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SOME REMARKS ON THE LOCATION OF FIXED POINTS

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Abstract

Several procedures for locating fixed points of nonexpansive selfmaps of a weakly compact convex subset of a Banach space are presented. Some of the results involve the notion of an asymptotic center or a Chebyshev center.

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1

Throughout this paper X denotes a uniformly convex Banach space, K a convex set and $f: K \to K$ a nonexpansive mapping. In [3] it is shown that the asymptotic center (see below) of each bounded sequence of iterates is a fixed point under f. In this paper we consider the fixed point properies associated with closely related centers; those that arise when sequences of subsets, rather than singletons, are considered.

2

Let $\{S_n\}$ be a sequence of sets in a Banach space Y with $\bigcup S_n$ bounded and let K be a nonempty closed convex subset of Y. For each $n = 1, 2, \ldots$ and each $x \in K$ set

$$(1) \qquad r_n(x) = \sup\left\{\|x - y\| : y \in \bigcup_{m \geqslant n} S_m\right\},$$

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$$r_n = \inf\{r_n(y): y \in K\}, \tag{2}$$

$$r(x) = \inf\{r_n(x): n = 1,2,\dots\}, \tag{3}$$

$$r = \inf\{r(y): y \in K\}. \tag{4}$$

The set $c_n = \{x \in K\colon r_n(x) = r_n\}$ is frequently referred to as the Chebyshev center (of $\bigcup_{m \geqslant n} S_m$) with respect to K. A familiar fact about Chebyshev centers is that they are nonempty convex sets whenever Y is reflexive or K is weakly compact. In certain Banach spaces (see [2]), uniformly convex ones included, the Chebyshev centers are known to be singleton sets.

More recently (see [8]) the set $c = \{x \in K\colon r(x) = r\}$ has been introduced and termed the asymptotic center with respect to K, whilst the number r is called the asymptotic radius (of the sequence of sets or singletons with respect to K). Clearly

$$r(c) < r(x) \tag{5}$$

whenever $x \in K \setminus \{c\}$.

To simplify notation we shall write $r(\{S_n\}, k)$ and $\mathscr{A}(\{S_n\}, K)$ for the asymptotic radius and center respectively of $\{\bigcup_{m \geqslant n} S_m\}$ with respect to K.

Arguments such as those used in [3] and [4] show that in a uniformly convex Banach space the Chebyshev centers $\{c_n\}$ converge to the asymptotic center $\mathscr{A}(\{S_n\}, K)$.

3

3.1 THEOREM 1. *Let S be a non-empty subset of a uniformly convex Banach space X, and let $f\colon X \to X$ be nonexpansive. If $\bigcup\{f^n[S]\}$ is bounded then c, the asymptotic center of $\{f^n[S]\colon n = 1,2\dots\}$ with respect to X, is a fixed point of f, that is $f(c) = c$.*

Furthermore, if ξ is any other fixed point of f, then

$$\begin{aligned}\inf\{\sup\{\|x - c\|: x \in f^n[S]\}: n = 1,2,\dots\}\\ < \inf\{\sup\{\|x - \xi\|: x \in f^n[S]\}: n = 1,2,\dots\}.\end{aligned} \tag{6}$$

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PROOF. For $n \geqslant 2$ we have

$$\begin{aligned} r_n(f(c)) &= \sup\Big\{\|f(c) - y\| : y \in \bigcup_{m \geqslant n} f^m[S]\Big\} \\ &= \sup\Big\{\|f(c) - f(z)\| : z \in \bigcup_{m \geqslant n} f^{m-1}[S]\Big\} \\ &\leqslant \sup\Big\{\|c - z\| : z \in \bigcup_{m \geqslant n-1} f^m[S]\Big\} \\ &= r_{n-1}(c). \end{aligned}$$

Thus $r_n(f(c)) \leqslant r_{n-1}(c)$ for each n, and hence $r(f(c)) \leqslant r(c)$. However as $r(c) = r(f^m[S], X)$ it follows that $f(c)$ is in $\mathscr{A}(f^m[S], X)$ and, by remarks in Section 2, that $f(c) = c$.

To prove (6) first observe that for any fixed point z of f,

$$\begin{aligned} \sup\{\|x - z\| : x \in f^n[S]\} &= \sup\{\|f(x) - f(z)\| : x \in f^{n-1}[S]\} \\ &\leqslant \sup\{\|x - z\| : x \in f^{n-1}[S]\}, \end{aligned}$$

implying that

$$\sup\Big\{\|x - z\| : x \in \bigcup_{n \geqslant m} f^n[S]\Big\} = \sup\{\|x - z\| : x \in f^m[S]\}.$$

Thus (6) is equivalent to $r(c) < r(\xi)$, which is true by (5).

Let X, S, and f be as in the above theorem.

3.2 If $\varnothing \neq S \subseteq X$ and S is bounded with $f[S] \subseteq S$, then obviously $\bigcup\{f^n[S] : n \geqslant m\} = f^m[S]$. We conclude that if c_m is the Chebyshev center of $f^m[S]$ then $c = \lim c_m$ exists and is a fixed point of f.

3.3 Theorem 1 can be restated in terms of a mapping $f: K \to K$ where K is a closed and bounded convex set in X. In this case the asymptotic center is taken with respect to K. One could also relax the nonexpansive condition on f in a manner similar to that employed in [3].

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4.1 THEOREM 2. *Let S be a nonempty subset of a uniformly convex Banach space X, and let $f: X \to X$ be nonexpansive. If $\bigcup\{f^n[S]\}$ is bounded and for some natural*

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k and l we have $r_k = r_{k+l}$, then $r_k = r_{k+1} = \cdots = r_{k+l}$ and $f(c_k) = c_k = \cdots = c_{k+l}$; hence also $f(c_{k+n}) = c_{k+n}$ for $n = 1, 2, \ldots, l$. (Here r_i and c_i are as in Section 2.)

PROOF. Clearly $r_k \geqslant r_{k+1} \geqslant \cdots \geqslant r_{k+l}$, so $r_k = r_{k+1} = \cdots = r_{k+l}$. It suffices then to prove the theorem for $l = 1$. Now c_{k+1} is the unique point with the property that $\bigcup_{n \geqslant k+1} f_n[S] \subseteq B(c_{k+1}, r_{k+1})$. However

$$f\Big(\bigcup_{n \geqslant k} f^n[S]\Big) = \bigcup_{n \geqslant k+1} f^n[S] \subseteq \bigcup_{n \geqslant k} f^n[S] \subseteq B(c_k, r_k) = B(c_k, r_{k+1})$$

showing that $c_k = c_{k+1}$. Since f is nonexpansive,

$$\sup\Big\{\|f(c_k) - f(y)\| : y \in \bigcup_{n \geqslant k} f^n[S]\Big\} \leqslant \sup\Big\{\|c_k - y\| : y \in \bigcup_{n \geqslant k} f^n[S]\Big\} = r_k.$$

Thus $\bigcup_{n \geqslant k+1} f^n[S] \subseteq B(f(c_k), r_k) = B(f(c_k), r_{k+1})$, showing that $f(c_k) = c_{k+1} = c_k$.

4.2 Under the hypothesis of Theorem 2, a finite (rather than infinite, as in Theorem 1) procedure leads to the location of a fixed point. As the following example shows the fixed point obtained may, or may not coincide with that obtained via the asymptotic center.

EXAMPLE. Let $K = \{(x, y, z) \in R^3 \colon z \geqslant 0,\ x^2 + y^2 + z^2 \leqslant 1\}$ and define $f\colon K \to K$ by $f((x, y, z)) = (y, 0, z)$. Clearly f is nonexpansive. The sets K and $f[K]$ both have Chebyshev radius 1 and Chebyshev center $(0, 0, 0)$. For $n \geqslant 2$ $f^n[K]$ has Chebyshev radius $1/2$ and center $(0, 0, 1/2)$, showing the asymptotic center of $f^n[K]$ is $(0, 0, 1/2)$. Thus Theorems 1 and 2 may both be used to locate a fixed point of f, however the fixed points obtained do not coincide.

4.3 The assumption that X is uniformly convex is used only to guarantee the existence of a unique asymptotic center for $\{f^n[S]\}$ in Theorem 1, and of a unique Chebyshev center for each $f^n[S]$, $k \leqslant n \leqslant k + l$, in Theorem 2. Any other condition leading to the existence and uniqueness of such centers may be substituted for uniform convexity; (see [2], [9]).

4.4 Theorem 2 can also be restated in terms of a mapping $f\colon K \to K$ where K is a closed bounded convex set in X. In this case the Chebyshev centers and radii would be taken with respect to K. A result of Floret [6], Theorem 2, shows that if $\overline{\mathrm{co}}(f[K]) = K$ (where $\overline{\mathrm{co}}$ denotes the closed convex hull) then the Chebyshev center of K is a fixed point of f. If $\overline{\mathrm{co}}(f[K]) = K$ the Chebyshev radii of $\overline{\mathrm{co}}(f[K])$ and K are equal, thus in view of the remarks in 4.3, his result follows from a theorem similar to Theorem 2.

5

The concept of asymptotic regularity is due to Browder and Petryshyn [1]. In [5] it was modified to uniform asymptotic regularity. If S is a subset of a Banach space Y, the mapping h: $S \to S$ is said to be uniformly asymptotically regular if $\|h^{n+1}(x) - h^n(x)\| \to 0$ uniformly over S. It was shown there that if K is convex and bounded and g: $K \to K$ is nonexpansive then $f = \frac{1}{2}(g + I)$ is uniformly asymptotically regular (here I is the identity mapping). The mappings f and g have identical fixed point sets. Procedures for locating fixed points are much simpler for uniformly asymptotically regular maps, as shown by the next result.

5.1 THEOREM 3. *Let h: $S \to S$ be a uniformly asymptotically regular self-mapping of a set S in a Banach space Y. Then $S_\omega = \bigcap\{h^n[S]: n = 1, 2, \ldots\}$ is the fixed point set of h.*

PROOF. Let F be the fixed point set of h. Certainly $F \subseteq S_\omega$. To show that $S_\omega \subseteq F$ let $\xi \in S_\omega$ and suppose, for a contradiction, that $h(\xi) \neq \xi$. Let $\varepsilon = \|h(\xi) - \xi\|$ and let n be sufficiently large to imply that $\|h^{n+1}(x) - h^n(x)\| < \varepsilon$ for all $x \in S$. Let $x \in S$ be such that $h^n(x) = \xi$. Then $\varepsilon = \|h(\xi) - \xi\| \leqslant \|h^{n+1}(x) - h^n\| < \varepsilon$, a contradiction.

5.2 COROLLARY 1. *Let K be a closed bounded convex set in a uniformly convex Banach space X, let f: $K \to K$ be nonexpansive and uniformly asymptotically regular, and let $\varnothing \neq S \subseteq K_\omega = \bigcap\{f^n[K]: n = 1, 2, \ldots\}$. Then the Chebyshev center of S with respect to K is a fixed point of f, and hence is in K_ω.*

PROOF. As $f[S] = S$ this follows from Theorem 2.

5.3 COROLLARY 2. *Let K be a weakly compact convex subset of a Banach space Y, and f: $K \to K$ be nonexpansive. Suppose in addition that K has normal structure (see [7]). By a theorem of Kirk [7] the set of fixed points F of f is non-empty. Then $g = \frac{1}{2}(f + I)$ is uniformly asymptotically regular, and applying Theorem 3 to g, $K_\omega = F$ is the fixed point set of both f and g.*

5.4 AN EXAMPLE. It is tempting to assume that in Theorem 3 the sets $f^n[K]$, in some sense, converge to K_ω and in particular that the Chebyshev radii of $f^n[K]$ converge to the Chebyshev radius of K_ω. The following example shows this is not the case.

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Let K be the unit ball in the Hilbert space l^2. For $x = (x_k) = \langle x_1, x_2, \ldots, x_k, \ldots\rangle \in K$ set

$$f(x) = \langle (1 - \tfrac{1}{k})x_k \rangle.$$

Clearly f is nonexpansive, $f(K) \subseteq K$, and $f^n(x) = \langle (1 - \frac{1}{k})^n x_k\rangle$. Also

$$\|f^{n+1}(x) - f^n(x)\|^2 = \sum (1 - \tfrac{1}{k})^{2n}((1 - \tfrac{1}{k})x_k - x_k)^2 \leqslant \sum (1 - \tfrac{1}{k})^{2n}(\tfrac{1}{k})^2.$$

It is readily seen that the above tends to 0 as $n \to \infty$.

Let $e_k = \langle \underbrace{0, 0, \ldots, 0}_{k-1 \text{ times}}, 1, 0, \ldots\rangle$. Then

$$\|f^n(e_k)\| = (1 - \tfrac{1}{k})^n.$$

Thus letting k' be such that for all $k \geqslant k'$, $(1 - \frac{1}{k})^n \geqslant 1 - \varepsilon$, it follows that $\|f^n(e_k)\| \geqslant 1 - \varepsilon$ for $k \geqslant k'$. It can now be concluded that the Chebyshev radius of $f^n(K)$ is 1 for all n. However $\bigcap f^n[K] = \{0\}$.

Note that this example can easily be modified so that $\bigcap f^n[K]$ is more than a singleton.

6

6.1 In this section we present a procedure for locating a fixed point o a nonexpansive map f from a weakly compact convex set K with normal structure to itself, without passing to an alternative map (for example, $\frac{1}{2}(I + f)$). In general this method involves a transfinite process.

Let $K_0 = K$ and suppose that for each ordinal $\beta < \alpha$, K_β has been defined such that $f(K_\beta) \subseteq K_\beta$ and K_β is a weakly compact convex subset of K_γ, for all $\gamma < \beta$.

$$K_\alpha = \begin{cases} \mathscr{A}(\{f^n(K_{\alpha-1})\}, K_{\alpha-1}), & \alpha \text{ not a limit ordinal,} \\ \bigcap\limits_{\beta<\alpha} K_\beta, & \alpha \text{ a limit ordinal.} \end{cases}$$

It is shown in [8] that if $x \in K$ then the asymptotic center of $\{f^n(x)\}$ is mapped by f into itself. With essentially the same proof it can be shown that the same is true for an arbitrary set $\varnothing \neq S \subseteq K$. That is

$$f(\mathscr{A}(\{f^n[S]\}, K)) \subseteq \mathscr{A}(\{f^n[S]\}, K)$$

We conclude (in either case) that $f(K_\alpha) \subseteq K_\alpha$. Also, by properties of the asymptotic center stated in Section 2 and properties of the intersection, in either case K_α is a weakly compact convex subset of K_β, for all $\beta \subset \alpha$.

It is also shown in [8] that if C is convex with normal structure, contains more than one point, and $C \supseteq B_n \supseteq B_{n+1}$, $n = 1, 2, \ldots$, then $\mathscr{A}(\{B_n\}, C)$ is a proper subset of C. It follows that for each ordinal α, K_α is a proper subset of K_β, $\beta < \alpha$ unless K_β is a singleton and hence there is a smallest ordinal λ such that K_λ is a singleton. We conclude with

THEOREM 4. *With f, and λ as above, the singleton K_λ is a fixed point of f.*

6.2 Theorem 4 can be restated in a form similar to that of Theorems 1 and 2. Let $K \subseteq Y$ be such that either K is closed and Y is reflexive or K is weakly compact. Let $f \colon K \to K$ and assume that $S \subseteq K$ is such that $\bigcup f^n[S]$ is bounded. In either case, the asymptotic center of $\{f^n[S]\}$, K_0, is a weakly compact subset of Y with $f(K_0) \subseteq K_0$. Hence the above procedure can be applied to this choice of K_0, rather than K itself.

6.3 As in Theorem 1, the fixed point obtained by the above procedure may be characterized as follows.

THEOREM 5. *Let f, K, K_α and K_λ be as above and let $K_\lambda = \{c\}$. Then for every fixed point ξ of f, and each ordinal α,*

$$\begin{aligned}(7)\qquad &\inf\left\{\sup\left\{\|x - c\| : x \in f^n[K_\alpha]\right\} : n = 1, 2, \ldots\right\} \\ &\qquad \leqslant \inf\left\{\sup\left\{\|x - \xi\| : x \in f^n[K_\alpha]\right\} : n = 1, 2, \ldots\right\}.\end{aligned}$$

PROOF. Let α be fixed. Then the left side of (7) is $r(c)$ and the right side is $r(\xi)$, where $r(x)$ is defined by (3) with $B_n = f^n[K_\alpha]$. As $c \in K_{\alpha+1} = \mathscr{A}(f^n[K_\alpha], K_\alpha)$, $r(c) = r$ and the theorem follows from (4).

References

[1] F. E. Browder and W. V. Petryshyn, 'The solution by iteration of nonlinear functional equations in Banach spaces', *Bull. Amer. Math. Soc.* **72** (1966), 571–575.

[2] M. M. Day, R. C. James and S. Swaminathan, 'Normed linear spaces that are uniformly convex in every direction', *Canad. J. Math* **23** (1971), 1051–1059.

[3] M. Edelstein, 'The construction of an asymptotic center with a fixed point property', *Bull. Amer. Math. Soc.* **78** (1972), 206–208.

[4] M. Edelstein, 'Fixed point theorems in uniformly convex Banach spaces', *Proc. Amer. Math. Soc.* **44** (1974), 369–374.

[5] M. Edelstein and R. C. O'Brien, 'Nonexpansive mappings, asymptotic regularity, and successive approximations', *J. London Math. Soc.* (2) **17** (1978), 547–554.

[6] Klaus Floret, 'Eine Bemerkung über a-priori-Fixpunkte nichtexpansiver Abbildungen', *Manuscripta Math.* **6** (1972), 321–326.
[7] W. A. Kirk, 'A fixed point theorem for mappings which do not increase distances', *Amer. Math. Monthly* **72** (1965), 1004–1006.
[8] T. C. Lim, 'Characterizations of normal structure', *Proc. Amer. Math. Soc.* **43** (1974), 313–319.
[9] J. Reinermann and R. Shoenberg, 'Some results in the fixed point theory of nonexpnsive mappings and generalized contractions', *Proc. Sem. Fixed Point Theory and its Applications*, Dalhousie University, Halifax, N. S., Canada, 1975 (Academic Press).

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