Dear Editor,

The joint distribution of the running maximum and its location of D-valued Markov processes

1. Introduction and main results

Let $Y = \{Y(u) : 0 \leqq u \leqq 1\}$ be a real-valued elementary Markovian process defined on a probability space $(\Omega, \mathscr{A}, \boldsymbol{P})$ with right-continuous trajectories also having left limits. We define for all $t \in (0, 1]$ the *running maximum* of the process Y,

$$M_t = \sup\{Y(u) : 0 \leqq u \leqq t\},$$

and its *location*

$$T_t = \min\{0 \leqq u \leqq t : \ Y(u) = M_t \text{ or } Y(u-) = M_t\}.$$

Since the set $\{0 \leqq u \leqq t : \ Y(u) = M_t \text{ or } Y(u-) = M_t\}$ is non-empty and closed, the random variable T_t is well defined. We give an explicit formula for the joint distribution of (T_t, M_t). Although the derivation of our result requires only a short argument, the obtained formula is very useful in connection with recent results of Durbin (1985), (1992). To be precise, let

$$M_{a,b} = \sup\{Y(u) : a \leqq u \leqq b\}, \qquad 0 \leqq a \leqq b \leqq t.$$

Then for all $0 \leqq x \leqq t$ and $y \in \mathbb{R}$

$$\begin{aligned} H_t(x, y) &:= \boldsymbol{P}(T_t \leqq x, \ M_t \leqq y) \\ &= \boldsymbol{P}(M_{x,t} \leqq M_{0,x} \leqq y) \\ &= \int \boldsymbol{P}(M_{x,t} \leqq M_{0,x} \leqq y \mid Y(x) = \xi) \mu_x(d\xi) \end{aligned}$$

with μ_x denoting the distribution of $Y(x)$. By the Markov property of Y it follows from Theorem 1, p. 36, of Gihman and Skorohod (1975) that $M_{x,t}$ and $M_{0,x}$ are independent with respect to the conditional probability $\boldsymbol{P}(\cdot \mid Y(x)=\xi)$. Consequently we obtain for all $\xi \in \mathbb{R}$

$$\begin{aligned}\boldsymbol{P}(M_{x,t} &\leqq M_{0,x} \leqq y \mid Y(x)=\xi) \\ &= \int_{(-\infty,y]} \boldsymbol{P}(M_{x,t} \leqq z \mid Y(x)=\xi) F(dz, x, \xi)\end{aligned}$$

with

$$F(z, x, \xi) = \boldsymbol{P}(M_{0,x} \leqq z \mid Y(x)=\xi), \qquad z \in \mathbb{R}.$$

If we put

$$G(z, x, t, \xi) = \boldsymbol{P}(M_{x,t} \leqq z \mid Y(x)=\xi), \qquad z \in \mathbb{R},$$

we therefore arrive at

$$H_t(x, y) = \int_{(-\infty,y]} \int_{(-\infty,y]} G(z, x, t, \xi) F(dz, x, \xi) \mu_x(d\xi), \tag{1.1}$$

upon noticing that $G(z, x, t, \xi) = F(z, x, \xi) = 0$ if $\xi > z$.

2. Applications to the Brownian bridge with general drift

In this section let

$$Y(u) = B_0(u) + \Delta(u), \qquad 0 \leqq u \leqq 1,$$

where B_0 is a Brownian bridge and the drift function

(2.1) $\Delta : [0, 1] \to \mathbb{R}$ is twice continuously differentiable and either convex on the whole interval $[0, t]$, or concave.

An application of Feller's (1971) criterion (8.13), p. 96, ensures that B_0 is a Markov process, whence Y is Markovian, too. By (1.1) we have to determine the functions G and F. Using (18), p. 38, in Shorack and Wellner (1986) we obtain that

$$G(z, x, t, \xi) = \boldsymbol{P}\left(B_0(s) \leqq a(s, z, x, \xi) \quad \forall 0 \leqq s \leqq \frac{t-x}{1-x}\right), \tag{2.2}$$

where

$$a(s, z, x, \xi) = (1-x)^{-1/2}[z - (1-s)(\xi - \Delta(x)) - \Delta(x + s(1-x))].$$

Similarly

$$F(z, x, \xi) = \boldsymbol{P}(B_0(s) \leqq A(s, z, x, \xi) \quad \forall 0 \leqq s \leqq 1), \tag{2.3}$$

where

$$A(s, z, x, \xi) = x^{-1/2}[z - s(\xi - \Delta(x)) - \Delta(sx)].$$

Our assumption (2.1) and Equations (2.2)–(2.3) enable us to apply Durbin's (1992) result. In sections four and five there explicit formulas are given for the non-crossing probabilities in (2.2)–(2.3). These formulas may be specified, but we omit them. However, they are of such a kind that one can use numerical methods to find $H_t(x, y)$. In the case of a *linear drift function*

$$\Delta(u) = au, \qquad a \in \mathbb{R},$$

we obtain comparatively simple analytical expressions. Namely, for $0 < x < t$ and $y \geqq 0$:

$$\begin{aligned} H_t(x, y) = {} & A \int_{-\infty}^{0} \int_{0}^{y} g(z, x, t, \xi)(2z - \xi) \exp\left\{ -\frac{2}{x} z(z - \xi) + B\xi^2 \right\} dz d\xi \\ & + A \int_{0}^{y} \int_{\xi}^{y} g(z, x, t, \xi)(2z - \xi) \exp\left\{ -\frac{2}{x} z(z - \xi) + B\xi^2 \right\} dz d\xi, \end{aligned} \tag{2.4}$$

where $A = 2(2\pi x^3(1-x))^{-1/2}$, $B = -(2x(1-x))^{-1}$ and

$$\begin{aligned} g(z, x, t, \xi) = {} & \Phi(C_0[(d-1)\xi + z - ad]) \\ & - \exp\{C_1(z-a)(z-\xi)\}\Phi(C_0[2d-1]z + (1-d)\xi) \end{aligned}$$

with Φ denoting the standard normal distribution and $C_0 = \{(1-t)(t-x)(1-x)^{-1}\}^{-1/2}$, $C_1 = -2(1-x)^{-1}$ and $d = (t-x)(1-x)^{-1}$.

For $x \in \{0, t\}$ we obtain for all $y \geqq 0$ that $H_t(0, y) = 0$ and

$$H_t(t, y) = \Phi\left(\frac{y - at}{\sqrt{t(1-t)}}\right) - \exp\{-2y(y-a)\}\Phi\left(\frac{2yt - y - at}{\sqrt{t(1-t)}}\right).$$

In the special case $t = 1$ and $a = 0$ we have that for all $0 \leqq x \leqq 1$ and $y \geqq 0$ (with the convention $\Phi(\infty) = 1$)

$$\begin{aligned} H_1(x, y) = {} & \Phi\left(\frac{y}{\sqrt{x(1-x)}}\right) - \exp\{-2y^2\}\Phi\left(\frac{y(2x-1)}{\sqrt{x(1-x)}}\right) \\ & - (1-x)\left(2\Phi\left(\frac{y}{\sqrt{x(1-x)}}\right) - 1\right). \end{aligned} \tag{2.5}$$

If $a \neq 0$ this simplification of (2.4) is unfortunately no longer possible. Notice that our formula (2.5) yields the marginal distributions

$$\boldsymbol{P}(T_1 \leqq x) = x, \qquad 0 \leqq x \leqq 1 \tag{2.6}$$

and

(2.7) $$P(M_1 \leqq y) = 1 - \exp\{-2y^2\}, \qquad y \geqq 0.$$

The distribution (2.7) is well known, but (2.6) may be less so.

2.1. *A numerical example*

We consider

$$Y(u) = B_0(u) + au, \qquad 0 \leqq u \leqq t,$$

with $a = 0.1$ and $t = 0.8$. Table 1 shows values of $H_t(x, y)$ for selected arguments $0 \leqq x \leqq 0.8$ and $y \geqq 0$. The required numerical integrals were computed using standard subroutines of MATHEMATICA 2.2 for MS – DOS (Enhanced Version).

TABLE 1

The probabilities $H_t(x, y)$ for $Y(u) = B_0(u) + au$ with $a = 0.1$ and $t = 0.8$

y \ x	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
0.0	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
0.1	0.000	0.013	0.023	0.028	0.032	0.035	0.037	0.038	0.040
0.2	0.000	0.027	0.048	0.061	0.071	0.080	0.086	0.091	0.099
0.3	0.000	0.040	0.075	0.097	0.115	0.132	0.150	0.172	0.178
0.5	0.000	0.059	0.126	0.172	0.211	0.249	0.289	0.339	0.378
0.7	0.000	0.068	0.161	0.233	0.297	0.360	0.427	0.506	0.593
1.0	0.000	0.071	0.183	0.284	0.379	0.475	0.636	0.691	0.840
1.5	0.000	0.071	0.188	0.302	0.417	0.535	0.659	0.796	0.985
3.0	0.000	0.071	0.188	0.303	0.420	0.540	0.667	0.806	0.999

References

DURBIN, J. (1985) The first-passage density of a continuous Gaussian process to a general boundary. *J. Appl. Prob.* **22**, 99–122.

DURBIN, J. (1992) The first-passage density of the Brownian motion process to a curved boundary. *J. Appl. Prob.* **29**, 291–304.

FELLER, W. (1971) *An Introduction to Probability Theory and Its Applications,* Volume II, 2nd edn. Wiley, New York.

GIHMAN, I. I. AND SHOROHOD (1975) *The Theory of Stochastic Processes* II. Springer, New York.

SHORACK, G. R. AND WELLNER, J. A. (1986) *Empirical Processes with Applications to Statistics.* Wiley, New York.

Yours sincerely

DIETMAR FERGER

Mathematisches Institut
Justus-Liebig-Universität
Arndtstrasse 2
D-35392 Giessen
Germany