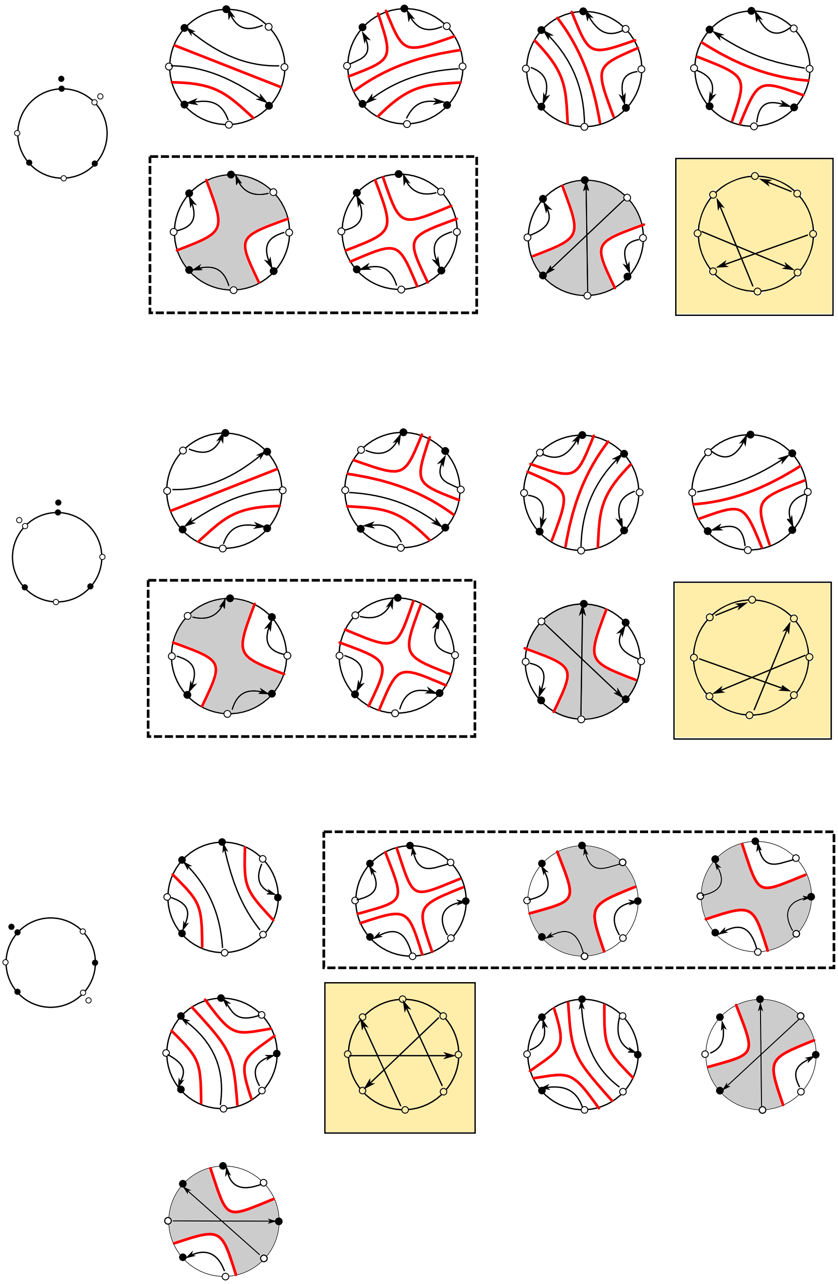

FIGURE A.3. Diagrams for $r = 4$ and $r' = 3$.

Up to conjugation, all of them can be realized by an infinite number of Brouwer mapping classes. For these diagrams, the tangle allows us to differentiate the different Brouwer mapping classes (see §§8.2 and 8.3).

We still denote the number of families of adjacency by $2r'$.

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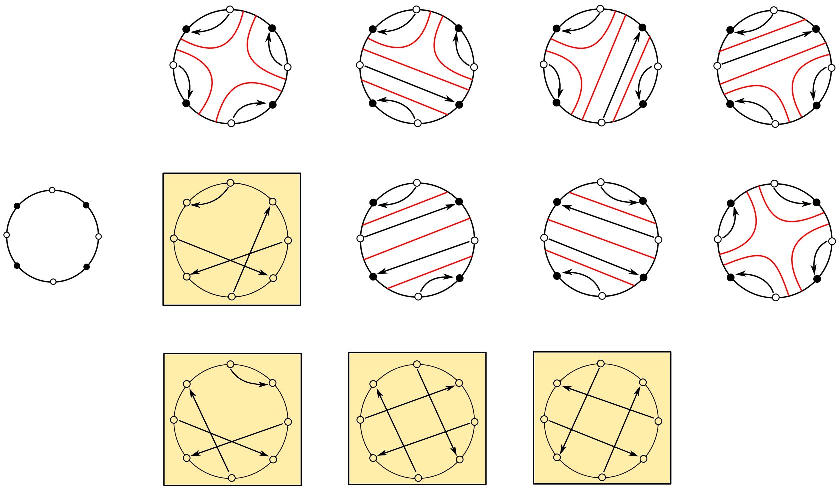

FIGURE A.4. Diagrams for $r = 4$ and $r' = 4$.

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