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TENSOR PRODUCTS OF CLEAN RINGS

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Abstract. A ring is called *clean* if every element is the sum of an idempotent and a unit. It is an open question whether the tensor products of two clean algebras over a field is clean. In this note we study the tensor product of clean algebras over a field and we provide some examples to show that the tensor product of two clean algebras over a field need not be clean.

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1. Introduction. Throughout this paper, R is commutative ring and we use $\mathrm{Min}(R)$ to denote the set of minimal prime ideals of R. We say R is quasi-local (resp. semi-local) if the set of maximal ideals of R has only one element (resp. finitely many elements). An element in R is called clean if it is the sum of a unit and an idempotent. Following Nicholson, cf. [**4**], we call the ring R clean if every element in R is clean. Examples of clean rings include all zero-dimensional rings (i.e. every prime ideal is maximal) and local rings. Clean rings have been studied by several authors, for example [**4**], [**2**], and [**1**]. It is an open question whether the tensor product of two clean algebras over a field is clean, cf. [**2**, Question 3]. The main purpose of this note is to prove Theorem 1, while Theorem 2 and Proposition 3 are used in the proof of Theorem 1. As an application of Theorem 1 we use it to give an example of two clean algebras A and B over a field F where the tensor product $A \otimes_F B$ is not clean, see Example 4. In this paper all algebras are unital.

THEOREM 1. *Let* F *be an algebraically closed field. Let* A *and* B *be algebras over* F. *If* A *and* B *have a finite number of minimal prime ideals* (*e.g.* A *and* B *Noetherian*) *then the following statements are equivalent:*

(*i*) $\mathrm{A} \otimes_{\mathrm{F}} \mathrm{B}$ *is clean.*
(*ii*) *The following hold*
  (*a*) A *and* B *are clean.*
  (*b*) A *or* B *is algebraic over* F.

To prove the above Theorem we first recall the following result from [**1**] and prove Proposition 3.

THEOREM 2. ([**1**, Theorem 5]) *Let* R *have a finite number of minimal prime ideals* (*e.g.*, R *is Noetherian*). *Then the following conditions are equivalent.*

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(i) R *is a finite direct product of quasi-local rings.*
(ii) R *is a clean ring.*
(iii) $\mathrm{R}/\mathfrak{p}$ *is quasi-local for each prime ideal* $\mathfrak{p}$ *of* R.

PROPOSITION 3. *Let* A *and* B *be algebras over a field* F. *Let* $\mathrm{Min}(\mathrm{A} \otimes_{\mathrm{F}} \mathrm{B})$ *be a finite set and assume that* $\mathrm{A} \otimes_{\mathrm{F}} \mathrm{B}$ *is clean. Then the following hold.*
(i) A *or* B *is algebraic over* F.
(ii) A *and* B *are clean.*
(iii) *For any* $\mathfrak{m} \in \mathrm{Max}(\mathrm{A})$ *and* $\mathfrak{n} \in \mathrm{Max}(\mathrm{B})$ *the ring* $\mathrm{A}/\mathfrak{m} \otimes_{\mathrm{F}} \mathrm{B}/\mathfrak{n}$ *is semi-local.*

Proof. (i) By Theorem 2 we know that $A \otimes_F B$ is semi-local and hence by [**3**, Theorem 6] A or B is algebraic over F.

(ii) Assume that A is algebraic over F. Then $\dim(A) = \dim(F) = 0$ and so A is clean, cf. [**1**, Corollary 11]. We know that $\varphi : B \to (A \otimes_F B)$ is integral. Assume that $\mathfrak{p}_2 \in \mathrm{Spec}\,(B)$. Since φ is faithfully flat there exists $\mathfrak{q} \in \mathrm{Spec}\,(A \otimes_F B)$ such that $\mathfrak{q} \cap B = \mathfrak{p}_2$. Since $\tilde{\varphi} : B/\mathfrak{p}_2 \to (A \otimes_F B)/\mathfrak{q}$ is integral and $(A \otimes_F B)/\mathfrak{q}$ is quasi-local, $B/\mathfrak{p}_2$ is quasi-local. On the other hand, since φ is faithfully flat and $\mathrm{Min}(A \otimes_F B)$ is finite, $\mathrm{Min}(B)$ is finite too. Therefore, by Theorem 2, B is clean.

(iii) By Theorem 2, $A \otimes_F B$ is semi-local and so $A/\mathfrak{m} \otimes_F B/\mathfrak{n} \cong (A \otimes_F B)/(\mathfrak{m} \otimes_F B + A \otimes_F \mathfrak{n})$ is semi-local. □

Proof of Theorem 1. (i) $\Longrightarrow$ (ii) First we show that $A \otimes_F B$ has a finite number of minimal prime ideals. Assume $\mathfrak{q} \in \mathrm{Min}(A \otimes_F B)$ and set $\mathfrak{q} \cap A = \mathfrak{p}_1$ and $\mathfrak{q} \cap B = \mathfrak{p}_2$. Since $A \to A \otimes_F B$ is a faithfully flat homomorphism we have that $\mathfrak{p}_1 \in \mathrm{Min}(A)$ and for the same reason $\mathfrak{p}_2 \in \mathrm{Min}(B)$. In addition, $\mathfrak{q} \in \mathrm{Min}(\mathfrak{p}_1 \otimes_F B + A \otimes_F \mathfrak{p}_2)$. Since F is algebraically closed $A \otimes_F B/(\mathfrak{p}_1 \otimes_F B + A \otimes_F \mathfrak{p}_2) \cong A/\mathfrak{p}_1 \otimes_F B/\mathfrak{p}_2$ is an integral domain. Therefore $\mathfrak{q} = \mathfrak{p}_1 \otimes_F B + A \otimes_F \mathfrak{p}_2$. Now the assertion follows from Proposition 3.

(ii) $\Longrightarrow$ (i). Assume that $\mathfrak{q} \in \mathrm{Spec}\,(A \otimes_F B)$ and set $\mathfrak{q} \cap A = \mathfrak{p}_1$ and $\mathfrak{q} \cap B = \mathfrak{p}_2$. Then $\mathfrak{p}_1 \otimes_F B + A \otimes_F \mathfrak{p}_2 \subseteq \mathfrak{q}$. Since A and B are clean and $\mathrm{Min}(A)$ and $\mathrm{Min}(B)$ are finite we have that $A/\mathfrak{p}_1$ and $B/\mathfrak{p}_2$ are quasi-local. Let $\mathfrak{m}/\mathfrak{p}_1$ (resp. $\mathfrak{n}/\mathfrak{p}_2$) be the unique maximal ideal of $A/\mathfrak{p}_1$ (resp. $B/\mathfrak{p}_2$). Since one of A or B is algebraic over F we have that one of $A/\mathfrak{p}_1$ or $B/\mathfrak{p}_2$ is algebraic over F. Since one of $A/\mathfrak{m}$ or $B/\mathfrak{n}$ is algebraic over F we have $\dim(A/\mathfrak{m} \otimes_F B/\mathfrak{n}) = 0$. On the other hand, F is algebraically closed so $A/\mathfrak{m} \otimes_F B/\mathfrak{n}$ is an integral domain. Therefore $A/\mathfrak{m} \otimes_F B/\mathfrak{n}$ is a field. Now by [**5**] the ring $A/\mathfrak{p}_1 \otimes_F B/\mathfrak{p}_2$ is quasi-local and hence $A \otimes_F B/(\mathfrak{p}_1 \otimes_F B + A \otimes_F \mathfrak{p}_2)$ is quasi-local. Now the assertion follows from Theorem 2. □

EXAMPLE 4. Assume that $F = \mathbb{C}$ and $A = B = \mathbb{C}[|x|]$. Then by [**1**, Proposition 12] A and B are clean. We claim that $A \otimes_F B$ is not clean. Otherwise, since $\mathbb{C}$ is an algebraically closed field and $A (= B)$ is Noetherian, by Theorem 1, we have that A or B is algebraic over $\mathbb{C}$ and hence $A (= B)$ is equal to $\mathbb{C}$. That is a contradiction.

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