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# Sublinearity and Other Spectral Conditions on a Semigroup

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*Abstract.* Subadditivity, sublinearity, submultiplicativity, and other conditions are considered for spectra of pairs of operators on a Hilbert space. Sublinearity, for example, is a weakening of the well-known property $L$ and means $\sigma(A + \lambda B) \subseteq \sigma(A) + \lambda\sigma(B)$ for all scalars $\lambda$. The effect of these conditions is examined on commutativity, reducibility, and triangularizability of multiplicative semigroups of operators. A sample result is that sublinearity of spectra implies simultaneous triangularizability for a semigroup of compact operators.

## 0 Introduction and Preliminaries

We say that spectrum is *sublinear* on a pair of operators $A$ and $B$ on a complex Hilbert space if

$$\sigma(A + \lambda B) \subseteq \sigma(A) + \lambda\sigma(B)$$

for every complex number $\lambda$. By $\sigma(A) + \lambda\sigma(B)$ is meant the set of all $\alpha + \lambda\beta$ with $\alpha$ and $\beta$ in $\sigma(A)$ and $\sigma(B)$ respectively. We say that spectrum is sublinear on a (multiplicative) semigroup $\mathcal{S}$ of operators if it is sublinear on every pair of its members. It is not part of the definition that $\sigma(\Sigma\lambda_i A_i) \subseteq \Sigma\lambda_i\sigma(A_i)$ for more than two summands. However, this complete sublinearity property will, in all the cases considered in this paper, follow automatically from the weaker hypothesis made on two summands.

We shall prove, among other things, that for certain semigroups, sublinearity of spectrum implies (simultaneous) triangularizability. This holds, *e.g.* for semigroups of compact operators. A special case is of course that of operators on a finite-dimensional space.

The following condition, apparently much stronger than sublinearity, has a long history for matrices (over a general field); it was first proposed and studied by Kac, Motzkin, Taussky, Wales, Wielandt (*cf.* [10], [11], [24]), then by Wales, Zassenhaus [22], [25], and more recently by Guralnick [2]. Two $n \times n$ matrices $A$ and $B$ are said to have *property L* ("L" for "linear") if the eigenvalues of $A$ and $B$ (in the algebraic closure of the underlying field) could be expressed as ordered sets $\alpha_1, \dots, \alpha_n$ and $\beta_1, \dots, \beta_n$ respectively, such that, for every scalar $\lambda$, the eigenvalues of $A + \lambda B$ are precisely $\lambda_i + \lambda\beta_i$, $i = 1, \dots, n$. The pairing $\langle\alpha_i, \beta_i\rangle$ is assumed to be independent of $\lambda$. A set of matrices is said to be an $L$-set if every pair of its members has property $L$; the terms $L$-group and $L$-semigroup have the obvious definitions.

Motzkin and Taussky [11] proved that a finite $L$-group of matrices over a field of characteristic zero is abelian. Wales and Zassenhaus [22] showed that an arbitrary $L$-group of matrices over any field is triangularizable so long as the field contains all the eigenvalues

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of members of the group. Zassenhaus [25] then extended this result to matrix semigroups with some restrictions in the case of characteristic two. Finally, Guralnick [2] proved it for all fields with more than two elements.

In Section 1 we consider sublinearity and the weaker condition of subadditivity. In Section 2 multiplicative analogues of these conditions are treated and Section 3 is devoted to "hybrid" polynomial conditions. Our aim is to determine whether these conditions, all necessary for triangularizability of a semigroup of operators, are also sufficient.

We confine ourselves to the field $\mathbf{C}$ of complex numbers except in the very brief final section of the paper. A semigroup $\mathcal{S}$ of operators is called *reducible* if there is a nontrivial subspace that is invariant under (every member of) $\mathcal{S}$. We say $\mathcal{S}$ is *triangularizable* if the lattice of its invariant subspaces contains a maximal subspace chain $\mathcal{C}$. (If $\mathcal{M}_1$ and $\mathcal{M}_2$ are two members of $\mathcal{C}$ with $\mathcal{M}_1 \subseteq \mathcal{M}_2$ and no other member between $\mathcal{M}_1$ and $\mathcal{M}_2$, then $\dim(\mathcal{M}_2 \ominus \mathcal{M}_1) \leq 1$.)

We should mention here that analogues of our results can be stated and proved for operators on Banach spaces. The slight modifications necessary for extensions will be clear to the interested reader. Thus in the interest of cleaner exposition we treat the Hilbert-space case only.

The following result from [13] is needed. By a "property defined on a semigroup of matrices" we mean a collection of equalities, inequalities, or set inclusions satisfied by the members of the semigroup.

***Lemma 0.1*** **(The Finiteness Lemma)** *Let $\mathcal{P}$ be a property, defined for semigroups of complex $n \times n$ matrices, such that whenever a semigroup $\mathcal{S}$ has the property, then so do the following semigroups:*

*(i)* $\mathbf{C}\mathcal{S} := \{cS : c \in \mathbf{C}, \mathbf{S} \in \mathcal{S}\}$,
*(ii)* $\overline{\mathcal{S}}$*, the norm-closure of* $\mathcal{S}$,
*(iii)* $\Phi(\mathcal{S})$*, for every ring automorphism $\Phi$ of $\mathcal{M}_n(\mathbf{C})$ which is induced by a field automorphism $\varphi$ of $\mathbf{C}$, i.e., $\Phi(M)$ is defined by applying $\varphi$ to $M$ entry-wise.*

*Assume $\mathcal{S}$ is a maximal semigroup with property $\mathcal{P}$. If $E$ is an idempotent in $\mathcal{S}$ whose rank is* $\min\{\operatorname{rank} S : 0 \neq S \in \mathcal{S}\}$*, then $E\mathcal{S}E = \mathbf{C}\mathcal{G}$ where $\mathcal{G}$ is a finite group with identity $E$.* ■

This lemma will be used mainly in the situations where $\mathcal{S}$ is assumed irreducible, in which case, an idempotent of the desired type exists automatically (*cf.* Lemma 0.4 below).

We shall also need the following simple lemmas (see, *e.g.*, [15]).

***Lemma 0.2*** *Each of the following conditions is sufficient for reducibility of a semigroup $\mathcal{S}$ of operators on a Hilbert space.*

1. *There exists a projection $P$ of rank at least two on the space, not necessarily in $\mathcal{S}$, such that the restriction of the collection $P\mathcal{S}P$ to the range of $P$ has a nontrivial invariant subspace.*
2. *A nonzero (semigroup) ideal of $\mathcal{S}$ is reducible.* ■

The next lemma is an easy consequence of Lomonosov's Theorem [8]. (See *e.g.*, [15].)

***Lemma 0.3*** *Let $\mathcal{S}$ be a semigroup of compact operators. Let $f$ be a nonzero linear functional on compact operators, continuous in the operator norm. If $f$ is zero on $\mathcal{S}$, then $\mathcal{S}$ is reducible.* ■

Throughout the paper we shall freely use the continuity properties of spectrum for compact operators. The next two lemmas are also easy to verify. (See, *e.g.*, [16].)

***Lemma 0.4*** *Let $\mathcal{S}$ be an irreducible semigroup of compact operators containing a nonquasinilpotent member. Then $\overline{\mathbf{C}\mathcal{S}}$ has an idempotent $E$ with*

$$\operatorname{rank} E = \min\{\operatorname{rank} S : S \in \mathcal{S}\}.$$ ■

***Lemma 0.5*** *Let $\mathcal{P}$ be a property of semigroups of operators such that*

*(i) every semigroup with property $\mathcal{P}$ is reducible, and*
*(ii) if $\mathcal{S}$ has property $\mathcal{P}$ and if $\mathcal{M}_1$ and $\mathcal{M}_2$ are invariant subspaces of $\mathcal{S}$ with $\mathcal{M}_1 \subset \mathcal{M}_2$, then the compression of $\mathcal{S}$ to the subspace $\mathcal{M}_2 \ominus \mathcal{M}_1$ has property $\mathcal{P}$.*

Then every semigroup with property $\mathcal{P}$ is triangularizable. ■

## 1 Sublinearity and Subadditivity

Our main result in this section is Theorem 1.8. We need the following lemma, which shows that for compact operators the sublinearity property of spectra is inherited by restrictions.

***Lemma 1.1*** *Let $A$ and $B$ be compact operators with a common invariant subspace $\mathcal{M}$. Let $A_1$ and $B_1$ be the respective restrictions of $A$ and $B$ to $\mathcal{M}$. If spectrum is sublinear on $A$ and $B$, then it is sublinear on $A_1$ and $B_1$.*

**Proof** For each ordered pair $\langle \alpha, \beta \rangle$ in $\sigma(A) \times \sigma(B) \setminus \sigma(A_1) \times \sigma(B_1)$ define

$$\Gamma_{\alpha,\beta} = \{\lambda : \alpha + \lambda\beta \in \sigma(A_1 + \lambda B_1)\}.$$

The set $\Gamma_{\alpha,\beta}$ is clearly closed. It is also nowhere dense. For otherwise $(A_1 - \alpha) + \lambda(B_1 - \beta)$ would be noninvertible for uncountably many values of $\lambda$. Now either $\alpha$ is not in $\sigma(A_1)$ or $\beta$ is not in $\sigma(B_1)$. Assume the latter. Then the spectrum of $T = (B_1 - \beta)^{-1}(A_1 - \alpha)$ is uncountable, which is a contradiction, because $T$ is a translate of a compact operator. The argument for the case $\alpha \in \sigma(A) \setminus \sigma(A_1)$ is analogous.

Since $\sigma(A) \times \sigma(B) \setminus \sigma(A_1) \times \sigma(B_1)$ is countable, we conclude by the Baire Category Theorem that $\Omega = \mathbf{C} \setminus \cup\Gamma_{\alpha,\beta}$ is a dense $G_\delta$ set and that

$$\sigma(A_1 + \lambda\beta_1) \subseteq \sigma(A_1) + \lambda\sigma(B_1)$$

for all $\lambda \in \Omega$. We will be done if we verify that $\Omega$ is closed. But this follows from the continuity of spectrum on compact operators: If $\lambda = \lim \lambda_n$, and $\lambda_n \in \Omega$, then every point $z$ in $\sigma(A_1 + \lambda B_1)$ is the limit of a sequence $\{z_n\}$, where $z_n \in \sigma(A_1 + \lambda_n B_1)$, *i.e.*,

$$z = \lim(\alpha_n + \lambda_n\beta_n)$$

for some $\{\alpha_n\} \subseteq \sigma(A_1)$ and some $\{\beta_n\} \subseteq \sigma(\beta_1)$. Pick a subsequence with convergent $\{\alpha_n\}$ and $\{\beta_n\}$ to obtain $z = \alpha + \lambda\beta$ with $\alpha$ in $\sigma(A_1)$ and $\beta$ in $\sigma(B_1)$. ■

Our next result shows that for rank-one operators the weaker condition of subadditivity for spectra of every pair in a semigroup is sufficient for triangularizability.

***Theorem 1.2*** *Let $\mathcal{S}$ be a semigroup of operators of* rank $\leq 1$. *If $\sigma(S+T) \subseteq \sigma(S) + \sigma(T)$ for every $S$ and $T$ in $\mathcal{S}$, then $\mathcal{S}$ is triangularizable.*

**Proof** If $\mathcal{S}$ consists of nilpotent operators, then this follows from generalizations of Levitzki's Theorem [7] to infinite dimensions (see, *e.g.* [12] or [21]). Thus we can assume, with no loss of generality, that $\mathcal{S}$ has a member of the form $T = tE$ where $E$ is an idempotent of rank one, $E = E^*$ and $t \neq 0$.

We first show that $\mathcal{S}$ is reducible (if the dimension of the underlying space is at least 2). Suppose, if possible, that $\mathcal{S}$ is irreducible. Then $\mathcal{S}T$ has a nonzero member $A$ whose range is different from the range of $T$. (Otherwise, the range of $T$ would be invariant under $\mathcal{S}$.) Form an orthonormal basis $\{e_1, e_2\}$ for the span of the ranges of $T$ and $A$ such that $Te_1 = te_1$. By irreducibility again, $T\mathcal{S}$ has a member $B$ such that the inner product $(Be_2, e_1)$ is nonzero. (Otherwise $(Se_2.e_1) = 0$ for all $S$ in $\mathcal{S}$, which contradicts Lemma 0.3.) Now the span $\mathcal{M}$ of $\{e_1, e_2\}$ is invariant under $T$, $A$, and $B$. Denote their respective restrictions to $\mathcal{M}$ by $T_0$, $A_0$, and $B_0$. Let $\mathcal{S}_0$ be the semigroup generated by these restrictions. Note that $\sigma(S_0) = \sigma(S)$ for every $S$ in $\mathcal{S}$ and its restriction $S_0$ to $\mathcal{M}$, because $S$ has rank at most 1 and $\mathcal{M}$ has dimensions 2. Relative to the basis $\{e_1, e_2\}$ we have

$$T_0 = \begin{pmatrix} t & 0 \\ 0 & 0 \end{pmatrix}, \quad A_0 = \begin{pmatrix} a & 0 \\ r & 0 \end{pmatrix}, \quad B_0 = \begin{pmatrix} b & s \\ 0 & 0 \end{pmatrix},$$

where $rst \neq 0$ and where spectrum is subadditive on $\mathcal{S}_0$. This will give us the desired contradiction as follows:

***Case 1*** Assume $ab + rs \neq 0$. By hypothesis,

$$\sigma(A_0 + B_0) \subseteq \sigma(A_0) + \sigma(B_0) = \{0, a, b, ab\}.$$

The characteristic polynomial of $A_0+B_0$ is $x^2-(a+b)x-rs$, and since $rs \neq 0$ and $ab+rs \neq 0$, the set of its zeros does not intersect $\{0, a, b, ab\}$, a contradiction.

***Case 2*** If $ab + rs = 0$, we use the subadditivity hypothesis on $A_0B_0$ and $T_0$:

$$\sigma(A_0B_0 + T_0) \subseteq \sigma(A_0B_0) + \sigma(T_0) = \{0, t\},$$

because $A_0B_0$ is nilpotent. But the characteristic equation, $x^2 - tx + rst = 0$, of $A_0B_0 + T_0$ is not satisfied by either 0 or $t$; a contradiction again.

We have shown that every semigroup of operators of rank $\geq 1$ with subadditive spectrum is reducible. To prove the triangularizability of $\mathcal{S}$ by Lemma 0.5 we need show that if $\mathcal{M}_1$ and $\mathcal{M}_2$ are any two subspaces invariant under $\mathcal{S}$ with $\mathcal{M}_1 \subset \mathcal{M}_2$ and $\dim(\mathcal{M}_2 \ominus \mathcal{M}_1) \geq 2$, then the compression of $\mathcal{S}$ to $\mathcal{M}_2 \ominus \mathcal{M}_1$ has subadditive spectrum. (If this dimension is 1 or 0, there is nothing to prove.) Let $S_0$ and $T_0$ be the corresponding compressions of two arbitrary members $S$ and $T$ of $\mathcal{S}$. If either $S_0$ or $T_0$ is zero, the assertion is trivial. Otherwise, note that $\sigma(S_0) = \sigma(S)$ and $\sigma(T_0) = \sigma(T)$. Thus

$$\sigma(S_0 + T_0) \subseteq \sigma(S + T) \subseteq \sigma(S) + \sigma(T) = \sigma(S_0) + \sigma(T_0). \qquad \blacksquare$$

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***Example 1.3*** Even in finite dimensions, the hypothesis on the ranks cannot be omitted in the statement of the result above. Let $E_{ij}$ denote the matrix whose $(i, j)$ entry is 1 and whose other entries are zero. Let $J$ be the $2 \times 2$ matrix $\operatorname{diag}(1, -1)$. The set

$$\{E_{11} \oplus I, E_{22} \oplus I, E_{12} \oplus J, E_{21} \oplus J, O \oplus I, O \oplus J\}$$

of $4 \times 4$ matrices is easily seen to be a semigroup. To verify that it has subadditive spectrum, we need only check noncommutative pairs. But

$$\sigma[(E_{12} \oplus J) + (E_{21} \oplus J)] = \{1, -1, 2, -2\} \subseteq \sigma(E_{12} \oplus J) + \sigma(E_{21} \oplus J)$$

and the remaining such pairs are of the following type:

$$\sigma[(E_{11} \oplus I) + (E_{12} \oplus J)] = \{0, 1, 2\} \subseteq \sigma(E_{11} \oplus I) + \sigma(E_{12} \oplus J). \qquad \blacksquare$$

Before giving the finite-group version of our main result we need the following lemma on minimal nonabelian matrix groups. Its proof could be shortened by appealing to results in representation theory or quoting the more general result of [2] on minimal nontriangularizable groups, but the more complete proof given here is more illuminating for our purposes.

***Lemma 1.4*** *Let $\mathcal{G}$ be a minimal finite nonabelian group of matrices. Then, up to simultaneous similarity, $\mathcal{G}$ has the following description: There is a diagonal subgroup $\mathcal{H}$ of $\mathcal{G}$, a prime number $p$, an integer $m \geq 1$, and a matrix*

$$G = G_1 \oplus \cdots \oplus G_m \oplus D,$$

*where $D$ is diagonal and $G_j$ is a $p \times p$ matrix of the form*

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & \alpha_j \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & & & & \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$

*for $j = 1, \ldots, m$, such that $\mathcal{G}$ is generated by $G$ and $\mathcal{H}$.*

**Proof** By minimality, every proper subgroup of $\mathcal{G}$ is abelian. It follows from the theorem of O. J. Schmidt [19] that $\mathcal{G}$ is solvable. This implies that it has a composition series with each factor group cyclic of prime order. In particular, $\mathcal{G}$ has a normal subgroup $\mathcal{H}$ of index $p$ for some prime $p$. Choose $G$ in $\mathcal{G} \setminus \mathcal{H}$, so that $G^p \in \mathcal{H}$.

After a simultaneous similarity, we can assume that $\mathcal{G}$ is a unitary group [20]. Now $\mathcal{H}$ is abelian by minimality of $\mathcal{G}$. Decompose the underlying space as $\mathcal{M}_1 \oplus \cdots \oplus \mathcal{M}_k$, where each $\mathcal{M}_i$ is a maximal subspace such that $\mathcal{H} \mid \mathcal{M}$ consists of scalars. It is easily verified that the

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set $\{G\mathcal{M}_i\}$ is just a permutation of $\{\mathcal{M}_i\}$. Since $\mathcal{G}$ is assumed nonabelian, there is at least one $\mathcal{M}_i$ with $G\mathcal{M}_i \neq \mathcal{M}_i$. For these $\mathcal{M}_i$, the orbit of subspaces $\{\mathcal{M}_i, G\mathcal{M}_i, \dots, G^{p-1}\mathcal{M}_i\}$ consists of distinct members, because $p$ is prime. Pick one member $\mathcal{M}_i$ from each such orbit and let $\mathcal{M}$ be their direct sum. Then the decomposition

$$\mathcal{M} \oplus G\mathcal{M} \oplus \cdots \oplus G^{p-1}\mathcal{M} \oplus \mathcal{N}$$

has the property that $D = G \mid \mathcal{N}$ commutes with every member of $\mathcal{H} \mid \mathcal{N}$ (since $\mathcal{N}$ is just the direct sum of those $\mathcal{M}_i$ which are invariant under $G$). With respect to the decomposition above $G$ has the form $G_0 \oplus D$, where $G_0$ is the $p \times p$ block matrix

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & A_p \\ A_1 & 0 & \cdots & 0 & 0 \\ 0 & A_2 & \cdots & 0 & 0 \\ \vdots & & & & \\ 0 & 0 & & A_{p-1} & 0 \end{pmatrix}.$$

Choosing suitable bases for $\mathcal{M}, G\mathcal{M}, \dots, G^{p-1}\mathcal{M}$, and $\mathcal{N}$ we can assume that $A_1 = \cdots = A_{p-1} = I$ and that $A_p$, $D$, and all members of $\mathcal{H}$ are diagonal. Let $D = \operatorname{diag}(\alpha_1, \dots, \alpha_m)$. Then an obvious permutation of the basis transforms $A$ to $G_1 \oplus \cdots \oplus G_m$, where $G$ and $\mathcal{H}$ have the desired form. ∎

***Proposition 1.5*** *A finite group of matrices with sublinear spectrum is abelian.*

**Proof** Suppose the assertion is false. Let $\mathcal{G}$ be a nonabelian group of minimal order with sublinear spectrum. Apply Lemma 1.4 to obtain $\mathcal{H}$, $G_i$, and $D$ as described there. All we need here is that $G = G_1 \oplus A_0$, that $\mathcal{H}$ is diagonal, and that the restriction of $\mathcal{H}$ to the range of $G_1$ contains a matrix $H_1$ which does not commute with $G_1$. This means that $H_1 = \operatorname{diag}(\beta_1, \dots, \beta_p)$ with at least two distinct eigenvalues. Let $H$ be in $\mathcal{H}$ with $H = H_1 \oplus B_0$. We shall use the sublinearity hypothesis for the spectra of the two matrices

$$G = G_1 \oplus A_0 \quad \text{and} \quad GH = G_1H_1 \oplus A_0B_0$$

to obtain a contradiction.

Note that if $\gamma$ is a root of unity, then the finite group generated by $\mathcal{G}$ and $\gamma I$ still has the sublinearity property. Thus we can adjoin a $p$-th root $\alpha_0$ of $\alpha_1$, the $(1, p)$ entry of $G_1$ as given in Lemma 1.4, and replace $G$ by $G/\alpha_0 I$ with no loss of generality. An obvious simultaneous similarity effected by a diagonal matrix, which does not disturb $\mathcal{H}$, transforms the direct summand $G_1$ of $G$ to a permutation matrix. Then

$$G_1 = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & & & & \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix} \quad \text{and} \quad G_1H_1 = \begin{pmatrix} 0 & 0 & \cdots & 0 & \beta_p \\ \beta_1 & 0 & \cdots & 0 & 0 \\ 0 & \beta_2 & \cdots & 0 & 0 \\ \vdots & & & & \\ 0 & 0 & \cdots & \beta_{p-1} & 0 \end{pmatrix},$$

where at least two of the numbers $\beta_i$ are distinct. Also note that the minimal polynomial of $\lambda G_1 + G_1 H_1$ is

$$x^p - \prod_{i=1}^{p}(\lambda + \beta_i),$$

so that $\sigma(\lambda G_1 + G_1 H_1)$ is the set of all $p$-th roots of $\prod_{i=1}^{p}(\lambda + \beta_i)$. Let $\mu$ be such a root.

By hypothesis,

$$\mu \in \sigma(\lambda G_1 + H_1 G_1) \subseteq \sigma(\lambda G + HG) \subseteq \lambda\sigma(G) + \sigma(GH)$$

for all $\lambda$. If $r$ is the exponent of $\mathcal{G}$, then the set $\Omega$ of all $r$-th roots of unity contains the spectrum of every member of $\mathcal{G}$. Thus $\mu \in \lambda\Omega + \Omega$. This implies that

$$\prod_{i=1}^{p}(\lambda + \beta_i) = \mu^p \in \{(\lambda\varphi + \psi)^p : \varphi, \psi \in \Omega\}$$

for every $\lambda$. Since $\Omega$ is a finite set, there exist fixed members $\varphi_0$ and $\psi_0$ of $\Omega$ such that

$$\prod_{i=1}^{p}(\lambda + \beta_i) = (\lambda\varphi_0 + \psi_0)^p = \varphi_0^p(\lambda + \varphi_0/\psi_0)^p$$

for infinitely many scalars $\lambda$. This shows that ($\varphi_0^p = 1$ and) $\beta_i = \varphi_0/\psi_0$ for every $i$, which is a contradiction. ■

***Example 1.6*** The hypothesis of sublinearity in the result above cannot be weakened to subadditivity. Let

$$J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad K = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and let $A = \operatorname{diag}(\xi, i\xi, -\xi, -i\xi)$, where $\xi$ is a primitive 8-th root of unity: $\xi^2 = i$. Note that $\{\pm I, \pm J, \pm K, \pm JK\}$ is an irreducible group of $2 \times 2$ matrices, generated by $J$ and $K$.

Let $\mathcal{G}$ be the group of $10 \times 10$ matrices generated by the two matrices

$$J \oplus A \oplus I \quad \text{and} \quad K \oplus I \oplus A,$$

where $I$ now denotes the $4 \times 4$ identity. Observe that every word in this finite group, in which $r$ letters with $J$ and $s$ letters with $K$ participate, has the form

$$\pm J^r K^s \oplus A^r \oplus A^s,$$

where "$\pm$" depends on the permutation of the letters $J$ and $K$ in the first direct summands. Thus every member of $\mathcal{G}$ is of one of the following types:

(1) $\pm I \oplus A^{2m} \oplus A^{2n}$,
(2) $\pm J \oplus A^{2m+1} \oplus A^{2n}$.

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(3) $\pm K \oplus A^{2m} \oplus A^{2n+1}$,
(4) $\pm JK \oplus A^{2m+1} \oplus A^{2n+1}$.

Since subadditivity is automatic for commuting pairs, we need only verify it for those pairs $S$ and $T$ which have different types and neither $S$ nor $T$ has type (1). In all these situations, both $S$ and $T$ have a direct summand $A^k$ with $k$ odd. Every odd power of $A$ is similar to $A$ itself. Thus for every such pair $S$ and $T$,

$$\{0, \sqrt{2}, -\sqrt{2}\} \subseteq \sigma(A) + \sigma(A) \subseteq \sigma(S) + \sigma(T).$$

If $S_1$ and $T_1$ denote the first $2 \times 2$ direct summands of $S$ and $T$ respectively (*i.e.*, the noncommuting parts of the two matrices), then it suffices to show that

$$\sigma(S_1 + T_1) \subseteq \{0, \sqrt{2}, -\sqrt{2}\}.$$

Of the twelve possible forms of $S_1 + T_1$ eight are nilpotent: those of the form $\pm K \pm JK$ and $\pm J \pm JK$. The remaining four, those of the form $\pm J \pm K$, have characteristic polynomial $x^2 - 2$. ■

A semigroup of compact quasinilpotent operators is the most obvious example of semigroups with sublinear spectra. The question of reducibility for such a semigroup, open for many years, has recently been settled in the affirmative by Turovskii [21]. This globalizes Lomonosov's result [8], which proved reducibility if the semigroup was in fact an algebra. Since the property of being quasinilpotent and compact is inherited by quotients (*cf.* Lemma 0.5), Turovskii has given the ultimate extension of Levitzki's finite-dimensional theorem [7]. Without the compactness hypothesis, a semigroup of quasinilpotent operators (even nilpotent operator of index two) may fail to be reducible [3]. For easy reference we include the following formal statement. Further results are proved in [21], *e.g.*, existence of hyperinvariant subspaces.

***Theorem 1.7*** **(Turovskii)** *Every semigroup of compact quasinilpotent operators is triangularizable.* ■

We now state and prove the main result of this section.

***Theorem 1.8*** *A semigroup of compact operators is triangularizable if and only if it has sublinear spectrum.*

**Proof** That the condition is necessary for triangularizability is nothing new; it follows, *e.g.*, from Ringrose's result [18] on diagonal coefficients. To show the converse, assume that the semigroup $\mathcal{S}$ has sublinear spectrum.

We first show that $\mathcal{S}$ is reducible. In view of Theorem 1.7, we can assume the existence of a member $A$ with $\sigma(A) \neq \{0\}$. By continuity and homogeneity of spectra, we assume, with no loss of generality, that $\mathcal{S} = \overline{\mathbf{C}\mathcal{S}}$. In view of Lemma 0.4, we can also assume that the minimal rank $r$ of $\mathcal{S}$ is achieved by an idempotent $P \in \mathcal{S}$. If $r = 1$, then the ideal $\mathcal{S}P\mathcal{S}$ is reducible by Theorem 1.2 and $\mathcal{S}$ is reducible by Lemma 0.2. Now assume $r > 1$. By Lemma 0.2, we must only show that the semigroup $P\mathcal{S}P$, when restricted to the range of

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$P$ is reducible. Thus we have reduced the problem to the finite-dimensional case of this restriction semigroup $\mathcal{S}_0$ of $r \times r$ matrices. Since $P\mathcal{S}P$ has sublinear spectrum, so does $\mathcal{S}_0$ by Lemma 1.1.

Embed $\mathcal{S}_0$ in a maximal semigroup $\mathcal{J}$ of $r \times r$ matrices with sublinear spectrum. It is easy to see that this property satisfies all the requirements of Lemma 0.1. In view of Lemma 0.4 again, we can assume that $\mathcal{J}$ contains an idempotent $E$ of rank $s$ with

$$s = \min\{\operatorname{rank} T : 0 \neq T \in \mathcal{J}\}.$$

(Note that, since induced isomorphisms of $\mathcal{S}$ as well as topological closures were involved in the embedding $\mathcal{S}_0 \subseteq \mathcal{S}$ there may well be noninvertible matrices in $\mathcal{J} \setminus \{0\}$, while there are none in $\mathcal{S}_0 \setminus \{0\}$.) Now Lemma 0.1 implies that the restriction of $E\mathcal{J}E$ to the range of $E$ is of the form $\mathbf{C}\mathcal{G}$, where $\mathcal{G}$ is a finite group of $s \times s$ matrices. Observe that $\mathcal{G}$ has sublinear spectrum by Lemma 1.1, and is thus abelian by Proposition 1.5.

If $s > 1$, then $\mathcal{J}$ is reducible by Lemma 0.2 and so is $\mathcal{S}$. If $s = 1$, then the ideal of $\mathcal{J}$ generated by its rank-one operators is reducible by Theorem 1.2. So is $\mathcal{S}$ by Lemma 0.2.

To prove that $\mathcal{S}$ is triangularizable by Lemma 0.5, let $\mathcal{M}_1$ and $\mathcal{M}_2$ be any invariant subspaces of $\mathcal{S}$ with $\mathcal{M}_1 \subset \mathcal{M}_2$. The restriction semigroup $\mathcal{S} \mid \mathcal{M}_2$ has sublinear spectrum by Lemma 1.1. Thus the semigroup $(\mathcal{S} \setminus \mathcal{M}_2)^*$ of its adjoints also has the property. By Lemma 1.1 again, the restriction of this semigroup to its invariant subspace $\mathcal{M}_2 \ominus \mathcal{M}_1$ has sublinear spectrum. By using adjoints once more, we see that the compression of $\mathcal{S}$ to $\mathcal{M}_2 \ominus \mathcal{M}_1$ has the property, so that Lemma 0.5 is applicable. ∎

The presence of compact members in general semigroups of operators leads to some positive results.

***Corollary 1.9*** *A semigroup of operators with sublinear spectrum is reducible if it contains a nonzero compact operator.*

**Proof** The compact members of the semigroup form an ideal $\mathcal{J}$ which is nonzero by hypothesis. Since $\mathcal{J}$ is reducible by Theorem 1.8, the proof is completed by Lemma 0.2. ∎

***Corollary 1.10*** *Every semigroup of operators with sublinear spectrum that contains a diagonalizable compact operator of finite nullity is triangularizable.*

**Proof** The reducibility follows from Theorem 1.9. To show that such a semigroup $\mathcal{S}$ is triangularizable, let $K$ a member of $\mathcal{S}$ as specified above. With no loss of generality, we can assume $K = K^*$. If $\mathcal{M}_1$ and $\mathcal{M}_2$ are invariant subspaces of $\mathcal{S}$ with $\mathcal{M}_1 \subset \mathcal{M}_2$, the compression $K_0$ of $K$ to $\mathcal{N} := \mathcal{M}_2 \ominus \mathcal{M}_1$ is also self-adjoint. Thus if $\mathcal{N}$ is infinite-dimensional, then $K_0 \neq 0$ and Theorem 1.9 can be applied to the compression of $\mathcal{S}$ to $\mathcal{N}$ (which has sublinear spectrum by Lemma 1.1). If $\mathcal{N}$ is finite-dimensional, then Theorem 1.8 is applied. Thus in both cases the compression semigroup is reducible if $\dim \mathcal{N} \geq 2$. Hence Lemma 0.5 is applicable (with property $\mathcal{P}$ defined as follows: $\mathcal{S}$ is a semigroup of compact operators with sublinear spectrum that either acts on a finite-dimensional space or contains a self-adjoint member with finite nullity). ∎

***Corollary 1.11*** *Let $\mathcal{S}$ be a self-adjoint semigroup of operators with sublinear spectrum. If $\mathcal{S}$ consists of compact operators or if $\mathcal{S}$ contains a compact operator of finite nullity, then it is diagonalizable and hence abelian. In particular, every unitary group in $\mathcal{M}_n(\mathbf{C})$ with sublinear spectrum is abelian.*

**Proof** Triangularizability follows from either Theorem 1.8 or Corollary 1.9 (observing that if $K$ has finite nullity, so does the diagonalizable operator $K^*K$). Diagonalizability needs only a little additional effort: Let $\mathcal{C}$ be a triangularizing chain for $\mathcal{S}$, so that $\mathcal{M}_1$ and $\mathcal{M}_2$ in $\mathcal{C}$ and $\mathcal{M}_2 \subset \mathcal{M}_1$ imply that $\mathcal{M}_2 \ominus \mathcal{M}_1$ is a 1-dimensional space invariant under $\mathcal{S}$. Each such gap $\mathcal{M}_2 \ominus \mathcal{M}_1$ and its orthocomplement are invariant under $\mathcal{S}$, and so is the direct sum $\mathcal{M}$ of these gaps. It is easily seen that $\mathcal{S} \mid \mathcal{M}$ is diagonal and $\mathcal{S} \mid \mathcal{M}^\top = \{0\}$. ■

For arbitrary semigroups of bounded operators on an infinite-dimensional Hilbert space, there are no analogues of the results above. The following example shows that sublinearity of spectrum for a unitary group does not imply commutativity, triangularizability, or even reducibility.

***Example 1.12*** Let $\{e_n\}_{n=-\infty}^{\infty}$ be a bilateral orthonormal basis for a Hilbert space $\mathcal{H}$ and let $U$ be the bilateral shift, *i.e.*, the operator defined by $Ue_n = e_{n+1}$ for all $n$. Choose an aperiodic $\omega$ of modulus one and define the diagonal operator $V$ by $Ve_n = \omega^n e_n$. Let $\mathcal{G}$ be the group generated by $U$ and $V$, so that

$$\mathcal{G} = \{\omega^r U^s V^t : r, s, t \in \mathbf{Z}\}.$$

It is not hard to verify that every nonscalar member of $\mathcal{G}$ has the unit circle $\mathbf{T}$ as its spectrum. Thus for every pair $A$ and $B$ of nonscalar operators in $\mathcal{G}$ and every $\lambda$,

$$\sigma(A) + \lambda\sigma(B) = \{z : 1 - |\lambda| \le |z| \le 1 + |\lambda|\}.$$

Now let $z \in \sigma(A+\lambda B)$. Then $|z| \le \|A+\lambda B\| \le 1+|\lambda|$. To prove sublinearity of spectrum we must also show that $1-|\lambda| \le |z|$. Assume otherwise. Then $\|A^{-1}(\lambda B - zI)\| < |\lambda| + |z| < 1$. This implies that $I + A^{-1}(\lambda B - zI)$ is invertible and, hence, so is $A + \lambda B - zI$, which is a contradiction. We have shown the sublinearity when $A$ and $B$ are both nonscalar. If at least one of them is scalar, then commutativity implies submultiplicativity of spectra. The irreducibility of $\mathcal{G}$ is a trivial consequence of the fact that the only operators that commute with every member of $\mathcal{G}$ are scalars. ■

One may conjecture that finiteness of spectrum in a group of operators with sublinear spectrum would yield triangularizability or at least reducibility. The next example shows that this is false even for singleton spectrum.

***Example 1.13*** It was shown in [3] that there exists an irreducible algebra $\mathcal{A}$ of nilpotent operators on an infinite-dimensional Hilbert space. Let

$$\mathcal{G} = \{I + A : A \in \mathcal{A}\}.$$

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Then $\mathcal{G}$ is an irreducible group with $\sigma(G) = \{1\}$ for every member $G$. Furthermore, for every $G$ and $H$ in $\mathcal{G}$ and scalar $\lambda$,

$$G + \lambda H = (1 + \lambda)I + T$$

for some $T$ in $\mathcal{A}$, so that

$$\sigma(G + \lambda H) = \{1 + \lambda\} = \sigma(G) + \lambda\sigma(H),$$

and spectrum is linear on $\mathcal{G}$. ■

We observe that in our results we did not use the sublinearity condition for all scalars $\lambda$. In fact, if the condition is assumed only for a sufficiently large set of scalars (which does not even have to be infinite in the case of $\mathcal{M}_n(\mathbf{C})$), we can draw the desired conclusions. A sample result is included in the following theorem.

***Theorem 1.14*** *For a semigroup $\mathcal{S}$ of compact operators the following assertions are mutually equivalent.*

*(i)* *$\mathcal{S}$ is triangularizable.*
*(ii)* *For every integer k, members $S_1, \ldots, S_k$ of $\mathcal{S}$ and scalars $\lambda_1, \ldots, \lambda_k$,*

$$\sigma(\lambda S_1 + \cdots + \lambda_k S_k) \subseteq \lambda_1\sigma(S_1) + \cdots + \lambda_k\sigma(S_k).$$

*(iii)* *Spectrum is sublinear on $\mathcal{S}$.*
*(iv)* *Spectrum is real-sublinear, i.e., $\sigma(S+T) \subseteq \sigma(S) + \lambda\sigma(T)$ for every real $\lambda$ and every pair $S$ and $T$ in $\mathcal{S}$.*

**Proof** The implication (i) $\Rightarrow$ (ii) is transparent in finite dimensions, and follows from Ringrose's Theorem [18] in the case of compact operators on an infinite-dimensional space. We need only show that (iv) implies triangularizability. The Baire-category argument given in Theorem 1.1 can be used *verbatim* to obtain the real-sublinearity of spectra for restrictions to invariant subspaces. Thus we will be done (as in the proof of Theorem 1.8) if we show that (iv) implies reducibility.

The second paragraph of the proof of Theorem 1.8 applies to the present situation; we just have to replace "sublinear" with "real-sublinear" and $\mathbf{C}\mathcal{S}$ with $\mathbf{R}\mathcal{S}$. Thus the problem is reduced to the finite-dimensional case. Of course, we cannot use the Finiteness Lemma, which requires the assumption $\mathcal{S} = \mathbf{C}\mathcal{S}$. But it is easy to verify that in finite dimensions real-sublinearity implies full sublinearity. In fact, if the relation $\sigma(A + \lambda B) \subseteq \sigma(A) + \lambda(B)$ holds for infinitely many values of $\lambda$ and a given pair $A$ and $B$ of matrices, then it holds for all complex $\lambda$. To see this, just observe that by hypothesis, for infinitely many $\lambda$, the characteristic polynomial $f$ of $A + \lambda B$ divides the polynomial $g^n$, where $g(z)$ is the product of all the factors $(z - \alpha - \lambda\beta)$ with $\alpha \in \sigma(A)$ and $\beta \in \sigma(B)$. It follows that $f$ divides $g^n$ for all values of $\lambda$. ■

We conclude this section by mentioning some results concerning a pair (as opposed to a semigroup) of operators with property $L$. Motzkin and Taussky [10] showed that any two hermitian matrices with property $L$ commute. Wiegmann [22] extended this result to normal operators and Kaplansky [4] to compact normal operators on an infinite-dimensional Hilbert space.

## 2 Submultiplicativity and Related Conditions

We say that spectrum is *submultiplicative* on a semigroup $\mathcal{S}$ if $\sigma(AB) \subseteq \sigma(A)\sigma(B)$ for every pair $A$ and $B$ in $\mathcal{S}$. By $\sigma(A)\sigma(B)$ is meant the set $\{\alpha\beta : \alpha \in \sigma(A), \beta \in \sigma(B)\}$. This condition, obviously necessary for triangularizability of $\mathcal{S}$, is not sufficient even for a finite group in finite dimensions [5]. Before discussing stronger conditions we present an affirmative result for operators of rank one.

***Theorem 2.1*** *A semigroup of operators of rank $\leq 1$ with submultiplicative spectrum is triangularizable.*

**Proof** Let $\mathcal{S}$ be such a semigroup and assume with no loss that $\mathcal{S} = \mathbf{C}\mathcal{S}$. First assume that $\mathcal{S}$ has no nonzero nilpotent members. Thus every member of $\mathcal{S}$ is a scalar multiple of a rank-one idempotent. Denote by $\mathcal{S}_0$ the set of idempotent elements in $\mathcal{S}$, so that $\mathcal{S} = \mathbf{C}\mathcal{S}_0$. Now if $E$ and $F$ are in $\mathcal{S}_0$, then it follows from the submultiplicativity condition that either $EF = 0$ or $1 \in \sigma(EF)$, which implies that $EF$ is idempotent, because its rank is at most one. Hence $\mathcal{S}_0$ is a subsemigroup and triangularizable by [14].

If $\mathcal{S}$ does have a nonzero nilpotent element, then its nilpotent elements form an ideal by submultiplicativity of spectrum and thus $\mathcal{S}$ is reducible by [14] and Lemma 0.2. To complete the proof by Lemma 0.5, let $\mathcal{M}_1$ and $\mathcal{M}_2$ be invariant subspaces of $\mathcal{S}$ and consider the compression of $\mathcal{S}_1$ of $\mathcal{S}$ to $\mathcal{M}_2 \ominus \mathcal{M}_1$. Let $A_1$ and $B_1$ be in $\mathcal{S}_1$. We must show that $\sigma(A_1B_1) \subseteq \sigma(A_1)\sigma(B_1)$. There is nothing to prove if $\mathcal{M}_2 \ominus \mathcal{M}_1$ has dimension one, or if one of the operators $A_1$ and $B_1$ is zero. So we assume the dimension is at least two and both $A_1$ and $B_1$ are nonzero. It follows that $\sigma(A_1) = \sigma(A)$ and $\sigma(B_1) = \sigma(B)$. Then

$$\sigma(A_1B_1) \subseteq \sigma(AB) \subseteq \sigma(A)\sigma(B) = \sigma(A_1)\sigma(B_1). \qquad \blacksquare$$

A condition much stronger than submultiplicativity, called property $G$, was introduced by Motzkin and Taussky [11]. This property and property $L$ described above are both weakenings of what McCoy [9] termed property $P$. A pair of $n \times n$ matrices $A$ and $B$ is said to have *property $P$* if their eigenvalues, counting multiplicities, can be ordered as $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_n$ respectively, such that for every (noncommutative) polynomial $p$, the eigenvalues of $p(A, B)$ are just $p(\alpha_i, \beta_i)$, $i = 1, \ldots, n$, (respecting multiplicities again). Clearly, this condition is necessary for commutativity, as first observed by Frobenius [1], and even for triangularizability. Property $L$ is obtained when this condition is assumed for all linear polynomials. We say that the pair has *property $G$* if the property is assumed for all *monomials $p$*, *i.e.*, all words in $A$ and $B$.

It was shown by Motzkin and Taussky [11] that a finite group in which every pair has property $G$ is abelian. The following example shows that even a slight weakening of the multiplicity condition in this property renders it insufficient for triangularizability in general. However, we shall obtain reducibility results with much weaker conditions.

***Example 2.2*** Let $\mathcal{T}$ be the semigroup in $\mathcal{M}_n(\mathbf{C})$ consisting of the zero matrix and all basic matrices, *i.e.*, those matrices with exactly one entry equal to one and all other entries zero. Let

$$\mathcal{S} = \left\{ \begin{pmatrix} 1 & X \\ 0 & T \end{pmatrix} : T \in \mathcal{T}, X \text{ arbitrary} \right\}.$$

If $n \geq 2$, then $\sigma(S) = \{0, 1\}$ for every $S$ in $\mathcal{S}$, where 1 has multiplicity one or two depending on $S$. Thus if $A$ and $B$ are any members of $\mathcal{S}$, we can order their eigenvalues as $\{1, 0, \alpha_3, 0, \ldots, 0\}$ and $\{1, \beta_2, 0, \ldots, 0\}$, so that

$$\sigma(A^{r_1} B^{s_1} \cdots A^{r_k} B^{s_k}) = \{\alpha_i^{\sum r_j} \beta_i^{\sum s_j} : 1 \leq i \leq n+1\}$$

for all nonnegative integers $r_j$ and $s_j$. The weakening is, of course, in disregarding multiplicities on the left hand side of the equality.

It is easily seen that $\mathcal{T}$ is irreducible, so that $\mathcal{S}$ has only one invariant subspace. Incidentally, this example also demonstrates that Theorem 2.1 fails to hold without the rank restriction. ■

The full force of property $G$, appropriately interpreted for the infinite-dimensional setting, does imply triangularizability. So does a weaker property, respecting multiplicities but requiring no preordering of eigenvalues as in property $G$, which we now introduce.

We say that spectrum is *strongly permutable* on a semigroup $\mathcal{S}$ if for all $R$, $S$, $T$ in $\mathcal{S}$, every nonzero element of $\sigma(RST)$ is an element of $\sigma(SRT)$ with the same multiplicity. We say a pair $\{A, B\}$ of operators has *strongly permutable* spectrum if the semigroup generated by $A$ and $B$ does.

Some observations are worth making. First of all, this property is a strengthening of permutability of spectra, *i.e.*, the equality of $\sigma(RST)$ and $\sigma(SRT)$ as sets, for compact operators. (In infinite dimensions zero is always a member of spectrum. In finite dimensions, if $0 \in \sigma(RST)$, then $0 \in \sigma(SRT)$ by enumeration of multiplicities). Secondly, a simple induction shows that if $\mathcal{S}$ is a semigroup of compact operators with strongly permutable spectrum, then for any $n$, any $S_1, \ldots, S_n$ in $\mathcal{S}$, and any permutation $\tau$ on $n$ letters,

$$S_1 S_2 \cdots S_n \quad \text{and} \quad S_{\tau(1)} S_{\tau(2)} \cdots S_{\tau(n)}$$

have the same nonzero spectrum with the same multiplicities. Thirdly, the pair $\{A, B\}$ has strongly permutable spectrum if and only if the operator

$$A^{r_1} B^{s_1} A^{r_2} B^{s_2} \cdots A^{r_m} B^{s_m}$$

has the same nonzero spectrum as $A^{\Sigma r_i} B^{\Sigma s_i}$ with the same multiplicities, for any choice of integer $m$ and nonnegative integers $r_i$ and $s_i$. Hence this condition is, at least in appearance, a substantial weakening of property $G$.

***Theorem 2.3*** *Let $\mathcal{S}$ be a semigroup of compact operators in which every pair has strongly permutable spectrum. Then $\mathcal{S}$ is triangularizable.*

**Proof** By Theorem 1.7, we can assume $\mathcal{S}$ contains nonquasinilpotent elements. Since we can also assume $\mathcal{S} = \overline{\mathbf{C}\mathcal{S}}$ by homogeneity and continuity properties of spectra, we deduce that the ideal $\mathcal{F}$ of finite-rank operators in $\mathcal{S}$ is nonzero.

The hypothesis applied to $\mathcal{F}$ implies that if $W_1$, $W_2$, and $W_3$ are arbitrary words in fixed members $A$ and $B$ of $\mathcal{F}$, then

$$\operatorname{tr}(W_1 W_2 W_3) = \operatorname{tr}(W_2 W_1 W_3),$$

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*i.e.*, the semigroup generated by $A$ and $B$ has permutable trace and is thus triangularizable [14]. Hence spectrum is sublinear on $A$ and $B$ for all pairs $A$ and $B$ in $\mathcal{F}$, so that $\mathcal{F}$ is triangularizable by Theorem 1.8. It follows that $\mathcal{S}$ is reducible by Lemma 0.2.

Choose a maximal chain $\mathcal{C}$ of invariant subspaces for $\mathcal{S}$. Let $\mathcal{C}_0$ be the subchain of those $\mathcal{M}$ in $\mathcal{C}$ for which

$$\mathcal{M}_- : \{\mathcal{N} \in \mathcal{C} : \mathcal{N} \subsetneqq \mathcal{M}\}$$

is distinct from $\mathcal{M}$; and define $\mathcal{E}$ as the set of all $\mathcal{M} \ominus \mathcal{M}_-$ for $\mathcal{M}$ in $\mathcal{C}_0$. Then $\mathcal{E}$ consists of mutually orthogonal subspaces $\mathcal{H}_i$ of $\mathcal{H}$, the so-called "irreducible gaps" in $\mathcal{C}$. If $S_i$ denotes the compression of $S$ to $\mathcal{H}_i$ for each $S$ in $\mathcal{S}$, then the homomorphism $S \longmapsto \bigoplus_i S_i$ preserves nonzero spectra with multiplicities, so that we can assume, with no loss of generality, that $S = \bigoplus_i S_i$ for all $S$. We must show that each $\mathcal{H}_i$ is one-dimensional.

By Theorem 1.8, we can dispose of those $\mathcal{H}_i$ for which $\mathcal{S}_i$ consists of quasinilpotent elements and still assume $\mathcal{H} = \bigoplus_i \mathcal{H}_i$. Let $\mathcal{L}_1$ be the (possibly zero) direct sum of one-dimensional $\mathcal{H}_i$ and let $\mathcal{L}_2 = \mathcal{H} \ominus \mathcal{L}_1$. Now $\mathcal{S}|_{\mathcal{L}_1}$ is a diagonal semigroup and thus it has strongly permutable spectrum. It follows that $\mathcal{S}|_{\mathcal{L}_2}$ has this property also. Now supposing, if possible, that $\mathcal{L}_2 \neq \{0\}$, we shall obtain a contradiction to the irreducibility of $\mathcal{S}|_{\mathcal{H}_i}$ for some $\mathcal{H}_i \subseteq \mathcal{L}_2$.

We can henceforth assume $\mathcal{L}_2 = \mathcal{H}$ (and still $\mathcal{S} = \overline{\mathbf{C}\mathcal{S}}$). Pick a nonzero finite-rank $F$ in $\mathcal{S}$ and let $\mathcal{H}_1, \ldots, \mathcal{H}_m$ be those $\mathcal{H}_i$ for which $F|_{\mathcal{H}_i} \neq 0$. Thus the ideal $\mathcal{J}$ of $\mathcal{S}$ generated by $F$ has permutable trace and is triangularizable as seen above. Now any given chain of invariant subspaces for a triangularizable collection of compact operators can be extended to a traingularizing chain [5]. Thus $\mathcal{J}|_{\mathcal{H}_1}$, which is the ideal of $\mathcal{S}|_{\mathcal{H}_1}$ generated by $F|_{\mathcal{H}_1}$, is triangularizable, implying that $\mathcal{S}|_{\mathcal{H}_1}$ is reducible by Lemma 0.2. ■

***Corollary 2.4*** *For a semigroup $\mathcal{S}$ of compact operators the following are mutually equivalent.*

*(i) $\mathcal{S}$ is triangularizable.*
*(ii) Spectrum is strongly permutable on $\mathcal{S}$.*
*(iii) Every pair in $\mathcal{S}$ has strongly permutable spectrum.*
*(iv) Spectrum is sublinear on $\mathcal{S}$.*

**Proof** The implications (i) $\Leftrightarrow$ (iv) and (iii) $\Leftrightarrow$ (i) are just Theorems 1.8 and 2.3. It is clear that (i) implies every other assertion by compactness of the operators. Since (ii) clearly implies (iii), the proof is complete. ■

If we settle for reducibility, much weaker conditions than the strong permutability of spectrum will do. One such property, used in the next result, is a direct generalization of the condition $\sigma(ABA^{-1}B^{-1}) = \{1\}$, which holds for a triangularizable group.

***Theorem 2.5*** *Let $\mathcal{S}$ be a semigroup in $\mathcal{M}_n(\mathbf{C})$ such that for every pair $A$ and $B$ in $\mathcal{S}$ and every positive integer $m$,*

$$\sigma(ABA^{m-1}B^{m-1}) \subseteq \sigma(A^m B^m).$$

*Then $\mathcal{S}$ is reducible.*

**Proof** For a finite group the condition implies commutativity [11]. (There is no harm in assuming that the group is unitary; then the hypothesis implies that $\sigma(ABA^{-1}B^{-1}) = \{1\}$ and thus $ABA^{-1}B^{-1} = I$ for every $A$ and $B$.)

It is easily verified that the hypothesized property satisfies the requirements of Lemma 0.1. Thus, assuming that $\mathcal{S}$ is maximal with the given property, we obtain an idempotent $E$ in $\mathcal{S}$ such that the restriction of $E\mathcal{S}E$ to the range of $E$ is $\mathbf{C}\mathcal{G}$ with a finite group $\mathcal{G}$ of $r \times r$ matrices. If $r > 1$, then the reducibility of $\mathcal{S}$ follows from that of $\mathbf{C}\mathcal{G}$ via Lemma 0.2. If $r = 1$, we shall show that the ideal $\mathcal{J} = \mathcal{S}E\mathcal{S}$ is triangularizable and this will complete the proof by Lemma 0.2 again. Note that every member of $\mathcal{J}$ has rank at most one.

By Theorem 2.1 it suffices to show that spectrum is submultiplicative on $\mathcal{J}$. Let $A$ and $B$ be any members of $\mathcal{J}$. If one of them is nilpotent, then $A^2B^2 = 0$ and using $m = 2$ in the hypothesis we get $\sigma\big((AB)^2\big) = 0$, which implies $AB$ is nilpotent and hence $\sigma(AB) \subseteq \sigma(A)\sigma(B)$. If neither $A$ nor $B$ is nilpotent, we can assume with no loss that they are both idempotent. Then

$$\sigma\big((AB)^2\big) \subseteq \sigma(A^2B^2) = \sigma(AB).$$

Since $AB$ has rank at most one, this implies that $\sigma(AB) \subseteq \{0,1\}$. But $\{0,1\} = \sigma(A)\sigma(B)$. ■

The reader may note that many other spectral conditions of the type given in Theorem 2.5 imply reducibility. For example,

$$\sigma(ABA^{m-1}B^{m-1}) \subseteq \sigma(A^m)\sigma(B^m).$$

The proof is similar to the one given above. The condition singled out in Theorem 2.5 is a special case of the much stronger condition of permutability of spectrum on a semigroup $\mathcal{S}$, *i.e.*, the requirement that $\sigma(ABC) = \sigma(BAC)$ for all $A$, $B$, and $C$ in $\mathcal{S}$. Permutability does suffice for triangularizability of a group; for semigroups it only yields reducibility in general [5].

We remark here that Theorem 2.5 and the assertion following its proof cannot be strengthened by replacing $ABA^{m-1}B^{m-1}$ with any "shorter-looking" word. In other words the relations of the form

$$\sigma(A^rB^sA^t) \subseteq \sigma(A^{r+t}B^s), \quad r,s,t \in \mathbf{N}$$

or

$$\sigma(A^rB^sA^t) \subseteq \sigma(A^{r+t})\sigma(B^s), \quad r,s,t \in \mathbf{N}$$

for all pairs $A$ and $B$, even in a finite group, do not yield reducibility. The first trivially holds true for any pair $A$ and $B$ of operators on a finite-dimensional space. The second is just submultiplicativity of spectrum which was shown not be sufficient for reducibility in [5].

***Corollary 2.6*** *If $\mathcal{S}$ is a semigroup of compact operators satisfying*

$$\sigma(ABA^{m-1}B^{m-1}) \subseteq \sigma(A^mB^m)$$

*for every pair $A$ and $B$ in $\mathcal{S}$ and every positive integer $m$, then $\mathcal{S}$ is reducible.*

**Proof** This is Theorem 2.5 if the underlying space is finite-dimensional. If $\mathcal{S}$ consists of quasinilpotent operators, we are done by Theorem 1.7. Otherwise, pick $A$ in $\mathcal{S}$ with $\sigma(A) \neq \{0\}$. Then the closure of the set $\{\mathbf{C}A^n : n \in \mathbf{N}\}$ contains a finite-rank operator. Thus there is a nonzero finite-rank operator $F$ in $\mathcal{S}$ of minimal rank. Let $\mathcal{S}_0$ be the restriction of $F\mathcal{S}$ to the range of $F$. Then $\mathcal{S}_0$ satisfies the hypotheses of Theorem 2.5 and is hence reducible. The reducibility of $\mathcal{S}$ follows from Lemma 0.2. ■

Example 2.2 shows that the hypotheses of Theorem 2.5 and Corollary 2.6 may yield no more than one invariant subspace in general.

There exist irreducible groups in $\mathcal{M}_n(\mathbf{C})$ on which spectrum is submultiplicative [5], but every such group is essentially finite [13], *i.e.*, it is contained in $\mathbf{C}\mathcal{G}$ for some finite group $\mathcal{G}$. In fact, such a group is even more special.

***Proposition 2.7*** *Let $\mathcal{S}$ be an irreducible semigroup in $\mathcal{M}_n(\mathbf{C})$ with submultiplicative spectrum. Then there is a finite, nilpotent group of unitary matrices such that $\mathbf{C}\mathcal{S} = \mathbf{C}\mathcal{G}$ up to a simultaneous similarity.*

**Proof** The existence of a finite unitary $\mathcal{G}$ was shown in [13]. To prove that $\mathcal{G}$ is nilpotent, we verify that it is the direct product of its Sylow subgroups. Submultiplicativity of spectrum implies that for each prime $p$, the set of all $G$ in $\mathcal{G}$ whose order is a power of $p$ is a subgroup and thus the Sylow $p$-subgroup of $\mathcal{G}$. We need only show that if $A$ and $B$ are members of $\mathcal{G}$ of order $p^r$ and $q^s$ respectively, where $p$ and $q$ are distinct primes, then $A$ commutes with $B$. But, by hypothesis,

$$\sigma(A \cdot BA^{-1}B^{-1}) \subseteq \sigma(A)\sigma(A^{-1}) \subseteq \Gamma(p^r)$$

and

$$\sigma(ABA^{-1} \cdot B^{-1}) \subseteq \sigma(B)\sigma(B^{-1}) \subseteq \Gamma(g^s)$$

where $\Gamma(m)$ denotes the group of $m$-th roots of unity. This implies that $\sigma(ABA^{-1}B^{-1}) = \{1\}$ and hence $AB^{-1}A^{-1}B^{-1} = I$ as desired. ■

This proposition can be used to formulate affirmative results in the presence of additional hypotheses. We omit the simple proof of the following sample corollary.

***Corollary 2.8*** *Let $\mathcal{S}$ be a semigroup in $\mathcal{M}_n(\mathbf{C})$ with submultiplicative spectrum. Each of the following conditions implies reducibility for $\mathcal{S}$.*

*(i)* *$\mathcal{S}$ contains a member whose positive powers are all nonscalar.*
*(ii)* *$\sigma(S)$ is real for every $S$ in $\mathcal{S}$.*
*(iii)* *For some prime p, the subsemigroup of $\mathbf{C}\mathcal{S}$ consisting of matrices whose orders are powers of p has a nonscalar member in its centre.* ■

For triangularizability stronger hypotheses are needed.

***Theorem 2.9*** *Let $\mathcal{S}$ be a semigroup in $\mathcal{M}_n(\mathbf{C})$ with submultiplicative spectrum. If $\mathcal{S}$ contains a member with n algebraically independent eigenvalues, then $\mathcal{S}$ is triangularizable.*

**Proof** Let $A$ be member of $\mathcal{S}$ as described. Reducibility follows easily from the preceding remarks. To apply Lemma 0.5, assume $\mathcal{M}$ and $\mathcal{N}$ are distinct invariant subspaces of $\mathcal{S}$ with $\mathcal{M} \subset \mathcal{N}$ and let $\mathcal{S}_1$ denote the compression of $\mathcal{S}$ to $\mathcal{N} \ominus \mathcal{M}$. Since the compression $A_1$ of $A$ has algebraically independent eigenvalues, we must only show that $\mathcal{S}_1$ has submultiplicative spectrum. Consider the block-upper-triangular form of $\mathcal{S}$ relative to the chain $0 \subseteq \mathcal{M} \subseteq \mathcal{N} \subseteq \mathbf{C}^n$ (which is $2 \times 2$ or $3 \times 3$ depending on whether or not one of the subspaces is trivial). Without loss of generality, we can assume that off-diagonal blocks are all zero, *i.e.*, the subspaces $\mathcal{N} \ominus \mathcal{M}$ and $\mathcal{M} \oplus (\mathbf{C}^n \ominus \mathcal{N})$ are both invariant under $\mathcal{S}$. Thus a typical member $S$ of $\mathcal{S}$ has the form $S_1 \oplus S_2$, where $S_1$ denotes the restriction to $\mathcal{N} \ominus \mathcal{M}$.

Let $\{\lambda_1, \ldots, \lambda_n\}$ be the eigenvalues of $A$ such that

$$\{\lambda_1, \ldots, \lambda_m\} = \sigma(A_1) \quad \text{and} \quad \{\lambda_{m+1}, \ldots, \lambda_n\} = \sigma(A_2).$$

Using successive field extensions and an inductive argument, we can find complex numbers $\{\mu_1, \ldots, \mu_n\}$ such that the $2n$ numbers $\lambda_j$ and $\mu_j$ form an algebraically independent set and, moreover,

$$|\mu_1| = \cdots = |\mu_m| = 1, \quad |\mu_j| < 1 \quad \text{for } j > m.$$

There exists a field automorphism $\varphi$ of $\mathbf{C}$ with $\varphi(\lambda_j) = \mu_j$ for $j = i, \ldots, n$. Let $\Phi$ be the ring automorphism of $\mathcal{M}_n(\mathbf{C})$ induced by $\varphi$ (as in Lemma 0.1). The semigroup $\Phi(\mathcal{S})$ has submultiplicative spectrum and so does its norm closure. Now in the decomposition $\Phi(A) = \Phi(A_1) \oplus \Phi(A_2)$, the first direct summand has eigenvalues $\mu_1, \ldots, \mu_m$ and the second $\mu_{m+1}, \ldots, \mu_n$. Note that $\Phi(A_1)$ and $\Phi(A_2)$ are both diagonalizable matrices (since they have distinct eigenvalues). By the choice of moduli for the $\mu_j$, we can assume $\Phi(A_1)$ is unitary and $\|\Phi(A_2)\| < 1$, after a simultaneous similarity. Thus there is a sequence $\{n_i\}$ of positive integers with

$$\lim_i \Phi(A_1)^{n_i} = I \quad \text{and} \quad \lim_i \Phi(A_2)^{n_i} = 0.$$

Hence $\overline{\Phi(\mathcal{S})}$ contains the projection $P$ with range $\mathcal{N} \ominus \mathcal{M}$. For every pair $S = S_1 \oplus S_2$ and $T = T_1 \oplus T_2$ in $\mathcal{S}$,

$$\begin{aligned}\sigma\big(\Phi(S_1)\Phi(T_1)\big) \cup \{0\} &= \sigma\big(P\Phi(S) \cdot P\Phi(T)\big)\\ &\subseteq \sigma\big(P\Phi(S)\big)\sigma\big(P\Phi(T)\big)\\ &= \sigma\big(\Phi(S_1)\big)\sigma\big(\Phi(T_1)\big) \cup \{0\}.\end{aligned}$$

Since $\Phi(S_1)\Phi(T_1)$ is invertible if and only if both $\Phi(S_1)$ and $\Phi(T_1)$ are, we infer from the relations above that spectrum is submultiplicative on $\Phi(\mathcal{S}_1)$. Applying $\Phi^{-1}$ to $\Phi(\mathcal{S}_1)$ we conclude the submultiplicativity of spectrum on $\mathcal{S}_1$ as desired. ∎

Our final example in this section will demonstrate that even for finite groups no weakening of property $G$ by restricting it to "short" words of the form $A^rB^s$ (and hence of the form $A^rB^sA^t$) guarantees reducibility. Recall that restricting to slightly longer-looking words $A^rB^sA^tB^u$ is more than enough; Theorem 2.5 has an even weaker condition sufficient for commutativity of finite groups.

***Example 2.10*** Let $p$ be a prime, $\omega$ a $p$-th root of unity, and $\omega \neq 1$. Let $\mathcal{G}_0$ be the group generated by the two operators $U$ and $V$ on $\mathbf{C}^p$ defined, relative to the standard basis $\{e_j\}$, by the relations

$$Ue_j = \omega^j e_j, \quad 1 \leq j \leq p;$$
$$Ve_j = e_{j+1}, \quad 1 \leq j \leq p-1; \quad ve_p = e_1.$$

Define $\mathcal{G} = \mathcal{G}_0 \otimes \mathcal{G}_0 := \{A \otimes A : A \in \mathcal{G}_0\}$. It was shown in [5] that $\mathcal{G}_0$ and $\mathcal{G}$, although irreducible, have submultiplicative spectrum.

We shall verify that $\mathcal{G}$ (but not $\mathcal{G}_0$) satisfies the following relations: If $A$ and $B$ are any pair in $\mathcal{G}$, then there are listings $\{\alpha_1, \dots, \alpha_{p^2}\}$ and $\{\beta_1, \dots, \beta_{p^2}\}$ for $\sigma(A)$ and $\sigma(B)$ respectively, such that

$$\{\alpha_1^r \beta_1^s, \dots, \alpha_{p^2}^r \beta_{p^2}^s\}$$

is a listing for $\sigma(A^r B^s)$ whatever the integers $r$ and $s$ may be. Since $A^p = B^p = I$, we can of course, assume that $0 \leq r \leq p-1$ and $0 \leq s \leq p-1$.

Each nonscalar member of $\mathcal{G}_0$ has simple spectrum $\Omega = \{1, \omega, \dots, \omega^{p-1}\}$. (This is easily seen by observing that

$$\mathcal{G}_0 = \{\omega^i U^j V^k : 0 \leq i, j, k, \leq p-1\}$$

as in [5].) This implies that if $G$ and $H$ are in $\mathcal{G}_0$ and are not both scalar, then $G \otimes H$ has spectrum $\Omega$ with (uniform) multiplicity $p$. Let $A$ and $B$ be given in $\mathcal{G}$. If they commute, there is nothing to prove; so assume otherwise. Then $A^r$ and $B^s$ fail to commute unless $r = 0$ or $s = 0$. Thus if $r$ and $s$ are not both zero, then $A^r B^s$ is nonscalar and its spectrum is $\Omega$ with multiplicity $p$. Consider the listings

$$\{\alpha_i\} = \{1, 1, \dots, 1; \omega, \omega, \dots, \omega; \dots; \omega^{p-1}, \omega^{p-1}, \dots, \omega^{p-1}\}$$

and

$$\{\beta_i\} = \{1, \omega, \dots, \omega^{p-1}; 1, \omega, \dots, \omega^{p-1}; \dots; 1, \omega, \dots, \omega^{p-1}\}$$

for $\sigma(A)$ and $\sigma(B)$ respectively. If $s \neq 0$, the listing $\{\alpha_i^r \beta_i^s\}$ is easily seen to be of the form $\{\Omega_1, \dots, \Omega_p\}$. where $\Omega_j$ is a permutation of $\{1, \dots, \omega^{p-1}\}$. This makes it a listing for $A^r B^s$. The case where $s$ is zero is trivial. Thus $\mathcal{G}$ is an irreducible group with property $G$ restricted to all words of the form $A^r B^s$.

## 3 Nilpotency of a Single Polynomial on a Semigroup

There are other interesting weakenings of the preceding section's property $P$. For example, if we consider the single noncommutative polynomial $xy - yx$, the corresponding weakening of property $P$ for a semigroup $\mathcal{S}$ is the hypothesis that $AB - BA$ be nilpotent for every pair $A$ and $B$ in $\mathcal{S}$. This indeed implies triangularizability, a fact proved in [2] for matrices over a general field. (See also [17].) Other triangularizability results on rings of matrices are also proved in [2], which use single polynomials in their hypotheses. A natural question is whether there are other examples of single noncommutative polynomials in $x$ and $y$

such that property $P$ assumed merely for $f$ would imply triangularizability for a semigroup. Among obvious candidates are polynomials $f$ such that $f(A,B)=0$ whenever $A$ and $B$ are two commuting matrices. A moment's reflection shows that such an $f$ can be assumed to be homogeneous in each variable and its coefficients must add up to zero. Not every polynomial of this sort will do of course. For example, the polynomial $x^p y - yx^p$ is identically zero on any finite group $\mathcal{G}$ of exponent $p$ and yet $\mathcal{G}$ can be irreducible (*cf.* Example 2.10).

We only consider polynomials that are linear in one of the variables, *i.e.*, those of the form

$$a_m x^m y + a_{m-1} x^{m-1} yx + \cdots + a_0 yx^m$$

and extend the result quoted above concerning the special case $m=1$. It turns out that in the case of groups, the only obstruction is that mentioned in the preceding paragraph in connection with the polynomial $x^p y - yx^p$. We first treat the finite subcase.

***Theorem 3.1*** *Let $g(x)=\sum_{i=0}^m a_i x^i$ be a polynomial not divisible by $x^p-1$ for any prime $p$, and define $f$ by*

$$f(x,y)=\sum_{i=0}^m a_i x^i yx^{m-i}.$$

*If $\mathcal{G}$ is a finite group in $\mathcal{M}_n(\mathbf{C})$ such that $f(A,B)$ is nilpotent for every pair $A$ and $B$ in $\mathcal{G}$, then $\mathcal{G}$ is abelian.*

**Proof** Assume the contrary. Let $\mathcal{G}$ be a nonabelian group of minimal order satisfying the hypothesis. We apply Lemma 1.4. Let $G, G_1, \ldots, G_k$, and $\mathcal{H}$ be as in that lemma. Since the restriction of $\mathcal{G}$ to an invariant subspace, if any, also satisfies the hypothesis, we can restrict ourselves to the case of irreducible $\mathcal{G}$ and assume, with no loss, that $k=1$ and $G=G_1$. By adjoining a $p$-th root of the scalar $\alpha_1$ of Lemma 1.4 to $\mathcal{G}$ as in the preceding proofs, we can assume that $G$ is the $p\times p$ matrix

$$G=\begin{pmatrix} 0 & 0 & \ldots & 0 & 1\\ 1 & 0 & \ldots & 0 & 0\\ \vdots & & & & \\ 0 & 0 & \ldots & 1 & 0\end{pmatrix},$$

and $\mathcal{G}$ contains a diagonal but nonscalar $H$ (in the normal subgroup $\mathcal{H}$).

By hypothesis, $f(G, H^k G^{-m})$ is nilpotent, but

$$f(G,H^kG^{-m}) = a_m G^m H^k G^{-m} + \cdots + a_1 GH^k G^{-1} + a_0 H^k$$

is diagonal and thus equal to zero for every $k$. We show that $x^p-1$ divides $g(x)$, obtaining a contradiction that completes the proof. To this end, observe that if $h$ is any polynomial, then

$$\sum_{i=1}^m a_i G^i h(H) G^{-i} = 0. \tag{1}$$

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Since $H$ is nonscalar, there is a polynomial $h$ such that $E = h(H)$ is a nontrivial idempotent. Viewing $E$ as a (column) vector $V$, we can rewrite (1) as

$$(2) \qquad \Big(\sum_{i=1}^{m} a_i G^i\Big) V = g(G)V = 0.$$

If $V_0$ represents the first coordinate vector, then $V = \gamma(G)V_0$, where $\gamma(x)$ is the polynomial $x^{r_1} + \cdots + x^{r_s}$ with suitable integers satisfying $0 \leq r_1 < \cdots < r_s < p$ and $s < p$. Now (2) implies $g(G)\gamma(G)V_0 = 0$. Since $V_0$ is a cyclic vector for $G$, we conclude that $g(G)\gamma(G) = 0$, and thus the minimal polynomial $x^p - 1$ of $G$ divides $g(x)\gamma(x)$. But, since $s < p$, $\gamma(x)$ has no common divisor with $x^p - 1$ (such a common divisor must have rational coefficients and must divide the polynomial $x^p - 1/(x-1)$, which is irreducible over $\mathbf{Q}$). ■

For general semigroups, the conclusion of the result above is not true as shown in Example 3.3 below, but an affirmative result holds for groups as the next proposition shows. The group hypothesis will also be relaxed later in Theorem 3.7.

The reduction of the case of general matrix groups to the finite subcase can be carried out using the purely algebraic methods of Guralnick [2]. However, since allowing fields of nonzero characteristics would take us too far afield, we continue to use our current techniques in the next result.

***Proposition 3.2*** *Let the polynomial $g(x) = \sum_{i=1}^{m} a_i x^i$ and its companion polynomial $f$ be as in Theorem 3.1. If $\mathcal{G}$ is any group of invertible matrices such that $f(S,T)$ is nilpotent for every pair $S$ and $T$ in $\mathcal{G}$, then $\mathcal{G}$ is triangularizable. In particular, $\mathcal{G}$ is abelian if it is a unitary group.*

**Proof** Assume with no loss that $a_m \neq 0$. If $g(0) = 0$, write $g(x) = x^r g_0(x)$, where $g_0(0) \neq 0$, and let $f_0(x,y)$ be the corresponding companion polynomial with $f = x^r f_0$. Now the nilpotency of $f(A,B)$ for all pairs $A$ and $B$ in $\mathcal{G}$ means that $f_0(A, A^r B)$ is nilpotent and, since $B$ can be replaced by $A^{-r}B$, we obtain the nilpotency of $f_0(A,B)$ for all pairs. We have thus shown that we can assume with no loss of generality that $g = g_0$ and $a_0 \neq 0$.

By Lemma 0.5, it suffices to prove that $\mathcal{G}$ is reducible. If the coefficients $a_i$ were rational, we could apply Lemma 0.1 directly. Since these coefficients can even be transcendental, we need a slightly strengthened version of that lemma, which fortunately holds and whose proof is the same as that given for the original lemma in [13]. This version is obtained by replacing the items (ii) and (iii) of Lemma 0.1 with a single item:

(ii)′ $\Phi^{-1}\big(\overline{\Phi(\mathcal{S})}\big)$ for every ring automorphism $\Phi$ of $\mathcal{M}_n(\mathbf{C})$ induced by a field automorphism $\varphi$ of $\mathbf{C}$.

Note that (ii)′ includes (ii) by taking $\varphi$ to be the identity automorphism of $\mathbf{C}$.

We now let $\mathcal{S}$ be a maximal semigroup (which may not be a group any longer) containing $\mathcal{G}$ such that $f(A,B)$ is nilpotent for all pairs $A, B$ in $\mathcal{S}$. This property clearly holds for $\Phi(\mathcal{S})$ but with the coefficients $a_i$ replaced by $\varphi(a_i)$, which do not change when passing to the norm closure of $\Phi(\mathcal{S})$. The original $a_i$ reappear for the semigroup $\Phi^{-1}\big(\overline{\Phi(\mathcal{S})}\big)$. Thus the strengthened finiteness lemma applies, and to show the reducibility of $\mathcal{G}$ we can now assume the existence of an idempotent $E$ of rank $r$ in $\mathcal{S}$ such that the restriction of $E\mathcal{S}E$ to the range of $E$ is $\mathbf{C}\mathcal{G}_0$ with $\mathcal{G}_0$ a finite group. Since $\mathcal{G}_0$ satisfies the hypothesis of Theorem 3.1, it is abelian. Thus the reducibility of $\mathcal{G}$ (and $\mathcal{S}$) follows from Lemma 0.2 if $r$ is at least two. For the remainder of the proof we assume $r = 1$.

Taking $S = T = E$ in the hypothesis, we note that $\sum a_i = 0$. Then taking $S = E$, we conclude that

$$a_0 TE - (a_0 + a_m)ETE + a_m ET$$

is nilpotent for every $T$ in $\mathcal{S}$. This implies that $ET(1 - E)TE = 0$ since $a_0 a_m \neq 0$. We shall use the equation $ET(1 - E)TE = 0$ for $T$ in the original group $\mathcal{G}$. If $ETE$ is nonzero for every such $T$, then the equation applied to $SET$ yields

$$ET(1 - E)SE = 0$$

for every pair $S$ and $T$ in $\mathcal{G}$. This implies reducibility: if $ET(1 - E) = 0$ for all $T$, there is nothing to prove; otherwise, pick $T_0$ in $\mathcal{G}$ with $R = ET_0(1 - E) \neq 0$, observe that

$$\operatorname{tr}(T_0 S) = \operatorname{tr}\big(ET_0(1 - E)SE\big) = 0$$

for every $S$ in $\mathcal{G}$, and apply Lemma 0.3.

We shall now obtain a contradiction by assuming that $ETE = 0$ for some $T$ in $\mathcal{G}$. It follows from the invertibility of $T$ that $ET^pE$ is nonzero for some positive integer $p$. Let $p$ be the smallest such integer and, replacing $T$ by an appropriate power if necessary, assume also that $p$ is prime. Moreover, since we can assume $\lambda G \in \mathcal{G}$ for every nonzero $\lambda$, we scale $T$ to get $ET^pE = E$. Then, for any integer $k$, $ET^kE$ is $E$ or zero according as $k$ is divisible by $p$ or not. To verify this, observe that if $r$ and $s$ are any integers with $ET^rE \neq 0$ and $ET^sE \neq 0$, then the equation $EG(1 - E)GE = 0$ applied to $T^rET^s$ implies

$$ET^{r+s}E = (ET^rE)(ET^sE) \neq 0.$$

Hence, also, $ET^{kr}E = (ET^rE)^k$ for every integer $k$. It follows that the set $\mathcal{J}$ of integers $j$ with $ET^jE \neq 0$ is an ideal of the ring $\mathbf{Z}$. We conclude at once that $\mathcal{J} = p\mathbf{Z}$ and $ET^jE = E$ for all $j \in \mathcal{J}$.

We now return to the hypothesis that $\sum_{i=1}^m a_i T^i S T^{m-i}$ is nilpotent for every $T$ and $S$ in $\mathcal{S}$ and apply it with $S = ET^{-m}$. This shows that the operator $M = \sum_{i=1}^m a_i T^i E T^{-i}$ is nilpotent. Let $e$ be a unit vector in the range of $E$ and let $k$ be the degree of the minimal polynomial $h$ with $h(T)E = 0$. Let $\mathcal{M}$ be the cyclic invariant subspace of $T$ containing $e$. Thus $\mathcal{M}$ is a $k$-dimensional space spanned by $e, Te, \dots, T^{k-1}e$ for some $k \neq 1$. (If $k$ were one then $ETE$ would not be zero.) Since the restriction of $M$ to $\mathcal{M}$ is still nilpotent we can assume with no loss of generality that $\mathcal{M}$ is the whole space, *i.e.*, $k = n$.

We write the matrix of $M$ with respect to the basis $T^ie$, $j = 0, \dots, n - 1$. Since

$$\begin{aligned} MT^je &= \sum_{i=1}^m a_i T^i E T^{-i+j} E e \\ &= \sum_{i \equiv j (\operatorname{mod} p)} a_i T^i e \end{aligned}$$

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for every $j$, we obtain

$$a_0 + a_p + a_{2p} + \cdots = \operatorname{tr} M = 0.$$

Similarly, considering the matrices of nilpotent matrices $T^r M T^{-r}$ for $r = 0, 1, \ldots, p-1$, we conclude that

$$a_r + a_{p+r} + a_{2p+r} + \cdots = 0$$

for every $r$. This means that the polynomial $x^p - 1$ divides $\sum a_i x^i$, giving the desired contradiction. ∎

The next example shows that for general semigroups, the only polynomial of the kind considered above that works is $f(x, y) = xy - yx$.

***Example 3.3*** Let $\mathcal{T}$ be the irreducible semigroup of all basic matrices together with zero (*cf.* Example 2.2). Then for every integer $m > 1$ and every choice of coefficients $a_i$ with $\sum_{i=0}^{m} a_i = 0$, the matrix

$$a_0 BA^m + a_1 ABA^{m-1} + \cdots + a_m A^m B$$

is nilpotent whenever $A$ and $B$ are in $\mathcal{T}$. To see this, note that if the two matrices $A$ and $B$ are simultaneously (lower or upper) triangular, the assertion is clearly true. In the remaining cases, observe that $A$ and $B$ are both nilpotent rank-one matrices with $ABA = A$. Since $A^2 = 0$, the polynomial above is zero if $m > 2$. For $m = 2$, it becomes $a_1 ABA = a_1 A$ which is nilpotent.

The semigroup $\mathcal{T}$ can be extended to include operators of all possible ranks. Just let $\mathcal{S}$ be the semigroup generated by $\mathcal{T}$ and the set $\mathcal{D}$ of all diagonal matrices. (In infinite dimensions take only suitably compact members of $\mathcal{D}$.) Observe that $\mathcal{S} = \mathcal{D} \cup \mathbf{C}\mathcal{T}$. Now every pair $A$ and $B$ in $\mathcal{S}$ satisfy the polynomial nilpotency condition above. For if they are not both in $\mathbf{C}\mathcal{T}$, then they are simultaneously triangularizable. ∎

As a consequence of Proposition 3.2 we obtain the following reducibility result for semigroups on a (not necessarily finite-dimensional) Hilbert space.

***Theorem 3.4*** *Let the polynomials $f$ and $g$ be as in Theorem 3.1. Let $\mathcal{S}$ be a semigroup of compact operators such that $\overline{\mathbf{C}\mathcal{S}}$ does not contain an idempotent of rank one. If $f(S, T)$ is quasinilpotent for every pair $S$ and $T$ in $\mathcal{S}$, then $\mathcal{S}$ is reducible.*

**Proof** We can assume that $\mathcal{S} = \overline{\mathbf{C}\mathcal{S}}$ and that, by Theorem 1.7, $\mathcal{S}$ contains a nonquasinilpotent member. Thus $\mathcal{S}$ has a nonzero operator of finite rank. Let $r$ be the minimal positive rank present. By Lemma 0.4 we can assume that there is an idempotent $E$ of rank $r$ in $\mathcal{S}$, so that $r > 1$ by hypothesis. By minimality of $r$, the set $E\mathcal{S}E \setminus \{0\}$ is actually a group of $r \times r$ matrices when restricted to the range of $E$. Theorem 3.2 then implies that this group is triangularizable. Hence $\mathcal{S}$ is reducible by Lemma 0.2. ∎

With a little more care, the preceding arguments give a chain of invariant subspaces for $\mathcal{S}$ whose length depends on the minimal rank of idempotents in $\mathcal{S}$.

***Theorem 3.5*** *Let $\mathcal{S}$ be a semigroup on Hilbert space $\mathcal{H}$ satisfying the hypotheses of Theorem 3.4 and let $r$ be the minimal rank present in $\overline{\mathbf{C}\mathcal{S}}$. If $\mathcal{S}$ contains an idempotent of rank $r$, then it has at least $r-1$ distinct nontrivial invariant subspacer $\mathcal{M}$ with*

$$\{0\} \subset \mathcal{M}_1 \subset \cdots \subset \mathcal{M}_{r-1} \subset \mathcal{H}.$$

**Proof** Assume with no loss that $\mathcal{S} = \overline{\mathbf{C}\mathcal{S}}$, pick an idempotent $E$ of rank $r$ in $\mathcal{S}$, and consider the triangularizable group $\mathcal{S}_0 = E\mathcal{S}E \mid E\mathcal{H} \setminus \{0\}$ as in the preceding proof. Let $\{e_1, \ldots, e_r\}$ be an orthonormal basis for $E\mathcal{H}$ that triangularizes $\mathcal{S}_0$, so that the span $\mathcal{V}_j$ of $\{e_1, \ldots, e_j\}$ is invariant under $\mathcal{S}_0$ for each $j$. Define $\mathcal{M}_j$ as the smallest invariant subspace of $\mathcal{S}$ containing $\mathcal{V}_j$ and let $\mathcal{V}_0 = \{0\}$. Clearly, $\mathcal{M}_{j-1} \subseteq \mathcal{M}_j$ for $j = 1, \ldots, m$. To see that these subspaces are distinct it suffices to show that $e_j$ is orthogonal to $\mathcal{M}_{j-1}$. But this follows from the relations

$$(Se_i, e_j) = (SEe_i, Ee_j) = (ESEe_i, e_j) = 0$$

for every $i < j$. ■

If $\mathcal{S}$ consists of operators of the form $T \oplus \cdots \oplus T$ with $r$ copies of $T$ from the irreducible semigroup $\mathcal{T}$ of Example 3.3, then $\mathbf{C}\mathcal{S}$ is closed and all its nonzero members have rank $r$. Whatever the dimension of the underlying space of $\mathcal{T}$ may be, no chain of invariant subspaces for $\mathcal{S}$ can have more than $r-1$ distinct members. Thus the length of the chain in the statement of Theorem 3.5 cannot be improved in general.

It is not a coincidence that the semigroup $\mathcal{T}$ of Example 3.3, or its extension given there, are irreducible among semigroups satisfying single-polynomial nilpotency conditions. The following proposition sheds more light on this matter and shows that $\mathcal{T}$ must be essentially contained in any counter-example.

***Proposition 3.6*** *Let $f$ be any polynomial $\sum_{i=0}^{m} a_i x^i y x^{m-i}$ with $a_0 a_m \neq 0$. Let $\mathcal{S}$ be a semigroup of operators on a Hilbert space $\mathcal{H}$ such that $f(S,T)$ is quasinilpotent for every pair $S$ and $T$ in $\mathcal{S}$. If $\mathcal{S}$ is irreducible with minimal rank one, then*

*(a) the rank-one idempotents of $\mathbf{C}\mathcal{S}$, together with zero, form an abelian semigroup $\mathcal{E}$ of operators whose ranges span $\mathcal{H}$, and*

*(b) every rank-one operator in $\mathbf{C}\mathcal{S}$ is of the form $ESF$ with $E$ and $F$ in $\mathcal{E}$.*

**Proof** Assume $\mathcal{S} = \mathbf{C}\mathcal{S}$ with no loss, so that it contains an idempotent $E$ of rank one. By irreducibility, we can pick a subset $\{S_i\}$ in $\mathcal{S}$ such that $\{S_i E\}$ is a maximal linearly independent set in $\mathcal{H}$. For each $i$, the set $S_i E\mathcal{S}$ must contain non-nilpotent members, because otherwise the relation $\operatorname{tr}(S_i ES) = 0$ holds for all $S$ in $\mathcal{S}$, which yields an invariant subspace for $\mathcal{S}$. Using this fact and the assumption $\mathcal{S} = \mathbf{C}\mathcal{S}$, we find $\{T_i\}$ in $\mathcal{S}$ such that $\{S_i E T_i\}$ consists of idempotents of rank one. Let $E_i = S_i E T_i$ for each $i$. We shall prove that $\{E_i\}$ is the desired semigroup.

If $P$ and $Q$ are any two idempotents of rank one in $\mathcal{S}$ with independent ranges, we claim that $PQ = QP = 0$. To prove this, pick nonzero vectors $e_1$ and $e_2$ in the ranges of $P$ and $Q$ respectively and consider the restrictions

$$A = \begin{pmatrix} 1 & \lambda \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 0 \\ \mu & 1 \end{pmatrix}$$

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of $P$ and $Q$ to the span $\mathcal{V}$ of $e_1$ and $e_2$. We must show that $\lambda = \mu = 0$. Since $f(A,B) = f(P,Q)|_{\mathcal{V}}$ is nilpotent and since $\sum a_i = 0$ (as verified by taking $S = T = E$ in the hypothesis), we have

$$\begin{aligned}\det f(A,B) &= \det \sum_{i=0}^{m} a_i A^i B A^{m-i} \\ &= \det\bigl(a_0 BA - (a_0 + a_m)ABA + a_m AB\bigr) \\ &= a_0 a_m \lambda\mu(\lambda\mu - 1) = 0,\end{aligned}$$

and hence $\lambda\mu(\lambda\mu - 1) = 0$. We next show that $\lambda\mu = 0$. Assume, if possible, that $\lambda\mu \neq 0$. Then $\lambda\mu = 1$ and a simultaneous diagonal similarity of $\mathcal{S}$ allows us to assume $\lambda = \mu = 1$. By the irreducibility of $\mathcal{S}$, there is a member of $P\mathcal{S}|_{\mathcal{V}}$ that is independent of $A$; its matrix is then of the form

$$C = \begin{pmatrix} \alpha & \beta \\ 0 & 0 \end{pmatrix}$$

with $\alpha \neq \beta$. The two equations $\det f(B,C) = 0$ and $\det f(A, BC) = 0$ yield

$$\beta(\alpha - \beta) = 0 \quad \text{and} \quad \beta(\alpha - \beta) = 0$$

or $C = 0$. This contradiction proves $\lambda\mu = 0$. Thus, by permuting the basis of $\mathcal{V}$ if necessary, we can assume $\mu = 0$.

Now suppose, if possible, that $\lambda \neq 0$. Then by irreducibility again, there is a member $S$ of $\mathcal{S}$ with $QSP|_{\mathcal{V}} \neq 0$. After scaling, if necessary, we can assume that this restriction has the matrix

$$B' = \begin{pmatrix} 0 & 0 \\ \frac{1}{\lambda} & 1 \end{pmatrix}$$

which satisfies the contradictory relation $2a_0 a_m = \det f(A, B') = 0$ as before. We have thus proved the claim. In particular, $\{E_i\}$ is an abelian semigroup with spanning ranges.

To complete the proof of (a) we need only demonstrate that if $P$ is any rank-one idempotent in $\mathcal{S}$ (and if the dimension of $\mathcal{H}$ is greater than one), then $P = E_i$ for some $i$. The range of $P$ is independent of that of $E_i$ for all but at most one $i$. By what we have proved, then, $PE_i = E_i P = 0$ for all $E_i$ except possibly one, say $E_1$. Since the ranges of $\{E_i\}$ span $\mathcal{H}$, this implies that $P = E_1$.

Observe that the preceding arguments can be applied to the irreducible semigroup $\mathcal{S}^*$ implying that the ranges of its rank-one idempotents, *i.e.*, members of $\mathcal{E}^*$, also span $\mathcal{H}$. To prove (b) it suffices to demonstrate that $\mathcal{S}\mathcal{E} = \mathcal{E}\mathcal{S}\mathcal{E}$. Fix $E$ in $\mathcal{E}$ and pick $S$ in $\mathcal{S}$. If $SE = 0$, there is nothing to prove. Thus assume $SE$ is nonzero, so that there is an $F$ in $\mathcal{E}$ with $FSE \neq 0$. We shall show that $SE = FSE$.

By irreducibility again, $FSETF \neq 0$ for some $T$ in $\mathcal{S}$. Replacing $T$ by a scalar multiple of it, we get $FSETF = F$, implying that $SETF$ is an idempotent which, by the uniqueness in (a), coincides with $F$. Thus $ETFSE$ is also an idempotent equal to $E$, and

$$FSE = (SETF)SE = S(ETFSE) = SE. \qquad \blacksquare$$

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The following triangularizability result shows, in particular, that in the statement of Proposition 3.2, "group" can be replaced by "semigroup" if $a_0 a_m \neq 0$. In the infinite-dimensional case, we only need the assumption that $\overline{S\mathcal{M}} = \mathcal{M}$ for every invariant subspace $\mathcal{M}$ of $S$, *i.e.*, $S$ is a *strong quasiaffinity*, for every $S$ in the semigroup.

***Theorem 3.7*** *Let $f$ be as in Theorem 3.1 with $a_0 a_m \neq 0$ and $\mathcal{S}$ a semigroup of compact strong quasiaffinities such that $f(S,T)$ is quasinilpotent for every pair $S$ and $T$ in $\mathcal{S}$. Then $\mathcal{S}$ is triangularizable.*

**Proof** If $K$ is a strong quasiaffinity in $\mathcal{S}$, then so is the compression of $K$ to $\mathcal{M} \ominus \mathcal{M}_1$ for any invariant subspaces $\mathcal{M}_1$ and $\mathcal{M}_2$. The quasinilpotency of $f(S,T)$ is also inherited by compressions. Thus, by Lemma 0.5, we need only show that $\mathcal{S}$ is reducible.

If $\overline{\mathbf{C}\mathcal{S}}$ does not contain an operator of rank one, we are done by Theorem 3.4. Suppose otherwise and assume, if possible, that $\mathcal{S}$ is irreducible. Let $\mathcal{E}$ be the abelian semigroup of rank-one idempotents obtained by applying Proposition 3.6 to the semigroup $\overline{\mathbf{C}\mathcal{S}}$. But we claim that $\mathcal{S}$ (and hence $\overline{\mathbf{C}\mathcal{S}}$) is commutative, so that the underlying space is one-dimensional and we are done. It is easily seen that this is equivalent to the claim that if $E$ is any member of $\mathcal{E}$, then $SE = ES$ for every $S$ in $\mathcal{S}$.

To obtain our final contradiction, we assume the existence of $E_0$ in $\mathcal{E}$ and $S$ in $\mathcal{S}$ with $E_0 S \neq SE_0$. By Proposition 3.6, there is an $E_1$ in $\mathcal{E}$ such that $SE_0 = E_1 SE_0$. Picking nonzero vectors $e_0$ and $e_1$ in the ranges of $E_0$ and $E_1$ respectively, we have $Se_0 = \lambda_1 e_1$. Similarly $SE_1 = E_2 SE_1$ for some $E_2$ in $\mathcal{E}$. Continuing in this fashion, we obtain a sequence $\{e_i\}$ of nonzero vectors in the ranges of $\{E_i\}$ such that $Se_i = \lambda_{i+1} e_{i+1}$, $\lambda_{i+1} \neq 0$ (since $S$ is injective). We now distinguish two cases.

(1) Assume that the members of the sequence $\{E_i\}$ are distinct. Pick $T$ in $\overline{\mathbf{C}\mathcal{S}}$ such that $E_0 T E_m = T \neq 0$. It can be easily verified that the operator $S^j T S^{m-j}$ is nonzero for $j = 0, \dots, m$, so that it has rank one. Furthermore, its range and the range of its adjoint coincide with those of $E_j$ and $E_j^*$ respectively. Thus

$$S^j T S^{m-j} = E_j S^j T S^{m-j} E_j = \mu_j E_j$$

with $\mu_j \neq 0$, and the quasinilpotency of $\sum a_j S^j T S^{m-j}$ implies that the diagonalizable operator $\sum a_j \mu_j E_j$ is nilpotent. Hence $a_j = 0$ for every $j$, which is a contradiction.

(2) It follows from Proposition 3.6 that the equality $E_i = E_j$ with $i > j$ is impossible unless $i = 0$. So the only remaining case to be considered is this: there is a smallest positive integer $p$ with $E_0 = E_p$. By passing to a power of $S$, if necessary, we can assume that $p$ is prime. By scaling $\mathcal{S}$ and the basis vectors $e_i$ we also assume with no loss that $\lambda_i = 1$ for $i \leq p$. The argument here is similar to that given in the proof of Proposition 3.2. Pick a sufficiently large integer $r$ such that $2rp \geq m$ and apply the hypothesis to the pair $S$ and $E_0 S^{2rp-m}$ to deduce that

$$T = f(S, E_0 S^{2rp-m}) = \sum_{i=0}^{m} a_i S^i E_0 S^{rp-i}$$

is quasinilpotent for all $j$. Thus the restriction of $T$ to the invariant subspace generated by $\{e_0, \dots, e_{p-1}\}$ is nilpotent. But this restriction is diagonalizable, so that $T = 0$ implying

$$a_j + a_{j+p} + a_{j+2p} + \cdots = 0$$

for every $j$, yielding the contradictory conclusion that $x^p - 1$ divides the polynomial $f(x)$. ■

We conclude this section with two reducibility results.

***Theorem 3.8*** *Let $f$ be a polynomial as in Theorem 3.1 with $a_0 a_m \neq 0$. Assume $\mathcal{S}$ is a semigroup of bounded operators containing a nonzero finite-rank member such that $f(S,T)$ is quasinilpotent for all $S$ and $T$ in $\mathcal{S}$. Then either of the following conditions is sufficient for reducibility of $\mathcal{S}$.*

*(i) The minimal positive rank in $\overline{\mathbf{C}\mathcal{S}}$ is greater than 1.*
*(ii) $\mathcal{S}$ contains an injective or surjective operator not commuting with all of the rank-one idempotents in $\overline{\mathbf{C}\mathcal{S}}$.*

**Proof** (i) The ideal of finite-rank operators in $\overline{\mathbf{C}\mathcal{S}}$ is reducible by Theorem 3.4, and so is $\mathcal{S}$ by Lemma 0.2.

(ii) Suppose $\mathcal{S}$ is irreducible and let $\mathcal{E}$ be the abelian semigroup of idempotents in $\mathcal{S}$ given by Proposition 3.6. In the last three paragraphs of the proof of Theorem 3.7, we only used the injectivity of $S$ to show that $S$ has to commute with every member of $\mathcal{E}$. Thus irreducibility contradicts (ii) in the injective case. The surjective case can be dealt with by considering $\mathcal{S}^*$. ■

In the next result we do not need the nondivisibility hypothesis on the polynomial $f$.

***Theorem 3.9*** *Let $f$ be any polynomial $\sum_{i=0}^{m} a_i x^i y x^{m-i}$ with $a_0 a_m \neq 0$, and assume $\mathcal{S}$ is a semigroup of bounded operators containing a nonzero rank-one operator such that $f(S,T)$ is quasinilpotent for all $S$ and $T$ in $\mathcal{S}$. Then each of the following conditions is sufficient for reducibility of $\mathcal{S}$.*

*(i) $\mathcal{S}$ contains an operator whose finite-dimensional invariant subspaces do not span the whole space (i.e., their linear span is not dense).*
*(ii) $\mathcal{S}$ contains an operator whose invariant subspaces of dimension $\leq m$ do not span the whole space.*
*(iii) The set $\{S^{m!} : S \in \mathcal{S}\}$ is not commutative.*

**Proof** Let $\mathcal{E}$ be the abelian subsemigroup of idempotents given by Proposition 3.6. We shall show that each of the conditions above results in a contradiction.

Since (i) implies (ii), we start with the latter. Thus assume (ii) and let $S$ be the member in question. Denoting by $\mathcal{M}(E)$ the invariant subspace of $S$ generated by the range of $E$ for each $E$ in $\mathcal{E}$, we observe that the underlying Hilbert space is spanned by $\{\mathcal{M}(E) : E \in \mathcal{E}\}$. By hypothesis, then, there exists $E_0$ in $\mathcal{E}$ such that $\mathcal{M}(E_0)$ has dimension $d > m$. Let $e_0$ be a nonzero vector in the range of $E_0$ and form the orbit $\{e_0, e_1, \dots\}$ as in the proof of Theorem 3.7, where $e_j$ is a vector in the range of $E_j$ with $SE_{j-1} = E_j S E_{j-1}$, so that $Se_{j-1} = \lambda_j e_j$. The inequality $d > m$ implies that $\lambda_j \neq 0$ for every $j \leq m$ and that the idempotents $E_0, \dots, E_m$ are distinct. Now the paragraph (1) of the proof of Theorem 3.7 can be repeated verbatim for the finite sequence $\{E_j\}$, for it did not use the nondivisibility hypothesis on $f$. Hence $a_j = 0$ for every $j$ giving the desired contradiction.

To complete the proof it suffices to show that the irreducibility assumption negates (iii). It suffices to prove that $S^{m!}E = ES^{m!}$ for every $S$ in $\mathcal{S}$ and every $E$ in $\mathcal{E}$. The preceding paragraph shows that the dimension $d$ of $\mathcal{M}(E)$ is at most $m$. Thus $S^dE$ is a scalar multiple of $E$ and so is $S^{m!}E$. ■

If the requirement that $x^p - 1$ divide $f(x)$ for no prime $p$ is added to the hypotheses of the theorem above, then the number $m!$ in the statement can be replaced by $m$.

## 4 Other Fields of Characteristic Zero in Finite Dimensions

All the results of the preceding sections hold for semigroups $\mathcal{S}$ contained in $\mathcal{M}_n(\mathbf{F})$, where $\mathbf{F}$ is an algebraically closed field of characteristic zero. The proofs are reduced to the complex case in view of the following simple observation.

Pick a maximal linearly independent subset $\{A_1, \dots, A_k\}$ of $\mathcal{S}$, *i.e.*, a basis for its linear span. Let $\mathcal{S}_0$ be the subsemigroup of $\mathcal{S}$ spanned by the $A_i$. Then it is easily seen that $\mathcal{S}_0$ has precisely the same invariant subspaces as $\mathcal{S}$. On the other hand, as was shown in [13], $\mathcal{S}_0$ can be thought of as a semigroup in $\mathcal{M}_n(\mathbf{C})$: Just consider the subfield $\mathbf{F}_0$ of $\mathbf{F}$ by adjoining all entries of all the $A_i$ to the rational subfield $\mathbf{Q}$ of $\mathbf{F}$. Since $\mathbf{F}_0$ is of finite transcendence degree over $\mathbf{Q}$, it can be embedded in $\mathbf{C}$. So can its algebraic closure.

## References


[1] G. Frobenius, *Über vertauschbare Matrizen*. Preuss. Akad. Wiss. Sitzungsber (1896), 601–604,

[2] R. M. Guralnick, *Triangularization of sets of matrices.* Linear and Multilinear Algebra **9**(1980), 133–140.

[3] H. Hadwin, E. Nordgren, M. Radjabalipour, H. Radjavi and P. Rosenthal, *A nil algebra of bounded operators on Hilbert space with semisimple norm closure.* Integral Equations Operator Theory **9**(1986), 239–243.

[4] I. Kaplansky, *Completely continuous normal operators with property L.* Pacific J. Math. **3**(1953), 721–724.

[5] M. Lambrou, W. Longstaff and H. Radjavi, *Spectral conditions and reducibility of operator semigroups.* Indiana University Math. J. **41**(1992), 449–464.

[6] C. Laurie, E. Nordgren, H. Radjavi and P. Rosenthal, *On triangularization of algebras of operators.* J. Reine Angew Math. **327**(1981), 143–155.

[7] J. Levitzki, *Über nilpotente Unterringe.* Math. Ann. **105**(1931), 620–627.

[8] V. I. Lomonosov, *Invariant subspaces of operators commuting with compact operators.* Functional Anal. Appl. **7**(1973), 213–214.

[9] N. H. McCoy, *On the characteristic roots of matrix polynomials.* Bull. Amer. Math. Soc. **42**(1936), 592–600.

[10] T. S. Motzkin and O. Taussky, *Pairs of matrices with property L.* Trans. Amer. Math. Soc. **73**(1952), 108–114.

[11] ———, *On representations of finite groups.* Nederl. Akad. Wetensch. Proc. Ser. A **55**—Indagationes Math. **14**(1952), 511–512.

[12] E. Nordgren, H. Radjavi and P. Rosenthal, *Triangularizing semigroups of compact operators.* Indiana University Math. J. **33**(1984), 271–275.

[13] M. Radjabalipour and H. Radjavi, *A finiteness lemma, Brauer's theorem and other irreducibility results.* Comm. Algebra **27**(1999), 301–319.

[14] H. Radjavi, *A trace condition equivalent to simultaneous triangularizability.* Canad. J. Math. **38**(1986), 376–386.

[15] ———, *On reducibility of semigroups of compact operators.* Indiana University Math. J. **39**(1990), 499–515.

[16] H. Radjavi and P. Rosenthal, *From local to global triangularization.* J. Funct. Anal. **147**(1997), 443–456.

[17] H. Radjavi, P. Rosenthal and V. Shulman, *Operator semigroups with nilpotent commutators.* Proc. Amer. Math. Soc., to appear.

[18] J. R. Ringrose, *Super-diagonal forms for compact linear operators.* Proc. London Math. Soc. **12**(1962), 367–384.

[19] O. J. Schmidt, *Über Gruppen, deren sämtliche Teiler spezielle Gruppen sind.* Math. Sbornik **31**(1924), 366–372.

[20] J. P. Serre, *Linear Representations of Finite Groups.* Springer-Verlag, New York, 1977.


https://doi.org/10.4153/CJM-2000-009-5 Published online by Cambridge University Press

[21] Yu. V. Turovskii, *Volterra semigroups have invariant subspaces.* J. Funct. Anal. **162**(1999), 313–322.
[22] D. B. Wales and H. J. Zassenhaus, *On L-groups.* Math. Ann. **198**(1972), 1–12.
[23] N. A. Wiegmann, *A note on pairs of normal matrices with property L.* Proc. Amer. Math. Soc. **4**(1953), 35–36.
[24] W. Wielandt, *Lineare Scharen von Matrizen mit reellen Eigenwerten.* Math. Zeit. **53**(1950), 219–225.
[25] H. J. Zaussenhaus, *On L-semigroups.* Math. Ann. **198**(1972), 13–22.

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