SUBAFFINE SCHEMES*

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Let an open, quasi-compact subscheme of an affine scheme be called subaffine. This note will centre on an elementary characterization of such schemes in terms of their topology and global sections. Thence one can obtain simplifications and generalizations of some well-known theorems, such as Serre's Criterion [2, Thm. 1].

In [1, II. 5.2.1], that criterion is stated for quasi-compact pre-schemes under the additional hypothesis that they be either separated or noetherian. This assumption (which we shall recognize to be superfluous) seems to enter into the theory via [1, I. 9.3], where it ensures that the pre-scheme in question is well-built, i.e., that it is the finite union of open, affine sets U_i whose pairwise intersections $U_i \cap U_j$ again are finite unions of open, affine sets. The chief virtue of this property is expressed in Lemma 1 below.

In the sequel, X will denote a quasi-compact pre-scheme, $A = \Gamma(X, O_X)$ its ring of global sections. As usual, for f in A, we write X_f for its domain of invertibility.

By the arguments of I. 9.3 of [1], we have

LEMMA 1. *If X is well-built, the canonical map $A_f \to \Gamma(X_f, O_X)$ is an isomorphism for all* $f \in A$.

A global section f will be called affine, if the set X_f is affine. Putting $Y = \operatorname{Spec} A$, the lemma shows that the canonical map $\phi : X \to Y$ induces an isomorphism $X_f \xrightarrow{\sim} Y_f$ for each affine f, provided that X is well-built. This, however, will be the case, if X can be covered by affine sets of the form X_f, because $X_f \cap U$ is affine whenever U is (f affine or not). Since moreover each X_f is the precise pre-image of Y_f, we can conclude:

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LEMMA 2. If X can be covered by affine sets of the form X_f, the canonical map $\phi : X \to Y = \mathrm{Spec}(A)$ is an open immersion. Its image is the union of all Y_f with f affine.

From this we deduce that X is subaffine, if and only if it has "many" global sections.

PROPOSITION 1. For a quasi-compact pre-scheme X the following statements are equivalent:

(i) Sets of the form X_f form a base of the topology of X.

(ii) X can be covered by affine sets of the form X_f.

(iii) X is subaffine.

Proof. (i) $\Rightarrow$ (ii) Every open, affine $U \subseteq X$ must contain an X_f. $X_f = X_f \cap U$ is affine.

(ii) $\Rightarrow$ (iii) The map ϕ of Lemma 2 identifies X with an open subscheme of $\mathrm{Spec}(A)$.

(iii) $\Rightarrow$ (i) Let X be open in $Y = \mathrm{Spec}(B)$. The topology of X is based on subsets of the form Y_b, $b \in B$. If $\rho : B \to \Gamma(X, O_X)$ is the restriction map, $Y_b = X_{\rho(b)}$.

Lemma 2 also allows an analogous characterization of affine schemes.

PROPOSITION 2. For a quasi-compact pre-scheme X, the following statements are equivalent:

(i) X is subaffine; for any covering composed of sets X_f, the corresponding global sections f generate no proper ideal in A.

(ii) The affine global sections generate no proper ideal in A.

(iii) X is affine.

Proof. (i) $\Rightarrow$ (ii) By hypothesis, we find an affine covering by sets of type X_f.

(ii) $\Rightarrow$ (iii) Ditto. But now the complement of $\phi(X)$ is empty: it consists of the zeros of all affine global sections.

(iii) $\Rightarrow$ (i) Trivial.

Using items (i) and (iii) of both propositions, and noting that (i) in each case depends on A only "modulo nilpotence", we get an extension of a well-known result (cf. [1 , I.5.1]).

COROLLARY. _Any pre-scheme_ X _is affine or subaffine_ (resp.), _if and only if_ X_{red} _is._

Proof. We note that if X_{red} is affine or subaffine, it (and hence X) is quasi-compact, so that Proposition 2 is applicable.

As for Serre's Criterion, we refer back to [1 , II. 5.2.1]. There it is shown that for any quasi-compact pre-scheme **X**, the cohomological triviality of quasi-coherent O_X-Modules implies condition (ii) of Proposition 2.

REFERENCES

1. A. Grothendieck, Éléments de géométrie algébrique. (Publ. Math. I.H.E.S., Paris, 1960, 1961).

2. J.P. Serre, Sur la cohomologie des variétés algébriques. Jour. de Math. 36 (1957) 1-16.

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