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A CHARACTERIZATION OF PROXIMAL SUBGRADIENT SET-VALUED MAPPINGS

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ABSTRACT. In this paper we tackle the problem of identifying set-valued mappings that are subgradient set-valued mappings. We show that a set-valued mapping is the proximal subgradient mapping of a lower semicontinuous function bounded below by a quadratic if and only if it satisfies a *monotone selection property*.

1. **Introduction.** In nonsmooth analysis, where one works with functions that are not differentiable in any classical sense, many types of *subgradients* have been introduced, *e.g. (Clarke) generalized* subgradients (see [1], [2], [13]), *approximate* subgradients (see [3]), *proximal* subgradients (see [6], [7], [10]), and *lower* subgradients (see [4]). The (Clarke) generalized subgradients are probably the best known among these different flavors of subgradients. To obtain the set of all (Clarke) generalized subgradients of a locally Lipschitzian function one takes the convex hull of the set of limiting proximal subgradients; for an arbitrary function one needs to consider in addition the *singular limiting* proximal subgradients; see [10]. For a lower semicontinuous extended-real-valued function f on $\mathbb{R}^n$ (*i.e.* $f: \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{\infty\}$), we say that a vector u is a proximal subgradient to f at $\bar{x}$ if, for some positive t,

$$f(x) \geq f(\bar{x}) + \langle u, x - \bar{x} \rangle - (t/2)||x - \bar{x}||^2 \text{ in a neighborhood of } \bar{x}.$$

The set of all proximal subgradients at $\bar{x}$ is denoted by $\partial_p f(\bar{x})$, the set of (Clarke) generalized gradients is denoted by $\partial f(x)$.

The expression proximal, comes from an equivalent characterization in terms of the *proximal normal cone*. For a closed set C of $\mathbb{R}^n$ and an element x of C, we say that y is a proximal normal to C at x if, for some positive t, x is the unique closest point of C to $x + ty$. The proximal normal cone is the set of all proximal normals, and is denoted by $\mathrm{PN}_C(x)$. The relationship between the set of proximal subgradients and the proximal normal cone is the following:

$$y \in \partial_p f(x) \Longleftrightarrow (y, -1) \in \mathrm{PN}_{\mathrm{epi} f}\big(x, f(x)\big).$$

(where $\mathrm{epi} f = \{(x, \alpha) \in \mathbb{R}^n \times \mathbb{R} \mid \alpha \geq f(x)\}$).

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When the function f is convex then the set of (Clarke) generalized subgradients to f at $\bar{x}$ is equal to the *subdifferential* to f at $\bar{x}$. (the same can be said for the set of proximal subgradients, in fact for the set of all subgradients mentioned previously). Recall that the subdifferential to f at $\bar{x}$, written $\partial f(\bar{x})$, is given by

$$\partial f(\bar{x}) = \{y \mid f(x) \geq f(\bar{x}) + \langle y, x - \bar{x}\rangle, \text{ for all } x\}.$$

It is well known that a set valued mapping Γ is the *subdifferential* of a lower semicontinuous proper (*i.e.* there exists $\bar{x}$ with $f(\bar{x}) < \infty$) convex function f if and only if Γ is maximal *cyclically monotone*; see [9]. Recall that a set-valued mapping $\Gamma\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is cyclically monotone if given $(x_i, y_i) \in \operatorname{gph}\Gamma$, $i = 0, \ldots, m$, where m is arbitrary and $\operatorname{gph}\Gamma$ is the graph of Γ, we have

$$\langle x_1 - x_0, y_0\rangle + \langle x_2 - x_1, y_1\rangle + \cdots + \langle x_0 - x_m, y_m\rangle \leq 0,$$

where $\langle x, y\rangle$ is the usual dot product. The set-valued mapping Γ is *monotone* if given $(x_i, y_i) \in \operatorname{gph}\Gamma$, $i = 1, 2$ we have $\langle x_1 - x_2, y_1 - y_2\rangle \geq 0$.

The next contribution in this area of identifying set-valued mappings that are subgradient mappings is due to Janin in 1984. In [5] he showed that a mapping Γ is *cyclically submonotone* if and only if Γ is the (Clarke) generalized subgradient mapping of a *lower-C^1* (locally Lipschitzian) function; see [11] for lower-C^1 functions. A set-valued mapping Γ is cyclically submonotone if for all $\bar{x}$ in the domain of Γ (i.e $\Gamma(\bar{x}) \neq \emptyset$) we have

$$\liminf_{\substack{x_1 \neq x_2 \\ x_i \to \bar{x} \\ y_i \in \Gamma(x_i)}} \frac{\langle x_1 - x_2, y_1 - y_2\rangle}{\|x_1 - x_2\|} \geq 0$$

This is not the definition given by Janin, but rather the equivalent one given in Spingarn [14].

Surprisingly enough very little is known beyond the cyclically monotone and the cyclically submonotone cases. How does one tell if a given set-valued mapping is a subgradient mapping? In this paper we give a necessary and sufficient condition for a set-valued mapping to be the proximal subgradient set-valued mapping of a lower semicontinuous function bounded below by a quadratic. In Theorem 2.3, we show that a set-valued mapping Γ is a proximal set-valued mapping if and only if it satisfies a *monotone selection* property, *i.e.* there exist $\bar{t}$ and for $t \geq \bar{t}$, set-valued mappings $M_t\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ with

(a) $M_t(x) \subset \Gamma(x)$ for all x.

(b) $M_t(x) \uparrow \Gamma(x)$ (*i.e.*, if $t_1 < t_2$, then $M_{t_1}(x) \subset M_{t_2}(x)$ and $\bigcup_{t \geq \bar{t}} M_t(x) = \Gamma(x)$).

(c) $M_t + tI$ is a monotone set-valued mapping (where I is the identity mapping).

(d) The set-valued mapping M defined by

$$M(z,t) = \begin{cases} \operatorname{con}\{(x, -(1/2)\|x\|^2) : x \in (M_t + tI)^{-1}(z)\} & t > \bar{t} \\ \operatorname{con}\{(x, -(1/2)\|x\|^2) : x \in (M_{\bar{t}} + \bar{t}I)^{-1}(z)\} + \{(0, -\lambda) \mid \lambda \in \mathbb{R}\} & t = \bar{t} \\ \emptyset & t < \bar{t} \end{cases}$$

(where $\operatorname{con} C$ is the convex hull of C) is maximal cyclically monotone.

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2. **Main result.** In this section we will assume that the lower semicontinuous function f *is strictly bounded below by a quadratic, with quadratic part* $(\bar{t}/2) \geq 0$ *i.e.*, there exist $\bar{a}, \bar{z}$ and $\bar{t}$, such that

$$f(x) > \bar{a} + \langle \bar{z}, x \rangle - (\bar{t}/2)\|x\|^2. \tag{2.1}$$

This occurs, for example, when $\operatorname{dom} f$ is a bounded set (where $\operatorname{dom} f = \{x \mid f(x) < \infty\}$), since in this case the function is bounded below (recall that f is lower semicontinuous).

In [8] the *quadratic conjugate function* was introduced as a tool for studying proximal subgradients; recall that for z in $\mathbb{R}^n$ and $t \geq \bar{t}$, the quadratic conjugate to f at (z, t) is given by

$$h_f(z,t) = \max_{x \in \mathbb{R}^n} \{\langle z, x \rangle - (t/2)\|x\|^2 - f(x)\}. \tag{2.2}$$

We are justified in writing max, since f is bounded below by a quadratic, with "quadratic part $\bar{t}$". Let argmax $h_f(z,t)$ be the set of points where the maximum is attained in (2.2), *i.e.*,

$$\operatorname{argmax} h_f(z,t) = \{x : \langle z, x \rangle - (t/2)\|x\|^2 - f(x) = h_f(z,t)\}.$$

NOTE. This is not the standard notation; to be more precise we have argmax $h_f(z,t)$ = argmax $h_f(z,t,\cdot)$ where $h_f(z,t,x) = \langle z, x \rangle - (t/2)\|x\|^2 - f(x)$.

The function h_f is lower semicontinuous proper and convex, with domain $\mathbb{R}^n \times [\bar{t}, \infty)$. We can express the subgradients of the quadratic conjugate function in the following way (please see [8] for details): For $t > \bar{t}$,

$$\partial h_f(z,t) = \operatorname{con}\left\{\left(x, -\frac{\|x\|^2}{2}\right) : x \in \operatorname{argmax} h_f(z,t)\right\}, \tag{2.3}$$

where con(S) is the convex hull of S, and for $t = \bar{t}$

$$\partial h_f(z,\bar{t}) = \operatorname{con}\left\{\left(x, -\frac{\|x\|^2}{2}\right) : x \in \operatorname{argmax} h_f(z,\bar{t})\right\} + \{(0, -\lambda) \mid \lambda \in \mathbb{R}\}. \tag{2.4}$$

We recall here some of the important properties of the conjugate quadratic function (these were all established in [8]).

THEOREM 2.1. *(a) For $z \in \mathbb{R}^n$ and $t \geq \bar{t}$, if $\bar{x} \in$ argmax $h_f(z,t)$, then $z - t\bar{x} \in \partial_p f(\bar{x})$.*
(b) If $u \in \partial_p f(\bar{x})$, then for t big enough, argmax $h_f(u + t\bar{x}, t) = \{\bar{x}\}$.
(c) If $\left(x, -(1/2)\|x\|^2\right) \in \partial h_f(z,t)$ and $t \geq \bar{t}$, then $x \in$ argmax $h_f(z,t)$.
(d) For all $\bar{x}$,

$$f(\bar{x}) = \sup_{\substack{(z,t) \\ t \geq \bar{t}}} \{\langle z, \bar{x} \rangle - (t/2)\|\bar{x}\|^2 - h_f(z,t)\}.$$

Hence, $f(\bar{x}) = h_f^*\left(\bar{x}, -(1/2)\|\bar{x}\|^2\right)$, where h_f^* is the convex conjugate of the function h_f; see [9].

We now give a characterization of the quadratic conjugate function. It is remarkable that a simple differentiability property and an adequate domain identifies a convex function as the quadratic conjugate of a lower semicontinuous function which is bounded below by a quadratic.

THEOREM 2.2. *Assume that* $h\colon \mathbb{R}^n \times \mathbb{R} \longrightarrow \mathbb{R}\cup\{\infty\}$ *is a lower semicontinuous proper convex function and that for some positive* $\bar{t}$, *the set* $\{(z,t) : t \geq \bar{t}\} \subset \operatorname{dom} h$. *In addition, we assume that if* h *is differentiable at* (z,t), *then* $\nabla h(z,t) = \big(x, -(1/2)\|x\|^2\big)$ *for some* x *in* $\mathbb{R}^n$. *Under these assumptions, there exists a lower semicontinuous extended-real-valued function on* $\mathbb{R}^n$, *bounded below by a quadratic, with quadratic conjugate* h. *Moreover, the function is given by*

$$\sup_{\substack{(z,t)\\ t\geq\bar{t}}}\{\langle z,x\rangle - (t/2)\|x\|^2 - h(z,t)\}. \tag{2.5}$$

PROOF. Let $f(x)$ be given by (2.5). Clearly f is lower semicontinuous, it is bounded below by a quadratic and we will show that $h_f(z,t) = h(z,t)$ for all $t \geq \bar{t}$.

CLAIM. Assume h is differentiable at (z_0,t_0) with $t_0 > \bar{t}$ and $\nabla h(z_0,t_0) = \big(x_0, -(1/2)\|x_0\|^2\big)$; then $f(x_0) = \langle z_0,x_0\rangle - (t_0/2)\|x_0\|^2 - h(z_0,t_0)$ and $h(z_0,t_0) = h_f(z_0,t_0)$.

PROOF OF CLAIM. Consider $L_{x_0}(z,t) = \langle z,x_0\rangle - (t/2)\|x_0\|^2 - h(z,t)$. The function L_{x_0} is concave with $\nabla L_{x_0}(z_0,t_0) = (0,0)$. Therefore, L_{x_0} attains a global maximum at (z_0,t_0). This means that $f(x_0) = \langle z_0,x_0\rangle - (t_0/2)\|x_0\|^2 - h(z_0,t_0)$. For all x, $h(z_0,t_0) \geq \langle z_0,x\rangle - (t_0/2)\|x\|^2 - f(x)$, therefore $h(z_0,t_0) \geq h_f(z_0,t_0)$. But, $h_f(z_0,t_0) \geq \langle z_0,x_0\rangle - (t_0/2)\|x_0\|^2 - f(x_0) = h(z_0,t_0)$. Hence, $h(z_0,t_0) = h_f(z_0,t_0)$ and this completes the proof of the claim.

By the previous claim and the fact that h is differentiable on a dense subset of the interior of its domain (see [9]) we conclude that $h(z,t) = h_f(z,t)$ for all $t > \bar{t}$ (a convex function is continuous on the interior of its domain). Because a convex function is completely determined by the values it assumes on the interior of its domain (see [9]) we conclude that $h(z,t) = h_f(z,t)$ for all $t \geq \bar{t}$. ∎

We end this section by giving our characterization of the proximal subgradient set-valued mapping of a function bounded below by a quadratic. For $t \geq \bar{t}$ (see (2.1)), let

$$M_t(x) = \{z - tx : x \in \operatorname{argmax} h_f(z,t)\}. \tag{2.6}$$

It is clear that $M_t + tI$ is a monotone set-valued mapping because $(M_t + tI)(x) = \partial h_{f,t}^{-1}(x)$, where $h_{f,t}\colon \mathbb{R}^n \longrightarrow \mathbb{R}$ is defined by $h_{f,t}(z) = h_f(z,t)$. In [8], it is shown that $h_{f,t}^*(x) - (t/2)\|x\|^2$ (where $h_{f,t}^*$ is the convex conjugate of $h_{f,t}$; see [9]) is the supremum of all quadratics functions majorized by f with quadratic part $-(t/2)$.

In addition $(M_t + tI)$ is a *selection* of $(\partial_p f + tI)$ in the sense that for all x

$$(M_t + tI)(x) \subset (\partial_p f + tI)(x).$$

(if $(M_t + tI)(x) = z$ then $x \in \operatorname{argmax} h_f(z,t)$, this implies that $z - tx \in \partial_p f(x)$ (Theorem 2.1(a)) *i.e.* $z \in \partial_p f(x) + tx = (\partial_p f + tI)(x)$.) There are examples where $(M_t + tI)$ is not a maximal monotone selection of $(\partial_p f + tI)$; one such example is

$$f(x) = \begin{cases} -1 & x \leq -1,\ x \geq 1 \\ 0 & -1 < x < 1 \end{cases}.$$

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The proximal set-valued mapping of this function is given by

$$\partial_p f(x) = \begin{cases} 0 & \text{if } x \neq -1 \text{ and } x \neq 1, \\ [0,\infty) & \text{if } x = -1, \\ (-\infty, 0] & \text{if } x = 1. \end{cases}$$

So that

$$(\partial_p f + tI)(x) = \begin{cases} tx & \text{if } x \neq -1 \text{ and } x \neq 1, \\ [-t,\infty) & \text{if } x = -1, \\ (-\infty, t] & \text{if } x = 1. \end{cases}$$

For $t \leq 2$ one easily calculates that

$$(M_t + tI)(x) = \begin{cases} tx & x \leq -1, x \geq 1 \\ [-t, 0] & x = -1 \\ [0, t] & x = 1. \end{cases}$$

However, adjoining $(0,0)$ to $(M_t + tI)$ still yields a monotone selection of $(\partial_p f + tI)$. By definition of $M_t(x)$, the set-valued mapping M defined on $\mathbb{R}^n \times [\bar{t}, \infty)$ by

$$M(z,t) = \operatorname{con}\big\{\big(x, -(1/2)\|x\|^2\big) : x \in (M_t + tI)^{-1}(z)\big\}$$

for $t > \bar{t}$ and for $t = \bar{t}$

$$M(z,\bar{t}) = \operatorname{con}\big\{\big(x, -(1/2)\|x\|^2\big) : x \in (M_{\bar{t}} + \bar{t}I)^{-1}(z)\big\} + \{(0,-\lambda) \mid \lambda \in \mathbb{R}\}$$

is maximal cyclically monotone (see [9]), because it is equal to $\partial h_f(z,t)$. Another property of the sets $M_t(x)$ is that they increase and the limit set is $\partial_p f(x)$. The sets $M_t(x)$ are the key to the characterization of the proximal subgradient mapping.

THEOREM 2.3. *Let $\Gamma\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a set-valued mapping. For all x, $\Gamma(x) = \partial_p f(x)$, where f is a lower semicontinuous extended-real-valued function on $\mathbb{R}^n$ bounded below by a quadratic, if and only if there exist $\bar{t} > 0$ and for $t \geq \bar{t}$, $M_t\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ with $M_t(x) \subset \Gamma(x)$ for all x, such that*

(a) $M_t(x) \uparrow \Gamma(x)$ (i.e., if $t_1 < t_2$, then $M_{t_1}(x) \subset M_{t_2}(x)$ and $\bigcup_{t\geq\bar{t}} M_t(x) = \Gamma(x)$).

(b) If M is the set-valued mapping defined by

$$M(z,t) = \begin{cases} \operatorname{con}\{(x, -(1/2)\|x\|^2) : x \in (M_t + tI)^{-1}(z)\} & t > \bar{t} \\ \operatorname{con}\{(x, -(1/2)\|x\|^2) : x \in (M_{\bar{t}} + \bar{t}I)^{-1}(z)\} + \{(0,-\lambda) \mid \lambda \in \mathbb{R}\} & t = \bar{t} \\ \emptyset & t < \bar{t} \end{cases}$$

then for all $t \geq \bar{t}$ and $z \in \mathbb{R}^n$, $M(z,t) \neq \emptyset$ and M is maximal cyclically monotone.

PROOF. $\Longrightarrow$ See the discussion preceding the Theorem.

$\Longleftarrow$ There exists $h\colon \mathbb{R}^n \times [\bar{t}, \infty) \longrightarrow \mathbb{R} \cup \{\infty\}$, lower semicontinuous proper convex with $\partial h(z,t) = M(z,t)$ (see [9]). Assume h is differentiable at (z,t). This implies that $\partial h(z,t)$ is a singleton, therefore $M(z,t) = \big(x, -(1/2)\|x\|^2\big)$ for some x in $\mathbb{R}^n$. By Theorem 2.2, if

$$f(x) = \sup_{\substack{(z,t) \\ t\geq\bar{t}}}\{\langle z, x\rangle - (t/2)\|x\|^2 - h(z,t)\},$$

then $h_f(z,t) = h(z,t)$. In addition,

(a) $\partial_p f(x) \subset \Gamma(x)$. To see this, let $u \in \partial_p f(x)$. By Theorem 2.1 (b), for t big enough, $\big(x, -(1/2)\|x\|^2\big) = \nabla h_f(u + tx, t)$. Therefore,

$$M(u + tx, x) = \big(x, -(1/2)\|x\|^2\big),$$

which implies that $u \in M_t(x) \subset \Gamma(x)$.

(b) $\Gamma(x) \subset \partial_p f(x)$. To see this, if $u \in \Gamma(x)$, then eventually $u \in M_t(x)$. Hence, $\big(x, -(1/2)\|x\|^2\big) \in M(u + tx, t) = \partial h_f(u + tx, t)$. By Theorem 2.1(c), $x \in \operatorname{argmax} h_f(u + tx, x)$. By Theorem 2.1(a), we know that $u \in \partial_p f(x)$, since $u = (u + tx) - tx$. ∎

We now wish to characterize the proximal normal cone mapping. The following corollary is an immediate consequence of Theorem 2.3 and the following obvious observation

$$y \in \mathrm{PN}_C(x) \Longleftrightarrow y \in \partial_p \delta_C(x),$$

where δ_C is the indicator function

$$\delta(x) = \begin{cases} 0 & \text{if } x \in C, \\ \infty & \text{if } x \notin C. \end{cases}$$

COROLLARY 2.4. *Let $\Gamma\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a set-valued mapping. For all x, $\Gamma(x) = \mathrm{PN}_C(x)$, where C is a closed nonempty subset of $\mathbb{R}^n$, if and only if there exist, for $t \geq 0$, $M_t\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ with $M_t(x) \subset \Gamma(x)$ for all x, such that*

(a) $M_t(x) \uparrow \Gamma(x)$ (i.e., if $t_1 < t_2$, then $M_{t_1}(x) \subset M_{t_2}(x)$ and $\bigcup_{t \geq 0} M_t(x) = \Gamma(x)$).

(b) If M is the set-valued mapping defined by

$$M(z,t) = \begin{cases} \operatorname{con}\{(x, -(1/2)\|x\|^2) : x \in (M_t + tI)^{-1}(z)\} & t > 0 \\ \operatorname{con}\{(x, -(1/2)\|x\|^2) : x \in (M_0)^{-1}(z)\} + \{(0, -\lambda) \mid \lambda \in \mathbb{R}\} & t = 0 \\ \emptyset & t < 0 \end{cases}$$

then for all $t \geq 0$ and $z \in \mathbb{R}^n$, $M(z,t) \neq \emptyset$ and M is maximal cyclically monotone.

We conclude this paper with a characterization of the generalized subgradient mapping of a locally Lipschitzian function.

COROLLARY 2.5. *Let $f\colon \mathbb{R}^n \longrightarrow \mathbb{R}$ be locally Lipschitzian and bounded below by a quadratic. Under these conditions there exist $\bar{t}$ and for $t \geq \bar{t}$, set-valued mappings $M_t\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^n$, where M_t is defined in (2.6), such that*

(1) $M_t(x) \subset \partial_p f(x)$ for all x.

(2) Parts (a) and (b) of Theorem 2.3 are satisfied (with $\Gamma(x) = \partial f(x)$).

(3) $\widetilde{M}_t(x) \uparrow \partial f(x)$, where

$$\widetilde{M}_t(x) = \operatorname{con}\{y \mid \exists x_i \longrightarrow x \text{ and } y_i \in M_t(x_i) \text{ with } y_i \longrightarrow y\}.$$

PROOF. Recall that the set of (Clarke) generalized subgradients is the convex hull of limiting proximal subgradients.

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