

# EXTENSIONS OF CHARACTERS IN TYPE D AND THE INDUCTIVE MCKAY CONDITION, I

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**Abstract.** This is a contribution to the study of  $\text{Irr}(G)$  as an  $\text{Aut}(G)$ -set for  $G$  a finite quasisimple group. Focusing on the last open case of groups of Lie type D and  ${}^2\text{D}$ , a crucial property is the so-called  $A'(\infty)$  condition expressing that diagonal automorphisms and graph-field automorphisms of  $G$  have transversal orbits in  $\text{Irr}(G)$ . This is part of the stronger  $A(\infty)$  condition introduced in the context of the reduction of the McKay conjecture to a question about quasisimple groups. Our main theorem is that a minimal counterexample to condition  $A(\infty)$  for groups of type D would still satisfy  $A'(\infty)$ . This will be used in a second paper to fully establish  $A(\infty)$  for any type and rank. The present paper uses Harish-Chandra induction as a parametrization tool. We give a new, more effective proof of the theorem of Geck and Lusztig ensuring that cuspidal characters of any standard Levi subgroup of  $G = D_{l,\text{sc}}(q)$  extend to their stabilizers in the normalizer of that Levi subgroup. This allows us to control the action of automorphisms on these extensions. From there, Harish-Chandra theory leads naturally to a detailed study of associated relative Weyl groups and other extendibility problems in that context.

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Received September 16, 2021. Revised June 12, 2023. Accepted July 3, 2023.

2020 Mathematics subject classification: Primary 20C20; Secondary 20C33, 20C34.

This material is partially based upon work supported by the National Science Foundation under Grant No. DMS-1440140 while the author was in residence at the Mathematical Sciences Research Institute in Berkeley, California, during the Spring 2018 semester. Some research was conducted in the framework of the research training group GRK 2240: *Algebra-Geometric Methods in Algebra, Arithmetic and Topology*, funded by the DFG.

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## §1. Introduction

After the classification of finite simple groups and with the knowledge on their representations having also greatly expanded in the last decades, it seems overdue to determine for each quasisimple group  $G$  the action of its outer automorphism group  $\text{Out}(G)$  on its set of irreducible (complex) characters  $\text{Irr}(G)$ . This is important in order to use our results on representations of simple groups to get theorems about arbitrary finite groups. A crucial example is the McKay conjecture asserting

$$|\text{Irr}_{p'}(X)| = |\text{Irr}_{p'}(N_X(P))|$$

for  $p$  a prime,  $X$  a finite group,  $P$  one of its Sylow  $p$ -subgroups, and  $\text{Irr}_{p'}(X)$  the set of irreducible characters of  $X$  of degree prime to  $p$ . It is fairly clear that once this is solved for a normal subgroup  $Y$  of  $X$ , the next step to deduce something for  $X$  is to determine the action of  $X$  on at least  $\text{Irr}_{p'}(Y)$ . The McKay conjecture has been reduced to a so-called inductive McKay condition about quasisimple groups by Isaacs–Malle–Navarro [IMN], and the first requirement is an  $\text{Out}(X)$ -equivariant bijection realizing McKay's equality. Knowing the action of  $\text{Out}(G)$  on  $\text{Irr}(G)$  for all quasisimple groups  $G$  would also have applications to other conjectures about characters with similar reductions such as the Alperin–McKay conjecture or the Dade conjecture (see [S5], [S6]) or even conjectures about modular characters (see [NT11]) through the unitriangularity of decomposition matrices (see [BDT]).

For alternating and sporadic groups, the action of  $\text{Out}(G)$  on  $\text{Irr}(G)$  is easy to deduce from the available description of  $\text{Irr}(G)$ . When  $G$  is the universal covering group of a finite simple group of Lie type, this is a question in [GM, §A.9]. Previous research on the subject has left open only the case of groups of type D (see [CS4, 2.5]). The present paper is the first part of a solution to that problem. A second part [S7] will finish the determination of  $\text{Irr}(G)$  as an  $\text{Out}(G)$ -set. The splitting is due to the quite different methods used here and in [S7]. A third part will focus on applications to the McKay conjecture [S8].

In order to be more specific about intermediate goals and results, let us introduce some notation. Let  $G = \mathbf{G}^F$  for  $F: \mathbf{G} \rightarrow \mathbf{G}$  be a Frobenius endomorphism of a simply connected simple algebraic group  $\mathbf{G}$ . Upon choosing an  $F$ -stable maximal torus and a Borel subgroup containing it, one can define a group  $E$  of so-called field and graph automorphisms of  $G$ . One can also define a reductive group  $\tilde{\mathbf{G}}$  realizing a regular embedding for  $\mathbf{G}$ , that is,  $\mathbf{G} = [\tilde{\mathbf{G}}, \tilde{\mathbf{G}}]$  with connected  $Z(\tilde{\mathbf{G}})$  and also assume that  $F$  extends to a Frobenius endomorphism of  $\tilde{\mathbf{G}}$

with  $E$  also acting on  $\tilde{G} := \tilde{\mathbf{G}}^F$ . Then  $\text{Aut}(G)$  is induced by the direct product  $\tilde{G} \rtimes E$  (see, e.g., [GLS, 2.5.12]).

The determination of the action of  $\tilde{G} \rtimes E$  on  $\text{Irr}(G)$  mostly relies on establishing that  $\tilde{G}$ -orbits and  $E$ -orbits are somehow transversal. More precisely, one aims at showing the following property:

$A'(\infty)$ : *There exists an  $E$ -stable  $\tilde{G}$ -transversal in  $\text{Irr}(G)$ .*

This, combined with the present knowledge of  $\text{Irr}(\tilde{G})$ , is enough to determine  $\text{Irr}(G)$  as an  $\text{Out}(G)$ -set (see [CS4, 2.5]). However, in order to deduce any valuable statement about representations of almost-simple groups, it is also important to answer extendibility questions. For instance, a difficult theorem of Lusztig essentially focusing on the case of type D shows that any element of  $\text{Irr}(G)$  extends to its stabilizer in  $\tilde{G}$  (see [L2], [L3]). This notably leads to the determination of the action of  $E$  on the set of  $\tilde{G}$ -orbits in  $\text{Irr}(G)$ .

The following strengthening of  $A'(\infty)$  was introduced in [S4] in order to check the inductive McKay condition for the defining characteristic.

$A(\infty)$ : *There exists an  $E$ -stable  $\tilde{G}$ -transversal  $\mathbb{T}$  in  $\text{Irr}(G)$  and any  $\chi \in \mathbb{T}$  extends to an irreducible character of its stabilizer  $G \rtimes E_\chi$ .*

The aim of the present paper and its sequel [S7] is to prove  $A(\infty)$  for  $G$  of type D and  ${}^2\text{D}$ . In the present paper,  $G$  will be indeed some  $D_{l,sc}(q)$  ( $l \geq 4$ ,  $q$  a power of an odd prime); the case of twisted types  ${}^2\text{D}$  will also be deduced in [S7].

Our main theorem here can be seen as showing that a putative counterexample to  $A(\infty)$  with minimal  $l$  still satisfies  $A'(\infty)$ .

**THEOREM A.** *Let  $G = D_{l,sc}(q)$  ( $l \geq 4$ ,  $q$  a power of an odd prime), and let  $\tilde{G}$  and  $E$  as above (see also Notation 2.2). If any  $D_{l',sc}(q)$  for  $4 \leq l' < l$  satisfies  $A(\infty)$ , then  $G$  satisfies  $A'(\infty)$ .*

More precisely, we assume Hypothesis 2.14, that is, that condition  $A(\infty)$  holds for the cuspidal characters of any  $G' = D_{l',sc}(q)$  with  $4 \leq l' < l$ .

Our proof uses as a starting point a theorem of Malle [Mal2] showing the existence part  $A'(\infty)$  of the above statement for cuspidal characters. Then, our strategy is through the parametrization of  $\text{Irr}(G)$  given by Harish-Chandra theory. In particular, we take the standard Levi supplement  $\mathbf{L}$  of an  $F$ -stable parabolic subgroup  $\mathbf{P}$  containing our chosen Borel subgroup and consider parabolic induction  $R_{\mathbf{L}}^{\mathbf{G}} \lambda$  of cuspidal characters  $\lambda \in \text{Irr}_{\text{cusp}}(\mathbf{L}^F)$ .

An essential ingredient of that parametrization is the deep theorem by Lusztig and Geck (see [L 1, 8.6] and [G, Cor. 2]) that any  $\lambda \in \text{Irr}_{\text{cusp}}(\mathbf{L}^F)$  extends to its stabilizer in  $N := N_{\mathbf{G}}(\mathbf{L})^F$ . In order to put that parametrization to use for our purpose of tracking automorphism actions, it is important to find an equivariant version of that statement. This does not seem possible from the available proofs, so we devise a new one in this paper, showing namely with the same notation for  $G = D_{l,sc}(q)$ ,  $\tilde{G}$ ,  $\mathbf{L}$ ,  $N$ ,  $E$ .

**THEOREM B.** *Let  $\lambda \in \text{Irr}_{\text{cusp}}(\mathbf{L}^F)$ . Assume Hypothesis 2.14 holds for  $D_{l',sc}(q)$  ( $4 \leq l' < l$ ). Then some  $(Z(\tilde{\mathbf{G}})\mathbf{L})^F$ -conjugate  $\lambda_0$  of  $\lambda$  has an  $(NN_E(\mathbf{L}))_{\lambda_0}$ -stable extension to  $N_{\lambda_0}$ .*

Studying the group structure of  $N$ , our proof uses essentially the Steinberg relations for the structure of  $G$ , not its realization as spin group, making probably more uniform a

case-by-case but effective proof of Geck–Lusztig’s theorem for other quasisimple groups of Lie type (see [BS, 4.3] and [CSS, 4.13] for types A and C).

We should point out that the above extendibility property is part of the following wider problem where  $(\mathbf{H}, F)$  is a reductive group defined over a finite field and  $F$  is its associated Frobenius endomorphism.

(P) *Let  $\mathbf{S}$  be an  $F$ -stable torus of  $\mathbf{H}$ . Does every  $\psi \in \text{Irr}(C_{\mathbf{H}}(\mathbf{S})^F)$  extend to its stabilizer in  $N_{\mathbf{H}}(\mathbf{S})^F$ ?*

This was answered in the affirmative in the case where  $\mathbf{S}$  is a Sylow  $d$ -torus ( $d \geq 1$ ) in the sense of [MT, 25.6] (see [S1], [S2], [S3]). Lusztig’s theorem on the case where  $\mathbf{S}$  is split and  $\psi$  is cuspidal was important in [L1] to turn Deligne–Lusztig theory into a parametrization of  $\text{Irr}(\mathbf{H}^F)$  when  $\mathbf{H}$  has connected center. It seems that even partial answers to (P) have quite interesting applications (see also [B1, §15] and [Mal1, 2.9]).

Let for now  $\text{Irr}_{\text{cusp}}(N)$  be the set of characters of  $N$  whose restriction to  $L$  is a sum of cuspidal characters. Theorem B then can be seen as the starting point of a specific parametrization of  $\text{Irr}_{\text{cusp}}(N)$  bearing similarities with the parametrizations of characters of normalizers of Sylow  $d$ -tori given in the author’s work just mentioned but with a special emphasis on outer automorphism actions.

Through preparations gathered at the start of the paper and similar to a method developed in [MS] where  $\mathbf{L}$  was a torus, our main goal Theorem A reduces to a weak analogue of it for  $\text{Irr}_{\text{cusp}}(N)$ . This is Theorem 6.1. It is checked through a strategy prescribed by Clifford theory. In particular, this entails a quite detailed analysis of the relative Weyl groups

$$W(\lambda) := N_{\lambda}/\mathbf{L}^F$$

and their various embeddings related to  $\tilde{G}$  and  $E$ .

### 1.1 Structure of the paper.

In §2, we recall notation on quasisimple groups of Lie type, their automorphisms, and the conditions  $A(\infty)$  and  $A'(\infty)$ . Then we collect the basic facts about cuspidal characters and Harish-Chandra theory for finite groups of Lie type. This leads to Theorem 2.8, which sums up the methods from [MS] to establish condition  $A(\infty)$  through Harish-Chandra theory. This is roughly the road map for the rest of the paper, in particular splitting the task into two halves that will be addressed in §§3 and 4 and §§6 and 7.

The rest of the paper is specific to type D (untwisted) in odd characteristic. After recalling a method from [CSS] for constructing extensions, the main objective of §§3 and 4 is Theorem B. Section 3 is a description of certain group theoretical aspects of the groups  $L := \mathbf{L}^F$  and  $N$ , using also the classic embedding  $\mathbf{G} \leq \overline{\mathbf{G}}$  of type  $D_l$  into type  $B_l$ . The root system  $\Phi'$  of  $\mathbf{L}$  is the direct product of irreducible root systems of types A and D. Roughly speaking, the factors of type  $A_{d-1}$  form a root system  $\Phi_d$  and the factor of type D gives  $\Phi_{-1}$ . Along the way, we introduce a set  $\mathbb{D}$  determining the types occurring as factors of  $\Phi'$ . This description will be used in the whole paper. For each  $d \in \mathbb{D}$ , we describe a normal inclusion  $H_d \triangleleft V_d \leq \overline{N} := N_{\overline{\mathbf{G}}}(\mathbf{L})^F$ , where  $H_d = L \cap V_d$  is an elementary abelian 2-group and  $L \langle V_d \mid d \in \mathbb{D} \rangle = \overline{N}$ . This normal inclusion  $H_d \triangleleft V_d$  concentrates the equivariant extendibility problem we have to solve.

In §4, we draw the consequences of the structure of  $N$  in terms of characters. One has to take care of all the factors involved and deal with the inclusion in type B, which provides

the graph automorphism specific to type D. Concerning the diagonal automorphisms, we avoid choosing a regular embedding  $\tilde{G}$  and instead consider inclusions  $L \triangleleft \mathcal{L}^{-1}(Z) \cap \mathbf{L}$  where  $Z \leq Z(\mathbf{G})$  and  $\mathcal{L}$  is the Lang map  $x \mapsto x^{-1}F(x)$  on  $\mathbf{G}$ .

Theorem B being proved, we study in §5 how automorphisms act on cuspidal characters in types A and D, making use in the latter case of Malle’s theorem [Mal2] mentioned above and some results about semisimple characters already used in the study of the McKay conjecture for the defining characteristic (see [Mas, §8]).

In §6, the most technical of the paper, the objective is to prove Theorem 6.1, showing that  $\text{Irr}_{\text{cusp}}(N)$  satisfies a version of  $A(\infty)$ . As already shown in §2, this translates into requirements on  $\text{Irr}(N_\lambda/L)$ , the characters of the relative Weyl group  $W(\lambda)$  associated with a cuspidal character  $\lambda$  of  $L$ . The comparison of the action of diagonal versus graph-field automorphisms on  $\text{Irr}_{\text{cusp}}(N)$  relates with the induced action of related characters of relative Weyl groups. The proof splits naturally into the various cases for the stabilizer of  $\lambda$  in  $L \cap \mathcal{L}^{-1}(Z(\mathbf{G}))/L$ . This leads to Propositions 6.28 and 6.35 describing the situation in the two main cases. In the proofs, graph-field automorphisms are taken care of by embedding the relative Weyl group  $W(\lambda)$  into overgroups  $K(\lambda)$  and  $\hat{K}(\lambda)$  (see Notation 6.4) for field automorphisms and the embedding into type B for the graph automorphism of order 2.

In §7, we essentially put together all the material of the preceding section to establish Theorem 6.1 and with some extra effort Theorem A.

**§2. Basic considerations**

We first gather here some notation around characters, recall Condition  $A(\infty)$ , and give a rephrasing that provides alternative approaches for the proof of Theorem A. In §2.2, we collect relevant results from Harish-Chandra theory. We conclude with general considerations on cuspidal characters in §2.3.

**2.1 Notation and Condition  $A(\infty)$**

In general, we follow the notation about characters as introduced in [I]. Additionally, we use some terminology from [S1], [S2], [S3] that is recalled in the following paragraph.

NOTATION 2.1. Let  $X \triangleleft Y$  be finite groups, and let  $\mathbb{T} \subseteq \text{Irr}(X)$ . An *extension map with respect to  $X \triangleleft Y$*  for  $\mathbb{T}$  is a map  $\Lambda : \mathbb{T} \rightarrow \coprod_{X \leq I \leq Y} \text{Irr}(I)$  such that every  $\lambda \in \mathbb{T}$  is mapped to an extension of  $\lambda$  to  $Y_\lambda$ , the inertia subgroup of  $\lambda$  in  $Y$ . We say that *maximal extendibility holds with respect to  $X \triangleleft Y$*  for  $\mathbb{T}$  if such an extension map exists (see also [CS2, Def. 5.7]). In such a case, the map can be chosen  $Y$ -equivariant, provided  $\mathbb{T}$  is  $Y$ -stable (see [CS2, Th. 4.1]). Whenever  $\mathbb{T} = \text{Irr}(X)$ , we omit to mention  $\mathbb{T}$ . For  $\lambda \in \text{Irr}(X)$  and  $\psi \in \text{Irr}(Y)$ , we write  $\lambda^Y$  for the character induced to  $Y$  and  $\psi \upharpoonright_X$  for the restricted character. For any generalized character  $\kappa$ , we denote by  $\text{Irr}(\kappa)$  the set of (irreducible) constituents of  $\kappa$ . If  $\sigma \in \text{Aut}(X)$  and  $\lambda \in \text{Irr}(X)$ , we write  $\lambda^\sigma = \sigma^{-1} \lambda$  for the character with  $\sigma^{-1} \lambda(x) = \lambda^\sigma(x) = \lambda(\sigma(x))$  for  $x \in X$ .

If two subgroups  $H_1, H_2 \leq Y$  satisfy  $[H_1, H_2] = 1$ , and  $\lambda_i \in \text{Irr}(H_i)$  for  $i = 1, 2$  with  $\text{Irr}(\lambda_1 \upharpoonright_{H_1 \cap H_2}) = \text{Irr}(\lambda_2 \upharpoonright_{H_1 \cap H_2})$ , then there exists a unique character  $\phi \in \text{Irr}(\langle H_1, H_2 \rangle)$  with  $\text{Irr}(\phi \upharpoonright_{H_i}) = \{\lambda_i\}$  according to [IMN, §5] and we write  $\lambda_1 \cdot \lambda_2$  for this character. Let  $\mathbb{I}$  be a finite set, and let  $Z, H$ , and  $H_i$  ( $i \in \mathbb{I}$ ) be finite groups with  $Z \leq H_i \leq H$ . If  $[H_i, H_{i'}] = 1$ , for every  $i, i' \in \mathbb{I}$  with  $i \neq i'$  and  $H_i \cap \langle H_j \mid j \in \mathbb{I} \setminus \{i\} \rangle = Z$ , we consider  $\langle H_i \mid i \in \mathbb{I} \rangle \leq H$  the central product of the groups  $H_i$ . Given  $\nu \in \text{Irr}(Z)$  and  $\lambda_i \in \text{Irr}(H_i \mid \nu)$ , we denote by

$\bigcirc_{i \in \mathbb{I}} \lambda_i \in \text{Irr}(\langle H_i \mid i \in \mathbb{I} \rangle)$  the character  $\phi \in \text{Irr}(\langle H_i \mid i \in \mathbb{I} \rangle)$  with  $\text{Irr}(\phi|_{H_i}) = \{\lambda_i\}$  for every  $i \in \mathbb{I}$  (see also [IMN, §5]).

Next, we introduce the groups and automorphisms considered in the following.

NOTATION 2.2 (Simple groups of Lie type). Let  $\mathbf{G}$  be a simple linear algebraic group of simply connected type over an algebraic closure  $\mathbb{F}$  of  $\mathbb{F}_p$  for  $p$  a prime. Additionally, let  $F : \mathbf{G} \rightarrow \mathbf{G}$  be a Frobenius endomorphism defining an  $\mathbb{F}_q$ -structure on  $\mathbf{G}$  for  $q$ , a power of  $p$ . The automorphisms of  $\mathbf{G}^F$  are restrictions to  $\mathbf{G}^F$  of bijective endomorphisms of  $\mathbf{G}$  commuting to  $F$  (see [GLS, §1.15]), so it makes sense to consider stabilizers  $\text{Aut}(\mathbf{G}^F)_{\mathbf{H}}$  for  $F$ -stable subgroups  $\mathbf{H} \leq \mathbf{G}$ . Let  $\mathbf{T}_0$  be an  $F$ -stable maximally split torus, and let  $\mathbf{B}$  be an  $F$ -stable Borel subgroup of  $\mathbf{G}$  with  $\mathbf{T}_0 \subseteq \mathbf{B}$  and  $\mathbf{N}_0 := \mathbf{N}_{\mathbf{G}}(\mathbf{T}_0)$ . According to [MT, Th. 24.11], the group  $G := \mathbf{G}^F$  has a split  $BN$ -pair with respect to  $B := \mathbf{B}^F$ ,  $T_0 := \mathbf{T}_0^F$ , and  $N_0 := \mathbf{N}_0^F$ . Let  $E(\mathbf{G}^F)$ , often just  $E$ , be the subgroup of  $\text{Aut}(\mathbf{G}^F)_{(\mathbf{B}, \mathbf{T}_0)}$  generated by the restrictions to  $\mathbf{G}^F$  of graph automorphisms and some Frobenius endomorphism  $F_0$  stabilizing  $\mathbf{T}_0$  and  $\mathbf{B}$  as in [GLS, Th. 2.5.1] and [CS4, §2.A].

Let  $\mathbf{G} \leq \tilde{\mathbf{G}}$  be a regular embedding, that is, a closed inclusion of algebraic groups with  $\tilde{\mathbf{G}} = \mathbf{Z}(\tilde{\mathbf{G}})\mathbf{G}$  and connected  $\mathbf{Z}(\tilde{\mathbf{G}})$ . Then  $\tilde{\mathbf{T}}_0 := \mathbf{Z}(\tilde{\mathbf{G}})\mathbf{T}_0$  is a maximal torus of  $\tilde{\mathbf{G}}$ . Let  $\tilde{T}_0 := \tilde{\mathbf{T}}_0^F$ . Assume that  $F : \tilde{\mathbf{G}} \rightarrow \tilde{\mathbf{G}}$  is a Frobenius endomorphism extending the one of  $\mathbf{G}$  (see also [MS, §2]). Then  $\tilde{\mathbf{G}}^F$  has again a split  $BN$ -pair with respect to the groups  $\tilde{B} := \tilde{T}_0 B$  and  $\tilde{N}'_0 := \tilde{T}_0 N_0$  (see [MT, Th. 24.11]). Often the action of  $\tilde{N}'_0$  on  $\mathbf{G}$  will be studied via the group  $\tilde{N}_0 := \{x \in \mathbf{N}_{\mathbf{G}}(\mathbf{T}_0) \mid x^{-1}F(x) \in \mathbf{Z}(\mathbf{G})\}$ , which will be shown to induce the same automorphisms on  $\mathbf{G}$  (see Remark 2.16).

Via the convention given in [MS, §2],  $E(\mathbf{G}^F)$  also acts on  $\tilde{\mathbf{G}}^F$  and the semi-direct product  $\tilde{\mathbf{G}}^F \rtimes E(\mathbf{G}^F)$  induces on  $\mathbf{G}^F$  the whole automorphism group  $\text{Aut}(\mathbf{G}^F)$ .

We recall the conditions  $A(\infty)$  and  $A'(\infty)$  from [CS4, Def. 2.2].

CONDITION 2.3 (On stabilizers of irreducible characters of  $\mathbf{G}^F$ ).

$A(\infty)$ : There exists some  $E$ -stable  $\tilde{\mathbf{G}}^F$ -transversal  $\mathbb{T}$  in  $\text{Irr}(\mathbf{G}^F)$ , such that every  $\chi \in \mathbb{T}$  extends to  $\mathbf{G}^F E_{\chi}$ .

$A'(\infty)$ : There exists some  $E$ -stable  $\tilde{\mathbf{G}}^F$ -transversal  $\mathbb{T}$  in  $\text{Irr}(\mathbf{G}^F)$ .

Condition  $A'(\infty)$  implies a weak version of [S4, Assum. 2.12(v)].

LEMMA 2.4. Let  $\tilde{Y}$  and  $\tilde{X}$  be two subgroups of a group  $Z$  with  $\tilde{X} \triangleleft Z$  and  $Z = \tilde{Y}\tilde{X}$ . For  $X := \tilde{X} \cap \tilde{Y}$ , let  $\mathcal{M} \subseteq \text{Irr}(X)$  be  $Z$ -stable. Then the following are equivalent:

- (i) There is a  $\tilde{Y}$ -stable  $\tilde{X}$ -transversal  $\mathcal{M}_0$  in  $\mathcal{M}$ .
- (ii) Every  $\zeta' \in \mathcal{M}$  is  $\tilde{X}$ -conjugate to some  $\zeta$  such that  $(\tilde{X}\tilde{Y})_{\zeta} = \tilde{X}_{\zeta}\tilde{Y}_{\zeta}$ .
- (iii) Every  $\zeta' \in \mathcal{M}$  satisfies  $(\tilde{X}\tilde{Y})_{\zeta'} = (\tilde{Y}^x)_{\zeta'}\tilde{X}_{\zeta'}$  for some  $x \in \tilde{X}$ .

*Proof.* This follows from [CSS, Rem. 3.3]. □

### 2.2 Action of $\text{Aut}(\mathbf{G})$ on Harish-Chandra-induced characters

Using a detailed analysis of Harish-Chandra induction, the results of [MS] describe the action of  $\text{Aut}(\mathbf{G}^F)$  in terms of cuspidal characters and their relative Weyl groups. The action is expressed in terms of the labels given by Howlett–Lehrer theory.

NOTATION 2.5. Let  $L$  be a standard Levi subgroup of  $G$  with respect to  $B$  and  $T_0$ , that is,  $L = \mathbf{L}^F$  for some standard Levi subgroup  $\mathbf{L}$  of  $\mathbf{G}$  such that  $\mathbf{T}_0 \leq \mathbf{L}$  and  $\mathbf{L}\mathbf{B}$  is an

$F$ -stable parabolic subgroup. We set  $N := N_{\mathbf{G}}(\mathbf{L})^F$ ,  $W := N/L$ , and we abbreviate

$$E_L = E(\mathbf{G}^F)_{\mathbf{L}}.$$

We write  $\text{Irr}_{\text{cusp}}(L)$  for the set of cuspidal characters of  $L$  as defined in [C, 9.1] and  $\text{Irr}_{\text{cusp}}(N) := \bigcup_{\lambda \in \text{Irr}_{\text{cusp}}(L)} \text{Irr}(\lambda^N)$ . Let us denote by  $R_L^G$  the Harish-Chandra induction from  $L$  to  $G$ . For  $\lambda \in \text{Irr}_{\text{cusp}}(L)$ , let

$$\text{Irr}(G \mid (L, \lambda)) := \text{Irr}(R_L^G(\lambda))$$

(sometimes denoted as  $\mathcal{E}(G, (L, \lambda))$  in the literature). Let also  $\text{Irr}(G \mid (L, \mathbb{T})) := \bigcup_{\lambda \in \mathbb{T}} \text{Irr}(G \mid (L, \lambda))$  for  $\mathbb{T} \subseteq \text{Irr}_{\text{cusp}}(L)$ .

2.6. Let  $\text{Aut}(\mathbf{G}^F)_{L, \text{HC}}$  be the subgroup of  $\text{Aut}(\mathbf{G}^F)$  generated by the automorphisms of  $\mathbf{G}^F$  induced by  $N$  and  $\text{Aut}(\mathbf{G}^F)_{(\mathbf{B}\mathbf{L}, \mathbf{L})}$ . Note  $E_L \leq \text{Aut}(\mathbf{G}^F)_{L, \text{HC}}$ . According to Howlett–Lehrer theory (see [C, §10]), fixing an extension  $\tilde{\lambda} \in \text{Irr}(N_\lambda)$  of  $\lambda \in \text{Irr}_{\text{cusp}}(L)$  defines a unique labeling of  $\text{Irr}(G \mid (L, \lambda))$  by  $\text{Irr}(W(\lambda))$  where  $W(\lambda) := N_\lambda/L$ . We write  $R_L^G(\lambda)_\eta$  for the character of  $\text{Irr}(G \mid (L, \lambda))$  associated with  $\eta \in \text{Irr}(W(\lambda))$  via the extension  $\tilde{\lambda}$ .

Accordingly, the parametrization of  $\text{Irr}(G \mid (L, \text{Irr}_{\text{cusp}}(L)))$  depends on an extension map  $\Lambda_L$  with respect to  $L \triangleleft N$  for  $\text{Irr}_{\text{cusp}}(L)$ . For  $\lambda \in \text{Irr}_{\text{cusp}}(L)$ , let  $R(\lambda) \triangleleft W(\lambda)$  be defined as in [C, Prop. 10.6.3]. If  $\lambda \in \text{Irr}_{\text{cusp}}(L)$  and  $\sigma \in \text{Aut}(\mathbf{G}^F)_{L, \text{HC}}$ , let  $\delta_{\lambda, \sigma}$  be the unique linear character of  $W(\sigma\lambda)$  satisfying

$$\sigma \Lambda_L(\lambda) = \Lambda_L(\sigma\lambda) \delta_{\lambda, \sigma}. \tag{2.1}$$

We only use the formula with some simplifying assumptions on  $R(\lambda)$  and  $\delta_{\lambda, \sigma}$ .

**THEOREM 2.7** (Malle–Späth [MS, Ths. 4.6 and 4.7]). *Let  $\sigma \in \text{Aut}(\mathbf{G}^F)_{L, \text{HC}}$  and  $\Lambda_L$  be an  $N$ -equivariant extension map with respect to  $L \triangleleft N$  for  $\text{Irr}_{\text{cusp}}(L)$ . Assume that  $R_L^G(\lambda)_\eta$  ( $\lambda \in \text{Irr}_{\text{cusp}}(L)$ ,  $\eta \in \text{Irr}(W(\lambda))$ ) is defined using  $\Lambda_L$  and*

$$R(\sigma\lambda) \leq \ker(\delta_{\lambda, \sigma}) \quad \text{for every } \lambda \in \text{Irr}_{\text{cusp}}(L). \tag{2.2}$$

*Then  $\sigma(R_L^G(\lambda)_\eta) = R_L^G(\sigma\lambda)_{\sigma\eta\delta_{\lambda, \sigma}^{-1}}$  for every  $\lambda \in \mathbb{T}$  and  $\eta \in \text{Irr}(W(\lambda))$ .*

In §5 of [MS], the analog of Theorem A was proved for characters in  $\text{Irr}(G \mid (T_0, \text{Irr}_{\text{cusp}}(T_0)))$  by studying  $\text{Irr}_{\text{cusp}}(N_{\mathbf{G}}(\mathbf{T}_0)^F)$ . For other standard Levi subgroups, the strategy from [MS] leads naturally to the following statement where we focus on a single  $L$  and its stabilizer in  $E$ . Sections 3–6 will ensure the assumptions for the groups from Notation 2.2 whenever  $\mathbf{G}^F = D_{l, \text{sc}}(q)$ .

**THEOREM 2.8.** *Let  $\tilde{L}' := \tilde{T}_0 L$ ,  $\tilde{N}' := \tilde{T}_0 N$  and  $\hat{N} := N E_L$ . Assume that there exist:*

- (i) *an  $\hat{N}$ -stable  $\tilde{L}'$ -transversal  $\mathbb{T}$  in  $\text{Irr}_{\text{cusp}}(L)$ , an  $N$ -equivariant extension map  $\Lambda_{L, \mathbb{T}}$  with respect to  $L \triangleleft N$  for  $\mathbb{T}$  such that any  $\lambda \in \mathbb{T}$  satisfies Equation (2.2); and*
- (ii) *some  $E_L$ -stable  $\tilde{N}'$ -transversal in  $\text{Irr}_{\text{cusp}}(N)$ .*

*Then there exists an  $E_L$ -stable  $\tilde{\mathbf{G}}^F$ -transversal in  $\text{Irr}(\mathbf{G}^F \mid (L, \text{Irr}_{\text{cusp}}(L)))$ .*

For the proof of Theorem 2.8, we parametrize  $\text{Irr}_{\text{cusp}}(N)$  via a set  $\mathcal{P}(L)$  using an extension map  $\Lambda_L$  with respect to  $L \triangleleft N$  for  $\text{Irr}_{\text{cusp}}(L)$  deduced from  $\Lambda_{L, \mathbb{T}}$ .

NOTATION 2.9. Assume that  $\mathbb{T}$  is an  $\widehat{N}$ -stable  $\widetilde{L}'$ -transversal in  $\text{Irr}_{\text{cusps}}(L)$ . For each  $\lambda \in \mathbb{T}$ , we denote by  $\mathcal{O}_\lambda$  its  $N$ -orbit in  $\text{Irr}_{\text{cusps}}(L)$ . Note  $\mathcal{O}_\lambda \subseteq \text{Irr}_{\text{cusps}}(L)$ . Let  $M(\lambda) \subseteq \widetilde{L}'$  be a set of representatives of the  $\widetilde{L}'_\lambda$ -cosets in  $\widetilde{L}'$ . We define an extension map  $\Lambda_L$  on  $\mathcal{O}_\lambda$  by

$$\Lambda_L(\lambda'^m) = \Lambda_{L,\mathbb{T}}(\lambda')^m \quad \text{for every } \lambda' \in \mathcal{O}_\lambda \text{ and } m \in M(\lambda).$$

Hence,  $\Lambda_L$  is defined, but depends on the choice of  $M(\lambda)$ . The map  $\Lambda'_L : \text{Irr}_{\text{cusps}}(L) \rightarrow \prod_{L \leq I \leq N} \text{Irr}(I)$  with  $\Lambda'_L(\mu) := \Lambda_L(\mu)|_{N_{\widetilde{\mu}}}$  for every  $\mu \in \text{Irr}_{\text{cusps}}(L)$  is well defined, where  $\widetilde{\mu} \in \text{Irr}(\widetilde{L}'_\mu)$  is an extension of  $\mu$ . In contrast to  $\Lambda_L$ , we see that  $\Lambda'_L$  is independent of the choice of  $M(\lambda)$ . Observe that  $[N/L, \widetilde{L}'/L] = 1$ . The map  $\Lambda'_L$  is even  $\widehat{N}\widetilde{L}'$ -equivariant since  $\Lambda_L$  is  $N$ -equivariant and  $\Lambda_{L,\mathbb{T}}$  is  $\widehat{N}$ -equivariant.

We write  $\mathcal{P}'(L)$  for the set of pairs  $(\lambda, \eta)$  with  $\lambda \in \text{Irr}_{\text{cusps}}(L)$  and  $\eta \in \text{Irr}(W(\lambda))$ . The groups  $N$  and  $W$  act naturally via conjugation on  $\mathcal{P}'(L)$ . We denote by  $\mathcal{P}(L)$  the set of  $N$ -orbits in  $\mathcal{P}'(L)$  and by  $\overline{(\lambda, \eta)}$  the  $N$ -orbit containing  $(\lambda, \eta)$ . Since  $L$  is mostly clear from the context, we omit it, writing  $\mathcal{P}'$  and  $\mathcal{P}$ .

The parametrization of  $\text{Irr}_{\text{cusps}}(N)$  is given by the following.

PROPOSITION 2.10. *Let  $\Lambda_L$ ,  $\mathcal{P}'$ , and  $\mathcal{P}$  be as in 2.8 and 2.9.*

(a) *Then the map*

$$\Upsilon : \mathcal{P} \longrightarrow \text{Irr}_{\text{cusps}}(N) \text{ with } \overline{(\lambda, \eta)} \longmapsto (\Lambda_L(\lambda)\eta)^N$$

*is a well-defined bijection.*

(b)  *$\sigma\Upsilon(\overline{(\lambda, \eta)}) = \Upsilon(\overline{(\sigma\lambda, \sigma\eta\delta_{\lambda, \sigma})})$  for every  $\sigma \in \text{Aut}(G)_{L, \text{HC}}$  and  $\overline{(\lambda, \eta)} \in \mathcal{P}$ , where  $\delta_{\lambda, \sigma} \in \text{Irr}(W(\sigma\lambda))$  is as given in 2.6.*

*Proof.* Clifford theory together with Gallagher’s lemma [I, 6.17] proves part (a). The definition of  $\delta_{\lambda, \sigma}$  in Equation (2.1) from 2.6 leads to part (b). □

In combination with Theorem 2.7, we obtain a proof of Theorem 2.8.

*Proof of Theorem 2.8.* For the application of Theorem 2.7, we have to ensure that under our assumptions, Equation (2.2) holds for characters  $\lambda \in \mathbb{T}$  and  $\sigma \in \text{Aut}(\mathbf{G}^F)_{L, \text{HC}}$ . For every  $\lambda \in \text{Irr}_{\text{cusps}}(L)$ , the character  $\Lambda_L(\lambda)$  is an extension of  $\Lambda'_L(\lambda)$ . Accordingly,  $\delta_{\lambda, \sigma}$  defined as the unique linear character of  $W(\sigma\lambda)$  such that  $\sigma\Lambda_L(\lambda) = \Lambda_L(\sigma\lambda)\delta_{\lambda, \sigma}$  satisfies as well  $\sigma\Lambda_L(\lambda)|_{N_{\widetilde{\lambda}\sigma}} = \Lambda_L(\sigma\lambda)\delta_{\lambda, \sigma}|_{N_{\widetilde{\lambda}\sigma}}$ . Since  $\Lambda'_L(\lambda)$  is  $\widetilde{N}'E_L$ -equivariant, we see that  $\delta_{\lambda, \sigma}|_{N_{\widetilde{\lambda}\sigma}}$  is trivial. Accordingly,  $\ker(\delta_{\lambda, \sigma}) \geq N_{\sigma\widetilde{\lambda}}/L$  for every  $\lambda \in \text{Irr}_{\text{cusps}}(L)$  and  $\sigma \in \text{Aut}(\mathbf{G}^F)_{L, \text{HC}}$  where  $\widetilde{\lambda}$  denotes an extension of  $\lambda$  to  $\widetilde{L}'_\lambda$ . Recall  $W(\sigma\widetilde{\lambda}) = N_{\sigma\widetilde{\lambda}}/L$ . In combination with the inclusion  $R(\sigma\lambda) \leq W(\sigma\widetilde{\lambda})$  from [CSS, Lem. 4.14], we obtain the required containment (2.2).

Via Harish-Chandra induction, the map

$$\Upsilon' : \mathcal{P} \longrightarrow \text{Irr}(G | (L, \text{Irr}_{\text{cusps}}(L))) \text{ with } \overline{(\lambda, \eta)} \longmapsto R_L^G(\lambda)_\eta$$

is well defined according to [MS, Th. 4.7] and bijective. Hence,  $\Upsilon' \circ \Upsilon^{-1}$  is a bijection between  $\text{Irr}_{\text{cusps}}(N)$  and  $\text{Irr}(G | (L, \text{Irr}_{\text{cusps}}(L)))$ . Via  $\Upsilon$  and  $\Upsilon'$ , the group  $\text{Aut}(\mathbf{G}^F)_{L, \text{HC}}$  and hence  $\widetilde{N}'E_L$  act on  $\mathcal{P}$ . By the description of this action given in Theorem 2.7 and Proposition 2.10, these actions coincide. Hence,  $\Upsilon' \circ \Upsilon^{-1}$  is  $\widetilde{N}'E_L$ -equivariant. By Assumption 2.8(ii), every  $\psi_0 \in \text{Irr}_{\text{cusps}}(N)$  has an  $\widetilde{L}'$ -conjugate  $\psi$  such that  $(\widetilde{N}'E_L)_\psi = \widetilde{N}'_\psi(E_L)_\psi$ . Hence, every  $\chi_0 \in \text{Irr}(G | (L, \text{Irr}_{\text{cusps}}(L)))$  has an  $\widetilde{N}'$ -conjugate  $\chi$  with  $(\widetilde{\mathbf{G}}^F E_L)_\chi = G(\widetilde{N}'E_L)_\chi = G(\widetilde{N}'_\chi)(E_L)_\chi = \mathbf{G}_\chi^F(E_L)_\chi$ . This implies the statement (see Lemma 2.4). □



In the following sections, we verify the assumptions of Theorem 2.8: We prove Assumption 2.8(ii), that is, that every  $\psi \in \text{Irr}_{\text{cusp}}(N)$  is  $\tilde{L}'$ -conjugate to some  $\psi_0$  with  $(\tilde{N}'E_L)_{\psi_0} = \tilde{N}'_{\psi_0}(E_L)_{\psi_0}$ , and prove the existence of an extension map as required in Assumption 2.8(i). Note that by Lusztig [L1] and Geck [G], an extension map exists. Their proofs are indirect, and we do not see how the required properties can be deduced from their proofs. In later sections, we give an independent explicit construction of the required extension map.

### 2.3 Action on characters of normalizers of Levi subgroups

In the following, we discuss some basic considerations that will be applied to ensure Assumption 2.8(ii). In the case where  $L = T_0$ , Assumption 2.8(ii) holds, whenever the underlying group  $\mathbf{G}^F$  is of simply connected type (see [MS, Proof of Cor. 5.3]). The assumption on the characters  $\text{Irr}_{\text{cusp}}(N)$  is very similar to the results [CS2, Prop. 5.13], [CS3, Th. 5.1], and [CS4, 5.E] on  $\text{Irr}(N_{\mathbf{H}}(\mathbf{S})^F)$  for Sylow  $\Phi_d$ -tori  $\mathbf{S}$  of  $(\mathbf{H}, F)$ , where  $\mathbf{H}$  is a simple simply connected group of type different from  $D_l$  and  $d$  is a positive integer. The proof there relies on [Th. 4.3], and we use here a similar strategy. The following proposition gives the road map for the verification of Assumption 2.8(ii).

We set  $W(\phi) = N_{\phi}/L$  for every  $L \leq M \leq \tilde{T}_0L$  and  $\phi \in \text{Irr}(M)$ .

PROPOSITION 2.11. *Let  $\hat{N}$ ,  $\tilde{L}' = \tilde{T}_0L$  be as in Theorem 2.8,  $\mathbb{T}$  and  $\Lambda_{L,\mathbb{T}}$  as in Assumption 2.8(i), and  $\Upsilon$  from Proposition 2.10. Let  $\lambda \in \mathbb{T}$ ,  $\tilde{\lambda} \in \text{Irr}(\tilde{L}'_{\lambda} | \lambda)$ ,  $\eta \in \text{Irr}(W(\lambda))$ , and  $\eta_0 \in \text{Irr}(\eta|_{W(\tilde{\lambda})})$ . We set  $\widehat{W} := \hat{N}/L = NE_L/L$  and  $\widehat{K}(\lambda) := \widehat{W}_{\lambda}$ . If  $\eta$  is  $\widehat{K}(\lambda)_{\eta_0}$ -stable, then*

$$(\widehat{N}\tilde{L}')_{\Upsilon(\overline{(\lambda, \eta)})} = \widehat{N}_{\Upsilon(\overline{(\lambda, \eta)})}\tilde{L}'_{\Upsilon(\overline{(\lambda, \eta)})}.$$

We adapt the arguments from the proof of [CS3, Th. 4.3], where  $\eta$  is assumed to be  $N_{W \rtimes E_L}(W(\tilde{\lambda}))_{\eta_0}$ -stable. Note that  $\widehat{K}(\lambda)$  normalizes  $W(\lambda)$ , but this group is in general different from  $N_{W \rtimes E_L}(W(\tilde{\lambda}))$ .

*Proof.* Recall  $\psi = \Upsilon(\overline{(\lambda, \eta)}) = (\Lambda_{L,\mathbb{T}}(\lambda)\eta)^N$ . By the assumptions on  $\mathbb{T}$ ,  $(\widehat{N}\tilde{L}')_{\lambda} = \widehat{N}_{\lambda}\tilde{L}'_{\lambda}$  for every  $\lambda \in \mathbb{T}$ .

Let  $\tilde{\lambda} \in \text{Irr}(\tilde{L}'_{\lambda} | \lambda)$  and  $\eta_0 \in \text{Irr}(\eta|_{W(\tilde{\lambda})})$ . According to [CE, 15.11],  $\tilde{\lambda}$  is an extension of  $\lambda$ . The group  $\tilde{L}'_{\lambda}/(LZ(\tilde{G}))$  acts by multiplication with linear characters of  $W(\lambda)/W(\tilde{\lambda})$  on  $\text{Irr}(W(\lambda) | \eta_0)$ . Computing the action of  $W(\lambda)/W(\tilde{\lambda})$  on  $\text{Irr}(\tilde{L}'_{\lambda} | \lambda)$  shows that the action of  $\tilde{L}'_{\lambda}/L$  on  $\text{Irr}(W(\lambda) | \eta_0)$  is transitive. Hence, the characters  $\{(\Lambda_L(\lambda)\eta')^N | \eta' \in \text{Irr}(W(\lambda) | \eta_0)\}$  are the  $\tilde{L}'_{\lambda}$ -conjugates of  $\psi$ .

Let  $l \in \tilde{L}'$  and  $\hat{n} \in \hat{N}$  with  $\psi^l = (\psi)^{\hat{n}}$ . Note that  $\psi^{\hat{n}} \in \text{Irr}(N | \mathbb{T})$  since  $\mathbb{T}$  is  $\hat{N}$ -stable. Then  $\text{Irr}(\psi^l|_L)$  is the  $N$ -orbit of  $\lambda^l$ . Recall  $\mathbb{T}$  is an  $\tilde{L}'$ -transversal. If  $l \notin \tilde{L}'_{\lambda}$ , then  $\lambda^l \neq \lambda$  and  $\lambda^l \notin \mathbb{T}$ , in particular  $\psi^l \notin \text{Irr}(N | \mathbb{T})$ . This implies  $l \in \tilde{L}'_{\lambda}$  and  $\psi^l = (\Lambda_L(\lambda)\eta\nu)^N$  for some linear character  $\nu$  of  $\text{Irr}(W(\lambda)/W(\tilde{\lambda}))$ . Accordingly,  $(\psi)^{\hat{n}} \in \text{Irr}(N | \lambda)$  and hence  $(\psi)^{\hat{n}} = (\psi)^{\hat{n}'}$  for some  $\hat{n}' \in \widehat{N}_{\lambda}$ . Note that

$$(\psi)^{\hat{n}'} = ((\Lambda_L(\lambda)\eta)^{\hat{n}'})^N = (\Lambda_L(\lambda)\eta^{\hat{n}'})^N.$$

The equality  $\psi^l = \psi^{\hat{n}'}$  implies  $\eta^{\hat{n}'} = \eta\nu$  and hence  $\hat{n}'L \in W(\lambda)\widehat{K}(\lambda)_{\eta_0}$ . As  $\eta$  is  $\widehat{K}(\lambda)_{\eta_0}$ -stable,  $\eta^{\hat{n}'} = \eta$  and hence  $\psi^{\hat{n}'} = \psi$ . This shows  $(\widehat{N}\tilde{L}')_{\psi} = \widehat{N}_{\psi}\tilde{L}'_{\psi}$ .  $\square$

The above proposition allows us to prove the following result showing how to construct an  $\widehat{N}$ -stable  $\widetilde{L}'$ -transversal in  $\text{Irr}_{\text{cusp}}(N)$ .

PROPOSITION 2.12. *In the situation of Lemma 2.8, assume that:*

- (i)  $(\widehat{N}\widetilde{L}')_\lambda = \widehat{N}_\lambda \widetilde{L}'_\lambda$  for every  $\lambda \in \mathbb{T}$ ,
- (ii) there exists an  $\widehat{N}$ -equivariant extension map  $\Lambda_{L, \mathbb{T}}$  with respect to  $L \triangleleft N$  for  $\mathbb{T}$ , and
- (iii) for every  $\lambda \in \mathbb{T}$ ,  $\widetilde{\lambda} \in \text{Irr}(\widetilde{L}'_\lambda \mid \lambda)$ , and  $\eta_0 \in \text{Irr}(W(\widetilde{\lambda}))$ , there exists some  $\widehat{K}(\lambda)_{\eta_0}$ -stable  $\eta \in \text{Irr}(W(\lambda) \mid \eta_0)$ .

Let  $\overline{\mathbb{T}} \subseteq \text{Irr}_{\text{cusp}}(L)$  be the set of characters that are  $\widetilde{L}'$ -conjugate to one in  $\mathbb{T}$ . Then there exists some  $\widehat{N}$ -stable  $\widetilde{L}'$ -transversal in  $\text{Irr}(N \mid \overline{\mathbb{T}})$ .

*Proof.* By the assumptions, there exists  $\mathcal{P}_1 \subseteq \mathcal{P}$  such that:

- if  $(\overline{\lambda, \eta}) \in \mathcal{P}_1$ , then  $\lambda \in \mathbb{T}$  and  $\eta$  is  $\widehat{K}(\lambda)_{\eta_0}$ -stable for some  $\widetilde{\lambda} \in \text{Irr}(\widetilde{L}'_\lambda \mid \lambda)$  and  $\eta_0 \in \text{Irr}(\eta \upharpoonright_{W(\widetilde{\lambda})})$ ; and
- for each  $\lambda \in \mathbb{T}$ ,  $\widetilde{\lambda} \in \text{Irr}(\widetilde{L}'_\lambda \mid \lambda)$  and  $\eta_0 \in \text{Irr}(W(\widetilde{\lambda}))$ , there exists some  $\eta \in \text{Irr}(W(\lambda) \mid \eta_0)$  with  $(\overline{\lambda, \eta}) \in \mathcal{P}_1$ .

Proposition 2.11 tells us that the characters  $\Upsilon(\mathcal{P}_1)$  can form part of an  $E_L$ -stable  $\widetilde{L}'$ -transversal.

According to Proposition 2.10, for every  $\lambda \in \mathbb{T}$  and  $\eta_0 \in \text{Irr}(W(\widetilde{\lambda}))$ , the group  $\widetilde{L}'_\lambda$  acts transitively on the set  $\text{Irr}(W(\lambda) \mid \eta_0)$ . Since for each  $\lambda \in \mathbb{T}$  and  $\eta_0 \in \text{Irr}(W(\widetilde{\lambda}))$  there exists some  $\eta \in \text{Irr}(W(\lambda) \mid \eta_0)$  such that  $(\overline{\lambda, \eta}) \in \mathcal{P}_1$ , each  $\widetilde{L}'$ -orbit has a nonempty intersection with  $\Upsilon(\mathcal{P}_1)$ . This implies that every character in  $\Upsilon(\mathcal{P}_1)$  has the property required (see Lemma 2.4). □

### 2.4 Reminder on cuspidal characters

The considerations of §2.2 explain how the action of automorphisms on non-cuspidal characters depends on the underlying cuspidal character and a character of the relative Weyl group associated with a cuspidal pair. For the proof of Theorem A, we require some general results on cuspidal characters that we collect here. By a theorem of Malle, stabilizers of cuspidal characters coincide with those of semisimple characters (see [B1, 15.A] for a definition of semisimple characters).

THEOREM 2.13. *Let  $\mathbf{H}$  be a simply connected simple linear algebraic group with an  $\mathbb{F}_q$ -structure given by a Frobenius map  $F : \mathbf{H} \rightarrow \mathbf{H}$ . Let  $\mathbf{H} \rightarrow \widetilde{\mathbf{H}}$  be a regular embedding, and let  $E(\mathbf{H}^F)$  be a group of automorphisms of  $\mathbf{H}^F$  generated by graph and field automorphisms as in 2.2. Then there exists some  $E(\mathbf{H}^F)$ -stable  $\widetilde{\mathbf{H}}^F$ -transversal in  $\text{Irr}_{\text{cusp}}(\mathbf{H}^F)$ .*

*Proof.* We abbreviate  $E(\mathbf{H}^F)$  as  $E$ . Let  $\chi \in \text{Irr}_{\text{cusp}}(\mathbf{H}^F)$ . According to Lemma 2.4, it is sufficient to prove that  $\chi$  has some  $\widetilde{\mathbf{H}}^F$ -conjugate  $\chi_0$  with  $(\widetilde{\mathbf{H}}^F E)_{\chi_0} = \widetilde{\mathbf{H}}^F_{\chi_0} E_{\chi_0}$ . By [Mal2, Th. 1], there exists a semisimple character  $\rho$  of  $\mathbf{H}^F$ , such that  $\rho$  and  $\chi$  have the same stabilizer. By [S4, Proof of 3.4(c)], the semisimple character  $\rho$  has some  $(\widetilde{\mathbf{H}})^F$ -conjugate  $\rho_0$  with  $(\widetilde{\mathbf{H}}^F E)_{\rho_0} = \widetilde{\mathbf{H}}^F_{\rho_0} E_{\rho_0}$ . □

In our considerations on  $D_{l,sc}(q)$ , we assume the following for all  $4 \leq l' < l$ , which amounts to  $A(\infty)$  for cuspidal characters in rank  $< l$ . This was called  $A_{\text{cusp}}$  in our Introduction. In [S7], we will see that it is actually always satisfied.

**HYPOTHESIS 2.14** (Extension of cuspidal characters of  $D_{l',sc}(q)$ ). Let  $\mathbf{H}$  be a simply connected simple group of type  $D_{l'}$  ( $l' \geq 4$ ), and let  $F : \mathbf{H} \rightarrow \mathbf{H}$  a standard Frobenius endomorphism. Then there exists some  $E(\mathbf{H}^F)$ -stable  $\tilde{\mathbf{H}}^F$ -transversal  $\mathbb{T}$  in  $\text{Irr}_{\text{cuspidal}}(\mathbf{H}^F)$  such that every  $\chi \in \mathbb{T}$  extends to  $\mathbf{H}^F E(\mathbf{H}^F)_\chi$ .

The following facts are well known (see also [B1, 12.1]).

**LEMMA 2.15.** *Let  $\mathbf{G}$  be a simply connected simple group with Frobenius endomorphism  $F : \mathbf{G} \rightarrow \mathbf{G}$ ,  $\mathbf{L}$  an  $F$ -stable Levi subgroup of  $\mathbf{G}$ ,  $L := \mathbf{L}^F$ ,  $L_0 := [\mathbf{L}, \mathbf{L}]^F$ , and  $\lambda \in \text{Irr}_{\text{cuspidal}}(L)$ .*

- (a) *Then  $\text{Irr}(\lambda|_{L_0}) \subseteq \text{Irr}_{\text{cuspidal}}(L_0)$ .*
- (b) *If  $[\mathbf{L}, \mathbf{L}]$  is a central product of  $F$ -stable semisimple groups  $\mathbf{H}_1$  and  $\mathbf{H}_2$ , then  $\text{Irr}(\lambda|_{\mathbf{H}_1^F}) \subseteq \text{Irr}_{\text{cuspidal}}(\mathbf{H}_1^F)$ .*
- (c) *Let  $\tilde{\mathbf{G}}$  be a reductive group with  $\mathbb{F}_q$ -structure given by  $F : \tilde{\mathbf{G}} \rightarrow \tilde{\mathbf{G}}$  extending  $F$  already defined on  $\mathbf{G}$  and such that  $[\tilde{\mathbf{G}}, \tilde{\mathbf{G}}] = \mathbf{G}$ , then every  $\tilde{\lambda} \in \text{Irr}((Z(\tilde{\mathbf{G}})\mathbf{L})^F | \lambda)$  is cuspidal.*

*Proof.* For a finite group  $H$  with a split BN-pair of characteristic  $p$ , a given  $\chi \in \text{Irr}(H)$  is cuspidal if and only if the corresponding representation space has no nonzero fixed point under any  $O_p(P)$  for any proper parabolic subgroup  $P$  of  $H$ . It is then clear that for any  $H' \triangleleft H$  with  $p'$ -index, one has  $\chi \in \text{Irr}_{\text{cuspidal}}(H)$  if and only if  $\chi|_{H'}$  has a cuspidal irreducible component (and then all are). This gives (a) and (c). For (b), note that  $\mathbf{H}_1 \cap \mathbf{H}_2$  is a group of semi-simple elements, so that the  $O_p(P)$ 's as above for  $H := \mathbf{L}_0^F$  are direct products of corresponding subgroups of  $\mathbf{H}_1^F$  and  $\mathbf{H}_2^F$ . □

**REMARK 2.16.**

- (a) Let  $\mathbf{G}$  be a simply connected simple group, and let  $\tilde{\mathbf{G}}$  be a connected algebraic group with  $\tilde{\mathbf{G}} = \mathbf{G}Z(\tilde{\mathbf{G}})$ . Let  $F : \tilde{\mathbf{G}} \rightarrow \tilde{\mathbf{G}}$  be a Frobenius endomorphism stabilizing  $\mathbf{G}$ . Then  $x \in \tilde{\mathbf{G}}^F$  can be written as  $x = gz$  with  $g \in \mathbf{G}$  and  $z \in Z(\tilde{\mathbf{G}})$ , such that  $g^{-1}F(g) = zF(z^{-1})$ . If  $\mathcal{L} : \mathbf{G} \rightarrow \mathbf{G}$  is defined by  $g \mapsto g^{-1}F(g)$  and  $\tilde{G} := \mathcal{L}^{-1}(Z(\mathbf{G}))$ , we see

$$\tilde{\mathbf{G}}^F \leq \tilde{G}\tilde{Z},$$

where  $\tilde{Z} := \{z \in Z(\tilde{\mathbf{G}}) \mid F(z) \in zZ(\mathbf{G})\}$ . Note that  $\tilde{G}$  by its construction is independent of the choice of  $\tilde{\mathbf{G}}$ . We also have  $\tilde{G} = N_{\tilde{\mathbf{G}}}(\mathbf{G}^F)$  as an easy consequence of [B1, Lem. 6.1].

- (b) From now on, we assume additionally that  $Z(\tilde{\mathbf{G}})$  is connected. Then the (outer) automorphisms of  $\mathbf{G}^F$  induced by conjugation by some element  $g \in \tilde{G}$  are called *diagonal automorphisms* and they are parametrized by  $\mathcal{L}(g)[Z(\mathbf{G}), F] \in Z(\mathbf{G})/[Z(\mathbf{G}), F]$  (see also [GM, 1.5.12]).

Note the difference with the convention used in the introduction where  $\tilde{G}$  was used to abbreviate  $\tilde{\mathbf{G}}^F$ . We still clearly have  $\tilde{G}/Z(\mathbf{G}^F) = \tilde{\mathbf{G}}^F/Z(\tilde{\mathbf{G}}^F)$ .

This allows the following conclusion for the above group  $\tilde{G}$ .

**THEOREM 2.17.** *Maximal extendibility holds with respect to  $\mathbf{G}^F \triangleleft \tilde{G}$ .*

*Proof.* Let  $\tilde{\mathbf{G}}$  be a group with connected centre, such that there exists a regular embedding  $\mathbf{G} \rightarrow \tilde{\mathbf{G}}$  that is also an  $\mathbb{F}_q$ -morphism as in 2.2. Then, according to a theorem of Lusztig (see [CE, 15.11]), maximal extendibility holds with respect to  $\mathbf{G}^F \triangleleft \tilde{\mathbf{G}}^F$ , and  $\chi$  has an extension  $\tilde{\chi}$  to  $\tilde{\mathbf{G}}_\chi^F$ . According to the above,  $\tilde{\mathbf{G}}^F \leq \tilde{G}\tilde{Z}$ . Clearly,  $\tilde{\chi}$  extends to  $\tilde{\mathbf{G}}_\chi^F \tilde{Z}$  since  $\tilde{Z}$  is abelian and  $[Z, \tilde{\mathbf{G}}^F] = 1$ . Now, we see that  $\tilde{\mathbf{G}}_\chi^F \tilde{Z} = \tilde{G}_\chi \tilde{Z}$  and hence  $\chi$  extends to  $\tilde{G}_\chi$  as well. □

PROPOSITION 2.18. *In the situation of Remark 2.16, let  $\mathbf{K} \leq \mathbf{G}$  be an  $F$ -stable reductive subgroup with  $\mathbf{T}_0 \leq \mathbf{K}$ . Let  $\tilde{\mathbf{K}} := \mathbf{K}Z(\tilde{\mathbf{G}})$  and  $\tilde{K} := \mathcal{L}^{-1}(Z(\mathbf{G})) \cap \mathbf{K}$ . Let  $\chi \in \text{Irr}(\mathbf{K}^F)$ ,  $\tilde{\chi} \in \text{Irr}(\tilde{\mathbf{K}}^F \mid \chi)$ , and  $\nu \in \text{Irr}(\tilde{\chi}|_{Z(\tilde{\mathbf{G}}^F)})$ . As said above,  $\chi$  extends to  $\tilde{\mathbf{K}}_\chi^F$ . Let  $\gamma \in E(\mathbf{G}^F)_{(\chi, \mathbf{K})}$  and  $\mu \in \text{Irr}(\tilde{\mathbf{K}}^F/\mathbf{K}^F)$  with  $\tilde{\chi}^\gamma = \tilde{\chi}\mu$ . Then the following are equivalent:*

- (i)  $\chi$  has a  $\gamma$ -stable extension to  $\tilde{K}_\chi$ .
- (ii) For  $\tilde{Z}' := \mathcal{L}(\tilde{K}_\chi)$ , there exists some extension  $\tilde{\nu} \in \text{Irr}(\tilde{Z}')$  of  $\nu$  such that  $\mu(tz) = \tilde{\nu}(z)^{-1}(\tilde{\nu}^\gamma(z))$  for every  $t \in \tilde{K}_\chi$  and  $z \in \tilde{Z}'$  with  $tz \in \tilde{\mathbf{K}}_\chi^F$ .

*Proof.* We prove the statement only in the case where  $\mathbf{K} = \mathbf{G}$ . The results transfer to a general  $\mathbf{K}$  as only the quotient groups are relevant to our considerations. Let  $\tilde{\chi}$  be a  $\gamma$ -stable extension to  $\tilde{G}_\chi$ , then there exists an extension  $\tilde{\nu} \in \text{Irr}(\tilde{Z}')$  of  $\nu$  such that  $\tilde{\chi} := (\tilde{\chi} \cdot \tilde{\nu}|_{\tilde{\mathbf{G}}_\chi^F})^{\tilde{\mathbf{G}}^F}$ . We observe  $(\tilde{\chi} \cdot \tilde{\nu})^\gamma = \tilde{\chi} \cdot \tilde{\nu}^\gamma$ . This leads to the given formula for  $\mu$  in (ii).

For the other direction, let  $\chi_0$  be the extension of  $\chi$  to  $\tilde{\mathbf{G}}_\chi^F$  such that  $\tilde{\chi} = \chi_0^{\tilde{\mathbf{G}}^F}$ . Then  $\chi_0^\gamma = \chi_0 \mu|_{\tilde{\mathbf{G}}_\chi^F}$  and  $\chi_0 \cdot \tilde{\nu}$  is an extension of  $\chi$  to  $\tilde{\mathbf{G}}_\chi^F \tilde{Z}' = \tilde{G}_\chi \tilde{Z}'$ . The character  $\tilde{\chi} := (\chi_0 \cdot \tilde{\nu})|_{\tilde{G}_\chi}$  satisfies

$$\tilde{\chi}^\gamma \cdot \tilde{\nu}^\gamma = (\tilde{\gamma} \cdot \tilde{\nu})^\gamma = (\chi_0 \cdot \tilde{\nu})^\gamma = \chi_0^\gamma \cdot \tilde{\nu}^\gamma = \chi_0 \mu|_{\tilde{\mathbf{G}}_\chi^F} \cdot \tilde{\nu}^\gamma.$$

There is some  $\kappa \in \text{Irr}(\tilde{G}_\chi/\mathbf{G}^F)$  with  $\tilde{\chi}^\gamma = \tilde{\chi}\kappa$ . According to [I, (6.17)], the above equality of characters implies  $\kappa(t)\tilde{\nu}^\gamma(z)(\tilde{\nu}(z))^{-1} = \mu(tz)$ , whenever  $t \in \tilde{G}_\chi$  and  $z \in \tilde{Z}'$  with  $tz \in \tilde{\mathbf{G}}_\chi^F$ . By the assumption on  $\mu$  and  $\tilde{\nu}$ , this leads to  $\kappa = 1$ . Then  $\chi$  has a  $\gamma$ -stable extension to  $\tilde{G}_\chi$ . □

For later, we restate  $A(\infty)$  for groups of type A (see [CS2]).

PROPOSITION 2.19. *Let  $G = \text{SL}_n(q)$ ,  $\tilde{G} := \text{GL}_n(q)$ , and write  $E(\text{SL}_n(q))$  for the group of field and graph automorphisms of  $G$  and  $\tilde{G}$  with regard to the usual BN-pair.*

- (a) *Then there exists an  $E(\text{SL}_n(q))$ -stable  $\text{GL}_n(q)$ -transversal  $\mathbb{T}$  in  $\text{Irr}(\text{SL}_n(q))$ , such that every  $\chi \in \mathbb{T}$  extends to  $\text{SL}_n(q)E(\text{SL}_n(q))_\chi$ .*
- (b) *Let  $\gamma'$  be the automorphism of  $\text{SL}_n(q)$  given by transpose-inverse, and let  $E'(\text{SL}_n(q)) \leq \text{Aut}(\text{SL}_n(q))$  be the subgroup generated by  $\gamma'$  and the field automorphisms described above. Then  $E'(\text{SL}_n(q))$  is abelian and there exists an  $E'(\text{SL}_n(q))$ -stable  $\text{GL}_n(q)$ -transversal  $\mathbb{T}$  in  $\text{Irr}(\text{SL}_n(q))$ , such that every  $\chi \in \mathbb{T}$  extends to  $\text{SL}_n(q)E'(\text{SL}_n(q))_\chi$ .*

*Proof.* Part (a) follows from [CS2, Th. 4.1] using Lemma 2.4.

Let  $\gamma \in E(\text{SL}_n(q))$  be the graph automorphism. Following the considerations in [CS2, 3.2], we see that  $\gamma'$  and  $v_0\gamma$  induce the same automorphism of  $\text{SL}_n(q)$ , where  $v_0 \in \text{SL}_n(p)$  is defined as in [CS2, 3.2] and  $p$  is the prime dividing  $q$ . This proves that  $\mathbb{T}$  is also  $E'(\text{SL}_n(q))$ -stable. For part (b), we have to prove that every  $\chi \in \mathbb{T}$  extends to its inertia group in  $\text{SL}_n(q)E'(\text{SL}_n(q))$ . This statement is clear whenever  $E'(\text{SL}_n(q))_\chi$  is cyclic (see [I, (9.12)]). If for  $\chi \in \mathbb{T}$  the group  $E'(\text{SL}_n(q))_\chi$  is noncyclic, we see  $\gamma' \in E'(\text{SL}_n(q))_\chi$ . Let  $F_{q'} \in E'(\text{SL}_n(q))$  be a field automorphism such that  $E'(\text{SL}_n(q))_\chi = \langle F_{q'}, \gamma' \rangle$ . By (a), there exists some  $\gamma$ -stable extension of  $\chi$  to  $G\langle F_{q'} \rangle$ . This extension is then also  $\gamma'$  and hence  $\gamma v_0$ -stable as  $[v_0, F_{q'}] = 1$ . From this, we deduce that  $\chi$  extends to  $\text{SL}_n(q)E'(\text{SL}_n(q))_\chi$ . □

**§3. The Levi subgroup and its normalizer**

In this and the following section, we reprove with quite different methods that for every standard Levi subgroup  $L$  of  $D_{l,sc}(q)$ , every  $\lambda \in \text{Irr}_{cusp}(L)$  extends to its stabilizer inside  $N_{\mathbf{G}^F}(\mathbf{L})$ , which follows from the mentioned results by Geck and Lusztig. For  $E(\mathbf{G}^F) \leq \text{Aut}(\mathbf{G}^F)$  from §2.2, we construct a  $\tilde{T}$ -transversal  $\mathbb{T}$  of  $\text{Irr}_{cusp}(L)$  and an  $N\text{Stab}_{E(\mathbf{G}^F)}(L)$ -equivariant extension map with respect to  $L \triangleleft N$  for  $\mathbb{T}$ .

**THEOREM 3.1.** *Let  $L$  be a standard Levi subgroup of  $\mathbf{G}^F = D_{l,sc}(q)$ . Let  $E_L := \text{Stab}_{E(\mathbf{G}^F)}(L)$ ,  $N$ ,  $\widehat{N} := NE_L$  and  $\tilde{L}' := \tilde{T}_0L$  be associated with  $L$  as in 2.8. If  $D_{l',sc}(q)$  is a direct factor of  $[L, L]$ , then assume Hypothesis 2.14 holds for  $D_{l',sc}(q)$ . Then:*

- (a) *There exists an  $\widehat{N}$ -stable  $\tilde{L}'$ -transversal  $\mathbb{T} \subseteq \text{Irr}_{cusp}(L)$ .*
- (b) *There exists an  $\widehat{N}$ -equivariant extension map  $\Lambda_{L \triangleleft N}$  with respect to  $L \triangleleft N$  for  $\mathbb{T}$ .*

This implies Theorem B and ensures Assumptions (i) and (ii) of Proposition 2.12. In [BS, Th. 4.3] and [CSS, Prop. 4.13], the analogous result was shown in the case where  $\mathbf{G}$  is of type  $A_l$  or  $C_l$ . The interested reader may notice that without assuming Hypothesis 2.14 for smaller ranks, the proof we give implies a version of the theorem without the equivariance statement.

Like in the proofs given in [BS] and [CSS], we essentially apply the following statement providing an extension map for nonlinear characters.

**PROPOSITION 3.2** [CSS, Prop. 4.1]. *Let  $K \triangleleft M$  be finite groups, let the group  $D$  act on  $M$ , stabilizing  $K$ , and let  $\mathbb{K} \subseteq \text{Irr}(K)$  be  $MD$ -stable. Assume that there exist  $D$ -stable subgroups  $K_0$  and  $V$  of  $M$  such that:*

- (i) *the groups satisfy:*
  - (i.1)  $K = K_0(K \cap V)$  and  $H := K \cap V \leq Z(K)$ ,
  - (i.2)  $M = KV$ ;
- (ii) *for  $\mathbb{K}_0 := \bigcup_{\lambda \in \mathbb{K}} \text{Irr}(\lambda|_{K_0})$ , there exist:*
  - (ii.1) *a  $VD$ -equivariant extension map  $\Lambda_0$  with respect to  $H \triangleleft V$ ; and*
  - (ii.2) *an  $\epsilon(V)D$ -equivariant extension map  $\Lambda_\epsilon$  with respect to  $K_0 \triangleleft K_0 \rtimes \epsilon(V)$  for  $\mathbb{K}_0$ , where  $\epsilon: V \rightarrow V/H$  denotes the canonical epimorphism.*

*Then there exists an  $MD$ -equivariant extension map with respect to  $K \triangleleft M$  for  $\mathbb{K}$ .*

In this section, we construct the set  $\mathbb{T}$  for Theorem 3.1(a) and introduce groups  $H, K, K_0$  (see Lemma 3.11),  $M, D$ , and  $V$  (in Corollary 3.23) for a later application of Proposition 3.2 in the proof of Theorem 3.1(b). Here, we show that the groups introduced satisfy the group-theoretic assumptions made in 3.2(i). Afterward, in §4, we ensure the character-theoretic assumptions, namely 3.2(ii) in order to prove Theorem 3.1(b).

**3.1 Subgroups of the Levi subgroup  $L$**

As a first step, we dissect the root system of  $\mathbf{L}$  and introduce subgroups of  $L$  with those new root systems. For a nonnegative integer  $i$ , let  $\underline{i} := \{1, \dots, i\}$ . For computations with elements of  $\mathbf{G}$ , we use the Steinberg generators satisfying the Chevalley relations together with an explicit embedding of  $D_{l,sc}(\mathbb{F})$  into  $\text{B}_{l,sc}(\mathbb{F})$ .

**NOTATION 3.3** (The groups  $\mathbf{G}$  and  $\overline{\mathbf{G}}$ , roots, and generators). In this and the following section, we assume that the simply connected simple group  $\mathbf{G}$  from 2.2 is of type  $D_l$  ( $l \geq 4$ )

over  $\mathbb{F}$  the algebraic closure of  $\mathbb{F}_p$  for  $p$  some odd prime. Hence,  $\mathbf{G} \cong D_{l,sc}(\mathbb{F})$ . Denote  $\underline{l} := \{1, \dots, l\}$ . Let  $\Phi := \{\pm e_i \pm e_j \mid i, j \in \underline{l}, i \neq j\}$  be the root system of  $\mathbf{G}$  with simple roots  $\alpha_2 := e_2 + e_1, \alpha_1 = e_2 - e_1$  and  $\alpha_i := e_i - e_{i-1} (i \geq 3)$ ,

$$\Delta := \{\alpha_i \mid i \in \underline{l}\}$$

(see [GLS, Rem. 1.8.8]), where the set  $\{e_i\}_{i \in \underline{l}}$  is an orthonormal basis of  $\mathbb{R}^l$  whose scalar product is denoted by  $(x, y)$ . The Chevalley generators  $\mathbf{x}_\alpha(t), \mathbf{n}_\alpha(t')$  and  $\mathbf{h}_\alpha(t')$  ( $\alpha \in \Phi, t, t' \in \mathbb{F}$  with  $t' \neq 0$ ) together with the Chevalley relations describe the group structure of  $\mathbf{G}$  (see [GLS, Th. 1.12.1]).

Let  $\overline{\Phi} := \{\pm e_i, \pm e_i \pm e_j \mid i, j \in \underline{l}, i \neq j\}, \overline{\mathbf{G}} := B_{l,sc}(\mathbb{F})$  with Chevalley generators  $\overline{\mathbf{x}}_\alpha(t), \overline{\mathbf{n}}_\alpha(t')$  and  $\overline{\mathbf{h}}_\alpha(t')$  ( $\alpha \in \overline{\Phi}, t, t' \in \mathbb{F}$  with  $t' \neq 0$ ). Assume that the structure constants of  $\mathbf{G}$  and  $\overline{\mathbf{G}}$  are chosen such that  $\mathbf{x}_\alpha(t) \mapsto \overline{\mathbf{x}}_\alpha(t)$  ( $\alpha \in \Phi, t \in \mathbb{F}$ ) defines an embedding  $\iota_D : \mathbf{G} \rightarrow \overline{\mathbf{G}}$ . For simplicity of notation, we write  $\mathbf{x}_\alpha(t), \mathbf{n}_\alpha(t') = \mathbf{x}_\alpha(t')\mathbf{x}_{-\alpha}(-t'^{-1})\mathbf{x}_\alpha(t')$ , and  $\mathbf{h}_\alpha(t') = \mathbf{n}_\alpha(t')\mathbf{n}_\alpha(1)^{-1}$  for the generators of  $\overline{\mathbf{G}}$  and thus identify  $\mathbf{G}$  with the corresponding subgroup of  $\overline{\mathbf{G}}$ . This is possible according to [S2, 10.1] (see also [MS, 2.C]). Among the relations between Chevalley generators, the following will be the most useful to us. For  $a, b \in \mathbb{R}^l \setminus \{0\}$ , recall  $\langle a, b \rangle = 2(a, b)/(b, b)$ . Let  $\alpha, \beta \in \overline{\Phi}, t \in \mathbb{F}, t' \in \mathbb{F}^\times$ . Then

$$\begin{aligned} \mathbf{h}_\alpha(t')\mathbf{h}_\beta(t') &= \mathbf{h}_{\alpha+\beta}(t') \text{ whenever } \alpha + \beta \in \overline{\Phi}, \\ \mathbf{n}_\alpha(t)\mathbf{h}_\beta(t') &= \mathbf{n}_\alpha(t'^{\langle \alpha, \beta \rangle}t), \\ \mathbf{h}_\alpha(t)\mathbf{n}_\beta(1) &= \mathbf{h}_{\alpha - \langle \alpha, \beta \rangle \beta}(c_{\alpha, \beta}t), \end{aligned}$$

where the first line is from [GLS, 1.12.1(e)], the second is easy from [GLS, 1.12.1(g)], and the third, along with the definition of  $c_{\alpha, \beta} \in \{\pm 1\}$ , is from [GLS, 1.12.1(i)].

DEFINITION 3.4. Let  $\mathbf{X}_\alpha := \langle \mathbf{x}_\alpha(t) \mid t \in \mathbb{F} \rangle$  for  $\alpha \in \overline{\Phi}, \mathbf{T} := \langle \mathbf{h}_\alpha(t') \mid \alpha \in \Phi, t' \in \mathbb{F}^\times \rangle$ , and  $\overline{\mathbf{T}} := \langle \mathbf{h}_\alpha(t') \mid \alpha \in \overline{\Phi}, t' \in \mathbb{F}^\times \rangle$ . Note  $\mathbf{T} = \overline{\mathbf{T}}$  is the image of the map

$$(\mathbb{F}^\times)^l \ni (t'_1, \dots, t'_l) \mapsto \mathbf{h}_{e_1}(t'_1) \dots \mathbf{h}_{e_l}(t'_l)$$

with kernel  $\{(t'_1, \dots, t'_l) \in \{\pm 1\}^l \mid t'_1 \dots t'_l = 1\}$  (see also [S2, 10.1]). The group  $\mathbf{T}$  can be chosen as the group  $\mathbf{T}_0$  from 2.2 and  $\mathbf{T}\langle \mathbf{X}_\alpha \mid \alpha \in \Delta \rangle$  as the group  $\mathbf{B}$ .

Denoting  $h_0 = \mathbf{h}_{e_1}(-1)$ , one has  $Z(\overline{\mathbf{G}}) = \langle h_0 \rangle$  of order 2 (see [GLS, 1.12.6]) with  $\overline{\mathbf{G}}/\langle h_0 \rangle = \text{SO}_{2l+1}(\mathbb{F}) \geq \text{SO}_{2l}(\mathbb{F}) = \mathbf{G}/\langle h_0 \rangle$ .

For every positive integer  $i$ , let  $F_{p^i} : \overline{\mathbf{G}} \rightarrow \overline{\mathbf{G}}$  be the Frobenius endomorphism given by  $\mathbf{x}_\alpha(t) \mapsto \mathbf{x}_\alpha(t^{p^i})$  for  $t \in \mathbb{F}$  and  $\alpha \in \overline{\Phi}$ . We write  $\gamma$  for the graph automorphism of  $\mathbf{G}$  given by  $\mathbf{x}_{\epsilon\alpha}(t) \mapsto \mathbf{x}_{\epsilon\gamma_0(\alpha)}(t)$  for  $t \in \mathbb{F}, \epsilon \in \{\pm 1\}$  and  $\alpha \in \Delta$ , where  $\gamma_0$  denotes the symmetry of the Dynkin diagram of  $\Delta$  of order 2 with  $\alpha_2 \mapsto \alpha_1$ . If  $l = 4$ , we denote by  $\gamma_3$  the graph automorphism of  $\mathbf{G}$  induced by the symmetry of the Dynkin diagram of  $\Delta$  with order 3 sending  $\alpha_2 \mapsto \alpha_1$  and  $\alpha_1 \mapsto \alpha_4$ . We assume that  $F = F_q$  for  $q := p^f$ , where  $f$  is a positive integer. Note that the group  $E(\mathbf{G}^F)$  from 2.2 satisfies accordingly  $E(\mathbf{G}^F) = \langle F_p \rfloor_{\mathbf{G}^F}, \gamma \rfloor_{\mathbf{G}^F} \rangle$  whenever  $l \geq 5$ ; otherwise,  $l = 4$  and  $E(\mathbf{G}^F) = \langle F_p \rfloor_{\mathbf{G}^F}, \gamma \rfloor_{\mathbf{G}^F}, \gamma_3 \rfloor_{\mathbf{G}^F} \rangle$ .

We recall that the graph automorphism  $\gamma$  of  $\mathbf{G}$  is induced by an element of  $\overline{\mathbf{G}}$  (see [GLS, 2.7] for the corresponding statement in classical groups). Let  $\varpi \in \mathbb{F}^\times$  such that  $\varpi^2 = -1$ . By [S2, 10.1] (see also [MS, 2.C]), the automorphism  $\gamma$  of  $\mathbf{G}$  is induced by conjugating with  $\mathbf{n}_{e_1}(\varpi) \in \overline{\mathbf{G}}$ .

NOTATION 3.5. Let  $\mathbf{L}$  be a Levi subgroup of  $\mathbf{G}$  such that  $\mathbf{BL}$  is a parabolic subgroup of  $\mathbf{G}$  and  $\mathbf{T} \subseteq \mathbf{L}$ . Let  $L := \mathbf{L}^F$ , and let  $\Phi'$  be the root system of  $\mathbf{L}$ , that is,  $\mathbf{L} = \mathbf{T}\langle \mathbf{X}_\alpha \mid \alpha \in \Phi' \rangle$ . As  $\Phi'$  is a parabolic root subsystem of  $\Phi$ , it has as basis  $\Delta' = \Delta \cap \Phi'$ . We assume that one of the following holds:

- (i)  $\Delta' \subseteq \{e_2 - e_1, e_3 - e_2, \dots, e_l - e_{l-1}\}$ , or
- (ii)  $\{e_2 - e_1, e_2 + e_1\} \subseteq \Delta'$ .

Recall that a split Levi subgroup of  $\mathbf{G}$  containing  $\mathbf{T}$  is called standard if it is generated by  $\mathbf{T}$  and the  $\mathbf{X}_\alpha$ 's such that  $\alpha \in \pm\Delta'$  for some subset  $\Delta' \subseteq \Delta$ . Recall that  $\gamma$  swaps  $e_2 - e_1$  and  $e_2 + e_1$  while fixing the other elements of  $\Delta$ . Then any subset  $\Delta' \subseteq \Delta$  is such that  $\Delta'$  or  $\gamma(\Delta')$  satisfies 3.5(i) or 3.5(ii). We then get that  $\mathbf{L}$  can be assumed to satisfy Notation 3.5.

LEMMA 3.6. *Every standard Levi subgroup of  $\mathbf{G}$  containing  $\mathbf{T}$  is  $\langle \gamma \rangle$ -conjugate to a standard Levi subgroup whose root system has a basis  $\Delta' \subseteq \Delta$  satisfying 3.5(i) or 3.5(ii).*

3.7 (Decomposing  $\Phi'$ ). In the following, we decompose  $\Phi'$  into smaller root systems, which are the disjoint union of irreducible root systems of the same type. By  $\text{type}(\Psi)$ , we denote the type of the root system  $\Psi$ . Whenever  $\Psi$  is a subset of  $\bar{\Phi}$ , we also denote by  $W_\Psi$  the subgroup of  $N_{\bar{\mathbf{G}}}(\mathbf{T})/\mathbf{T}$  generated by reflections defined by elements of  $\Psi$ .

Since  $\Phi'$  is a parabolic root subsystem of  $\Phi$ ,  $\Phi'$  decomposes as a disjoint union of indecomposable root systems of types D and A, that are called components of  $\Phi'$ .

If  $\Delta'$  satisfies Assumption 3.5(i), let  $\Phi_d$  be the union of the components of  $\Phi'$  of type  $A_{d-1}$  ( $d \geq 2$ ). If  $\Delta'$  satisfies Assumption 3.5(ii), let  $\Phi_{-1}$  be the union of components of  $\Phi'$  that have a nontrivial intersection with  $\{e_2 - e_1, e_1 + e_2\}$  and let  $\Phi_d$  be the union of components of  $\Phi' \setminus \Phi_{-1}$  of type  $A_{d-1}$  ( $d \geq 2$ ). If  $\Delta'$  satisfies Assumption 3.5(ii),  $\text{type}(\Phi_{-1}) \in \{A_3, A_1 \times A_1, D_m \mid m \geq 4\}$ .

Let  $\mathbb{D}'$  be the set of integers  $d$  such that  $\Phi_d$  is defined and nonempty, that is,  $SL_d(\mathbb{F})$  is a summand of  $[\mathbf{L}, \mathbf{L}]$ . Then  $\Phi' = \bigsqcup_{d \in \mathbb{D}'} \Phi_d$ , a disjoint union.

Recall that  $W_{\bar{\Phi}}$ , the group generated by the reflections along the roots of  $\bar{\Phi}$  coincides with  $W_0 := \bar{N}_0/T_0$ , can be identified with the permutations of  $\underline{l} \cup -\underline{l}$  that commute with the sign change, and hence acts on  $\underline{l}$  (see [GLS, Rem. 1.8.8]).

3.8 (Orbits of  $W_{\Phi'}$  on  $\underline{l}$ ). Let  $\mathcal{O}$  be the set of orbits of  $W_{\Phi'}$  on  $\underline{l}$ , let  $\mathcal{O}_1 \subseteq \mathcal{O}$  be the set of singletons in  $\mathcal{O}$ , and let  $\mathcal{O}_d$  be the set of orbits of  $W_{\Phi_d}$  on  $\underline{l}$  contained in  $\mathcal{O} \setminus \mathcal{O}_1$ , whenever  $d \in \mathbb{D}'$ . We define

$$\mathbb{D}(L) = \mathbb{D} = \begin{cases} \mathbb{D}' \cup \{1\}, & \text{if } \mathcal{O}_1 \neq \emptyset, \\ \mathbb{D}', & \text{otherwise.} \end{cases}$$

For  $d \in \mathbb{D} \setminus \{-1\}$ , let  $a_d := |\mathcal{O}_d|$  and note that  $|I| = d$  for any  $I \in \mathcal{O}_d$ .

For  $I \subseteq \underline{l}$ , let  $\Phi_I := \Phi' \cap \langle e_k \mid k \in I \rangle$  and  $\bar{\Phi}_I := \bar{\Phi} \cap \langle e_k \mid k \in I \rangle$ . For  $d \in \mathbb{D}$ , let  $J_d := \bigcup_{o \in \mathcal{O}_d} o$ , and  $\bar{\Phi}_d := \bar{\Phi}_{J_d}$ .

Next, we introduce groups  $K, K_0$ , and  $H$  that will later be proved to satisfy Assumption 3.2(i) with a group  $M$ .

NOTATION 3.9 (Subgroups of  $\mathbf{L}$  and  $L$ ). Let  $\varpi \in \mathbb{F}^\times$  and  $h_0$  as in Definition 3.4. Define  $\mathbf{h}_I(t) := \prod_{i \in I} \mathbf{h}_{e_i}(t)$  for  $I \subseteq \underline{l}$  and  $t \in \mathbb{F}^\times$ . For  $I \in \mathcal{O}$ , let  $\mathbf{G}_I = \langle \mathbf{X}_\alpha \mid \alpha \in \Phi_I \rangle$  and  $G_I := \mathbf{G}_I^F$ . Note that for  $I \in \mathcal{O}_1$ , the group  $\mathbf{G}_I$  is trivial. Let  $H_0 :=$

$\langle h_0, \mathbf{h}_{e_i}(\varpi)\mathbf{h}_{e_{i'}}(-\varpi) \mid i, i' \in \underline{l} \rangle = \langle h_\alpha(-1) \mid \alpha \in \Phi \rangle$ . For  $d \in \mathbb{D}$ , let  $\tilde{H}_d := \langle h_0, \mathbf{h}_I(\varpi) \mid I \in \mathcal{O}_d \rangle$ ,  $H_d := \langle h_0, \mathbf{h}_I(\varpi)\mathbf{h}_{I'}(-\varpi) \mid I, I' \in \mathcal{O}_d \rangle$  and

$$H := \langle \tilde{H}_d \mid d \in \mathbb{D} \rangle \cap H_0.$$

LEMMA 3.10. *Let  $\mathbb{D}_{\text{even}} := \mathbb{D} \cap 2\mathbb{Z}$  and  $\mathbb{D}_{\text{odd}} := \mathbb{D} \setminus \mathbb{D}_{\text{even}}$ . If  $H_\epsilon := \langle \tilde{H}_d \mid d \in \mathbb{D}_\epsilon \rangle \cap H_0$  for  $\epsilon \in \{\text{odd}, \text{even}\}$ , then  $H = H_{\text{even}} \cdot H_{\text{odd}}$ .*

*Proof.* An element  $t \in \mathbf{T}$  can be written as  $\prod_{i=1}^l \mathbf{h}_{e_i}(t_i)$  ( $t_i \in \mathbb{F}^\times$ ). We have  $t \in H_0$  if  $t_i \in \langle \varpi \rangle$  and  $\prod_{i=1}^l t_i^2 = 1$ . In particular,  $\mathbf{h}_I(\varpi) \in H_0$  if and only if  $|I|$  even. This implies  $H_d \leq H_0$  whenever  $2 \mid d$ . On the other hand,  $\tilde{H}_d \not\leq H_0$  for every  $d \in \mathbb{D}_{\text{odd}}$ .  $\square$

With this notation,  $Z(\mathbf{G}) = \langle h_0, \mathbf{h}_I(\varpi) \rangle$  (see [GLS, Table 1.12.6]).

LEMMA 3.11.  $H \leq Z(\mathbf{L})$ .

*Proof.* We see that  $[\mathbf{h}_I(\varpi), \mathbf{G}_I] = 1$  by the Chevalley relations and this implies the statement by the definition of  $H$ .  $\square$

The groups  $K_0 := \langle G_I \mid I \in \mathcal{O} \rangle$  and  $K := K_0 H$  then satisfy Assumption 3.2(i.1) for  $H$ .

To understand later the action of  $N_{\mathbf{G}^F}(\mathbf{L})$  on  $\text{Irr}(K)$ , we analyze the structure of  $L$  by introducing several subgroups.

3.12 (Structure of  $\mathbf{L}$ ). We note that the Levi subgroup  $\mathbf{L}$  satisfies  $\mathbf{L} = \mathbf{T} \langle \mathbf{G}_I \mid I \in \mathcal{O} \rangle$ . Let  $\mathbf{T}_I := \langle \mathbf{h}_{e_i}(t) \mid i \in I, t \in \mathbb{F}^\times \rangle$  for  $I \in \mathcal{O}$ . For  $I, I' \in \mathcal{O}$  with  $I \neq I'$ , we see that no nontrivial linear combination of a root in  $\Phi_I$  and one in  $\Phi_{I'}$  is a root in  $\Phi$  as well. Therefore,  $[\mathbf{G}_I, \mathbf{G}_{I'}] = 1$  according to Chevalley’s commutator formula. By the Steinberg relations, we see  $[\mathbf{G}_I, \mathbf{T}_{I'}] = 1$ . The group  $\mathbf{G}_I$  is either trivial or a simply connected simple group unless  $I = \mathcal{O}_{-1}$  and  $\text{type}(\Phi_{-1}) = A_1 \times A_1$ . Accordingly,  $[\mathbf{L}, \mathbf{L}] = \langle \mathbf{G}_I \mid I \in \mathcal{O} \rangle$ .

We observe that  $\mathbf{G}_I \cap \mathbf{T} \leq \mathbf{T}_I$  and computations with the coroot lattices prove that  $\mathbf{T}$  is the central product of the groups  $\mathbf{T}_I$  ( $I \in \mathcal{O}$ ) over  $\langle h_0 \rangle$ . This implies that  $\mathbf{L}$  is the central product of the groups  $\mathbf{L}_I$  ( $I \in \mathcal{O}$ ) where  $\mathbf{L}_I := \mathbf{T}_I \mathbf{G}_I$ .

Analogously, we see that  $\mathbf{L}$  is the central product of the groups  $\mathbf{L}_d$  ( $d \in \mathbb{D}$ ) over the group  $\langle h_0 \rangle$ , where  $\mathbf{L}_d := \langle \mathbf{L}_I \mid I \in \mathcal{O}_d \rangle$ .

The structure of  $\mathbf{L}$  studied above implies the following results on  $L$ . Recall  $K_0 := \langle G_I \mid I \in \mathcal{O} \rangle$  from Lemma 3.11.

LEMMA 3.13. *Recall  $\mathcal{L} : \mathbf{G} \rightarrow \mathbf{G}$  the Lang map defined by  $g \mapsto g^{-1}F(g)$ , let  $\widehat{L} := \mathbf{L} \cap \mathcal{L}^{-1}(\langle h_0 \rangle)$ , and let  $\widetilde{L} := \mathbf{L} \cap \mathcal{L}^{-1}(Z(\mathbf{G}))$ .*

- (a) *If  $L_I := \mathbf{L}_I^F$  for every  $I \in \mathcal{O}$  and  $L_0 := \langle L_I \mid I \in \mathcal{O} \rangle$ , then  $L_0 \triangleleft L$ .*
- (b) *Let  $I \in \mathcal{O}$  and  $\widehat{L}_I := \mathbf{L}_I \cap \widehat{L}$ . Then  $\widehat{L}_I = \langle L_I, t_I \rangle$  for every  $t_I \in \mathbf{T}_I \cap \mathcal{L}^{-1}(\langle h_0 \rangle)$ . We assume chosen such a  $t_I$  for each  $I \in \mathcal{O}$ .  
The group  $\widehat{L}$  is the central product of  $\widehat{L}_I$  ( $I \in \mathcal{O}$ ) and for  $d \in \mathbb{D}$ ,  $\widehat{L}_d := \widehat{L} \cap \mathbf{L}_d$  is the central product of  $\widehat{L}_I$  ( $I \in \mathcal{O}_d$ ).*
- (c)  $L = \langle \mathbf{L}_I^F, t_I t_{I'} \mid I, I' \in \mathcal{O} \rangle$ .
- (d)  $K_0$  is the direct product of all  $G_I$ ,  $K_0 \triangleleft \widetilde{L}$  and  $\widetilde{L}/K_0$  is abelian.
- (e) *If  $\zeta \in \mathbb{F}^\times$  with  $\zeta^{(q-1)^2} = \varpi$  and  $t_{Q,2} := \mathbf{h}_Q(\zeta)$  for every  $Q \subseteq \underline{l}$ , then  $\widetilde{L} = \widehat{L} \langle t_{\underline{l},2} \rangle$ .*

The arguments of Remark 2.16 show that  $\widetilde{L}'$  from 2.8 and  $\widetilde{L}$  induce the same automorphisms on  $\mathbf{G}$ .



*Proof.* Recall that  $\mathbf{L}$  is the central product of the groups  $\mathbf{L}_I$ , where each  $\mathbf{L}_I$  is  $F$ -stable. Every  $x \in \mathbf{L}$  can be written as  $\prod_{I \in \mathcal{O}} x_I$  with  $x_I \in \mathbf{L}_I$ . Clearly,  $x \in L$  if and only if  $\mathcal{L}(x) = 1$ . We see that  $\mathcal{L}(x) = \prod_{I \in \mathcal{O}} \mathcal{L}(x_I)$  by the structure of  $\mathcal{L}$  and hence  $x \in L$  implies  $\mathcal{L}(x_I) \in \langle h_0 \rangle$ . The group  $L_0$  is the group of elements  $\prod_{I \in \mathcal{O}} x_I$  with  $x_I \in L_I := \mathbf{L}_I^F$ . The group  $\widehat{L} := \mathcal{L}(\langle h_0 \rangle) \cap \mathbf{L}$  is the group of elements  $\prod x_I$  with  $x_I \in \mathbf{L}_I$  and  $\mathcal{L}(x_I) \in \langle h_0 \rangle$ . Hence,  $\widehat{L}$  is the central product of  $\widehat{L}_I$  ( $I \in \mathcal{O}$ ) over  $\langle h_0 \rangle$ . Clearly,  $L_0 \triangleleft \widehat{L}$ ,  $\widehat{L}_I = \langle L_I, t_I \rangle$  and  $L = L_0 \langle t_I t_{I'} \mid I, I' \in \mathcal{O} \rangle$ . This ensures the parts (a)–(c).

Part (d) follows from the fact that  $\mathbf{L}/\langle \mathbf{G}_I \mid I \in \mathcal{O} \rangle$  is isomorphic to a quotient of  $\mathbf{T}$  and hence abelian. For part (e), we observe  $\mathcal{L}(\mathbf{h}_Q(\zeta)) = \mathbf{h}_Q(\varpi)$  for every  $Q \subseteq \underline{l}$  and recall that  $Z(\mathbf{G}) = \langle h_0, \mathbf{h}_l(\varpi) \rangle$ . □

### 3.2 The structure of $N/L$

We analyze  $N := N_{\mathbf{G}^F}(\mathbf{L})$  and  $\overline{N} := N_{\overline{\mathbf{G}}^F}(\mathbf{L})$ . In the following, we identify  $W_{\mathbb{F}}$  with certain permutation groups  $\mathcal{S}_{\pm \underline{l}}$  via the action on  $\{\pm e_i \mid i \in \underline{l}\}$  and  $W_{\mathbb{F}}$  with  $\mathcal{S}_{\pm \underline{l}}^D$ . We generalize the notation of those permutation groups in order to describe  $N/L$ .

NOTATION 3.14 (Young-like subgroups,  $\mathcal{S}_M$  and  $\mathcal{Y}_J$ ). Let  $M$  be a set. Given a map  $\|\cdot\| : M \rightarrow \mathbb{Z}$  with  $m \mapsto \|m\|$ , we define  $\mathcal{S}_M$  to be the group of bijections  $\pi : M \rightarrow M$  with  $\|\pi(m)\| = \|m\|$  for every  $m \in M$  and we write  $\mathcal{S}_{\pm M}$  for the bijections  $\pi : \{\pm 1\} \times M \rightarrow \{\pm 1\} \times M$  satisfying  $\pi(-1, m) = (-\epsilon, m')$  and  $\|m\| = \|m'\|$ , whenever  $m, m' \in M$  with  $\pi(1, m) = (\epsilon, m')$ . When no map  $\|\cdot\|$  is specified, we assume it is a constant map.

In order to denote the elements of  $\mathcal{S}_M$  and  $\mathcal{S}_{\pm M}$ , we fix a bijection  $f : M \rightarrow \{1, \dots, |M|\}$ . This induces a canonical embedding  $\iota : \mathcal{S}_M \rightarrow \mathcal{S}_{|M|}$  and an embedding  $\iota_{\pm} : \mathcal{S}_{\pm M} \rightarrow \mathcal{S}_{\pm |M|}$ . For  $r$  pairwise distinct elements  $m_1, m_2, \dots, m_r \in M$ , we write  $(m_1, m_2, \dots, m_r) \in \mathcal{S}_M$  for the element  $\iota^{-1}(f(m_1), f(m_2), \dots, f(m_r))$ . Via  $\iota_{\pm}$ , we obtain also a cycle notation for elements of  $\mathcal{S}_{\pm M}$ .

If  $J$  is a partition of  $M$ , we write  $J \vdash M$  for short. For  $J \vdash M$ , we set

$$\begin{aligned} \mathcal{Y}_J &:= \{ \pi \in \mathcal{S}_M \mid \pi(J') = J' \text{ for every } J' \in J \}, \text{ and} \\ \mathcal{Y}_{\pm J} &:= \{ \pi \in \mathcal{S}_{\pm M} \mid \pi(\{\pm 1\} \times J') = \{\pm 1\} \times J' \text{ for every } J' \in J \}. \end{aligned}$$

Let  $M_{\text{odd}} := \{ m \in M \mid \|m\| \text{ odd} \}$  and

$$\mathcal{S}_{\pm M}^D = \{ \pi \in \mathcal{S}_{\pm M} \mid |(\{1\} \times M_{\text{odd}}) \cap \pi^{-1}(\{-1\} \times M_{\text{odd}})| \text{ is even} \}.$$

We use the above notation for permutation groups on the set  $\mathcal{O}$  from 3.8.

DEFINITION 3.15. Let  $\|\cdot\| : \mathcal{O} \rightarrow \mathbb{Z}$  be given by  $\|I\| = d$  for every  $I \in \mathcal{O}_d$ , and let  $\mathcal{S}_{\pm \mathcal{O}}$ ,  $\mathcal{S}_{\pm \mathcal{O}}^D$ , and  $\mathcal{S}_{\mathcal{O}}$  be the permutation groups on  $\mathcal{O}$  defined as in 3.14 with respect to  $\|\cdot\|$ .

Recall that we have chosen a maximal torus  $\mathbf{T}$  of  $\mathbf{G}$  and that  $\mathbf{L}$  is a standard Levi subgroup of  $\mathbf{G}$  with  $\mathbf{T} \subseteq \mathbf{L}$  (see 3.3 and 3.5). For  $\overline{\mathbf{N}}_0 := N_{\overline{\mathbf{G}}}(\mathbf{T})$ , we identify the Weyl group  $\overline{\mathbf{N}}_0/\mathbf{T}$  with  $\mathcal{S}_{\pm \underline{l}}$ , the epimorphism  $\rho_{\mathbf{T}} : \overline{\mathbf{N}}_0 \rightarrow \mathcal{S}_{\pm \underline{l}}$  is given by

$$\rho_{\mathbf{T}}(\mathbf{n}_{e_i}(-1)) = (i, -i) \text{ and } \rho_{\mathbf{T}}(\mathbf{n}_{e_i - e_j}(-1)) = (i, j)(-i, -j).$$

With this notation, we can compute the relative Weyl group of  $L$  in  $G$ . Recall  $N := N_{\mathbf{G}}(\mathbf{L})^F$ .

PROPOSITION 3.16. Let  $N_0 := N_{\mathbf{G}^F}(\mathbf{T})$ ,  $\overline{N}_0 := N_{\overline{\mathbf{G}}^F}(\mathbf{T})$ , and  $\overline{N} := N_{\overline{\mathbf{G}}^F}(\mathbf{L})$ . Then

$$\rho_{\mathbf{T}}(\overline{N} \cap N_0) / \rho_{\mathbf{T}}(L \cap N_0) \cong \mathcal{S}_{\pm \mathcal{O}} \text{ and } \rho_{\mathbf{T}}(N \cap N_0) / \rho_{\mathbf{T}}(L \cap N_0) \cong \mathcal{S}_{\pm \mathcal{O}}^D.$$

*Proof.* According to the considerations in [C, 9.2],  $\rho_{\mathbf{T}}(N \cap N_0)/\rho_{\mathbf{T}}(L \cap N_0) \cong N_{\overline{W}_0}(W_{\Phi'})/W_{\Phi'}$ , where  $\overline{W}_0 := \overline{N}_0/\overline{T}_0$ . We then make routine considerations inside  $\overline{W}_0$  (see, e.g., [H]). Note that  $N_{\overline{W}_0}(W_{\Phi'}) = \text{Stab}_{\overline{W}_0}(\Phi') = W_{\Phi'} \text{Stab}_{\overline{W}_0}(\Delta')$ .

From the definition of  $\Phi_{-1}$ , one can check that  $\text{Stab}_{\overline{W}_0}(\Delta')$  stabilizes  $\Phi_{-1} \cap \Delta'$ . This implies that  $\text{Stab}_{\overline{W}_0}(\Delta')$  stabilizes  $\Phi_d \cap \Delta'$  for every  $d \in \mathbb{D}$ , and

$$\text{Stab}_{W_{\overline{\Phi}_d}}(\Phi_d \cap \Delta') = \mathcal{S}_{\pm \mathcal{O}_d}.$$

We have  $N_{\overline{W}_0}(W_{\Phi'}) = W_{\overline{\Phi}_1} \times \prod_{d \in \mathbb{D}} \text{Stab}_{W_{\overline{\Phi}_d}}(\Phi_d) = \text{Stab}_{W_{\overline{\Phi}_{-1}}}(\Phi_{-1}) \times W_{\overline{\Phi}_1} \times \prod_{\substack{d \in \mathbb{D} \\ d > 1}} \text{Stab}_{W_{\overline{\Phi}_d}}(\Phi_d)$  with

$$\text{Stab}_{W_{\overline{\Phi}_{-1}}}(\Phi_{-1}) = W_{\Phi_{-1}} \langle (1, -1) \rangle = W_{\overline{\Phi}_{-1}},$$

and

$$\text{Stab}_{W_{\overline{\Phi}_d}}(\Phi_d) = W_{\Phi_d} \rtimes \mathcal{S}_{\pm \mathcal{O}_d}$$

for  $d \in \mathbb{D}$  with  $d > 1$ . Hence,  $\rho_{\mathbf{T}}(N \cap N_0)/\rho_{\mathbf{T}}(L \cap N_0) \cong \mathcal{S}_{\pm \mathcal{O}}^{\mathbb{D}}$ . □

By the proof, we see that  $\mathcal{S}_{\pm \mathcal{O}}$  corresponds to  $\text{Stab}_{\overline{W}_0}(\Phi' \cap \Delta)$  and hence there exists some embedding of  $\mathcal{S}_{\pm \mathcal{O}}$  into  $\mathcal{S}_{\pm \underline{l}}$ . We fix some more notation to describe explicitly the permutations in  $\mathcal{S}_{\pm \underline{l}}$  corresponding to  $\text{Stab}_{\overline{W}_0}(\Delta')$ .

NOTATION 3.17. For  $d \in \mathbb{D} \setminus \{-1\}$ , we fix orderings on  $\mathcal{O}_d$  and the sets  $I \in \mathcal{O}_d$ : we write  $I_{d,j}$  ( $j \in \underline{a}_d$ ) for the sets in  $\mathcal{O}_d$  and  $I_{d,j}(k) \in I_{d,j}$  ( $j \in \underline{a}_d, k \in \underline{d}$ ) for the elements of  $I_{d,j}$ .

For each  $k \in \underline{d}$ , let  $f_k^{(d)} : \underline{l} \rightarrow \underline{l}$  be a bijection such that  $f_k^{(d)}(j) = I_{d,j}(k)$  for every  $j \in \underline{a}_d$  and  $f_k^{(d)}$  has the maximal number of fixed points. Then  $f_k^{(d)}$  defines an element of  $\mathcal{S}_{\pm \underline{l}}$  without sign changes, that we also denote by  $f_k^{(d)}$  by abuse of notation.

In the following, we use that for every  $Q \subseteq \underline{l}$ ,  $\mathcal{S}_{\pm Q}$  can be seen naturally as a subgroup of  $\mathcal{S}_{\pm \underline{l}}$ .

LEMMA 3.18.

- (a) Let  $d \in \mathbb{D} \setminus \{-1\}$  and  $\overline{\kappa}_d : \mathcal{S}_{\pm \underline{a}_d} \rightarrow \mathcal{S}_{\pm J_d}$  be given by  $\pi \mapsto \prod_{k \in \underline{d}} \pi^{f_k^{(d)}}$  the latter a product of conjugates of  $\pi$  in  $\mathcal{S}_{\pm \underline{l}}$ . Then  $\overline{\kappa}_d$  is injective and  $\text{Stab}_{\mathcal{S}_{\pm J_d}}(\Phi_d) = W_{\Phi_d} \rtimes \overline{\kappa}_d(\mathcal{S}_{\pm \underline{a}_d})$
- (b) If  $-1 \in \mathbb{D}$ , let  $\overline{\kappa}_{-1} : \mathcal{S}_{\pm \underline{l}} \rightarrow \mathcal{S}_{\pm J_{-1}}$  be the morphism with  $\overline{\kappa}_{-1}(\mathcal{S}_{\pm \underline{l}}) = \langle (1, -1) \rangle$ . Let  $\overline{W}_d := \overline{\kappa}_d(\mathcal{S}_{\pm \underline{a}_d})$  and  $W^\circ(L) := \prod_{d \in \mathbb{D}} \overline{W}_d$ . Then  $\text{Stab}_{\overline{W}_0}(\Phi') = W_{\Phi'} W^\circ(L)$ .

*Proof.* For (a), we observe that the sets  $\bigcup_{j \in \underline{a}_d} I_{d,j}(k)$  ( $k \in \underline{d}$ ) form a partition of  $J_d$ . This implies that the groups  $\mathcal{S}_{\pm \underline{a}_d}^{f_k^{(d)}}$  and  $\mathcal{S}_{\pm \underline{a}_d}^{f_{k'}^{(d)}}$  commute and are disjoint. We see that  $\overline{\kappa}_d(\mathcal{S}_{\pm \underline{a}_d})$  stabilizes  $\mathcal{O}_d$ . This proves (a). Part (b) is clear from the definitions. □

We can choose  $I_{d,j}(k)$  ( $d \in \mathbb{D} \setminus \{\pm 1\}, j \in \underline{a}_d$ , and  $k \in \underline{d}$ ) such that  $e_{I_{d,j}(k+1)} - e_{I_{d,j}(k)} \in \Delta'$  for every  $j \in \underline{a}_d$  and  $k \in \underline{d} - 1$ . With this choice,  $\overline{\kappa}_d(\mathcal{S}_{\pm \underline{a}_d})$  stabilizes  $\Delta'$  and hence coincides with  $\text{Stab}_{\overline{W}_{\overline{\Phi}_d}}(\Delta')$ .

### 3.3 A supplement of $L$ in $\overline{N}$

In the following, we determine a supplement  $\overline{V} \leq \overline{N}_0$  with  $\overline{N} = L\overline{V}$  and  $\rho_{\mathbf{T}}(\overline{V}) = W^\circ(L)$  where  $W^\circ(L)$  is the group from Lemma 3.18. We construct the group  $\overline{V}$  using extended Weyl groups  $\overline{V}'_{Weyl}$  (see 3.19). Extended Weyl groups are known to be supplements of  $T_0$  in  $\overline{N}_0$ .

In a first step, we define for every  $d \in \mathbb{D}$  a subgroup  $\bar{V}_d \leq \bar{N}_0$  with  $\rho_{\mathbf{T}}(\bar{V}_d) = \bar{\kappa}_d(\mathcal{S}_{\pm a_d})$ . We construct  $\kappa_d$ , a lifting of  $\bar{\kappa}_d$  via  $\rho_{\mathbf{T}}$ . This construction will later simplify some arguments by providing a tool to transfer results from [MS].

By definition, the group  $\bar{\mathbf{N}}_0$  is an extension of  $\bar{W}_0$  by  $\mathbf{T}$ . It has proved to be more convenient to work with an extension of  $\bar{W}_0$  by an elementary abelian 2-group, the extended Weyl group (introduced first by Tits), here denoted by  $\bar{V}'_{Weyl}$ . (Note that if  $2 \mid q$ , the group  $\bar{\mathbf{N}}_0$  is the semi-direct product of  $\mathbf{T}$  and a group isomorphic to the Weyl group.) In consideration of Definition 3.4, we work here with the group  $\bar{V}_0$ , a  $\mathbf{T}$ -conjugate of  $\bar{V}'_{Weyl}$ . Then the graph automorphism of  $G$  is induced by an element of  $\bar{V}_0$  (see Definition 3.4).

NOTATION 3.19 (The groups  $\bar{V}_0, \bar{V}_I$ , and  $V_I$ ). The group  $\bar{V}'_{Weyl} := \langle \bar{\mathbf{n}}'_i \mid i \in \underline{l} \rangle$  with  $\bar{\mathbf{n}}'_1 := \mathbf{n}_{e_1}(1)$  and  $\bar{\mathbf{n}}'_i := \mathbf{n}_{\alpha_i}(-1)$ , where  $\alpha_i = e_i - e_{i-1}$  ( $2 \leq i \leq l$ ) is known as the extended Weyl group of type  $B_l$ .

Let  $\zeta_8 \in \mathbb{F}$  with  $\zeta_8^2 = \varpi$ . The group  $\bar{V}_0 := (\bar{V}'_{Weyl})^{\mathbf{h}_l(\zeta_8)}$  is accordingly generated by  $\bar{\mathbf{n}}_1 := (\bar{\mathbf{n}}'_1)^{\mathbf{h}_l(\zeta_8)} = \mathbf{n}_{e_1}(\varpi)$  and  $\bar{\mathbf{n}}_i := (\bar{\mathbf{n}}'_i)^{\mathbf{h}_l(\zeta_8)} = \mathbf{n}_{\alpha_i}(-1)$ . The group  $\bar{V}_0$  satisfies  $\bar{V}_0 \cap \bar{T}_0 = H_0$  where  $H_0$  is defined as  $\langle \mathbf{h}_{\alpha}(-1) \mid \alpha \in \Phi \rangle$  in 3.19. According to Definition 3.4,  $\bar{\mathbf{n}}_1 \in \bar{V}_0$  and  $\gamma$  induce the same automorphism of  $\mathbf{G}$ .

For  $I \subseteq l$ , we set

$$V_I := \langle h_0, \mathbf{n}_{\pm e_i \pm e_{i'}}(1) \mid i, i' \in I \text{ with } i \neq i' \rangle \text{ and } \bar{V}_I := V_I \langle \mathbf{n}_{e_i}(\varpi) \mid i \in I \rangle. \tag{3.1}$$

Let  $\tilde{H}_I := \langle \mathbf{h}_{e_i}(\varpi) \mid i \in I \rangle$  and  $\tilde{H}_0 := \tilde{H}_l$ .

3.20 (Facts around  $H_I \triangleleft \bar{V}_I$ ). Maximal extendibility holds with respect to  $H_I \triangleleft \bar{V}_I$ , since those groups are conjugate to those considered in [MS, Prop. 3.8] for the case where the underlying root system is of type  $B_{|I|}$ . For  $H_I := \langle h_0, \mathbf{h}_{\pm e_i \pm e_{i'}}(-1) \mid i, i' \in I \rangle$ , we obtain  $\bar{V}_I \cap \mathbf{T} = H_I$ .

For disjoint sets  $I, I' \subseteq l$ , the Steinberg relations imply

$$[\bar{V}_I, V_{I'}] = 1 \text{ and } [\bar{V}_I, \bar{V}_{I'}] = \langle h_0 \rangle. \tag{3.2}$$

We introduce maps  $\kappa_d : \tilde{H}_{a_d} V_{a_d} \rightarrow \tilde{H}_0 \bar{V}_0$  with  $\rho_{\mathbf{T}} \circ \kappa_d = \bar{\kappa}_d \circ \rho_{a_d}$  for the canonical epimorphism  $\rho_{a_d} : \bar{V}_{a_d} \rightarrow \mathcal{S}_{\pm a_d}$ .

The following defines a lift of  $\bar{W}_d := \bar{\kappa}_d(\mathcal{S}_{\pm a_d})$  that is a subgroup of  $\bar{V}_0$ . In 3.17, we introduced the elements  $f_k^{(d)} \in \mathcal{S}_{\pm l}$  ( $d \in \mathbb{D} \setminus \{-1\}, k \in a_d$ ) without sign changes.

LEMMA 3.21. Let  $d \in \mathbb{D} \setminus \{-1\}, m_k^{(d)} \in \bar{V} \cap \rho_{\mathbf{T}}^{-1}(f_k^{(d)})$  ( $k \in a_d$ ), and

$$\kappa_d : \tilde{H}_{a_d} \bar{V}_{a_d} \rightarrow \tilde{H}_0 \bar{V}_0 \text{ with } x \mapsto \prod_{k=1}^d x^{m_k^{(d)}}$$

for a fixed order in  $a_d$ . Set  $\bar{V}_d := \langle \kappa_d(\bar{V}_{a_d}) \rangle$ . Then:

- (a)  $\kappa_d|_{V_{a_d}}$  is a morphism of groups;
- (b)  $\kappa_d(v^x) = \kappa_d(v)^{\kappa_d(x)}$  for every  $x \in \bar{V}_{a_d}$  and  $v \in V_{a_d}$ ;
- (c)  $\kappa_d(H_{a_d}) = \langle h_0^d, \mathbf{h}_I(\varpi) \mathbf{h}_{I'}(-\varpi) \mid I, I' \in \mathcal{O}_d \rangle \leq H_d$ ;
- (d)  $\kappa_d(\tilde{H}_{a_d}) = \langle h_0^d, \mathbf{h}_I(\varpi) \mid I \in \mathcal{O}_d \rangle$ ;
- (e)  $\rho_{\mathbf{T}} \circ \kappa_d = \bar{\kappa}_d \circ \rho_{a_d}$  for the canonical epimorphism  $\rho_{a_d} : \bar{V}_{a_d} \rightarrow \mathcal{S}_{\pm a_d}$ , in particular  $\rho_{\mathbf{T}}(\bar{V}_d) = \bar{W}_d = \kappa_d(\mathcal{S}_{\pm a_d})$ .

*Proof.* The sets  $J_d(k) := f_k^{(d)}(a_d)$  form a partition of  $J_d$ . For  $x \in V_{a_d}$ , we see  $x^{m_k^{(d)}} \in V_{J_d(k)}$  and hence  $\kappa_d|_{V_{a_d}}$  is independent of the order chosen in  $\underline{d}$ . Then  $\kappa_d|_{V_{a_d}}$  is a diagonal embedding of  $V_{a_d}$  into the central product of the groups  $V_{J_d(k)}$  ( $k \in \underline{d}$ ) over  $\langle h_0 \rangle$ . This implies that  $\kappa_d|_{V_{a_d}}$  is a morphism of groups. This proves (a).

By part (a), it is enough to prove part (b) for  $x = \bar{\mathbf{n}}_1$  and  $v \in \{\bar{\mathbf{n}}_2^{\bar{\mathbf{n}}_1}, \bar{\mathbf{n}}_2, \bar{\mathbf{n}}_3, \dots, \bar{\mathbf{n}}_{a_d}\}$ , since  $V_{a_d}$  is generated by  $\{\bar{\mathbf{n}}_2^{\bar{\mathbf{n}}_1}, \bar{\mathbf{n}}_2, \bar{\mathbf{n}}_3, \dots, \bar{\mathbf{n}}_{a_d}\}$ . The equation  $\kappa_d(\bar{\mathbf{n}}_i^{\bar{\mathbf{n}}_1}) = \kappa_d(\bar{\mathbf{n}}_i)^{\kappa_d(\bar{\mathbf{n}}_1)}$  for  $i \geq 3$  is clear since no nontrivial linear combination of those roots is a root. Computations show  $\kappa_d(\bar{\mathbf{n}}_2^{\bar{\mathbf{n}}_1}) = \kappa_d(\bar{\mathbf{n}}_2)^{\kappa_d(\bar{\mathbf{n}}_1)}$  and hence part (b).

For part (c), we note that  $\ker(\kappa_d|_{H_{a_d}}) = \langle h_0^{d-1} \rangle$ . The equation  $\rho_{\mathbf{T}} \circ \kappa_d = \bar{\kappa}_d \circ \rho_{a_d}$  in (e) follows from  $\rho_{\mathbf{T}}(m_k^{(d)}) = f_k^{(d)}$ . □

Recall that the group  $H$  from Notation 3.9 is a subgroup associated with  $\mathbf{L}$ . To understand the above construction, we consider the following statement.

**THEOREM 3.22.** *If  $-1 \in \mathbb{D}$ , set  $\bar{V}_{-1} := \langle H_{-1}, \bar{\mathbf{n}}_1 \rangle$ . Let  $\bar{V} := H \langle \bar{V}_d \mid d \in \mathbb{D} \rangle$  and  $V_{\mathbb{D}} := \bar{V} \cap \mathbf{G}$ .*

- (a)  $N = LV_{\mathbb{D}}$ .
- (b) If  $\gamma \in E_L$ , then  $\bar{\mathbf{n}}_1 \in \bar{V}$ .

*Proof.* Because of  $\rho_{\mathbf{T}}(\bar{V}_d) = \bar{W}_d$ , we see  $\rho_{\mathbf{T}}(\bar{V}) = W^\circ(L)$ . Clearly,  $V_{\mathbb{D}} \leq N$ . Additionally, we see that  $\bar{V}$  normalizes  $L$  and  $\mathbf{L}$  by definition. If  $\mathbf{L}$  is  $\gamma$ -stable, then  $\bar{\mathbf{n}}_1 \in \bar{V}$ . According to Definition 3.4,  $\bar{\mathbf{n}}_1$  and  $\gamma$  induce the same automorphism of  $\mathbf{G}$ . □

**COROLLARY 3.23.** *The groups  $K, K_0$ , and  $H$  from Notation 3.9 and Lemma 3.11 together with  $V := V_{\mathbb{D}}, M := KV$ , and  $D := E_L$  satisfy Assumption 3.2(i).*

*Proof.* According to Lemma 3.11,  $K = HK_0$  and  $H \leq Z(K)$ . This is Assumption 3.2(i.1). The equality  $M = KV$  follows from the definition of  $M$ . In order to prove  $H = V \cap K$ , we show  $\bar{V}_d \cap \mathbf{L} \leq H_d$  for every  $d \in \mathbb{D}$ . Since  $\bar{V}_d \leq \bar{V}_0$  by construction and  $\bar{V}_d := \langle \kappa_d(\bar{V}_{a_d}) \rangle = \bar{W}_d$ , we observe that  $\kappa_d(H_{a_d}) \leq H_d$  according to Lemma 3.21(c).

By the construction,  $V_{\mathbb{D}}$  is  $\bar{V}\langle F_p \rangle$ -stable and hence  $E_L$ -stable. By the construction, we also see that  $K_0$  and  $V_{\mathbb{D}}$  are  $D$ -stable. □

### §4. Extending cuspidal characters of Levi subgroups

This section now focuses on the character theory of our groups. We ensure the character-theoretic Assumption 3.2(ii) and apply Proposition 3.2 in the proof of Theorem 3.1(b). We analyze the action of  $\bar{V}$  on  $K_0$  and consider subgroups of  $\bar{V}$  and  $L$  associated with each  $d \in \mathbb{D}$ . For every  $d \in \mathbb{D}$ , we define subgroups  $K_{0,d}$  and  $K_d$  and study them separately for  $d = 1, d \geq 2$ , and  $d = -1$ .

#### 4.1 The inclusion $H_1 \triangleleft \bar{V}_1$

We recall here some results on the extended Weyl groups. If  $1 \in \mathbb{D}$ , then  $H_1 \cong H_{J_1}$  and  $\bar{V}_1 \cong \bar{V}_{J_1}$  for the group  $\bar{V}_{J_1}$  from 3.19. (Recall  $J_d = \bigcup_{I \in \mathcal{O}_d} I$  for  $d \in \mathbb{D}$  [see Notation 3.9].) We set  $K_{0,1} := 1$  and  $K_1 := H_1$ . In order to apply Proposition 3.2, we investigate the Clifford theory for  $H_1 \triangleleft \bar{V}_1$ . The results are also relevant for studying  $H_d \triangleleft \bar{V}_d$  for  $d \geq 1$ .

**PROPOSITION 4.1.** *Let  $l' \leq l$  be some positive integer,  $\tilde{H}' := \tilde{H}_{l'}$ ,  $H' := \tilde{H}' \cap H_0$ ,  $\bar{V}' := \bar{V}_{l'}$ , and  $\rho' : \bar{V}' \rightarrow \mathcal{S}_{\pm l'}$  the canonical epimorphism.*

- (a) *Maximal extendibility holds with respect to  $H' \triangleleft \bar{V}'$ .*
- (b) *Let  $\lambda \in \text{Irr}(H')$  with  $h_0 \notin \ker(\lambda)$ . Then some  $\bar{V}'$ -conjugate  $\lambda'$  of  $\lambda$  has an extension  $\tilde{\lambda}'$  to  $\tilde{H}'$  such that  $\rho'(\bar{V}'_{\tilde{\lambda}'}) = S_{l'}$  and  $\bar{V}'_{\tilde{\lambda}'} = \bar{V}'_{\tilde{\lambda}'}\langle c \rangle$  for some  $c \in \bar{V}'$  with  $\rho_{\mathbf{T}}(c) = \prod_{i \in l'}(i, -i)$ .*

*Proof.* By [MS, Prop. 3.10], maximal extendibility holds with respect to  $H' \triangleleft \bar{V}'$ . Note that  $\bar{V}'$  coincides with the group  $\bar{V}^t$  considered in [MS]. This proves part (a).

Let  $\lambda \in \text{Irr}(H')$  with  $\lambda(h_0) = -1$  and  $\tilde{\lambda} \in \text{Irr}(\tilde{H}' \mid \lambda)$ . Note that  $\tilde{H}'$  is the  $l'$ -fold central product of the cyclic groups  $\langle \mathbf{h}_{e_i}(\varpi) \rangle$  ( $i \in l'$ ) over  $\langle h_0 \rangle$ . The group  $\bar{V}$  acts by permutation and inversion on the factors. It is then easy to see that some  $\bar{V}'$ -conjugate  $\lambda'$  of  $\lambda$  has an extension  $\tilde{\lambda}' \in \text{Irr}(\tilde{H}')$  such that

$$\tilde{\lambda}'(\mathbf{h}_{e_i}(\varpi)) = \tilde{\lambda}'(\mathbf{h}_{e_{i'}}(\varpi)) \text{ for every } i, i' \in l'.$$

The other extension of  $\lambda'$  to  $\tilde{H}$  is  $(\tilde{\lambda}')^{-1}$ . Observe that  $(\tilde{\lambda}')^2$  is the character with kernel  $H'$ . (Recall  $\tilde{H}'/H' \cong C_2$  and hence there is exactly one character with this property.) The element  $c \in \bar{V}'$  with  $\rho'(c) = \prod_{i=1}^{l'}(i, -i)$  satisfies  $(\tilde{\lambda}')^c = (\tilde{\lambda}')^{-1}$  and hence  $\bar{V}'_{\lambda'} = \bar{V}'_{\tilde{\lambda}'}\langle c \rangle$ . According to (a), there exists some extension  $\phi_0$  of  $\lambda'$  to  $\bar{V}_{\lambda'}$ . Then  $\phi_0|_{\bar{V}'_{\lambda'}}$  and  $\tilde{\lambda}'$  determine a common extension  $\phi$  to  $\tilde{H}'\bar{V}'_{\lambda'}$  (see [S3, 4.1(a)]). By this construction,  $\phi|_{\bar{V}'_{\lambda'}}$  is  $c$ -stable. □

#### 4.2 The inclusion $K_d \triangleleft K_d \bar{V}_d$ for $d \geq 2$

In the following, we investigate the groups  $K_d := H_d \langle G_I \mid I \in \mathcal{O}_d \rangle$  and  $K_d \bar{V}_d$  for  $d \in \mathbb{D} \setminus \{\pm 1\}$ , where  $\mathbf{G}_I = \langle \mathbf{X}_\alpha \mid \alpha \in \Phi_I \rangle$  and  $G_I = \mathbf{G}_I^F$  (see Notation 3.9 and Lemma 3.13).

LEMMA 4.2. *Let  $I \in \mathcal{O} \setminus (\{J_{-1}\} \cup \mathcal{O}_1)$  and  $\mathbf{Z}_I := \mathbf{h}_I(\mathbb{F}^\times)$ . Then:*

- (a)  $\mathbf{G}_I \cong \text{SL}_{|I|}(\mathbb{F})$  and  $G_I \cong \text{SL}_{|I|}(q)$ ;
- (b)  $\mathbf{L}_I = \mathbf{G}_I \cdot \mathbf{Z}_I$ ,  $\mathbf{L}_I / \langle h_0 \rangle \cong \text{GL}_{|I|}(\mathbb{F})$  and  $|\mathbf{G}_I \cap \mathbf{Z}_I| = \frac{|I|_{p'}}{\gcd(2, |I|)}$ ; and
- (c)  $L_I \cong \text{GL}_{|I|}(q)$  if  $2 \nmid |I|$ .

*Proof.* By the assumptions,  $d := |I| \geq 1$  and  $\Phi_I$  is a root system of type  $A_d$ . One has  $\mathbf{G}_I = [\mathbf{T}\mathbf{G}_I, \mathbf{T}\mathbf{G}_I]$  where  $\mathbf{T}\mathbf{G}_I$  is a Levi subgroup, so  $\mathbf{G}_I$  is simply connected  $\cong \text{SL}_{|I|}(\mathbb{F})$  by [MT, 12.14]. Note  $I \neq J_{-1}$ . This gives (a).

Any element of  $\mathbf{T}_I$  can be written as  $\prod_{i \in I} \mathbf{h}_{e_i}(t_i)$  for some  $t_i \in \mathbb{F}^\times$ . Let  $\kappa \in \mathbb{F}$  with  $\kappa^{|I|} = \prod_{i \in I} t_i$  and fix  $j \in I$ . Then, by the Chevalley relations in  $\bar{\mathbf{G}}$ ,

$$\begin{aligned} \prod_{i \in I} \mathbf{h}_{e_i}(t_i) &= \mathbf{h}_{e_j}(t_j \kappa^{-1}) \mathbf{h}_{e_j}(\kappa) \prod_{\substack{i \in I \\ i \neq j}} (\mathbf{h}_{e_j}(t_i^{-1} \kappa)^{-1} \mathbf{h}_{e_j}(t_i^{-1} \kappa) \mathbf{h}_{e_i}(t_i \kappa^{-1}) \mathbf{h}_{e_i}(\kappa)) = \\ &= \left( \mathbf{h}_{e_j}(\kappa^{-|I|} \prod_{i \in I} t_i) \right) \mathbf{h}_{e_j}(\kappa) \prod_{\substack{i \in I \\ i \neq j}} (\mathbf{h}_{e_i - e_j}((t_i \kappa^{-1})^2) \mathbf{h}_{e_i}(\kappa)) = \\ &= \left( \prod_{\substack{i \in I \\ i \neq j}} \mathbf{h}_{e_i - e_j}((t_i \kappa^{-1})^2) \right) \mathbf{h}_I(\kappa). \end{aligned}$$

Accordingly,  $\mathbf{T}_I = (\mathbf{T}_I \cap \mathbf{G}_I) \mathbf{Z}_I$ . We note that  $\mathbf{G}_I \cong \text{SL}_{|I|}(\mathbb{F})$  and  $G_I \cong \text{SL}_{|I|}(q)$  as  $F$  acts on  $\mathbf{G}_I$  as standard Frobenius endomorphism. By the Chevalley relations,  $\mathbf{Z}_I \leq C_{\mathbf{L}}(\mathbf{G}_I)$  and  $\mathbf{L}_I = \mathbf{T}_I \mathbf{G}_I = \mathbf{Z}_I \mathbf{G}_I$ .

The calculations above show that an element of  $\mathbf{Z}_I \cap \mathbf{G}_I$  can be written as  $\prod_{i \in I} \mathbf{h}_{e_i}(t)$  with  $t^{|I|} = 1$ . For  $d \in \mathbb{D}_{\text{even}}$ , the element  $\prod_{i=1}^l \mathbf{h}_{e_i}(-1)$  is trivial and hence  $|\mathbf{Z}_I \cap \mathbf{G}_I| = \frac{|I|_{p'}}{\gcd(2, |I|_{p'})}$ .

If  $2 \mid d$ , then with similar considerations as above, we see

$$h_0 = \mathbf{h}_I(\zeta) \prod_{\substack{i \in I \\ i \neq j}} \mathbf{h}_{e_i - e_j}(\zeta^{-2}), \tag{4.1}$$

where  $\zeta \in \mathbb{F}^\times$  has order  $2|I|_2$ . This implies that  $\mathbf{L}_I / \langle h_0 \rangle$  is the central product of the one-dimensional torus  $\mathbf{Z}_I / \langle h_0 \rangle$  with  $\mathbf{G}_I / \langle h_0 \rangle$  over  $Z(\mathbf{G}_I)$ . Accordingly,  $\mathbf{L}_I / \langle h_0 \rangle \cong \text{GL}_{|I|}(\mathbb{F})$ .

For odd  $d$ , this implies analogously  $\mathbf{L}_I \cong \text{GL}_d(\mathbb{F})$  and  $L_I \cong \text{GL}_d(q)$ . This is the statement in (b) and (c). We could also have argued on Levi subgroups of  $\mathbf{G} / \langle h_0 \rangle = \text{SO}_{2l}(\mathbb{F})$ .  $\square$

Next, we study how  $\tilde{L}$  acts on  $K_{0,d}$ , which includes the action induced by  $t_I$  ( $I \in \mathcal{O}$ ) and  $t_{l,2}$  from Lemma 3.13. Recall that  $\widehat{L}_I := \mathbf{L}_I \cap \mathcal{L}^{-1}(\langle h_0 \rangle)$  satisfies  $\widehat{L}_I = \langle L_I, t_I \rangle$  for some  $t_I \in \mathbf{T}_I \cap \mathcal{L}^{-1}(h_0)$ , and  $\mathcal{L}^{-1}(\mathbf{h}_l(\varpi)) \cap \mathbf{L} = \langle \widehat{L}, t_{l,2} \rangle$  with  $t_{l,2} = \mathbf{h}_l(\zeta)$ . According to Remark 2.16(b), diagonal automorphisms of  $G_I$  are parametrized by  $Z(\mathbf{G}_I) / [Z(\mathbf{G}_I), F]$ .

LEMMA 4.3. *Let  $I \in \mathcal{O} \setminus (\mathcal{O}_1 \cup \{J_{-1}\})$ .*

- (a) *If  $2 \nmid |I|$ , then  $\widehat{L}_I = L_I \mathbf{C}_{\mathbf{T}_I}(L_I)$ , in particular  $t_I$  from Lemma 3.13(b) can be chosen such that  $t_I \in \mathbf{C}_{\mathbf{T}_I}(L_I)$ .*
- (b) *For  $2 \mid |I|$ , the element  $t_I$  induces on  $G_I$  a diagonal automorphism corresponding to  $g[Z(\mathbf{G}_I), F]$  with  $g \in Z(\mathbf{G}_I)$  of order  $|Z(\mathbf{G}_I)|_2$ .*
- (c)  *$t_{l,2} \in \mathbf{C}_{\tilde{L}}(L_I)$ .*

*Proof.* Keep  $d := |I|$ . According to the theorem of Lang, we can choose  $t_I \in \mathbf{T}_I$  such that  $t_I^{-1} F(t_I) = h_0$  as  $\mathbf{T}_I$  is connected.

If  $2 \nmid |I|$ , we see that  $h_0 = \mathbf{h}_I(-1)$  and hence  $h_0 \in \mathbf{Z}_I$ . Since  $\mathbf{Z}_I$  is again connected  $t_I$  can be chosen in  $Z(\mathbf{L}_I)$ , whence (a).

Following (4.1),  $h_0 = z_1 z_2$  for some  $z_1 \in \mathbf{Z}_I$  and  $z_2 \in Z(\mathbf{G}_I)$ . Here,  $z_2$  is an element of order  $|I|_2 = d_2$ . Then the element  $t_I$  can be analogously written as  $z g$  with  $z \in Z(\mathbf{G}_I)$  and  $g \in \mathbf{G}$  such that  $\mathcal{L}(z) = z_1$  and  $\mathcal{L}(g) = z_2$ . As  $g$  induces on  $\mathbf{G}$  a diagonal automorphism associated with  $z_2[Z(\mathbf{G}_I), F]$ , the element  $t_I \in \widehat{\mathbf{G}}_I$  with  $x^{-1} F(x) = h_0$  induces the same diagonal automorphism. This gives (b).

The element  $t_{l,2} = \mathbf{h}_l(\zeta)$  from Lemma 3.13(e) centralizes  $\mathbf{G}_I$  since the Weyl group of  $\mathbf{G}_I$  centralizes  $t_{l,2}$ .  $\square$

Recall the groups  $\tilde{H}_d = \langle h_0, \mathbf{h}_I(\varpi) \mid I \in \mathcal{O}_d \rangle$ ,  $H_0 = \langle \mathbf{h}_\alpha(-1) \mid \alpha \in \Phi \rangle$ , and  $H_d = \tilde{H}_d \cap H_0$  defined in Notation 3.9 for every  $d \in \mathbb{D}$ . Using the groups  $G_I$  from Lemma 3.13, let  $K_{0,d} := \langle G_I \mid I \in \mathcal{O}_d \rangle$  and  $K_d := H_d K_{0,d}$ . If  $\mathbb{D} = \{d\}$ , then  $K_0 = K_{0,d}$  and  $K = K_d$ . As  $\overline{V}_d \cap K_d \leq \mathbf{C}_{\mathbf{L}}(\mathbf{G}_I)$  as a consequence of Lemma 3.11, there is a well-defined action of  $\overline{V}_d / H_d$  on  $K_d$ .

LEMMA 4.4 (The action of  $V_d$  on  $K_{0,d}$ ). *Let  $d \in \mathbb{D}$ . Let  $\epsilon_d : \overline{V}_d \rightarrow \overline{V}_d / H_d$  be the canonical morphism and  $\overline{\mathbf{n}}_1^{(d)} := \kappa_d(\overline{\mathbf{n}}_1)$ . Then:*

- (a)  $K_{0,d} \rtimes \epsilon(\overline{V}_d) \cong (G_{I_{d,1}} \rtimes \langle \epsilon(\overline{\mathbf{n}}_1^{(d)}) \rangle) \wr \mathcal{S}_{a_d}$ .
- (b) *Then  $\overline{\mathbf{n}}_1^{(d)}$  induces the graph automorphism transpose-inverse on  $G_{I_{d,1}}$ .*
- (c) *If  $2 \nmid d$ ,  $\overline{\mathbf{n}}_1^{(d)}$  induces on  $L_{I_{d,1}}$  a product of a nontrivial graph and an inner automorphism via the isomorphism  $L_{I_{d,1}} \cong \text{GL}_d(q)$  from Lemma 4.2(c).*

*Proof.* Part (a) follows from the Steinberg presentation.

For part (b), we see that  $G_{I_{d,1}}^{\bar{\mathbf{n}}_1^{(d)}} = G_{I_{d,1}}$  and  $G_{I_{d,1}} \cap \langle h_0 \rangle = \{1\}$  from the Chevalley relations. We compute the action of  $\bar{\mathbf{n}}_1^{(d)}$  on  $G_{I_{d,1}}$  in the quotient  $G/\langle h_0 \rangle$  or  $G_{I_{d,1}} \times \langle h_0 \rangle / \langle h_0 \rangle$ , respectively. In [GLS, 2.7], the group  $\mathbf{G}/\langle h_0 \rangle$  and its Steinberg generators are given explicitly as subgroup and elements of  $\mathrm{SO}_{2l}(q)$ . The element  $\bar{\mathbf{n}}_1^{(d)}$  acts on  $\mathbf{G}_{I_{d,1}} \langle h_0 \rangle / \langle h_0 \rangle$  by transpose-inverse via  $G_{I_{d,1}} \cong \mathrm{SL}_d(q)$ . Computations in that group show part (b).

The element  $\bar{\mathbf{n}}_1^{(d)}$  acts by inversion on  $\mathbf{Z}_{I_{d,1}}$  and hence  $\bar{\mathbf{n}}_1^{(d)}$  satisfies the statement in part (c) as  $\mathbf{L}_{I_{d,1}} = \mathbf{G}_{I_{d,1}} \mathbf{Z}_{I_{d,1}}$ . □

Next, we study an analog of  $\tilde{L}$  from Lemma 3.13 associated with  $d \in \mathbb{D}$ , that is defined using the Lang map  $\mathcal{L}$  from there. Note that  $\mathbf{h}_l(\varpi) \notin \mathbf{T}_{J_d}$  whenever  $\mathbb{D} \neq \{d\}$ , but  $\mathbf{h}_l(\varpi) = \prod_{d \in \mathbb{D}} \mathbf{h}_{J_d}(\varpi)$ .

**PROPOSITION 4.5.** *Let  $d \in \mathbb{D} \setminus \{\pm 1\}$ , let  $\epsilon_d : \bar{V}_d \rightarrow \bar{V}_d/H_d$  be as in Lemma 4.4, let  $\tilde{T}_d := \mathbf{T}_{J_d} \cap \mathcal{L}^{-1}(\langle h_0, \mathbf{h}_{J_d}(\varpi) \rangle)$ , and let  $\tilde{L}_d := \tilde{T}_d K_{0,d}$ . Then:*

- (a) *There exists some  $\bar{V}_d \langle F_p \rangle$ -stable  $\tilde{L}_d$ -transversal  $\mathbb{T}_d^\circ$  in  $\mathrm{Irr}_{\mathrm{cusp}}(K_{0,d})$ .*
- (b) *There exists an  $\epsilon_d(\bar{V}_d) \times \langle F_p \rangle$ -equivariant extension map  $\Lambda_{\epsilon_d}$  with respect to  $K_{0,d} \triangleleft K_{0,d} \rtimes \epsilon_d(V_d)$  for  $\mathbb{T}_d^\circ$ .*
- (c) *Maximal extendibility holds with respect to  $K_{0,d} \triangleleft \tilde{L}_d$  and  $K_d \triangleleft \tilde{L}_d$ .*

For the proof of part (b), we require a strengthening of a result on wreath products that can, for example, be found in [K, Th. 2.10].

**LEMMA 4.6.** *Let  $X \rtimes Y$  be a finite group, and let  $A$  be a group of automorphisms of  $X \rtimes Y$ , stabilizing  $X, Y$  and some  $\mathbb{K} \subseteq \mathrm{Irr}(X)$ . Let  $a$  be a positive integer. Note that  $A$  acts on  $X^a \triangleleft (X \rtimes Y) \wr \mathcal{S}_a$  by diagonally acting on  $(X \rtimes Y)^a$  and trivially on  $\mathcal{S}_a$ . In this context, we write then  $\Delta A$  for that group. If there exists an  $(X \rtimes Y) \rtimes A$ -equivariant extension map with respect to  $X \triangleleft X \rtimes Y$  for  $\mathbb{K}$ , then there exists an  $((X \rtimes Y) \wr \mathcal{S}_a) \rtimes \Delta A$ -equivariant extension map with respect to  $X^a \triangleleft (X \rtimes Y) \wr \mathcal{S}_a$  for  $\mathbb{K}^a := \{\chi_1 \otimes \dots \otimes \chi_a \mid \chi_i \in \mathbb{K}\}$ .*

*Proof.* This follows by the considerations in the proof of [K, Th. 2.10] using the construction of representations of wreath products given in [N, 10.1]. □

*Proof of Proposition 4.5.* Let  $I_1 := I_{d,1}$ . Via the isomorphism  $G_{I_1} \cong \mathrm{SL}_d(q)$  from Lemma 4.2, the  $E(\mathrm{SL}_d(q))$ -stable  $\mathrm{GL}_d(q)$ -transversal in  $\mathrm{Irr}(\mathrm{SL}_d(q))$  from Proposition 2.19(b) determines a subset  $\mathbb{T}_{I_1} \subseteq \mathrm{Irr}_{\mathrm{cusp}}(G_{I_1})$ . According to Lemma 4.4(b), this set is  $\bar{\mathbf{n}}_1^{(d)}$ -stable. The  $E(\mathrm{SL}_d(q))$ -stable  $\mathrm{GL}_d(q)$ -transversal in  $\mathrm{Irr}(\mathrm{SL}_d(q))$  can even be chosen such that each character extends to its inertia group in  $\mathrm{SL}_d(q) \rtimes E(\mathrm{SL}_d(q))$ . Accordingly, maximal extendibility holds with respect to  $G_{I_1} \triangleleft G_{I_1} \rtimes \langle F_p, \epsilon_d(\bar{\mathbf{n}}_1^{(d)}) \rangle$  for  $\mathbb{T}_{I_1}$ .

Note that  $\mathbb{T}_{I_1}$  is  $N_{\bar{V}_d}(G_{I_1})$ -stable, as  $N_{\bar{V}_d}(G_{I_1})$  acts as  $\langle \bar{\mathbf{n}}_1^d \rangle$ . Accordingly, via conjugation with elements of  $\bar{V}_d$ , the set  $\mathbb{T}_{I_1}$  determines characters  $\mathbb{T}_I \subseteq \mathrm{Irr}_{\mathrm{cusp}}(G_I)$  for every  $I \in \mathcal{O}_d$ . Recall that by Lemma 3.13(d), the group  $K_0$  is the direct product of the groups  $G_I$  ( $I \in \mathcal{O}$ ). Analogously,  $K_{0,d}$  is the direct product of the groups  $G_I$  ( $I \in \mathcal{O}_d$ ).

The product  $\mathbb{T}_d^\circ$  of these characters  $\prod_{I \in \mathcal{O}_d} \mathbb{T}_I$  defines a  $\bar{V}_d \langle F_p \rangle$ -stable set in  $\mathrm{Irr}_{\mathrm{cusp}}(K_{0,d})$ . By this construction,  $\mathbb{T}_d^\circ$  is  $\bar{V}_d \langle F_p \rangle$ -stable. Following the description of the action of  $\tilde{L}$  on  $K_{0,d}$  given in Lemma 4.3, we see that  $\mathbb{T}_d^\circ$  is an  $\tilde{L}_d$ -transversal in  $\mathrm{Irr}_{\mathrm{cusp}}(K_{0,d})$ . This proves part (a).

Recall  $K_{0,d} \rtimes \epsilon_d(\overline{V}_d) \cong (G_{I_1} \rtimes \langle \epsilon(\overline{\mathbf{n}}_1^{(d)}) \rangle) \wr \mathcal{S}_{a_d}$  from Lemma 4.4. As stated above, maximal extendibility holds with respect to  $G_{I_1} \triangleleft G_{I_1} \rtimes \langle F_p, \epsilon(\overline{\mathbf{n}}_1^{(d)}) \rangle$  for  $\mathbb{T}_{I_1}$ . According to Lemma 4.6, this implies by the choice of  $\mathbb{T}_d^\circ$  that there is an  $\epsilon_d(\overline{V}_d)\langle F_p \rangle$ -equivariant extension map  $\Lambda_{\epsilon_d}$  with respect to  $K_{0,d} \triangleleft K_{0,d} \rtimes \epsilon_d(\overline{V}_d)$  for  $\mathbb{T}_d^\circ$ .

According to Theorem 2.17, maximal extendibility holds with respect to  $G_I \triangleleft \tilde{G}_I$  where  $\tilde{G}_I := \mathbf{G}_I \cap \mathcal{L}^{-1}(\mathbf{Z}(\mathbf{G}_I))$ . Additionally,  $[\mathbf{G}_I, \mathbf{Z}_I] = 1$  for  $\mathbf{Z}_I := \mathbf{h}_I(\mathbb{F}^\times)$  from Lemma 4.2. We observe that  $\tilde{L}_d \leq \langle \tilde{G}_I \mid I \in \mathcal{O}_d \rangle \langle \mathbf{Z}_I \mid I \in \mathcal{O}_d \rangle$ , even more precisely

$$\tilde{L}_d \leq \langle \tilde{G}_I \mid I \in \mathcal{O}_d \rangle \langle \hat{\mathbf{Z}}_I \mid I \in \mathcal{O}_d \rangle,$$

where  $\hat{\mathbf{Z}}_I := \mathcal{L}^{-1}(\mathbf{Z}(\mathbf{G}_I) \cap \mathbf{Z}_I) \cap \mathbf{Z}_I$ . We see that maximal extendibility holds with respect to  $K_{0,d} \triangleleft \langle \tilde{G}_I \mid I \in \mathcal{O}_d \rangle \langle \hat{\mathbf{Z}}_I \mid I \in \mathcal{O}_d \rangle$ . Hence, maximal extendibility holds with respect to  $K_{0,d} \triangleleft \tilde{L}_d$  and  $K_d \triangleleft \tilde{L}_d$ , as  $\tilde{L}_d/K_{0,d}$  is abelian. □

LEMMA 4.7. *Let  $d \in \mathbb{D} \setminus \{\pm 1\}$ .*

- (a) *Maximal extendibility holds with respect to  $H_d \triangleleft \overline{V}_d$ .*
- (b) *If  $2 \nmid d$ ,  $\lambda \in \text{Irr}(H_d)$  with  $\lambda(h_0) = -1$ , and  $\tilde{\lambda} \in \text{Irr}(\tilde{H}_d \mid \lambda)$ , then  $(\overline{V}_d)_{\tilde{\lambda}} \leq V_D$  and  $(\overline{V}_d)_\lambda = (\overline{V}_d)_{\tilde{\lambda}} \langle c_d \rangle$  for some  $c_d \in \overline{V}_d$ .*

*Proof.* Recall that by [MS, Prop. 3.8] maximal extendibility holds with respect to  $H_{a_d} \triangleleft \overline{V}_{a_d}$ . Via the map  $\kappa_d: \overline{V}_{a_d} \rightarrow \overline{V}_d$  from Lemma 3.21, the maximal extendibility with respect to  $H_{a_d} \triangleleft \overline{V}_{a_d}$  gives a  $\overline{V}_d$ -equivariant extension map for  $\kappa_d(H_{a_d}) \triangleleft \kappa_d(\overline{V}_{a_d})$ . This implies part (a) according to [S2, 4.1(a)].

In part (b), we assume  $2 \nmid d$  and hence  $\kappa_d(H_{a_d}) = H_d$ . The character  $\lambda \in \text{Irr}(H_d)$  with  $\lambda(h_0) = -1$  corresponds via  $\kappa_d$  to some  $\lambda_0 \in \text{Irr}(\overline{H}_{a_d})$  with  $\lambda_0(h_0) = -1$ . Proposition 4.1(b) implies that via  $\kappa_d$  there is some  $\overline{V}_d$ -conjugate  $\lambda'$  of  $\lambda$  with  $\rho_{\mathbf{T}}(\overline{V}_{\tilde{\lambda}'}) = \mathcal{S}_{\mathcal{O}_d}$  for any  $\tilde{\lambda}' \in \text{Irr}(\tilde{H}_d \mid \lambda')$  and  $(\overline{V}_d)_{\lambda'} = \overline{V}_{\tilde{\lambda}'} \langle c'_d \rangle$  for some  $c'_d \in \text{Irr}(\overline{V}_d)$  with  $\rho_{\mathbf{T}}(c'_d) = \prod_{i \in J_d} (i, -i)$ . We observe that  $(\overline{V}_d)_{\tilde{\lambda}'} \leq V_D$ . Because of  $V_D \triangleleft \overline{V}$ , this implies  $(\overline{V}_d)_{\tilde{\lambda}} \leq V_D$  and  $(\overline{V}_d)_\lambda = (\overline{V}_d)_{\tilde{\lambda}} \langle c_d \rangle$  for some  $c_d \in \overline{V}_d$ . This proves part (b). □

### 4.3 Consideration of $K_{-1} \triangleleft K_{-1} \overline{V}_{-1}$

The group structure of  $G_{J_{-1}}$  depends on  $\text{type}(\Phi_{-1})$ . By its definition,  $\text{type}(\Phi_{-1}) \in \{A_1 \times A_1, A_3, D_{|J_{-1}|}\}$ . For the application of Proposition 3.2, we prove the following statement. Recall  $\overline{V}_{-1} = \langle \mathbf{n}_{e_1}(\varpi), h_0 \rangle$ ,  $\tilde{H}_{-1} = \langle \mathbf{h}_{J_{-1}}(\varpi), h_0 \rangle$ ,  $H_{-1} = \tilde{H}_{-1} \cap H_0$ , and  $G_{J_{-1}} = \langle \mathbf{X}_\alpha \mid \alpha \in \Phi_{-1} \rangle^F$ . As before, we set  $K_{0,-1} := G_{J_{-1}}$  and  $K_{-1} := H_{-1} G_{J_{-1}}$ .

PROPOSITION 4.8. *Assume Hypothesis 2.14 holds for  $\mathbf{G}_{J_{-1}}^F$  if  $\Phi_{-1}$  is of type D. Let  $\epsilon_{-1}: \overline{V}_{-1} \rightarrow \overline{V}_{-1}/H_{-1}$  be the canonical epimorphism, and let  $\tilde{L}_{-1} := (\mathbf{G}_{J_{-1}} \mathbf{T}_{J_{-1}}) \cap \mathcal{L}^{-1}(\langle h_0, \mathbf{h}_{J_{-1}}(\varpi) \rangle)$ . Then:*

- (a) *There exists some  $\overline{V}_{-1}\langle F_p \rangle$ -stable  $\tilde{L}_{-1}$ -transversal  $\mathbb{T}_{-1}^\circ$  in  $\text{Irr}_{\text{cusp}}(K_{0,-1})$ .*
- (b) *There exists an  $\epsilon_{-1}(\overline{V}_{-1})\langle F_p \rangle$ -equivariant extension map  $\Lambda_{\epsilon_{-1}}$  with respect to  $K_{0,-1} \triangleleft K_{0,-1} \rtimes \epsilon_{-1}(\overline{V}_{-1})$  for  $\mathbb{T}_{-1}^\circ$ .*
- (c) *Maximal extendibility holds with respect to  $K_{0,-1} \triangleleft \tilde{L}_{-1}$  and  $K_{-1} \triangleleft \tilde{L}_{-1}$ .*



*Proof.* As in the proof of Lemma 3.13, we see that  $\tilde{L}_{-1} = T_{-1}G_{J_{-1}}\langle t_{J_{-1}}, t_{J_{-1},2} \rangle$ , where  $T_{-1} := \mathbf{T}_{J_{-1}}^F$ ,  $\zeta \in \mathbb{F}^\times$  with  $\zeta^{(q-1)^2} = \varpi$ ,  $t_{J_{-1}} := \mathbf{h}_{e_1}(\zeta^2)$  and  $t_{J_{-1},2} := \mathbf{h}_{J_{-1}}(\zeta)$ . Note that the action of  $\tilde{L}$  on  $G_{J_{-1}}$  coincides with the one of  $\langle T_{-1}, t_{J_{-1}}, t_{J_{-1},2} \rangle$  up to inner automorphisms. By the definition of  $G_{J_{-1}}$ , we see

$$G_{J_{-1}} \cong \begin{cases} D_{|J_{-1}|,sc}(q), & \text{if } \text{type}(\Phi_{-1}) = D_{|J_{-1}|}, \\ \text{SL}_4(q), & \text{if } \text{type}(\Phi_{-1}) = A_3, \\ \text{SL}_2(q) \times \text{SL}_2(q), & \text{if } \text{type}(\Phi_{-1}) = A_1 \times A_1. \end{cases}$$

Assume  $\text{type}(\Phi_{-1}) = D_{l_{-1}}$  with  $l_{-1} := |J_{-1}|$  and  $l_{-1} > 3$ . Then  $T_{-1} \leq G_{J_{-1}}$ . The elements  $t_{J_{-1}}$  and  $t_{J_{-1},2}$  act as diagonal automorphisms on  $G_{J_{-1}}$ . Part (a) follows from Theorem 2.13. By Hypothesis 2.14, we can choose a  $\bar{V}_{-1}\langle F_p \rangle$ -stable  $\tilde{L}_{-1}$ -transversal  $\mathbb{T}_{-1}^\circ$  in  $\text{Irr}_{\text{cusp}}(K_{0,-1})$  such that maximal extendibility holds with respect to  $K_{0,-1} \triangleleft K_{0,-1} \rtimes \langle \gamma, F_p \rangle$  for  $\mathbb{T}_{-1}$ . Note that  $K_{0,-1} \rtimes \langle \gamma, F_p \rangle = K_{0,-1} \rtimes (\epsilon_{-1}(\bar{V}_{-1}) \times \langle F_p \rangle)$ . By this choice, we see that an extension map  $\Lambda_{e_1}$  as required in part (b) exists. Note that the actions on  $G_{J_{-1}}$  induced by  $\gamma$  and  $\mathbf{n}_{e_1}(\varpi)$  coincide by 3.3. According to Theorem 2.17, maximal extendibility holds with respect to  $G_{J_{-1}} \triangleleft \tilde{L}_{-1}$ . This proves part (c) in the case where  $\text{type}(\Phi_{-1}) = D_{l_{-1}}$  with  $l_{-1} > 3$ .

Assume  $\text{type}(\Phi_{-1}) = A_1 \times A_1$ , then  $t_{J_{-1}}$  induces on both factors a noninner diagonal automorphism, while  $t_{J_{-1},2}$  induces a noninner diagonal automorphism only on one factor, since  $h_0 = \mathbf{h}_{e_2-e_1}(-1)\mathbf{h}_{e_1+e_2}(-1)$  and  $\mathbf{h}_{e_1}(\varpi)\mathbf{h}_{e_2}(\varpi) = \mathbf{h}_{e_1+e_2}(-1)$ . Clearly,  $\bar{V}_{-1}$  acts by permutation of the two factors. Let  $\mathbb{T}(\text{SL}_2(q))$  be an  $\langle F_p \rangle$ -stable  $\text{GL}_2(q)$ -transversal in  $\text{Irr}_{\text{cusp}}(\text{SL}_2(q))$  (see Proposition 2.19). Then  $\mathbb{T}_{-1}^\circ := \mathbb{T}(\text{SL}_2(q)) \times \mathbb{T}(\text{SL}_2(q))$  is a  $\bar{V}_{-1}\langle F_p \rangle$ -stable  $\tilde{L}_{-1}$ -transversal in  $\text{Irr}_{\text{cusp}}(G_{J_{-1}})$ . This proves part (a) in that case. Part (b) follows from the fact that  $K_{0,-1} \rtimes \epsilon_{-1}(\bar{V}_{-1}) \cong \text{SL}_2(q) \wr C_2$  (see also Lemma 4.6). Part (c) follows again from the fact that  $\tilde{L}_{-1}$  is  $(\widehat{\text{SL}}_2(q))^2$ , where

$$\widehat{\text{SL}}_2(q) := \{x \in \text{SL}_2(\mathbb{F}) \mid F_q(x) = \pm x\}.$$

Assume  $\text{type}(\Phi_{-1}) = A_3$ . Recall  $\alpha_2 = e_2 + e_1$ ,  $\alpha_1 = e_2 - e_1$ , and  $\alpha_i := e_i - e_{i-1} (i \geq 3)$  for the simple roots in  $\Delta$ . In this case,  $G_{J_{-1}} \cong \text{SL}_4(q)$  and  $\mathbf{n}_{e_1}(\varpi)$  acts on  $G_{J_{-1}}$  as a nontrivial graph automorphism. In order to see the automorphisms induced by  $t_{J_{-1}}$  and  $t_{J_{-1},2}$ , we use again the equation  $h_0 = \mathbf{h}_{\alpha_1}(-1)\mathbf{h}_{\alpha_2}(-1)$  and additionally the equation

$$\mathbf{h}_3(\varpi) = \mathbf{h}_{\alpha_1}(-\varpi)\mathbf{h}_{\alpha_2}(-\varpi)\mathbf{h}_{\alpha_3}(-1).$$

This implies that  $t_{J_{-1}}$  induces on  $G_{J_{-1}}$  some noninner diagonal automorphism of  $\text{SL}_4(q)$  corresponding via the Lang map (see Remark 2.16(b)) to the central involution, while  $t_{J_{-1},2}$  induces a diagonal automorphism of  $\text{SL}_4(q)$  associated with a generator of the center. Let  $E(\text{SL}_4(q))$  be the subgroup of  $\text{Aut}(\text{SL}_4(q))$  from Proposition 2.19. According to Proposition 2.19(a), there exists a  $\text{GL}_4(q)$ -transversal  $\mathbb{T}(\text{SL}_4(q))$  in  $\text{Irr}(\text{SL}_4(q))$ , that is stable under the group  $E(\text{SL}_4(q))$  generated by graph and field automorphisms of  $\text{SL}_4(q)$  and such that maximal extendibility holds with respect to  $\text{SL}_4(q) \triangleleft \text{SL}_4(q) \rtimes E(\text{SL}_4(q))$  for  $\mathbb{T}(\text{SL}_4(q))$ . This choice guarantees part (b). As  $\tilde{L}_{-1}/G_{J_{-1}}$  is cyclic, part (c) holds in that case, as well. □

Recall  $\tilde{H}_1 := \langle h_0, \mathbf{h}_{J_{-1}}(\varpi) \rangle$ .

LEMMA 4.9.

- (a) *There exists some  $\bar{V}_{-1}$ -equivariant extension map  $\Lambda_{0,-1}$  with respect to  $H_{-1} \triangleleft \bar{V}_{-1}$ .*
- (b) *If  $\lambda \in \text{Irr}(H_{-1})$  with  $\lambda(h_0) = -1$  and  $\tilde{\lambda} \in \text{Irr}(\tilde{H}_{-1} \mid \lambda)$ , then  $(\bar{V}_{-1})_{\tilde{\lambda}} = H_{-1}$ .*

*Proof.* As  $\bar{V}_{-1}/H_{-1}$  is cyclic, there exists an extension map as required in (a). For the proof of (b), note that the equality  $[\bar{\mathbf{n}}_1, \mathbf{h}_{J_{-1}}(\varpi)] = h_0$  implies  $\tilde{\lambda}^{\bar{\mathbf{n}}_1} \neq \tilde{\lambda}$  as  $\tilde{\lambda}(h_0) = -1$ .  $\square$

**4.4 Proof of Theorem B**

We now finish the proof of Theorem 3.1 and therefore Theorem B. The above allows us now to verify the character-theoretic assumptions from Proposition 3.2 for the groups  $K$ ,  $K_0$ ,  $K_{0,d}$ , and  $V_D$ , introduced in Lemma 3.11 and Theorem 3.22. From the definitions of  $K_{0,d}$  before Lemma 4.4, we see  $K_0 = \langle K_{0,d} \mid d \in \mathbb{D} \rangle$ , even more  $K_0$  is the central product of the groups  $K_{0,d}$  ( $d \in \mathbb{D}$ ).

By abuse of notation, we write  $\text{Irr}_{\text{cusp}}(K)$  for  $\bigcup_{\chi \in \text{Irr}_{\text{cusp}}(L)} \text{Irr}(\chi \upharpoonright_K) \subseteq \text{Irr}(K)$ .

PROPOSITION 4.10. *There exists a  $\bar{V}\langle F_p \rangle$ -stable  $\tilde{L}$ -transversal  $\mathbb{K}_0$  in  $\text{Irr}_{\text{cusp}}(K_0)$ . Moreover  $\mathbb{K} := \text{Irr}(K \mid \mathbb{K}_0)$  and  $\mathbb{T} = \text{Irr}(L \mid \mathbb{K})$  are  $NE_L$ -stable  $\tilde{L}$ -transversals in  $\text{Irr}_{\text{cusp}}(K)$  and  $\text{Irr}_{\text{cusp}}(L)$ , respectively.*

Note that this implies Theorem 3.1(a).

*Proof.* For  $d \in \mathbb{D} \setminus \{1\}$ , let  $\mathbb{T}_d^\circ$  be the  $\bar{V}_d \langle F_p \rangle$ -stable  $\tilde{L}_d$ -transversal in  $\text{Irr}_{\text{cusp}}(K_{0,d})$  from Propositions 4.5 and 4.8. Note that  $K_{0,1} = 1$ . The group  $K_0$  is a central product of the groups  $K_{0,d}$  ( $d \in \mathbb{D}$ ) according to Lemma 3.13. Hence, the irreducible characters of  $K_0$  are obtained as the products of the irreducible characters of  $K_{0,d}$ . The central product of the characters in  $\mathbb{T}_d^\circ$  form a subset  $\mathbb{K}_0 \subseteq \text{Irr}(K_0)$ . We see that  $\mathbb{K}_0$  is  $V_D E_L$ -stable since  $V_D E_L$  and  $\bar{V} \langle F_p \rangle$  act on each factor  $K_{0,d}$  as  $\bar{V}_d \langle F_p \rangle$ . Let  $\hat{T} := \mathbb{T} \cap \mathcal{L}^{-1}(\langle h_0 \rangle)$ . The automorphisms of  $\tilde{L}$  induced on  $K_0$  are induced by  $K_0$ ,  $\hat{T} = \prod_{d \in \mathbb{D}} \hat{T}_d$  and  $t_{L,2} = \prod_{d \in \mathbb{D}} t_{J_d,2}$  (see Lemma 4.2). According to Lemma 4.3, the element  $t_{J_d,2}$  acts trivially on  $G_d$ , whenever  $d \geq 1$ . Hence,  $\mathbb{K}_0$  is an  $\tilde{L}$ -transversal of  $\text{Irr}_{\text{cusp}}(K_0)$  as well. According to Propositions 4.5 and 4.8, maximal extendibility holds with respect to  $K_{0,d} \triangleleft \tilde{L}_d$ . Since  $[\tilde{L}_d, \tilde{L}_{d'}] = 1$  for every  $d, d' \in \mathbb{D}$  with  $d \neq d'$ , this implies that maximal extendibility holds also with respect to  $K_0 \triangleleft \tilde{L}$  as  $\tilde{L} \leq \langle \tilde{L}_d \mid d \in \mathbb{D} \rangle$ . Since  $\tilde{L}/K_0$  is abelian by Lemma 3.13,  $\mathbb{K}$  and  $\mathbb{T}$  are again  $NE_L$ -stable  $\tilde{L}$ -transversals in  $\text{Irr}_{\text{cusp}}(K)$  and  $\text{Irr}_{\text{cusp}}(L)$ , respectively.  $\square$

We apply the following statement in order to construct some extension map with respect to  $L \triangleleft N$  for  $\text{Irr}_{\text{cusp}}(L)$  satisfying Equation (2.2) from Theorem 2.7.

PROPOSITION 4.11. *There exists a  $V_D \rtimes E_L$ -equivariant extension map  $\Lambda_0$  with respect to  $H \triangleleft V_D$ .*

This ensures Assumption 3.2(ii.1) with the choice made in Lemma 3.11.

*Proof.* Recall  $\bar{V} := H \langle \bar{V}_d \mid d \in \mathbb{D} \rangle$  and  $V_D := \bar{V} \cap \mathbf{G}$  (see Theorem 3.22). Let  $\tilde{H}_\epsilon := \langle \tilde{H}_d \mid d \in \mathbb{D}_\epsilon \rangle$ . We apply the extension maps from Proposition 4.1, Lemma 4.7, and Proposition 4.8(c) for constructing a  $\bar{V}$ -equivariant extension map for  $H \triangleleft V_D$ . Note that by the definition of  $\bar{V}$ ,  $\bar{\mathbf{n}}_1 \in \bar{V} \setminus V_D$  whenever  $\gamma \in E_L$ , and then  $\bar{\mathbf{n}}_1$  and  $\gamma$  induce the same automorphism on  $\mathbf{G}$  according to Definition 3.4. By Theorem 3.22,  $[F_p, V_D] = 1$ . Altogether, it is sufficient to prove that maximal extendibility holds with respect to  $H \triangleleft \bar{V}$ .

Let  $\lambda \in \text{Irr}(H)$  and  $\tilde{\lambda} \in \text{Irr}(\tilde{H} \mid \lambda)$ . Then  $\tilde{\lambda} = \bigoplus_{d \in \mathbb{D}} \tilde{\lambda}_d$  for some  $\tilde{\lambda}_d \in \text{Irr}(\tilde{H}_d)$  ( $d \in \mathbb{D}$ ). Let  $\psi_d$  be the extension of  $\lambda_d := \lambda \upharpoonright_{H_d}$  to  $(V_d)_{\lambda_d}$  given by Proposition 4.1, Lemma 4.7, and Proposition 4.8(c).

Assume  $\lambda(h_0) = 1$ . Let  $\bar{\lambda} \in \text{Irr}(H/\langle h_0 \rangle)$  be associated with  $\lambda$ . It is sufficient to show that  $\bar{\lambda}$  extends to  $\bar{V}_{\bar{\lambda}}/\langle h_0 \rangle$ . Since  $[\bar{V}_d/\langle h_0 \rangle, \bar{V}_{d'}/\langle h_0 \rangle] = 1$  according to (3.2), the group  $\bar{V}/\langle h_0 \rangle$  is the central product of the groups  $\bar{V}_d/\langle h_0 \rangle$ . The characters  $\psi_d$  ( $d \in \mathbb{D}$ ) define extensions  $\bar{\psi}_d$  of  $\bar{\lambda}_d$  to  $(\bar{V}_d)_{\lambda_d}/\langle h_0 \rangle$  and  $\bar{\psi} := \bigoplus_{d \in \mathbb{D}} \bar{\psi}_d$  lifts to a character  $\psi^\circ$  of  $\langle (\bar{V}_d)_{\lambda_d} \mid d \in \mathbb{D} \rangle$ . Recall  $H \geq \langle H_d \mid d \in \mathbb{D} \rangle$ . According to [S3, 4.1(a)] we see that  $\lambda$  has an extension  $\psi$  to  $\bar{V}_\lambda$  such that  $\psi \upharpoonright_{\langle (V_d)_{\lambda_d} \mid d \in \mathbb{D} \rangle_\lambda} = \psi^\circ \upharpoonright_{\langle (V_d)_{\lambda_d} \mid d \in \mathbb{D} \rangle_\lambda}$ . The extension map with respect to  $H \triangleleft V_D$  for  $\text{Irr}(H \mid 1_{\langle h_0 \rangle})$  obtained this way is then automatically  $\bar{V} \rtimes \langle F_p \rangle$ -equivariant.

Assume otherwise  $\lambda(h_0) = -1$ . As in Lemma 3.10, let  $\mathbb{D}_{\text{odd}} := \{i \in \mathbb{D} \mid i \text{ odd}\}$  and  $\mathbb{D}_{\text{even}} := \{i \in \mathbb{D} \mid i \text{ even}\}$ . For  $\epsilon \in \{\text{odd}, \text{even}\}$ , recall

$$\tilde{H}_\epsilon := \langle \tilde{H}_d \mid d \in \mathbb{D}_\epsilon \rangle, \quad H_\epsilon := \tilde{H}_\epsilon \cap H_0,$$

and  $H = H_{\text{even}} \times H_{\text{odd}}$  (see Lemma 3.10). Analogously, we define

$$\bar{V}_\epsilon := H_\epsilon \langle \bar{V}_d \mid d \in \mathbb{D}_\epsilon \rangle \quad \text{and} \quad \tilde{V}_\epsilon := \tilde{H}_\epsilon V_\epsilon.$$

Notice that by this definition  $V_{\text{even}} \leq V_D$  and hence  $V_D = H(V_{\text{even}} \cdot (V_{\text{odd}} \cap V_D))$ .

Let  $\tilde{\lambda}_\epsilon := \tilde{\lambda} \upharpoonright_{\tilde{H}_\epsilon}$  and  $\lambda_\epsilon := \lambda \upharpoonright_{H_\epsilon}$ . Since  $[\bar{V}_d, \bar{V}_{d'}] = 1$  for every  $d \in \mathbb{D}_{\text{even}}$  and  $d' \in \mathbb{D}$  by (3.2), the extensions  $\psi_d$  ( $d \in \mathbb{D}_{\text{even}}$ ) allow us to define an extension of  $\lambda_{\text{even}}$  to  $(\bar{V}_{\text{even}})_{\lambda_{\text{even}}}$ .

Now,  $H_{\text{even}}$  is the central product of the groups  $H_d$  ( $d \in \mathbb{D}_{\text{even}}$ ) and  $(V_{\text{even}})_{\lambda_{\text{even}}}$  is analogously the central product of the groups  $(\bar{V}_d)_{\lambda_d}$ . Hence, the product of the characters  $\psi_d$  ( $d \in \mathbb{D}_{\text{even}}$ ) defines an extension  $\hat{\lambda}_{\text{even}} \in \text{Irr}((V_{\text{even}})_{\lambda_{\text{even}}})$  of  $\lambda_{\text{even}}$ .

In order to extend  $\lambda_{\text{odd}}$  to  $(V_{\text{odd}})_{\lambda_{\text{odd}}}$ , we first extend  $\tilde{\lambda}_{\text{odd}} := \tilde{\lambda} \upharpoonright_{\tilde{H}_{\text{odd}}}$ . Again,  $\tilde{\lambda}_{\text{odd}}$  is the central product of characters  $\tilde{\lambda}_d$  ( $d \in \mathbb{D}_{\text{odd}}$ ). According to Proposition 4.1(b) and Lemma 4.7(b), we have  $(\bar{V}_d)_{\tilde{\lambda}_d} \leq V_D$ . The same holds also for  $d = -1$  by straight-forward calculations.

Let  $\nu \in \text{Irr}(\tilde{H}_{\text{odd}})$  with  $\ker(\nu) = H_{\text{odd}}$ . According to Lemma 4.7(b), there exists some element  $c_d \in \bar{V}_d$  such that  $(\bar{V}_d)_{\lambda_d} = \langle (\bar{V}_d)_{\tilde{\lambda}_d}, c_d \rangle$ , which satisfies  $\tilde{\lambda}_d^{c_d} = \tilde{\lambda}_d \nu \upharpoonright_{\tilde{H}_d}$ . The extensions  $\psi_d \upharpoonright_{(\bar{V}_d)_{\tilde{\lambda}_d}}$  define easily extensions  $\psi'_d$  of  $\tilde{\lambda}_d$  to  $\tilde{H}_d(\bar{V}_d)_{\tilde{\lambda}_d}$ . The restriction  $\psi'_d \upharpoonright_{(V_d)_{\tilde{\lambda}_d}}$  is then  $c_d$ -stable. Since  $(\bar{V}_d)_{\tilde{\lambda}_d}$  is contained in  $V_D$ , the group  $(\bar{V}_{\text{odd}})_{\tilde{\lambda}_{\text{odd}}}$  is the central product of the groups  $\tilde{H}_d(\bar{V}_d)_{\tilde{\lambda}_d}$  ( $d \in \mathbb{D}_{\text{odd}}$ ). The product  $\psi' := \prod_{d \in \mathbb{D}_{\text{odd}}} \psi'_d$  determines uniquely an extension  $\psi''$  of  $\tilde{\lambda}_{\text{odd}}$  to  $\tilde{H}_{\text{odd}}(\bar{V}_{\text{odd}})_{\tilde{\lambda}_{\text{odd}}}$ . Routine calculations show that  $(\bar{V}_{\text{odd}})_{\lambda_{\text{odd}}} = (\bar{V}_{\text{odd}})_{\tilde{\lambda}_{\text{odd}}} \langle c_{\text{odd}} \rangle$  where  $c_{\text{odd}} = \prod_{d \in \mathbb{D}_{\text{odd}}} c_d$ . The character  $\psi'' \upharpoonright_{H_{\text{odd}}(\bar{V}_{\text{odd}})_{\tilde{\lambda}_{\text{odd}}}}$  is then  $c_{\text{odd}}$ -stable and extends to  $(\bar{V}_{\text{odd}})_{\lambda_{\text{odd}}}$ . This way we obtain an extension  $\hat{\lambda}_{\text{odd}}$  of  $\lambda_{\text{odd}}$  to  $(\bar{V}_{\text{odd}})_{\lambda_{\text{odd}}}$ .

Recall  $[\bar{V}_{\text{odd}}, \bar{V}_{\text{even}}] = 1$ . Hence, the extensions  $\hat{\lambda}_{\text{odd}}$  and  $\hat{\lambda}_{\text{even}}$  determine an extension of  $\lambda$  to  $\bar{V}_\lambda$  (see [S1, Lem. 4.2]). □

In the next step, we show that there exists an extension map with respect to  $K_0 \triangleleft K_0 \rtimes \epsilon(V_D)$  for the set  $\mathbb{K}_0$  from Proposition 4.10 as required in Proposition 3.2.

PROPOSITION 4.12. *There exists a  $V_D E_L$ -equivariant extension map  $\Lambda_\epsilon$  with respect to  $K_0 \triangleleft K_0 \rtimes \epsilon(V_D)$  for  $\mathbb{K}_0$ , where  $\epsilon : V_D E_L \rightarrow V_D E_L/H$  is the canonical morphism.*

*Proof.* By Propositions 4.5 and 4.8, there exist  $\overline{V}_d \langle F_p \rangle$ -equivariant extension maps  $\Lambda_{\epsilon_d}$  with respect to  $K_{0,d} \triangleleft K_{0,d} \rtimes \epsilon(\overline{V}_d)$  for  $\mathbb{T}_d^\circ$ , whenever  $d \in \mathbb{D}$  with  $d \neq 1$ . Note that the case  $d = 1$  is trivial since  $K_{0,1} = 1$ . The group  $K_0 \rtimes \epsilon(\overline{V})$  is the direct product of the groups  $K_{0,d} \rtimes \epsilon_d(\overline{V}_d)$ . Using the maps  $\Lambda_{\epsilon_d}$  ( $d \in \mathbb{D}$ ), we therefore obtain an extension map  $\Lambda_\epsilon$  as required.  $\square$

This leads to the following statement. We use the set

$$\mathbb{K} := \text{Irr}(K \mid \mathbb{K}_0)$$

with  $\mathbb{K}_0$  from Proposition 4.10. For the application of Proposition 3.2, we use the group  $M = K V_D$  (see also Corollary 3.23).

PROPOSITION 4.13. *There exists a  $V E_L$ -equivariant extension map  $\Lambda_{K \triangleleft M}$  with respect to  $K \triangleleft M$  for  $\mathbb{K}$ .*

*Proof.* By the above, all the assumptions of Proposition 3.2 are satisfied. The groups satisfy the required assumptions in Proposition 3.2(i) according to Corollary 3.23. Using the set  $\mathbb{K}$ , given as  $\text{Irr}(K \mid \mathbb{K}_0)$  from Proposition 4.10, the set  $\mathbb{K}_0$  coincides with  $\bigcup_{\lambda \in \mathbb{K}} \text{Irr}(\lambda \upharpoonright_{K_0})$ . With the  $V_D E_L$ -equivariant extension map  $\Lambda_0$  for  $H \triangleleft V_D$  from Proposition 4.11 and the extension map  $\Lambda_\epsilon$  for  $K_0 \triangleleft K_0 \rtimes \epsilon(\overline{V})$  from Proposition 4.12, Assumption 3.2(ii) is satisfied. The application of this statement implies the result.  $\square$

For the set  $\mathbb{T}$  defined as  $\text{Irr}(L \mid \mathbb{K})$  in Proposition 4.10, we verify that there exists an  $N E_L$ -equivariant extension map with respect to  $L \triangleleft N$  for  $\mathbb{T}$ .

*Proof of Theorems 3.1(b) and B.* For the proof, it is sufficient to construct for every  $\lambda \in \mathbb{T} = \text{Irr}(L \mid \mathbb{K})$  some  $N E_L$ -stable extension to  $N_\lambda$ . A character  $\lambda \in \mathbb{T}$  lies above a unique  $\lambda_0 \in \mathbb{K} = \text{Irr}(K \mid \mathbb{K}_0)$ . Moreover, some extension  $\tilde{\lambda}_0 \in \text{Irr}(L_{\lambda_0})$  to  $L_{\lambda_0}$  satisfies  $(\tilde{\lambda}_0)^L = \lambda$ . By the properties of  $\mathbb{K}$ , we see  $N_{\lambda_0} = L_{\lambda_0} M_{\lambda_0}$ , which is normalized by  $(N E_L)_{\tilde{\lambda}_0}$ . By Proposition 4.13, the character  $\lambda_0$  has a  $(\overline{V} \langle F_p \rangle)_{\lambda_0}$ -stable extension to  $M_{\lambda_0}$ . According to [S3, 4.1], this defines an extension  $\phi$  of  $\tilde{\lambda}_0$  to  $N_{\tilde{\lambda}_0}$  since  $N_{\tilde{\lambda}_0} \leq L_{\lambda_0} M_{\lambda_0}$ . By the construction, we see that  $\phi^{N_\lambda}$  is an extension of  $\lambda$ .

As  $\mathbb{T}$  is an  $M$ -stable  $\tilde{L}$ -transversal,  $\tilde{N}_{\lambda_0} = \tilde{L}_{\lambda_0} M_{\lambda_0}$  and  $(\tilde{N} E_L)_{\lambda_0} = \tilde{L}_{\lambda_0} \widehat{M}_{\lambda_0}$ . Hence, this extension of  $\lambda_0$  defines an extension of  $\lambda$  as required.  $\square$

Later this ensures Assumption 2.12(ii).

REMARK 4.14. While Theorem 3.1(b) assumes  $q$  to be odd, the proof would give a similar conclusion in the other case. For even  $q$ , every  $\chi \in \text{Irr}(G)$  satisfies  $(\tilde{G} E)_\chi = \tilde{G}_\chi E_\chi$  since  $\tilde{G} = G$  in the notation of 2.2. Nevertheless, the conclusion of Theorem 3.1(b) holds as well. We observe that the arguments from before prove that there exists some  $N E_L$ -equivariant extension map  $K_0 \triangleleft K_0 V_D$  for  $\text{Irr}_{\text{cusp}}(L)$ , where  $V_D$  is isomorphic to  $N/L$  and is defined as before with  $1 = \pm \varpi$  in the argument of the Chevalley generators.

### §5. More on cuspidal characters

In order to prove our main theorem, we need more specific results on cuspidal characters, especially with regard to automorphisms. We keep  $q$  a power of an odd prime.

PROPOSITION 5.1. *Let  $n \geq 3$ ,  $\chi \in \text{Irr}_{\text{cus}}(\text{GL}_n(q))$ , and  $\gamma \in \text{Aut}(\text{GL}_n(q))$  given by transpose-inverse up to some inner automorphism.*

- (a) *If  $\chi^\gamma = \chi$ , then  $2 \mid n$  and  $Z(\text{GL}_n(q)) \leq \ker(\chi)$ .*
- (b) *If  $\chi^\gamma = \chi\delta$  for  $\delta \in \text{Irr}(\text{GL}_n(q))$ , a linear character of multiplicative order 2, then  $2 \mid n$ .*

*Proof.* Let us recall the form of elements of  $\text{Irr}_{\text{cus}}(\text{GL}_n(q))$  (see also [B1, 16.1]). We write  $\mathbf{K} := \text{GL}_n(\mathbb{F})$  and  $\mathbf{K}^* := \text{GL}_n(\mathbb{F})$  as the dual with  $\mathbb{F}_q$ -structures given by  $F$ . Let  $s \in (\mathbf{K}^*)^F = \text{GL}_n(q)$  be such that the Lusztig series  $\mathcal{E}(\mathbf{K}^F, (s))$  associated with  $s$  contains a cuspidal character. Combining [GM, 3.2.22] and the fact that the group  $C_{\mathbf{K}^*}(s)^F$  of type A can have cuspidal unipotent characters only when it is a torus (see, e.g., [GM, Exam. 2.4.20]), we get that  $s$  is regular and  $C_{\mathbf{K}^*}(s)$  is a Coxeter torus. This can be summed up in the fact that the spectrum of  $s$  is a single orbit of length  $n$  under  $F$ , or equivalently  $\mathbb{F}_q[\zeta] = \mathbb{F}_{q^n}$  for any eigenvalue  $\zeta$  of  $s$ . Concerning the action of  $\gamma$ , note that an element of  $\mathcal{E}(\mathbf{K}^F, (s))$  is sent to an element of  $\mathcal{E}(\mathbf{K}^F, (s^{-1}))$  (apply [CS1, 3.1]).

For the proof of (a), let  $\chi \in \mathcal{E}(\mathbf{K}^F, (s))$  be invariant under  $\gamma$ . Then  $s$  and  $s^{-1}$  have the same spectrum. If 1 or  $-1$  is an eigenvalue of  $s$ , then  $s \in \{\text{Id}_n, -\text{Id}_n\}$  and  $n = 1$  since the eigenvalues of  $s$  form a single  $F$ -orbit. This is impossible, so inversion is without fixed point on the spectrum of  $s$ . This implies that  $n$  is even and that the product of the eigenvalues of  $s$  is 1. So  $s \in [\mathbf{K}^*, \mathbf{K}^*]^F$  and this implies that all characters of  $\mathcal{E}(\mathbf{K}^F, (s))$  have  $Z(\text{GL}_n(q))$  in their kernel (see [CE, p. 207]).

For the proof of part (b), note that by the assumptions  $q$  is odd and  $\text{SL}_n(q)$  is perfect (see [MT, 24.17]). By the correspondence induced by duality between (linear) characters of  $\mathbf{K}^F/[\mathbf{K}, \mathbf{K}]^F$  and elements of  $Z(\mathbf{K}^*)^F$  (see, e.g., [DM, 11.4.12]), we have  $\delta\mathcal{E}(\mathbf{K}^F, (s)) = \mathcal{E}(\mathbf{K}^F, (-s))$ . Assuming  $\chi^\gamma = \chi\delta$ , the same considerations as above show that  $s$  and  $-s^{-1}$  have the same eigenvalues. The spectrum of  $s$  is of the form  $\{F(\zeta), F^2(\zeta), \dots, F^n(\zeta) = \zeta\}$  with  $\mathbb{F}_q[\zeta] = \mathbb{F}_{q^n}$ . Since  $s$  and  $-s^{-1}$  have the same eigenvalues, then  $-\zeta^{-1} = F^a(\zeta)$  for some  $1 \leq a \leq n$ . We have  $F^{2a}(\zeta) = -F^a(\zeta)^{-1} = \zeta$  and therefore  $\mathbb{F}_{q^{2a}} \supseteq \mathbb{F}_q[\zeta] = \mathbb{F}_{q^n}$ . Then  $n$  divides  $2a$ . Assume now that  $n$  is odd. This implies that  $n$  divides  $a \leq n$ . So  $a = n$  and  $-\zeta^{-1} = F^n(\zeta) = \zeta$ . But then,  $\zeta^2 = -1$  and  $\mathbb{F}_q[\zeta] \subseteq \mathbb{F}_{q^2}$ , which contradicts  $n \geq 3$ . So we get our claim that  $2 \mid n$ . □

The following statement is used later for computing the relative Weyl groups associated with cuspidal characters of a Levi subgroup of  $D_{l,\text{sc}}(q)$ .

PROPOSITION 5.2. *Let  $n \geq 2$ ,  $\psi \in \text{Irr}_{\text{cus}}(\text{SL}_n(q))$ , and  $\gamma \in \text{Aut}(\text{GL}_n(q))$  given by transpose-inverse up to some inner automorphism. If  $|\text{GL}_n(q) : \text{GL}_n(q)_\psi|$  is even and  $\psi^\gamma = \psi$ , then  $n = 2$  and  $\psi$  is one of the two characters  $R'_\sigma(\theta_0)$  ( $\sigma \in \{\pm 1\}$ ) of degree  $\frac{q-1}{2}$  from [B2, Table 5.4].*

*Proof.* According to [B2, Table 5.4], the two characters  $R'_\sigma(\theta_0)$  ( $\sigma \in \{\pm 1\}$ ) are the only characters of  $\text{SL}_2(q)$  that are cuspidal and not  $\text{GL}_2(q)$ -stable. The characters  $R(\theta)$  given there are  $\text{GL}_2(q)$ -stable and the other characters  $R_\sigma(\alpha_0)$  ( $\sigma \in \{\pm 1\}$ ) are in the principal Harish-Chandra series. Note that  $\gamma$  then restricts to an inner automorphism of  $\text{SL}_2(q)$ .

Now, consider  $n \geq 3$ . Let  $\psi$  be as in the proposition, and let  $\chi \in \text{Irr}(\text{GL}_n(q) \mid \psi)$ , so that  $\chi$  is cuspidal thanks to Lemma 2.5(c). We keep the notation of the proof of Proposition 5.1 with  $\chi \in \mathcal{E}(\text{GL}_n(q), (s))$  and  $\zeta$  some eigenvalue of  $s$ .

By Clifford theory,  $\chi$  is induced from a character of  $GL_n(q)_\psi$ . Then the assumption  $2 \mid |GL_n(q) : GL_n(q)_\psi|$  implies  $\chi = \nu_1 \chi$  for  $\nu_1 \in \text{Irr}(GL_n(q))$  the linear character of order 2 with kernel containing  $SL_n(q)$ . Hence,  $s$  is  $GL_n(q)$ -conjugate to  $-s$ . Then  $-\zeta \in \{F(\zeta), F^2(\zeta), \dots, F^n(\zeta) = \zeta\}$  since this is the spectrum of  $s$ .

Clifford theory also tells us that the assumption  $\psi^\gamma = \psi$  implies  $\chi^\gamma = \nu_2 \chi$  for some linear character  $\nu_2$  of  $GL_n(q)$  with  $SL_n(q)$  in its kernel. Then  $s^{-1}$  is conjugate to  $\lambda s$  for some  $\lambda \in \mathbb{F}_q^\times$ . As before, we obtain  $\zeta^{-1} \in \{\lambda F(\zeta), \lambda F^2(\zeta), \dots, \lambda F^n(\zeta) = \lambda \zeta\}$ .

We can now write  $-\zeta = F^a(\zeta)$  and  $\lambda \zeta^{-1} = F^b(\zeta)$  for  $1 \leq a, b \leq n$ . The first equality gives  $F^{2a}(\zeta) = -F^a(\zeta) = \zeta$  and the second  $F^{2b}(\zeta) = \lambda F^b(\zeta)^{-1} = \zeta$  since  $\lambda \in \mathbb{F}_q$ . So  $\zeta \in \mathbb{F}_{q^{2a}} \cap \mathbb{F}_{q^{2b}}$ , but since  $\mathbb{F}_q[\zeta] = \mathbb{F}_{q^n}$ , we get that  $n$  divides both  $2a$  and  $2b$ . The latter are at most  $2n$ , so  $2a, 2b \in \{n, 2n\}$ . Having  $a = n$  would imply  $-\zeta = F^n(\zeta) = \zeta$ , which is impossible because  $q$  is odd. So  $n$  is even and  $a = \frac{n}{2}$ . On the other hand, if  $b = n$ , then  $\zeta = F^n(\zeta) = \lambda \zeta^{-1}$  and therefore  $\zeta^2 \in \mathbb{F}_q$ . Then  $\mathbb{F}_q[\zeta] \subseteq \mathbb{F}_{q^2}$  and this implies  $n = 2$ .

There remains the case when  $b = \frac{n}{2} = a$ . Then  $\lambda \zeta^{-1} = F^a(\zeta) = -\zeta$  and again  $\zeta^2 \in \mathbb{F}_q$ . This yields  $n = 2$  as seen before. □

We complement the above by a result on cuspidal characters of  $D_{l,sc}(q)$ , which follows from a combination of results from [Mal2] and [S4]. We use  $\mathbf{G}$ ,  $F$ ,  $\gamma$  from Notation 3.3 and  $h_0$  from Notation 3.9. Recall the Lang map  $\mathcal{L}$  defined on  $\mathbf{G}$  by  $\mathcal{L}(g) = g^{-1}F(g)$ . Note that  $\mathcal{L}^{-1}(\langle h_0 \rangle) / \langle h_0 \rangle = (\mathbf{G} / \langle h_0 \rangle)^F = SO_{2l}(\mathbb{F}_q)$ .

**PROPOSITION 5.3.** *Recall  $\tilde{G} := \mathcal{L}^{-1}(Z(\mathbf{G})) = N_{\mathbf{G}}(\mathbf{G}^F)$  (see Remark 2.16). If  $\lambda \in \text{Irr}_{\text{cusp}}(\mathbf{G}^F \mid 1_{\langle h_0 \rangle})$  with  $\tilde{G}_\lambda \leq \mathcal{L}^{-1}(\langle h_0 \rangle)$ , then  $\gamma$  acts trivially on  $\lambda$  and  $\text{Irr}(\mathcal{L}^{-1}(\langle h_0 \rangle) \mid \lambda)$ .*

*Proof.* Recall that a character of  $\tilde{\mathbf{G}}^F$  is called semisimple when it corresponds to a trivial unipotent character through the Jordan decomposition of characters. The components of their restrictions to  $\mathbf{G}^F$  are also called the semisimple characters of  $\mathbf{G}^F$ . In particular, both are of degree prime to  $p$  (see [GM, 2.6.11]).

According to [Mal2, Th. 1], there exists a semisimple character  $\rho \in \text{Irr}(\mathbf{G}^F)$  with  $(\tilde{G}E)_\rho = (\tilde{G}E)_\lambda$ , where  $\rho$  and  $\lambda$  lie in the same rational Lusztig series. We use now results from [S4] to investigate  $\rho$  further. In a first step, we prove that  $\gamma$  acts trivially on  $\rho$  and  $\text{Irr}(\mathcal{L}^{-1}(\langle h_0 \rangle) \mid \rho)$ .

We assume that  $\mathbf{G}$ ,  $\mathbf{T}$ , and  $\Delta$  are as given in Notation 3.3, and let  $\mathbf{U} := \langle \mathbf{X}_\alpha \mid \alpha \in \Delta \rangle$  and  $\mathbf{B} := \mathbf{TU}$ . As group  $\tilde{\mathbf{G}}$  introduced in 2.2, we use the particular choice from [S4, 3.1]. Then  $\tilde{\mathbf{G}}^F$  and  $\tilde{G}$  induce the same automorphisms on  $\mathbf{G}^F$ . Let  $\tilde{\mathbf{B}} := \mathbf{B}Z(\tilde{\mathbf{G}})$ . Let  $\tilde{\Omega} : \text{Irr}_{p'}(\tilde{\mathbf{G}}^F) \rightarrow \text{Irr}_{p'}(\tilde{\mathbf{B}}^F)$  be the  $\text{Irr}(\tilde{\mathbf{G}}^F/\mathbf{G}^F) \times E(\mathbf{G}^F)$ -equivariant bijection with  $\text{Irr}(\tilde{\psi} \Big|_{Z(\tilde{\mathbf{G}}^F)}) = \text{Irr}(\tilde{\Omega}(\tilde{\psi}) \Big|_{Z(\tilde{\mathbf{B}}^F)})$  for every  $\psi \in \text{Irr}_{p'}(\tilde{\mathbf{G}}^F)$  from [S4, 3.3(a)].

Let  $\tilde{\rho} \in \text{Irr}(\tilde{\mathbf{G}}^F \mid \rho)$ ,  $\tilde{\phi} := \tilde{\Omega}(\tilde{\rho})$ , and  $\phi \in \text{Irr}(\tilde{\phi} \Big|_{\mathbf{B}^F})$ . Let  $C$  be the Cartan matrix associated with  $\Delta$  and  $C^{-1} = (c'_{\alpha\beta})$  its inverse. Let  $\zeta \in \mathbb{F}^\times$  be a root of unity of order  $\det(C)(q-1) = 4(q-1)$ . For  $\alpha \in \Delta$ , we set

$$t_\alpha^{(0)} := \prod_{\beta \in \Delta} \mathbf{h}_\beta(\zeta^{\det(C)c'_{\alpha\beta}})$$

(see also [Mas, 8.1]). Then we can choose elements  $t_\alpha \in t_\alpha^{(0)} Z(\tilde{\mathbf{G}}) \cap \tilde{\mathbf{T}}^F$  such that  $\tilde{\mathbf{T}}^F = Z(\tilde{\mathbf{G}}^F) \langle t_\alpha \mid \alpha \in \Delta \rangle$  (see [Mas, §8]). Assume that  $\Delta$  is given as in Notation 3.3, and let  $\alpha \in \{e_2 \pm e_1\}$ . The entries of  $C^{-1}$  can be found in [OV, p. 296]. We see  $\mathcal{L}(t_\alpha^{(0)}) = (t_\alpha^{(0)})^{q-1} \in$

$\mathbf{h}_l(\varpi)\langle h_0 \rangle$ . Hence,  $t_\alpha$  induces a diagonal automorphism of  $\mathbf{G}^F$  associated with some element in  $\mathbf{h}_l(\varpi)\langle h_0 \rangle$  in the notation of Remark 2.16(b).

We abbreviate  $\widehat{G} := \mathcal{L}^{-1}(\langle h_0 \rangle)$ . The assumption  $\widetilde{G}_\lambda \leq \widehat{G}$  implies  $\lambda^{t_\alpha} \neq \lambda$ . Via the construction, we have  $(\widetilde{G}E)_\rho = (\widetilde{G}E)_\lambda$  and hence  $\rho^{t_\alpha} \neq \rho$ . By Clifford theory, the character  $\widetilde{\rho}$  satisfies  $\widetilde{\rho}(t_\alpha \mathbf{G}^F) = 0$  and is stable under multiplying with linear characters with kernel

$$\{g \in \widetilde{\mathbf{G}}^F \mid g \text{ induces diagonal autom. of } \mathbf{G}^F \text{ associated with an element of } \langle h_0 \rangle\}$$

(see 2.16(b)). As  $\widetilde{\Omega}$  is an  $\text{Irr}(\widetilde{\mathbf{G}}^F/\mathbf{G}^F)$ -equivariant bijection, the character  $\widetilde{\phi}$  has to satisfy  $\widetilde{\phi}(t_\alpha) = 0$  as well. As in Remark 2.16,  $\widetilde{\phi}$  can be extended to some character  $\kappa$  on  $\widetilde{\mathbf{B}}^F \widetilde{Z} = \widetilde{B}_0 \widetilde{Z}$ , where  $\widetilde{\mathcal{L}}$  is the Lang map on  $\widetilde{\mathbf{G}}$ ,  $\widetilde{Z} := \widetilde{\mathcal{L}}^{-1}(Z(\mathbf{G})) \cap Z(\widetilde{\mathbf{G}})$ , and  $\widetilde{B}_0 := \mathcal{L}^{-1}(Z(\mathbf{G})) \cap \mathbf{B}$ . Note that  $\widetilde{B}_0 = Z(\mathbf{G}^F) \langle t_\beta^{(0)} \mid \beta \in \Delta \rangle \mathbf{U}^F$ . Then  $\kappa(t_\alpha^{(0)}) = 0$ . The character  $\kappa|_{\widetilde{B}_0}$  is  $\gamma$ -stable, since  $\text{Irr}(\kappa|_{Z(\mathbf{G}^F)}) = \text{Irr}(\phi|_{Z(\mathbf{G}^F)})$  is  $\gamma$ -stable because of  $h_0 \in \ker(\phi)$  and  $t_\beta^{(0)}$  is  $\gamma$ -fixed for every  $\beta \in \Delta \setminus \{e_2 \pm e_1\}$  according to the explicit value of  $C^{-1}$ .

As  $\kappa$  is  $\gamma$ -stable,  $\phi \in \text{Irr}(\kappa|_{\mathbf{B}^F}) = \text{Irr}(\widetilde{\phi}|_{\mathbf{B}^F})$  can be assumed to be  $\gamma$ -stable (see [S4, 3.6(a)]). By Clifford theory,  $\kappa|_{\widetilde{B}_0}$  is of the form  $\widehat{\phi}^{\widetilde{B}_0}$  for a unique  $\widehat{\phi} \in \text{Irr}((\widetilde{B}_0)_\phi)$ . As  $\phi$  extends to  $\widetilde{\mathbf{B}}_\phi^F$  according to [S4, Th. 3.5(a)], the character  $\widehat{\phi}$  is an extension of  $\phi$ . As  $\kappa$  and  $\phi$  are  $\gamma$ -stable,  $\widehat{\phi}$  is  $\gamma$ -stable. Note that  $(\widetilde{B}_0)_\phi \leq \widehat{B} := \mathcal{L}^{-1}(\langle h_0 \rangle) \cap \mathbf{B}$ .

Via the statement given in Proposition 2.18, some  $\widetilde{G}$ -conjugate  $\rho'$  of  $\rho$  is  $\gamma$ -stable and has also a  $\gamma$ -stable extension to  $\widehat{G}$ . If  $\rho' \neq \rho$ , we observe that  $(\widehat{G}\langle \gamma \rangle)^{t'} = \widehat{G}\langle \gamma \rangle$  for every  $t' \in \mathcal{L}^{-1}(\mathbf{h}_l(\varpi))$ . Hence, the character  $\rho$  extends to  $\widehat{G}\langle \gamma \rangle$ , as well.

We deduce from this result on  $\rho$  the analogous property of  $\lambda$ . Recall that  $\lambda$  and  $\rho$  are in the same rational Lusztig series and that  $(\widetilde{\mathbf{G}}^F E)_\rho = (\widetilde{\mathbf{G}}^F E)_\lambda$ , in particular  $\widetilde{\mathbf{G}}_\rho^F = \widetilde{\mathbf{G}}_\lambda^F$ . Recall that  $\widetilde{\rho} \in \text{Irr}(\widetilde{\mathbf{G}}^F \mid \rho)$  and  $\text{Irr}(\widetilde{\mathbf{G}}^F/\mathbf{G}^F)$  acts on  $\text{Irr}(\widetilde{\mathbf{G}}^F)$  by multiplication with linear characters. As  $\widetilde{\mathbf{G}}^F/\mathbf{G}^F$  is abelian and maximal extendibility holds with respect to  $\mathbf{G}^F \triangleleft \widetilde{\mathbf{G}}^F$ , we see

$$\text{Irr}(\widetilde{\mathbf{G}}^F/\mathbf{G}^F)_{\widetilde{\lambda}} = \text{Irr}(\widetilde{\mathbf{G}}^F/\widetilde{\mathbf{G}}_\lambda^F) = \text{Irr}(\widetilde{\mathbf{G}}^F/\widetilde{\mathbf{G}}_\rho^F) = \text{Irr}(\widetilde{\mathbf{G}}^F/\mathbf{G}^F)_{\widetilde{\rho}}.$$

Let  $\mathcal{E}(\widetilde{\mathbf{G}}^F, s)$  be the rational Lusztig series containing  $\widetilde{\rho}$ . The character  $\widetilde{\rho}$  is semisimple. The series  $\mathcal{E}(\widetilde{\mathbf{G}}^F, s)$  contains exactly one regular character (see [DM, 12.4.10]). By the definition of semisimple and regular in [DM, 12.4.1], we see that there exists also a unique regular character in that series. Let  $\widetilde{\rho}' \in \text{Irr}(\widetilde{\mathbf{G}}^F)$  be the Alvis–Curtis dual of  $\widetilde{\rho}$  up to a sign (see [DM, 7.2]). Then  $\{\widetilde{\rho}'\} = \text{Irr}(\Gamma(\widetilde{\mathbf{G}}^F)) \cap \mathcal{E}(\widetilde{\mathbf{G}}^F, s)$ , where  $\Gamma(\widetilde{\mathbf{G}}^F)$  denotes the Gelfand–Graev character of  $\widetilde{\mathbf{G}}^F$ . Since it vanishes outside unipotent elements, the Gelfand–Graev character is stable under  $\text{Irr}(\widetilde{\mathbf{G}}^F/\mathbf{G}^F)$ . Hence,  $\text{Irr}(\widetilde{\mathbf{G}}^F/\mathbf{G}^F)_{\widetilde{\rho}'}$  coincides with the stabilizer of  $\mathcal{E}(\widetilde{\mathbf{G}}^F, s)$  in  $\text{Irr}(\widetilde{\mathbf{G}}^F/\mathbf{G}^F)$ . This group is called  $B(s)$  in [CE, 15.13]. By the construction of Alvis–Curtis duality, this implies  $\text{Irr}(\widetilde{\mathbf{G}}^F/\mathbf{G}^F)_{\widetilde{\rho}} = B(s)$ . The characters  $\widetilde{\rho}^\gamma$  and  $\widetilde{\lambda}^\gamma$  belong to  $\mathcal{E}(\widetilde{\mathbf{G}}^F, \gamma^{-1}(s))$ . As  $\lambda$  and  $\rho$  are  $\gamma$ -stable,  $\widetilde{\rho}^\gamma = \widetilde{\rho}\mu$  and  $\widetilde{\lambda}^\gamma = \widetilde{\lambda}\mu'$  for linear characters  $\mu, \mu' \in \text{Irr}(\widetilde{\mathbf{G}}^F)$ . Since  $\widetilde{\rho}^\gamma$  and  $\widetilde{\lambda}^\gamma$  are in the same rational series,  $\mu \in \mu' B(s)$  or equivalently  $\mu|_{\widetilde{\mathbf{G}}_\rho^F} = \mu'|_{\widetilde{\mathbf{G}}_\lambda^F}$ .

Because of  $\text{Irr}(\widetilde{\rho}|_{Z(\widetilde{\mathbf{G}}^F)}) = \text{Irr}(\widetilde{\lambda}|_{Z(\widetilde{\mathbf{G}}^F)})$ , Proposition 2.18 allows to conclude that  $\lambda$  has a  $\gamma$ -stable extension to  $\widehat{G}$  as  $\rho$  has such an extension. □

§6. Character theory for the relative inertia groups  $W(\lambda)$

The aim of this section is to ensure Assumption 2.8(ii), namely to prove (a main step toward) the following statement.

**THEOREM 6.1.** *Let  $l \geq 4$ . Let  $\mathbf{G}^F = D_{l,sc}(q)$  with odd  $q$ , and let  $L = \mathbf{L}^F$  be a standard Levi subgroup of  $\mathbf{G}^F$  (see Notation 3.3). Let  $N, \tilde{N}' := \tilde{T}_0 N$ , and  $E_L := \text{Stab}_{E(\mathbf{G}^F)}(\mathbf{L}^F)$  be associated with  $L$  as in Lemma 2.8. If Hypothesis 2.14 holds for every  $l'$  with  $4 \leq l' < l$ , then there exists some  $E_L$ -stable  $\tilde{N}'$ -transversal in  $\text{Irr}_{cusp}(N)$ .*

Some technicalities (mainly in the case where  $\mathbf{G} = D_{4,sc}(\mathbb{F})$ ) delay the complete proof until §7. We construct the  $E_L$ -stable  $\tilde{N}'$ -transversal as a subset of  $\text{Irr}(N | \mathbb{T})$ , where  $\mathbb{T}$  is the  $\tilde{N}$ -stable  $\tilde{N}'$ -transversal from Theorem 3.1(a). In Lemma 6.3, we find some  $E_L$ -stable  $\tilde{N}'$ -transversal of  $\text{Irr}(N | \{\lambda \in \text{Irr}_{cusp}(L) \mid \tilde{L}'_\lambda = L\})$  where  $\tilde{L}' = \tilde{T}_0 L$  as in Theorem 2.8.

In order to find the transversal of  $\text{Irr}(N | \{\lambda \in \text{Irr}_{cusp}(L) \mid \tilde{L}'_\lambda \neq L\})$  with the required properties, we apply the strategy mapped out by Proposition 2.12, itself based on the parametrization of Proposition 2.10. Thanks to Theorem 3.1, the two first assumptions of Proposition 2.12 can be assumed, in particular there exist some  $\hat{N}$ -equivariant extension map  $\Lambda_{L,\mathbb{T}}$  with respect to  $L \triangleleft N$  for  $\mathbb{T}$ , where  $\hat{N} = NE_L$ . We have to ensure the remaining Assumption 2.12(iii) and study the characters of the relative Weyl groups and their Clifford theory.

As already discussed in §2.3, characters in such a transversal have a stabilizer in  $\tilde{N}'E_L$  with a specific structure, namely such a  $\psi \in \text{Irr}(N)$  satisfies

$$(\hat{N}\tilde{L}')_\psi = \hat{N}_\psi \tilde{L}'_\psi \tag{6.1}$$

(see also Lemma 2.4). For studying a character  $\psi \in \text{Irr}(N | \mathbb{T})$ , we apply the parametrization  $\Upsilon$  from Proposition 2.10(a) and the extension map  $\Lambda_{L,\mathbb{T}}$  with respect to  $L \triangleleft N$  for  $\mathbb{T}$ . Then  $\psi = \Upsilon(\overline{(\lambda, \eta)}) = (\Lambda_{L,\mathbb{T}}(\lambda)\eta)^N$  with  $\lambda \in \mathbb{T}$  and  $\eta \in \text{Irr}(W(\lambda))$ . According to Proposition 2.11, the character  $\psi = \Upsilon(\overline{(\lambda, \eta)})$  satisfies Equation (6.1) if

$$\eta \text{ is } \hat{K}(\lambda)_{\eta_0}\text{-stable, where } \tilde{\lambda} \in \text{Irr}(\tilde{L}'_\lambda \mid \lambda) \text{ and } \eta_0 \in \text{Irr}(W(\tilde{\lambda})),$$

where  $\hat{W} = NE_L/L$  and  $\hat{K}(\lambda) = \hat{W}_\lambda$ . The aim of this section is Corollary 6.36, namely to prove that for every  $\lambda \in \mathbb{T}$ ,  $\tilde{\lambda} \in \text{Irr}(\tilde{L}'_\lambda \mid \lambda)$ , and  $\eta_0 \in \text{Irr}(W(\tilde{\lambda}))$ ,

$$\text{there exists some } K(\lambda)_{\eta_0}\text{-stable } \eta \in \text{Irr}(W(\lambda) \mid \eta_0),$$

where  $K(\lambda)$  is the group from Notation 6.4. According to Lemma 6.5, such a character  $\eta$  is also  $\hat{K}(\lambda)_{\eta_0}$ -stable, whenever  $\mathbf{G}$  is not of type  $D_4$ .

In the proof, some arguments depend on the group  $\tilde{L}'_\lambda$ . As in Remark 2.16, we relate the group  $\tilde{L}'$  to subgroups of  $\mathbf{G}$ .

**NOTATION 6.2.** Recall the definitions  $\tilde{L} := \mathcal{L}^{-1}(Z(\mathbf{G})) \cap \mathbf{L}$  and  $\hat{L} := \mathcal{L}^{-1}(\langle h_0 \rangle) \cap \mathbf{L}$  from Lemma 3.13, where  $\mathcal{L} : \mathbf{G} \rightarrow \mathbf{G}$  is given by  $x \mapsto x^{-1}F(x)$ . Recall  $\tilde{N}' := \tilde{T}_0 N$  and set analogously  $\tilde{N} := \tilde{L}N$ . Then  $\tilde{L}'$  and  $\tilde{N}'$  induce on  $\mathbf{G}$  the same automorphisms as  $\tilde{L}$  and  $\tilde{N}$ , respectively.

Note that by an application of Lang’s theorem,  $\mathcal{L}(\mathbf{L}) = \mathbf{L} \supseteq Z(\mathbf{G})$  so that  $L \leq \hat{L} \leq \tilde{L}$ .



For  $\lambda \in \mathbb{T}$ , the characters of  $W(\lambda)$  and  $W(\tilde{\lambda})$  defined as above will be investigated depending on the value of  $\tilde{L}_\lambda$ . The set  $\text{Irr}_{\text{cus}}(L)$  can be partitioned in the following way:

$$\text{Irr}_{\text{cus}}(L) = \mathbb{M}^{(L)} \sqcup \mathbb{M}^{(\hat{L})} \sqcup \mathbb{M}^{(\tilde{L})} \sqcup \mathbb{M}_0,$$

where  $\mathbb{M}^{(X)} := \{\lambda \in \text{Irr}_{\text{cus}}(L) \mid \tilde{L}_\lambda = X\}$  for any subgroup  $L \leq X \leq \tilde{L}$  and  $\mathbb{M}_0 := \text{Irr}_{\text{cus}}(L) \setminus (\mathbb{M}^{(L)} \cup \mathbb{M}^{(\hat{L})} \cup \mathbb{M}^{(\tilde{L})})$ . (In case of  $|\mathbb{Z}(\mathbf{G}^F)| = 2$ , one has  $\mathbb{M}^{(L)} = \emptyset$ .) Note that the sets are by definition  $\tilde{N}E_L$ -stable as  $L, \hat{L}$ , and  $\tilde{L}$  are  $\tilde{N}E_L$ -stable. In the following, we construct an  $E_L$ -stable  $\tilde{N}$ -transversal in  $\text{Irr}_{\text{cus}}(N \mid \mathbb{M}')$  for each of those four given  $NE_L$ -stable subsets  $\mathbb{M}' \subseteq \text{Irr}_{\text{cus}}(L)$ .

**LEMMA 6.3.** *Let  $\mathbb{T}^{(L)} := \mathbb{T} \cap \mathbb{M}^{(L)}$ . Then  $\text{Irr}(N \mid \mathbb{T}^{(L)})$  is an  $E_L$ -stable  $\tilde{N}'$ -transversal in  $\text{Irr}(N \mid \mathbb{M}^{(L)})$ .*

*Proof.* The set  $\mathbb{T}$  is by construction  $NE_L$ -stable, and no two elements are  $\tilde{L}$ -conjugate. Hence, for  $\lambda \in \mathbb{T}^{(L)}$ , we have  $(NE_L \tilde{L})_\lambda = (NE_L)_\lambda$  by Lemma 2.4. By Clifford theory,  $\text{Irr}(N \mid \mathbb{T}^{(L)})$  is an  $\tilde{N}$ -transversal in  $\text{Irr}(L \mid \mathbb{M}^{(L)})$  and is  $NE_L$ -stable.  $\square$

Determining an  $NE_L$ -stable  $\tilde{L}$ -transversal in  $\text{Irr}(N \mid \mathbb{M}')$  for the other sets  $\mathbb{M}'$  is more involved. We start by some general descriptions of  $W(\lambda)$  and related groups for  $\lambda \in \text{Irr}_{\text{cus}}(L)$  (see Proposition 6.28). Afterward, we collect some particular results on cuspidal characters. In the following two subsections, we verify for characters of  $W(\lambda)$  the above condition under the assumption that  $\lambda \in \mathbb{M}^{(\tilde{L})} \cup \mathbb{M}_0$  or  $\lambda \in \mathbb{M}^{(\hat{L})}$ .

In §6.4, we ensure a closely related condition on characters of  $W(\lambda)$  for  $\lambda \in \text{Irr}_{\text{cus}}(L)$  with  $\tilde{L}_\lambda = \hat{L}$ . In §7, we show how these considerations prove Theorem 6.1 and how this implies Theorem A.

**6.1 Understanding  $\text{Irr}_{\text{cus}}(N)$  via characters of subgroups of  $W$**

We start by recalling some basic notation and introducing subgroups of  $\widehat{W} := \widehat{N}/L = NE_L/L$  as in 2.8. Additionally, let  $\overline{N} := N_{\mathbf{G}^F}(\mathbf{L})$  and  $\overline{W} = \overline{N}/L$  (see also Proposition 3.16).

**NOTATION 6.4.** Let  $\mathbf{G}$  and  $F : \mathbf{G} \rightarrow \mathbf{G}$  be as in Notation 3.3 with odd  $q$ . Let  $\mathbf{L}$  be the standard Levi subgroup of  $(\mathbf{G}, F)$  such that  $L = \mathbf{L}^F$ . For any  $J$  with  $L \leq J \leq \tilde{L}$  and  $\lambda \in \mathbb{Z}\text{Irr}(J)$ , we set  $W(\lambda) := N_\lambda/L$ . If additionally  $J$  is  $E_L$ -stable,  $\overline{W}$  acts on  $\text{Char}(J)$ ; hence, we can define  $\overline{W}(\lambda) := \overline{W}_\lambda$  and  $K(\lambda) := \overline{W}_{\lambda^{J\langle F_p \rangle}} \Big|_J$ .

The groups  $K(\lambda)$  and  $\widehat{K}(\lambda)$  are strongly related; in particular, by the following result, it is sufficient to consider  $K(\lambda)$  instead of  $\widehat{K}(\lambda)$  if  $\mathbf{G}$  is not of type  $D_4$ . Recall that  $\gamma$  is the graph automorphism of  $\mathbf{G}$  of order 2 swapping  $\alpha_1$  and  $\alpha_2$ .

**LEMMA 6.5.** *Let  $E^\circ := \langle F_p, \gamma \rangle$ ,  $E_L^\circ := E^\circ \cap E_L$ ,  $\lambda \in \mathbb{T}$ ,  $\tilde{\lambda}$  defined as above,  $\eta_0 \in \text{Irr}(W(\tilde{\lambda}))$ , and  $\eta \in \text{Irr}(W(\lambda))$ . Then  $\eta$  is  $\widehat{K}(\lambda)_{\eta_0} \cap (W \rtimes E_L^\circ)$ -stable if and only if it is  $K(\lambda)_{\eta_0}$ -stable.*

*Proof.* Note that  $F_p \in \mathbb{Z}(\widehat{W})$ ; hence,  $\eta$  and  $\eta_0$  are  $F_p$ -stable. Recall that  $\overline{W}$  can be identified with the quotient  $(\widehat{W} \cap (W \rtimes E_L^\circ)) / \langle F_p \rangle$ . The group  $\widehat{K}(\lambda) \cap (W \rtimes E_L^\circ)$  then projects to  $K(\lambda)$ , that is, for every  $w \in \overline{W}$  and  $e \in \langle F_p \rangle$  with  $\lambda^{we} = \lambda$ , we see  $(\lambda^{LE_L^\circ} \Big|_L)^w = \lambda$ . This implies  $\widehat{K}(\lambda)\langle F_p \rangle \cap \overline{W} = K(\lambda)$  and  $\widehat{K}(\lambda)_{\eta_0}\langle F_p \rangle \cap \overline{W} = K(\lambda)_{\eta_0}$ . As  $F_p$  stabilizes  $\eta$ , this implies the statement.  $\square$

For  $\lambda \in \mathbb{T}$  and  $\tilde{\lambda} \in \text{Irr}(\tilde{L}_\lambda | \lambda)$ , the group  $W(\tilde{\lambda})$  is determined by  $\tilde{\lambda} \Big|_L$ , as  $W$  acts trivially on the characters of  $\tilde{L}_\lambda/L$ . Note that  $W(\tilde{\lambda}) \neq W(\tilde{\lambda} \Big|_L)$  in general. We can work with the group  $\tilde{L}$  instead of  $\tilde{L}'$  because of the following observation.

LEMMA 6.6. *Let  $\lambda \in \text{Irr}_{\text{cusp}}(L)$ ,  $\hat{\lambda} \in \text{Irr}(\hat{L} | \lambda)$ , and  $\tilde{\lambda} \in \text{Irr}(\tilde{L} | \hat{\lambda})$ .*

- (a) *Then  $\overline{W}(\tilde{\lambda}) = \overline{W}(\tilde{\lambda}')$  and  $W(\tilde{\lambda}) = W(\tilde{\lambda}')$  for every  $\tilde{\lambda}' \in \text{Irr}(\tilde{L}' | \lambda)$ .*
- (b) *Then  $\overline{W}(\tilde{\lambda}) \leq \overline{W}(\hat{\lambda}) \leq \overline{W}(\lambda)$  and  $W(\tilde{\lambda}) \leq W(\hat{\lambda}) \leq W(\lambda)$ .*

*Proof.* By the construction of  $\mathbb{T}$ , the character  $\lambda \in \mathbb{T}$  satisfies  $(\hat{N}\tilde{L})_\lambda = \hat{N}_\lambda \tilde{L}_\lambda$ . Because of  $\tilde{\mathbf{G}} = \mathbf{Z}(\tilde{\mathbf{G}})\tilde{\mathbf{G}}$ , the group  $\tilde{L}'$  is a subgroup of  $\tilde{L}\mathbf{Z}(\tilde{\mathbf{G}})$  and  $\tilde{L}' = \mathbf{L} \cap \tilde{L}\mathbf{Z}(\tilde{\mathbf{G}})$ . This implies  $(NE_L)_{\tilde{\lambda}} = (NE_L)_{\tilde{\lambda}'}$  and hence part (a) (see Remark 2.16 for a similar argument).

As  $h_0$  is centralized by  $\hat{N}$ , the group  $\hat{L}$  is normalized by  $N$  and  $\overline{N}$ . The containments from part (b) follow from this by straightforward considerations. □

For  $\lambda \in \mathbb{T}$  and  $\tilde{\lambda} \in \text{Irr}(\tilde{L} | \lambda)$ , we compute  $\overline{W}(\tilde{\lambda})$  as an approximation of  $W(\tilde{\lambda})$ . For  $I \in \mathcal{O}$  and  $d \in \mathbb{D}$ , we use the groups  $\mathbf{G}_I, L_I, \hat{L}_I$ , and  $\mathbf{L}_d$  from Notation 3.9 and Lemmas 3.12 and 3.13. In Lemma 3.13, the structure of  $\mathbf{L}$  and some of its subgroups was already studied. Additionally, we use the following properties of  $\hat{L}$ .

PROPOSITION 6.7 (The structure of  $\hat{L}$ ). *For  $d \in \mathbb{D}$  and  $I \in \mathcal{O}$ , let  $\hat{L}_d := \hat{L} \cap \mathbf{L}_d$ .*

- (a)  *$L_1$  is a split torus of rank  $|J_1|$ .*
- (b)  *$\hat{L}$  is the central product of  $\hat{L}_d$  ( $d \in \mathbb{D}$ ) over  $\langle h_0 \rangle$ .*
- (c)  *$\hat{L}_d$  is the central product of  $\hat{L}_I$  ( $I \in \mathcal{O}_d$ ) over  $\langle h_0 \rangle$ .*
- (d)  *$[\hat{L}_I, \hat{L}_{I'}] = 1$  for all  $I, I' \in \mathcal{O}$  with  $I \neq I'$ .*

*Proof.* The first three parts follow from Lemma 3.13(c).

Part (d) is clear if  $I \in \mathcal{O}_1$  or  $I' \in \mathcal{O}_1$ . Note that the groups  $\hat{L}_I$  and  $\hat{L}_{I'}$  contain the root subgroups for  $\Phi_I$  and  $\Phi_{I'}$ , which are orthogonal to each other. At least one of them is of type  $A_l$ . Hence, no nontrivial linear combination of roots from  $\Phi_I$  and  $\Phi_{I'}$  is a root itself. Hence, by Chevalley’s commutator formula, we see that the commutator of the groups is trivial. □

We continue using the groups  $\overline{V}_d$  from Lemma 3.21 for the description of  $\overline{W}(\hat{\lambda})$ . We write  $\text{Irr}_{\text{cusp}}(\hat{L})$  for  $\text{Irr}(\hat{L} | \text{Irr}_{\text{cusp}}(L))$ .

LEMMA 6.8 (Characters of  $\hat{L}$ ). *Let  $\hat{\lambda} \in \text{Irr}_{\text{cusp}}(\hat{L})$ ,  $\hat{\lambda}_d \in \text{Irr}(\hat{\lambda} \Big|_{\hat{L}_d})$  for every  $d \in \mathbb{D}$  and  $\hat{\lambda}_I \in \text{Irr}(\hat{\lambda} \Big|_{\hat{L}_I})$  for every  $I \in \mathcal{O}$ . Then:*

- (a)  *$\hat{\lambda} = \odot_{d \in \mathbb{D}} \hat{\lambda}_d$  and  $\hat{\lambda}_d = \odot_{I \in \mathcal{O}_d} \hat{\lambda}_I$  for every  $d \in \mathbb{D}$ ,*
- (b)  *$\hat{\lambda}_d \in \text{Irr}_{\text{cusp}}(\hat{L}_d)$  and  $\hat{\lambda}_I \in \text{Irr}_{\text{cusp}}(\hat{L}_I)$ ,*
- (c)  *$\overline{W}(\hat{\lambda})$  is the direct product of the groups  $\overline{W}_d(\hat{\lambda}) := (\overline{V}_d)_{\hat{\lambda}}/H_d$  ( $d \in \mathbb{D}$ ), and*
- (d)  *$(\overline{V}_d)_{\hat{\lambda}}/H_d = (\overline{V}_d)_{\hat{\lambda}_d}/H_d$ .*

*Proof.* The description of  $\hat{\lambda}$  and  $\hat{\lambda}_d$  in (a) follows from the structure of  $\hat{L}$  and  $\hat{L}_d$  given in Proposition 6.7. The characters  $\hat{\lambda}_d$  and  $\hat{\lambda}_I$  cover a cuspidal character of  $L_d$  and  $L_I$ , respectively, by Lemma 2.15, which then also gives (b). Considering the roots underlying  $V_d$  and  $\mathbf{L}_{d'}$ , we see that the Chevalley relations imply  $[V_d, \mathbf{L}_{d'}] = 1$  for  $d, d' \in \mathbb{D}$  with  $d \neq d'$ . This implies the parts (c) and (d). □

For a more explicit description of the groups  $\overline{W}(\widehat{\lambda})$ , we introduce some elements of  $\overline{V}$  using the maps  $\kappa_d$  ( $d \in \mathbb{D}$ ) from Lemma 3.21. For  $d \in \mathbb{D} \setminus \{-1\}$ , recall  $\mathcal{O}_d = \{I_{d,1}, \dots, I_{d,a_d}\}$  from Notation 3.17.

NOTATION 6.9. Let  $d \in \mathbb{D} \setminus \{-1\}$  and  $c_{I_{d,j}} := \kappa_d(\mathbf{n}_{e_j}(\varpi)) \in \overline{V}_d$  for every  $j \in \underline{a}_d$ . Note that for every  $I \in \mathcal{O}_d$ ,  $c_I$  is some  $\overline{V}_d$ -conjugate of  $\overline{\mathbf{n}}_1^{(d)}$  and  $\rho_{\mathbf{T}}(c_I) = \prod_{i \in I} (i, -i)$ , where  $\rho_{\mathbf{T}} : \overline{N}_0 \rightarrow \mathcal{S}_{\pm \underline{l}}$  is the natural epimorphism (see before Proposition 3.16). If  $2 \nmid |I|$  and  $I \notin \mathcal{O}_{-1} \cup \mathcal{O}_1$ , then by considerations as in the proof of Lemma 4.4(b),  $c_I$  acts as transpose-inverting on  $L_I$  via the identification of  $L_I$  with  $\text{GL}_{|I|}(q)$ .

We define additionally the subgroups

$$\overline{V}_{d,S} := H_d \langle \kappa_d(\mathbf{n}_{e_i - e_{i+1}}(-1)) \mid i \in \underline{a}_d - 1 \rangle \tag{6.2}$$

and  $V_S := \langle V_{d,S} \mid d \in \mathbb{D} \setminus \{-1\} \rangle$ . Then  $\rho_{\mathbf{T}}(V_S(L \cap N_0)) / \rho_{\mathbf{T}}(L \cap N_0) = \mathcal{S}_{\mathcal{O}} \leq \mathcal{S}_{\pm \mathcal{O}}$ .

If  $-1 \in \mathbb{D}$ , then we set  $c_{J_{-1}} := \overline{\mathbf{n}}_1$  from 3.19.

Using the notation of permutation groups given in Notation 3.14, we identify the group  $\overline{W} = \overline{N}/L$  with  $\mathcal{S}_{\pm \mathcal{O}}$ . Computations in  $\overline{W}$  show that  $\overline{V} = H \langle c_I \mid I \in \mathcal{O} \rangle V_S$ .

DEFINITION 6.10. Let  $\widehat{\lambda} \in \text{Irr}(\widehat{L})$ . We call  $\widehat{\lambda}$  *standardized* if for every  $I, I' \in \mathcal{O}$  the characters  $\widehat{\lambda}_I$  and  $\widehat{\lambda}_{I'}$  are either  $V_S$ -conjugate or not  $\overline{V}$ -conjugate. For such  $\widehat{\lambda}$ , we call the characters in  $\text{Irr}(\widehat{\lambda} \upharpoonright_L)$  also *standardized*.

Computations show that every standardized character  $\widehat{\lambda}$  satisfies  $\overline{V}_{\widehat{\lambda}} = H \langle c_I \mid I \in \mathcal{O} \rangle_{\widehat{\lambda}}$   $(V_S)_{\widehat{\lambda}}$  and every  $\overline{N}$ -orbit in  $\text{Irr}(\widehat{L})$  contains a standardized character. For a more explicit description of  $\overline{W}_d(\widehat{\lambda})$ , we introduce the following notation.

NOTATION 6.11. Let  $E$  be a set, and let  $M$  be a subset of  $2^E$ , the set of all subsets of  $E$ . For  $m' \subseteq E$ , we write  $m' \subset\subset M$  if  $m' \subseteq m$  for some  $m \in M$ .

Using the notation of permutation groups given in Notation 3.14, we identify the group  $\overline{W} = \overline{N}/L$  with  $\mathcal{S}_{\pm \mathcal{O}}$ . In the following, we describe  $\overline{W}_d(\widehat{\lambda}_d)$  as a subgroup of  $\mathcal{S}_{\pm \mathcal{O}_d}$ . We use the Young-like subgroups of  $\mathcal{S}_{\pm \mathcal{O}_d}$  from Notation 3.14 that are associated with a partition of  $\mathcal{O}_d$ .

LEMMA 6.12. Let  $\lambda \in \text{Irr}_{\text{cusp}}(L)$  be standardized. We set

$$\mathcal{O}_c(\widehat{\lambda}) := \{I \in \mathcal{O} \mid (\widehat{\lambda}_I)^{c_I} = \widehat{\lambda}_I\}.$$

Let  $Y(\widehat{\lambda}) \vdash \mathcal{O}_c(\widehat{\lambda})$  and  $Y'(\widehat{\lambda}) \vdash (\mathcal{O}(\widehat{\lambda}) \setminus \mathcal{O}_c(\widehat{\lambda}))$  be the partitions such that  $\{I, I'\} \subset\subset Y(\widehat{\lambda})$  or  $\{I, I'\} \subset\subset Y'(\widehat{\lambda})$  if and only if  $\widehat{\lambda}_I$  and  $\widehat{\lambda}_{I'}$  are  $V_S$ -conjugate. Then

$$\overline{W}(\widehat{\lambda}) = \mathcal{Y}_{\pm Y(\widehat{\lambda})} \times \mathcal{Y}_{Y'(\widehat{\lambda})},$$

where  $\mathcal{Y}_{\pm Y(\widehat{\lambda})}$  and  $\mathcal{Y}_{Y'(\widehat{\lambda})}$  are defined as in Notation 3.14.

*Proof.* Note  $Y'(\widehat{\lambda}) \cup Y(\widehat{\lambda}) \vdash \mathcal{O}$ . As  $\widehat{\lambda}$  is standardized,

$$\overline{W}_d(\widehat{\lambda}) = \langle (I, -I) \mid I \in \mathcal{O}_d \rangle_{\widehat{\lambda}} \rtimes \langle (I, I')(-I, -I') \mid I, I' \in \mathcal{O}_d \rangle_{\widehat{\lambda}} \text{ for every } d \in \mathbb{D}.$$

This gives our claim. □

Let  $\zeta \in \mathbb{F}^\times$  with  $\varpi = \zeta^{(q-1)^2}$  and  $t_{I,2} := \mathbf{h}_I(\zeta)$  for every  $I \subseteq \underline{l}$  as in Lemma 3.13. For  $I \in \mathcal{O} \setminus \{J_{-1}\}$ , the element  $t_{I,2}$  satisfies  $[\mathbf{L}_I, t_{I,2}] = 1$ .

LEMMA 6.13 (Structure of  $\overline{W}(\tilde{\lambda})$ ). *Let  $\lambda \in \text{Irr}_{\text{cusp}}(L)$ ,  $\hat{\lambda} \in \text{Irr}(\hat{L} \mid \lambda)$ ,  $\tilde{\lambda} \in \text{Irr}(\tilde{L} \mid \hat{\lambda})$ ,  $\hat{\lambda}_I \in \text{Irr}(\hat{L}_I \mid \hat{\lambda})$ , and  $\tilde{\lambda}_I \in \text{Irr}(\tilde{L}_I \mid \hat{\lambda}_I)$  ( $I \in \mathcal{O}_c(\hat{\lambda})$ ). Assume that  $\hat{\lambda}$  is standardized and  $\tilde{L}_{\hat{\lambda}} = \tilde{L}$ . We set*

$$\mathcal{O}_{c,1}(\hat{\lambda}) := \{I \in \mathcal{O}_c(\hat{\lambda}) \mid (\tilde{\lambda}_I)^{c_I} = \tilde{\lambda}_I\} \text{ and } \mathcal{O}_{c,-1}(\hat{\lambda}) := \{I \in \mathcal{O}_c(\hat{\lambda}) \mid (\tilde{\lambda}_I)^{c_I} \neq \tilde{\lambda}_I\}.$$

(a) *For  $I \in \mathcal{O}_c(\hat{\lambda}) \setminus \{J_{-1}\}$  and  $\epsilon = \pm 1$ , we have  $I \in \mathcal{O}_{c,\epsilon}(\hat{\lambda}) \Leftrightarrow \hat{\lambda}_I(t_{I,2}^2) = \epsilon \hat{\lambda}_I(1)$ .*

(b)  *$\overline{W}(\tilde{\lambda}) \leq \mathcal{S}_{\pm \mathcal{O}_{c,1}(\hat{\lambda})} \times \mathcal{S}_{\pm \mathcal{O}_{c,-1}(\hat{\lambda})}^D$ , more precisely*

$$\overline{W}(\tilde{\lambda}) = \left( \langle (I, -I) \mid I \in \mathcal{O}_{c,1}(\hat{\lambda}) \rangle \langle (I, -I)(I', -I') \mid I, I' \in \mathcal{O}_{c,-1}(\hat{\lambda}) \rangle \right) \rtimes \mathcal{Y}_{Y(\hat{\lambda}) \cup Y'(\hat{\lambda})}.$$

If the character  $\hat{\lambda}$  is clear from the context, we write  $\mathcal{O}_{c,\epsilon}$  instead of  $\mathcal{O}_{c,\epsilon}(\hat{\lambda})$ .

*Proof.* Note that the description of  $\tilde{L}$  given in Lemma 3.13(e) shows that  $\hat{\lambda}$  extends to  $\tilde{L}$  if and only if  $\hat{\lambda}_I$  extends to  $\tilde{L}_I := \hat{L}_I \langle t_{I,2} \rangle$  for every  $I \in \mathcal{O}$ .

We have  $t_{I,2} := \prod_{I \in \mathcal{O}} t_{I,2}$ ,  $\mathcal{L}(t_{I,2}) = \mathbf{h}_I(\varpi)$ , and  $\tilde{L} = \langle \hat{L}, t_{I,2} \rangle$  (see Lemma 3.13). This implies  $\tilde{L} \leq \langle \tilde{L}_I \mid I \in \mathcal{O} \rangle$ . By the Chevalley relations, we see  $[V_S, t_{I,2}] = 1$ . Let  $I, I' \in \mathcal{O}$  such that  $\hat{\lambda}_I$  and  $\hat{\lambda}_{I'}$  are  $V_S$ -conjugate. Then we can choose their extensions  $\tilde{\lambda}_I$  and  $\tilde{\lambda}_{I'}$  to  $\tilde{L}_I$  and  $\tilde{L}_{I'}$  such that they are  $V_S$ -conjugate, as  $N_{V_S}(\hat{L}_I) = HC_{V_S}(\hat{L}_I)$ , and therefore  $\tilde{\lambda}_{I'}$  is uniquely determined by  $\tilde{\lambda}_I$ .

Let  $\phi \in \text{Irr}(\langle \tilde{L}_I \mid I \in \mathcal{O} \rangle)$  with  $\phi|_{\tilde{L}_I} = \tilde{\lambda}_I$  for every  $I \in \mathcal{O}$ . Without loss of generality, we may assume  $\phi|_{\tilde{L}} = \tilde{\lambda}$ . By the above construction, we have  $(V_S)_\phi = (V_S)_{\tilde{\lambda}}$ . Because of  $\overline{V}_{\tilde{\lambda}} = H \langle c_I \mid I \in \mathcal{O} \rangle_{\tilde{\lambda}} (V_S)_{\tilde{\lambda}}$ , it is sufficient to determine  $\langle c_I \mid I \in \mathcal{O} \rangle_{\tilde{\lambda}}$  for computing  $\overline{V}_{\tilde{\lambda}}$ .

Let  $\mu_I \in \text{Irr}(\tilde{L}_I)$  be the linear character with  $\ker(\mu_I) = \hat{L}_I$ . For any  $Q \subseteq \mathcal{O}$ , let  $\mu_Q \in \text{Irr}(\langle \tilde{L}_I \mid I \in \mathcal{O} \rangle)$  be the linear character with  $\langle L_I \mid I \in \mathcal{O} \rangle \leq \ker(\mu_Q)$  such that for every  $I \in \mathcal{O}$ , the inclusion  $\tilde{L}_I \leq \ker(\mu_Q)$  holds if and only if  $I \notin Q$ . Note that  $\mu_Q(t_{I,2}) = 1$  if and only if  $|Q|$  is even.

For  $Q \subseteq \mathcal{O}$ , let  $c_Q := \prod_{I \in Q} c_I \in \overline{V}$ . If  $Q' \subseteq \mathcal{O}_{c,-1}(\hat{\lambda})$ , then  $c_{Q'} \in \overline{V}_{\tilde{\lambda}}$  and we see that  $\phi^{c_{Q'}} = \phi_{\mu_{Q'} \cap \mathcal{O}_{c,-1}}$ . As  $\mu_{Q' \cap \mathcal{O}_{c,-1}}(t_{I,2}) = (-1)^{|Q' \cap \mathcal{O}_{c,-1}|}$ , this leads to a proof of part (b), in particular

$$\overline{W}(\tilde{\lambda}) = (\langle (I, -I) \mid I \in \mathcal{O}_{c,1} \rangle \langle (I, -I)(I', -I') \mid I, I' \in \mathcal{O}_{c,-1} \rangle) \rtimes \mathcal{Y}_{Y(\hat{\lambda}) \cup Y'(\hat{\lambda})}.$$

Let  $I \in \mathcal{O}_c(\hat{\lambda}) \setminus \{J_{-1}\}$ . Then  $c_I$  acts by inverting on  $\mathbf{T}_I$ , in particular  $t_{I,2}^{c_I} = t_{I,2}^{-1}$  and  $[c_I, t_{I,2}] = t_{I,2}^{-2}$ . Because of  $t_{I,2} \in Z(\mathbf{L}_I)$ , we see that  $[t_{I,2}, \overline{V}_d] \subseteq Z(\mathbf{L})$ . Any extension  $\tilde{\lambda}_I$  of  $\hat{\lambda}_I$  to  $\langle \hat{L}_I, t_{I,2} \rangle$  satisfies  $\tilde{\lambda}_I(t_{I,2}) \neq 0$  since  $t_{I,2} \in Z(\tilde{L}_I)$ .

Note that  $\nu_I \in \text{Irr}(\tilde{\lambda}|_{Z(\tilde{L}_I)})$  is linear. As  $\hat{\lambda}_I$  is  $c_I$ -stable,  $\nu_I$  has multiplicative order 1 or 2. We observe that  $t_{I,2}^{c_I} = t_{I,2}^{-1} \in Z(\tilde{L}_I)$  and hence

$$\tilde{\lambda}_I(t_{I,2}^{c_I}) = \lambda_I(1)\nu_I(t_{I,2}^{c_I}) = \tilde{\lambda}(t_{I,2})\nu_I([t_{I,2}, c_I]) = \tilde{\lambda}(t_{I,2})\nu_I(t_{I,2}^{-2}).$$

Accordingly,  $\tilde{\lambda}_I$  is  $c_I$ -invariant if and only if  $[c_I, t_{I,2}] \in \ker(\nu_I) = \ker(\hat{\lambda}) \cap Z(\hat{L}_I)$ . This proves (a). □

The group  $W(\lambda)$  is then generated by  $W(\hat{\lambda})$  and an element that is described below.

LEMMA 6.14. *Let  $\mu \in \text{Irr}(\widehat{L})$  with  $\ker(\mu) = L$  and  $\widehat{\lambda} \in \text{Irr}(\widehat{L})$ . Additionally, for every  $I \in \mathcal{O}$ , let  $\mu_I \in \text{Irr}(\widehat{L}_I)$  with  $\ker(\mu_I) = L_I$ ,  $\widehat{\lambda}_I \in \text{Irr}(\widehat{\lambda}|_{\widehat{L}_I})$  and  $\lambda_I \in \text{Irr}(\widehat{\lambda}_I|_{L_I})$ .*

- (a) *Let  $x \in \overline{W} \setminus \overline{W}(\widehat{\lambda})$  and  $\lambda \in \text{Irr}(\widehat{\lambda}|_L)$ . Then  $x \in \overline{W}(\lambda)$  if and only if, for every  $I \in \mathcal{O}$ , the equality  $(\widehat{\lambda}_I)^x = \widehat{\lambda}_{I'}\mu_{I'}$  holds, where  $I' \in \mathcal{O}$  with  $(\widehat{L}_I)^x = \widehat{L}_{I'}$ .*
- (b) *We set  $\mathcal{O}_{ext} := \{I \in \mathcal{O} \mid \widehat{\lambda}_I|_{L_I} = \lambda_I\}$  and  $\mathcal{O}_{ind} := \mathcal{O} \setminus \mathcal{O}_{ext}$ . Then  $W(\lambda)$  stabilizes  $\mathcal{O}_{ext}$  and  $\mathcal{O}_{ind}$ .*

*Proof.* Since  $\widehat{L}/L$  has order 2, we see that  $\mu_{\mathcal{O}}$ , the product of the characters  $\widehat{\mu}_I$  ( $I \in \mathcal{O}$ ) defined as in the proof of Lemma 6.13, is an extension of  $\mu$ . This implies part (a).

For part (b), we observe that for  $I \in \mathcal{O}$ ,  $\sigma \in W(\lambda)$  and  $I' := \sigma^{-1}(I)$  the characters  $\widehat{\lambda}_I|_{L_I}$  and  $(\widehat{\lambda}_I)^\sigma|_{L_{I'}} = \widehat{\lambda}_{I'}\mu_{I'}|_{L_{I'}}$  have the same number of constituents. This proves part (b) since  $I \in \mathcal{O}_{ind}$  if and only if  $\widehat{\lambda}_I|_{L_I}$  is reducible. □

### 6.2 Cuspidal characters of $L_I$

The aim here is to describe the structure of  $\overline{W}(\widehat{\lambda})$  by analyzing  $\mathcal{O}_{c,-1}(\widehat{\lambda})$  (see Lemma 6.13). We show in this section that for some  $I \in \mathcal{O}$  there exist no or only few  $c_I$ -stable cuspidal characters of  $L_I$  and study the kernel of those characters (see Corollary 6.22).

For  $I \in \mathcal{O}$ , let  $\text{Irr}_{cusp}(\widehat{L}_I) := \text{Irr}(\widehat{L}_I \mid \text{Irr}_{cusp}(L_I))$  and call those characters *cuspidal* as well.

LEMMA 6.15. *Let  $I \in \mathcal{O}_d$  for some  $d \in \mathbb{D}_{\text{odd}} \setminus \{\pm 1\}$ . There exists no  $c_I$ -stable character in  $\text{Irr}_{cusp}(\widehat{L}_I)$ .*

*Proof.* According to Lemma 4.2(c),  $L_I \cong \text{GL}_d(q)$  and the element  $c_I$  defined in Notation 6.9 induces on  $G_I$  a combination of an inner automorphism and the nontrivial graph automorphism according to Lemma 4.4(b). The element  $c_I$  acts on the torus  $\mathbf{Z}_I := \mathbf{h}_I(\mathbb{F}^\times)$  from Lemma 4.2 by inverting. Hence, via the isomorphism  $L_I \cong \text{GL}_d(q)$ , the element  $c_I$  induces on  $L_I$  a combination of an inner automorphism and the nontrivial graph automorphism.

According to Proposition 5.1(a), there is no cuspidal character of  $\text{GL}_d(q)$  that is invariant under transpose-inverse. So no character in  $\text{Irr}_{cusp}(L_I)$  is  $c_I$ -stable. Now, the element  $t_I$  from Lemma 3.13 can be chosen such that  $[t_I, \mathbf{L}_I] = 1$  (see Lemma 4.3). This implies that every cuspidal character of  $\widehat{L}_I$  is an extension of a cuspidal character of  $L_I$ . This proves that there is no  $c_I$ -stable character in  $\text{Irr}_{cusp}(\widehat{L}_I)$ . □

With the following statement, the above shows that  $\mathcal{O}_{c,1}(\widehat{\lambda}) \cap \mathcal{O}_d = \emptyset$  for every  $d \in \mathbb{D} \setminus \{\pm 1\}$  and  $\widehat{\lambda} \in \text{Irr}_{cusp}(\widehat{L})$  with  $h_0 \in \ker(\widehat{\lambda})$ .

PROPOSITION 6.16. *Let  $I \in \mathcal{O}_d$  for some  $d \in \mathbb{D} \setminus \{\pm 1\}$ . Then every  $\psi \in \text{Irr}_{cusp}(\widehat{L}_I \mid 1_{\langle h_0 \rangle})$  with  $\psi^{c_I} = \psi$  satisfies  $\mathbf{Z}_I^F \leq \ker(\psi)$ , where  $\mathbf{Z}_I := \mathbf{h}_I(\mathbb{F}^\times)$  is as in Lemma 4.2.*

*Proof.* Under the isomorphism  $\mathbf{L}_I/\langle h_0 \rangle \cong \text{GL}_d(\mathbb{F})$  from Lemma 4.2, we obtain  $\widehat{L}_I/\langle h_0 \rangle \cong \text{GL}_d(q)$ . Via this isomorphism,  $\mathbf{Z}_I^F$  is mapped to  $\mathbf{Z}(\text{GL}_d(q))$ . Let  $\psi \in \text{Irr}_{cusp}(\widehat{L}_I \mid 1_{\langle h_0 \rangle})$ . If  $\psi$  is  $c_I$ -invariant, then it corresponds to a cuspidal character of  $\text{GL}_d(q)$  that is invariant under transpose-inverse (see Lemma 4.4(b)). According to Proposition 5.1(a), such a character is trivial on the center. This implies  $\mathbf{Z}_I^F \leq \ker(\psi)$ . □

**THEOREM 6.17.** *Let  $\nu \in \text{Irr}(\langle h_0 \rangle)$  be nontrivial,  $d \in \mathbb{D}_{\text{even}}$ ,  $I \in \mathcal{O}_d$ , and let  $t_{I,2}$  be as defined before Lemma 6.13.*

- (a) *If  $d \geq 4$ , every  $\psi \in \text{Irr}_{\text{cusp}}(\widehat{L}_I | \nu)$  with  $\psi^{c_I} = \psi$  satisfies  $t_{I,2}^2 \in \ker(\psi)$ .*
- (b) *If  $d = 2$  and  $4 \mid (q - 1)$ , there is a unique  $\psi \in \text{Irr}_{\text{cusp}}(\widehat{L}_I | \nu)$  with  $\psi^{c_I} = \psi$  and  $t_{I,2}^2 \notin \ker(\psi)$ .*

The proof goes through the next three lemmas. We keep  $\nu$  the nontrivial irreducible character of  $\langle h_0 \rangle$ . As a first step toward a proof of the above, we determine the inertia group in  $\widehat{L}_I$  of cuspidal  $c_I$ -stable characters of  $L_I$ .

**LEMMA 6.18.** *Let  $d \in \mathbb{D}_{\text{even}}$ ,  $I \in \mathcal{O}_d$ ,  $\psi \in \text{Irr}_{\text{cusp}}(L_I | \nu)$  with  $\psi^{c_I} = \psi$ , and  $t_{I,2}^2 \notin \ker(\psi)$ . Then  $(\widehat{L}_I)_\psi = L_I$ .*

*Proof.* For the proof, it is sufficient to show that a character  $\psi$  with the above properties and  $(\widehat{L}_I)_\psi = \widehat{L}_I$  cannot exist. Recall  $t_{I,2}^2 = \mathbf{h}_I(\zeta')$ , where  $\zeta' \in \mathbb{F}^\times$  is a root of unity of order  $2(q - 1)_2$ .

Let  $\mathbf{G}' := D_{2d,sc}(\mathbb{F})$  with an  $\mathbb{F}_q$ -structure given by a standard Frobenius endomorphism  $F_1 : \mathbf{G}' \rightarrow \mathbf{G}'$ . Let  $\mathbf{L}'$  be the Levi subgroup of  $\mathbf{G}'$  of type  $A_{d-1} \times A_{d-1}$  such that  $\mathcal{O}(\mathbf{L}') = \mathcal{O}_d(\mathbf{L}') = \{I_1, I_2\}$  be defined by  $\mathbf{L}'$  as in 3.8. Then  $\psi$  defines cuspidal characters  $\lambda_{I_1} \in \text{Irr}_{\text{cusp}}(L_{I_1})$  and  $\lambda_{I_2} \in \text{Irr}_{\text{cusp}}(L_{I_2})$  that have extensions to  $\widehat{L}_{I_1}$  and  $\widehat{L}_{I_2}$  and are  $V'_S$ -stable, where  $V'_S$  is associated with  $\mathbf{G}'$  and  $\mathbf{L}'$  as in Definition 6.10. We can choose  $\widehat{\lambda}_{I_j} \in \text{Irr}(\widehat{L}_{I_j} | \lambda_{I_j})$  ( $j = 1, 2$ ) such that they are not  $V'_S$ -conjugate. The group  $\widehat{L}' := \widehat{L}_{I_1} \cdot \widehat{L}_{I_2}$  is a central product of the groups  $\widehat{L}_{I_j}$  ( $j \in \underline{2}$ ) over  $\langle h_0 \rangle$ . Let  $\widehat{\lambda}' := \widehat{\lambda}_{I_1} \cdot \widehat{\lambda}_{I_2} \in \text{Irr}(\widehat{L}')$ ,  $\lambda' = \widehat{\lambda}'|_{L'}$  and  $\widetilde{\lambda}' \in \text{Irr}(\widetilde{L}' | \lambda')$  where  $L' := (\mathbf{L}')^F$ ,  $\widetilde{L}' := \mathcal{L}'^{-1}(\langle h_0 \rangle) \cap \mathbf{L}'$  and  $\widetilde{L}' := \mathcal{L}'^{-1}(Z(\mathbf{G}')) \cap \mathbf{L}'$  for the Lang map  $\mathcal{L}' : x \mapsto x^{-1}F'(x)$  of  $\mathbf{G}'$ .

Defining  $W, \overline{W}$  from the above for  $\mathbf{G}'$  and  $\mathbf{L}'$ , note that  $W(\widehat{\lambda}') = \overline{W}(\widehat{\lambda}') = \langle (I_1, -I_1), (I_2, -I_2) \rangle$  and  $W(\lambda') = \overline{W}(\lambda') = \mathcal{S}_{\pm \mathcal{O}(\mathbf{L}'})$ . Note also that  $W(\widetilde{\lambda}') = \langle (I_1, -I_1)(I_2, -I_2) \rangle \leq Z(W(\lambda')) = [W(\lambda'), W(\lambda')]$ .

Now, observe that the nontrivial character of  $W(\widetilde{\lambda}')$  is  $W(\lambda')$ -stable but does not extend to  $W(\lambda')$  as the kernel of any linear character of  $W(\lambda')$  contains  $Z(W(\lambda')) = [W(\lambda'), W(\lambda')]$ . This also implies that for some character  $\eta \in \text{Irr}(W(\lambda'))$ , the constituent  $\eta_0$  of  $\eta|_{W(\widetilde{\lambda}' )}$  has multiplicity 2 in  $\eta|_{W(\widetilde{\lambda}' )}$ . The character  $R_{\widetilde{L}'}^{\widetilde{G}'}(\widetilde{\lambda}')_{\eta_0}$  restricts to  $(\mathbf{G}')^F$  and has only constituents with multiplicity 1 according to [CE, 15.11].

Like in other places, these results are considering first the situation of Harish-Chandra induction for a group  $(\widetilde{\mathbf{G}}')^{F_1}$  that comes from a regular embedding of  $\mathbf{G}'$  into a group with connected center. These results can then be applied to the groups  $\widetilde{G}' := \mathcal{L}'^{-1}(Z(\mathbf{G}'))$  and the subgroup  $\widetilde{L}'$ .

On the other hand, according to [B1, 13.9(b)], the character  $R_{L'}^{\mathbf{G}'}(\lambda')_\eta$  has multiplicity 2 in  $R_{\widetilde{L}'}^{\widetilde{G}'}(\widetilde{\lambda}')_{\eta_0}$ . This is a contradiction. This implies that a character  $\psi$  with the above properties cannot exist and proves the statement.  $\square$

In the next step, we continue to consider the case where  $I \in \mathcal{O}_d$  with  $2 \mid d$ .

**LEMMA 6.19.** *Let  $\nu \in \text{Irr}(\langle h_0 \rangle)$  be nontrivial, and let  $I \in \mathcal{O}_d$  for some  $d \in \mathbb{D}_{\text{even}}$  with  $d > 2$ . Then every  $\psi \in \text{Irr}_{\text{cusp}}(L_I | \nu)$  with  $\psi^{c_I} = \psi$  satisfies  $t_{I,2}^2 \in \ker(\psi)$ .*

*Proof.* Let  $z := t_{I,2}^2 = \mathbf{h}_I(\zeta')$  for some  $\zeta' \in \mathbb{F}^\times$  a root of unity of order  $2(q - 1)_2$ , and  $\psi \in \text{Irr}_{\text{cusp}}(L_I | \nu)$  with  $\psi^{c_I} = \psi$  and  $z \notin \ker(\psi)$ . According to Lemma 6.18,  $\psi^{\widehat{L}_I}$  is irreducible.

Note  $z \in Z(\mathbf{L}_I)$ . Since  $d \geq 4$ , it is sufficient to show the statement in the case where  $I = \underline{l}$  and hence  $L_I = L$ .

The group  $L_0 := [\mathbf{L}, \mathbf{L}]^F$  satisfies  $L_0 \cong \mathrm{SL}_l(q)$  (see Lemma 4.2). Let  $\psi_0 \in \mathrm{Irr}(\psi|_{L_0})$ . According to Lemma 2.15,  $\psi_0$  is cuspidal. Following Lemma 4.3, the automorphisms of  $L_0$  induced by  $\widehat{L}$  are diagonal automorphisms of  $L_0$ . Since  $C_{\widehat{L}}(L_0) \leq Z(\mathbf{L})$  and  $\widehat{L}/(C_{\widehat{L}}(L_0)L_0)$  is cyclic, we can see that maximal extendibility holds with respect to  $L_0 \triangleleft L$ . As  $\psi^{\widehat{L}}$  is irreducible,  $\widehat{L}\psi = L$ . As  $\widehat{L}/L_0$  is abelian, this implies  $\widehat{L}\psi_0 \leq L$ .

We now use the fact  $L_0 \cong \mathrm{SL}_l(q)$ . Let  $\mathbf{H} := \mathrm{GL}_l(\mathbb{F})$ , and let  $F' : \mathbf{H} \rightarrow \mathbf{H}$  be a Frobenius endomorphism giving an  $\mathbb{F}_q$ -structure such that  $\mathbf{H}^{F'} \cong \mathrm{GL}_l(q)$ . Via  $[\mathbf{H}, \mathbf{H}] \cong [\mathbf{L}, \mathbf{L}]$ , we identify  $[\mathbf{H}, \mathbf{H}]^F$  with  $L_0$ . Hence,  $\psi_0$  defines  $\psi'_0 \in \mathrm{Irr}_{\mathrm{cusp}}([\mathbf{H}, \mathbf{H}]^F)$ . By the above, this implies

$$2 \mid |\mathbf{H}^F : \mathbf{H}_{\psi'_0}^F|.$$

The character  $\psi^{\widehat{L}}$  is  $c_I$ -stable. Hence,  $\psi|_{L_0}$  is  $c_I$ -stable. Following Notation 6.9,  $c_I$  acts on  $L_0$  by a graph automorphism and  $\widehat{L}$  acts on  $L_0$  as diagonal automorphisms.

As  $\psi|_{L_0}$  is  $c_I$ -stable, we can choose  $\psi'$  to be stable under the graph automorphism of  $\mathrm{SL}_l(q)$  and it is cuspidal according to Lemma 2.15. In this situation,  $\psi_0$  only exists if  $l = 2$  (see Proposition 5.2). By the assumption  $d > 2$ , so we get a contradiction. This implies our claim that any  $c_I$ -stable character  $\psi$  satisfies  $t_{I,2}^2 \in \ker(\psi)$ .  $\square$

LEMMA 6.20. *Let  $I \in \mathcal{O}_2$  and  $\nu$  as in Lemma 6.19. There are exactly two characters  $\psi \in \mathrm{Irr}_{\mathrm{cusp}}(L_I \mid \nu)$  with  $\psi^{c_I} = \psi$  and  $t_{I,2}^2 \notin \ker(\psi)$ . Those characters are  $\widehat{L}$ -conjugate.*

*Proof.* From the proof of Lemma 6.19 and Proposition 5.2, we see that there are two  $\mathrm{GL}_2(q)$ -conjugate characters  $\psi_0 \in \mathrm{Irr}(\psi|_{[\mathbf{L}_I, \mathbf{L}_I]^F})$ , that are the only possible constituents of  $\psi$ . If  $I = \{i, i'\}$ , then  $\psi_0(\mathbf{h}_{e_i - e_{i'}}(-1)) = (-1)^{\frac{q+1}{2}}\psi_0(1)$  according to [B 2, Table 5.4]. Then  $\mathbf{L}^F \cong \mathrm{SL}_2(q) \times \mathbf{Z}_I^F$  by Lemma 4.2, in particular  $h_0 = \mathbf{h}_I(\varpi)\mathbf{h}_{e_i - e_{i'}}(-1)$ . Because of  $\psi(h_0) = -\psi(1)$ , this implies  $\psi(\mathbf{h}_I(\varpi)) = -(-1)^{\frac{q+1}{2}}\psi(1) = (-1)^{\frac{q-1}{2}}\psi(1)$ .

Let  $\kappa \in \mathrm{Irr}(\mathbf{Z}_I^F)$  such that  $\psi = \psi_0 \times \kappa$ . As  $c_I$  acts by inverting on  $\mathbf{Z}_I$ ,  $\kappa$  has multiplicative order 1 or 2. The assumption  $t_{I,2}^2 \notin \ker(\psi)$  implies that  $\kappa$  has order 2. This proves that given  $\psi_0$ , the character  $\kappa$  is uniquely determined by the fact that  $\psi$  is  $c_I$ -stable and  $t_{I,2}^2 \notin \ker(\psi)$ . Hence, the only characters with the given properties are  $\widehat{L}$ -conjugate.  $\square$

Thanks to the above three statements, we can now show Theorem 6.17.

*Proof of Theorem 6.17.* Let  $\nu \in \mathrm{Irr}(\langle h_0 \rangle)$  be nontrivial,  $d \in \mathbb{D}_{\mathrm{even}}$ ,  $I \in \mathcal{O}_d$ , and  $\psi \in \mathrm{Irr}_{\mathrm{cusp}}(\widehat{L}_I \mid \nu)$  with  $\psi^{c_I} = \psi$ . If  $d > 2$ , then  $t_{I,2}^2 \in \ker(\psi)$  according to Lemma 6.19. This shows part (a).

Assume  $d = 2$  and  $t_{I,2}^2 \notin \ker(\psi)$ . The set  $\mathrm{Irr}(\psi|_{L_I})$  contains two characters according to Lemma 6.18. Following Lemma 4.4(b) together with Proposition 2.19(b), the character  $\psi' \in \mathrm{Irr}(\psi|_{L_I})$  is cuspidal, and satisfies  $(\psi')^{c_I} = \psi'$  and  $t_{I,2}^2 \notin \ker(\psi')$ . Then there are exactly two  $\widehat{L}_I$ -conjugate characters in  $\mathrm{Irr}_{\mathrm{cusp}}(L_I \mid \nu)$  with those properties (see Lemma 6.20). Since  $|\widehat{L}_I : L_I| = 2$ , this implies that there is only one character  $\psi$  with the given properties. This proves (b).  $\square$

LEMMA 6.21. *If  $\nu \in \mathrm{Irr}(\langle h_0 \rangle)$  is nontrivial, then every  $\psi \in \mathrm{Irr}_{\mathrm{cusp}}(\widehat{L}_{J-1} \mid \nu)$  satisfies  $\psi^{c_{J-1}} \neq \psi$ .*

*Proof.* Note  $\mathbf{h}_{J-1}(\varpi) \in Z(\mathbf{L}_{-1})$  and  $[c_{J-1}, \mathbf{h}_{J-1}(\varpi)] = h_0$ . No extension  $\widehat{\nu} \in \mathrm{Irr}(Z(\mathbf{L}_{-1})^F \mid \nu)$  is  $c_{J-1}$ -stable. This implies that  $\psi_{-1}$  is not  $c_{J-1}$ -stable.  $\square$

The above leads to the following statement on the sets  $\mathcal{O}_c(\widehat{\lambda})$ ,  $\mathcal{O}_{c,-1}(\widehat{\lambda})$  and  $Y(\widehat{\lambda})$  introduced earlier in Lemmas 6.12 and 6.13. We use the notation  $o(\mu)$  to denote the multiplicative order of a linear character  $\mu$  of a finite group.

**COROLLARY 6.22.** *Let  $\lambda \in \text{Irr}_{\text{cusp}}(L)$ ,  $\widehat{\lambda}$  and  $\widehat{\lambda}_I$  associated with  $\lambda$  as in Lemma 6.13. If  $\widehat{\lambda}$  is standardized, then:*

- (a)  $\mathcal{O}_c(\widehat{\lambda}) \subseteq \bigcup_{d \in \mathbb{D}_{\text{even}} \cup \{1, -1\}} \mathcal{O}_d$ .
- (b) If  $h_0 \in \ker(\lambda)$ , then  $\mathcal{O}_{c,-1}(\widehat{\lambda}) \subseteq \{J_{-1}\} \cup \{I \in \mathcal{O}_1 \mid o(\widehat{\lambda}_I) \mid 2\}$ .
- (c) If  $h_0 \notin \ker(\lambda)$ , then  $\mathcal{O}_{c,-1}(\widehat{\lambda}) \subseteq \mathcal{O}_2$  and all  $\{\widehat{\lambda}_I \mid I \in \mathcal{O}_{c,-1}(\widehat{\lambda})\}$  are  $V_S$ -conjugate, that is,  $\mathcal{O}_{c,-1}(\widehat{\lambda}) \in Y(\widehat{\lambda})$ .

*Proof.* Lemma 6.15 implies that  $\mathcal{O}_{c,-1}(\widehat{\lambda}) \cap \mathcal{O}_d = \emptyset$  for every  $d \in \mathbb{D}_{\text{odd}} \setminus \{\pm 1\}$ . This gives (a).

For the proof of (b), assume  $h_0 \in \ker(\lambda)$ . Then Corollary 6.16 implies  $\mathcal{O}_{c,-1}(\widehat{\lambda}) \subseteq \mathcal{O}_{-1} \cup \mathcal{O}_1$ . For  $I \in \mathcal{O}_1$ , the character  $\widehat{\lambda}_I$  is  $c_I$ -stable if and only if  $o(\widehat{\lambda}_I) \mid 2$ .

For the proof of (c), assume  $h_0 \notin \ker(\lambda)$ . Then  $\mathcal{O}_{c,-1}(\widehat{\lambda}) \cap \mathcal{O}_1 = \emptyset$  and  $\mathcal{O}_{c,-1}(\widehat{\lambda}) \subseteq \mathcal{O}_2$  according to Theorem 6.17. Lemma 6.20 proves that  $\{\widehat{\lambda}_I \mid I \in \mathcal{O}_{c,-1}(\widehat{\lambda})\}$  are  $V_S$ -conjugate. Hence, the partition  $Y(\widehat{\lambda})$  from Lemma 6.12 contains  $\mathcal{O}_{c,-1}(\widehat{\lambda})$ .  $\square$

Recall  $K(\lambda) := \overline{W}_{\lambda^{L(F_p)}} \Big|_L$ . For any  $\overline{W}$ -stable  $L \leq J \leq \tilde{L}$ ,  $\kappa \in \text{Irr}(J)$ , and  $Q \subseteq \mathcal{O}$ , let  $\overline{W}^Q(\kappa) := \overline{W}(\kappa) \cap \mathcal{S}_{\pm Q}$  and  $W^Q(\kappa) := \overline{W}^Q(\kappa) \cap W$ .

**PROPOSITION 6.23.** *Let  $\lambda \in \mathbb{T}$ ,  $\widehat{\lambda} \in \text{Irr}_{\text{cusp}}(\widehat{L} \mid \lambda)$ ,  $\widehat{\lambda}_I \in \text{Irr}(\widehat{\lambda} \Big|_{\widehat{L}_I})$  ( $I \in \mathcal{O}$ ), and  $\tilde{\lambda} \in \text{Irr}(\tilde{L} \mid \widehat{\lambda})$ . Assume that  $\widehat{\lambda}$  is standardized in the sense of 6.10. We set*

$$Q^1(\widehat{\lambda}) := \begin{cases} \{I \in \mathcal{O}_1 \mid o(\widehat{\lambda}_I) \mid 2\} \cup \mathcal{O}_{-1}, & \text{if } h_0 \in \ker(\lambda), \\ \mathcal{O}_{c,-1}(\widehat{\lambda}), & \text{otherwise.} \end{cases}$$

*Then:*

- (a)  $Q^1(\widehat{\lambda})$  is  $K(\lambda)$ -stable, and
- (b)  $W(\tilde{\lambda}) = W^1(\tilde{\lambda}) \times W^2(\tilde{\lambda})$ , where  $Q^2(\widehat{\lambda}) := \mathcal{O} \setminus Q^1(\widehat{\lambda})$ , and  $W^j(\tilde{\lambda}) := W^{Q^j(\widehat{\lambda})}(\tilde{\lambda})$  for  $j \in \mathbb{Z}$ .

*Proof.* Let  $e \in \langle F_0 \rangle$  such that  $\lambda$  and  $\lambda^e$  are  $\overline{N}$ -conjugate. As  $\widehat{\lambda}$  is standardized, then  $\widehat{\lambda}^e$  is also standardized. As the orders of  $\widehat{\lambda}_I$  and  $(\widehat{\lambda}_I)^e$  coincide for every  $I \in \mathcal{O}$ , we see that  $Q^1(\widehat{\lambda}) = Q^1(\widehat{\lambda}^e)$  from the definition, whenever  $h_0 \in \ker(\lambda) = \ker(\widehat{\lambda}^e)$  and hence  $h_0 \in \ker(\widehat{\lambda}^e)$ .

Assume  $h_0 \notin \ker(\lambda)$ . Then  $h_0 \notin \ker(\lambda^e)$ . Let  $I \in \mathcal{O} \setminus \{J_{-1}\}$ . Because of  $c_I^e \in c_I \langle h_0 \rangle$ , we see

$$\widehat{\lambda}_I^{c_I^e} = \widehat{\lambda}_I \Leftrightarrow \text{Irr} \left( \widehat{\lambda}^e \Big|_{\widehat{L}_I} \right)^{c_I^e} = \text{Irr} \left( \widehat{\lambda}^e \Big|_{\widehat{L}_I} \right).$$

In the case of  $\widehat{\lambda}_I^{c_I^e} = \widehat{\lambda}_I$ , the character  $\widehat{\lambda}_I$  has some  $c_I$ -stable extension to  $\tilde{L}_I$  if and only if the unique character in  $\text{Irr}(\widehat{\lambda}^e \Big|_{\widehat{L}_I})$  has some  $c_I$ -stable extension to  $\tilde{L}_I$ . (The set  $\text{Irr}(\phi \Big|_{\widehat{L}_I})$  is a singleton for every  $\phi \in \text{Irr}_{\text{cusp}}(\widehat{L})$ , since  $\widehat{L}$  is the central product of the groups  $\widehat{L}_I$  over  $\langle h_0 \rangle$ .) This shows  $\mathcal{O}_{c,-1}(\widehat{\lambda}) = \mathcal{O}_{c,-1}(\widehat{\lambda}^e)$  and  $Q^1(\widehat{\lambda}) = Q^1(\widehat{\lambda}^e)$  by the definition of those sets. Let  $w \in \overline{W}$  and  $e \in \langle F_p \rangle$  with  $we \in \widehat{K}(\lambda)$ . Then  $\widehat{\lambda}^w$  is standardized and  $Q^1(\widehat{\lambda})^w = Q^1(\widehat{\lambda}^w)$ . Accordingly,  $w \in K(\lambda)$  stabilizes  $Q^1(\widehat{\lambda})$ . This implies part (a).



For part (b), recall the description of  $\overline{W}(\tilde{\lambda})$  from Lemma 6.6:

$$\overline{W}(\tilde{\lambda}) = \left( \left\langle (I, -I) \mid I \in \mathcal{O}_{c,1}(\hat{\lambda}) \right\rangle \left\langle (I, -I)(I', -I') \mid I, I' \in \mathcal{O}_{c,-1}(\hat{\lambda}) \right\rangle \right) \rtimes \mathcal{Y}_{Y(\hat{\lambda}) \cup Y'(\hat{\lambda})}.$$

First assume  $h_0 \in \ker(\lambda)$ . By construction,  $\mathcal{O}_c(\hat{\lambda}) \subseteq Q^1(\hat{\lambda}) \cup \bigcup_{d \in \mathbb{D}_{\text{even}}} \mathcal{O}_d$  and hence  $\overline{W}(\tilde{\lambda}) = \overline{W}^1(\tilde{\lambda}) \times \overline{W}^2(\tilde{\lambda})$ . According to Corollary 6.22(a), we observe  $(I, -I) \in W$  for every  $I \in \mathcal{O}_c(\hat{\lambda}) \setminus Q^1(\hat{\lambda})$ . This implies  $\overline{W}^2(\tilde{\lambda}) \leq W$  and  $W(\tilde{\lambda}) = W^1(\tilde{\lambda}) \times W^2(\tilde{\lambda})$  by the definition of  $W$ .

It remains to consider the case where  $h_0 \notin \ker(\lambda)$ . Then  $Q^1(\hat{\lambda}) = \mathcal{O}_{c,-1}(\hat{\lambda}) \subseteq \mathcal{O}_2$  by Corollary 6.22(c) and hence  $\overline{W}^1(\tilde{\lambda}) \leq W$ . By the structure of  $\overline{W}(\tilde{\lambda})$  described in Lemma 6.6, we see  $W(\tilde{\lambda}) = W^1(\tilde{\lambda}) \times W^2(\tilde{\lambda})$ . □

### 6.3 Clifford theory for $W(\tilde{\lambda}) \triangleleft W(\lambda)$ in the case of $\widehat{L}\tilde{L}_\lambda = \tilde{L}$

In this section, we study the characters of  $W(\tilde{\lambda})$ , in particular their Clifford theory with respect to  $K(\lambda)$ . Assuming  $\widehat{L}\tilde{L}_\lambda = \tilde{L}$ , we prove maximal extendibility with respect to  $W(\tilde{\lambda}) \triangleleft K(\lambda)$ . This result is required for a later application of Proposition 2.11. We consider the following situation.

NOTATION 6.24. Let  $\lambda \in \text{Irr}_{\text{cusp}}(L)$ ,  $\hat{\lambda} \in \text{Irr}(\widehat{L} \mid \lambda)$ , and  $\tilde{\lambda} \in \text{Irr}(\tilde{L} \mid \hat{\lambda})$  such that  $\hat{\lambda}$  is standardized and  $\widehat{L}\tilde{L}_\lambda = \tilde{L}$  (or equivalently  $\tilde{\lambda} \Big|_{\tilde{L}} = \hat{\lambda}$ ).

For further computations, we use the groups  $K^j(\lambda)$  associated with the subsets  $Q^j(\lambda) \subseteq \mathcal{O}$  from Proposition 6.23, where  $K^j(\lambda) := (K(\lambda)\mathcal{S}_{\pm(\mathcal{O} \setminus Q^j(\lambda))}) \cap \mathcal{S}_{\pm Q^j(\lambda)}$  for  $j \in \underline{2}$ .

LEMMA 6.25. *If maximal extendibility holds with respect to  $W^j(\tilde{\lambda}) \triangleleft K^j(\lambda)$  for every  $j \in \underline{2}$ , then maximal extendibility holds with respect to  $W(\tilde{\lambda}) \triangleleft K(\lambda)$ , in particular for every  $\eta_0 \in \text{Irr}(W(\tilde{\lambda}))$  there exists some  $K(\lambda)_{\eta_0}$ -stable  $\eta \in \text{Irr}(W(\lambda) \mid \eta_0)$ .*

In that situation, the above statement will ensure Assumption 2.12(iii) for  $(\overline{\lambda}, \eta) \in \mathcal{P}(L)$  via Lemma 6.5.

*Proof.* Recall  $K(\lambda) := \overline{W} \Big|_{\lambda L \langle F_p \rangle} \Big|_L$  by the definition in 6.4. As  $K(\lambda)$  stabilizes  $Q^1(\hat{\lambda})$  and  $Q^2(\hat{\lambda})$  by Proposition 6.23,  $K(\lambda) \leq \mathcal{S}_{\pm Q^1(\hat{\lambda})} \times \mathcal{S}_{\pm Q^2(\hat{\lambda})}$ . This rewrites as  $K(\lambda) \leq K^1(\lambda) \times K^2(\lambda)$ . Recall that  $\mathcal{O}_{c,-1}(\hat{\lambda}) \subseteq Q^1(\hat{\lambda})$ .

Since maximal extendibility holds with respect to  $W^j(\tilde{\lambda}) \triangleleft K^j(\lambda)$  for  $j \in \underline{2}$  by assumption, maximal extendibility holds with respect to

$$W(\tilde{\lambda}) = W^1(\tilde{\lambda}) \times W^2(\tilde{\lambda}) \triangleleft K^1(\lambda) \times K^2(\lambda).$$

This implies the statement as  $K(\lambda) \leq K^1(\lambda) \times K^2(\lambda)$ . □

For  $\lambda \in \mathbb{T}$  with  $\tilde{L}_\lambda = \tilde{L}$ , we study first the Clifford theory of  $W^2(\tilde{\lambda}) \triangleleft K^2(\lambda)$  for the groups from Lemma 6.25.

LEMMA 6.26. *Let  $W^1(\tilde{\lambda})$ ,  $W^2(\tilde{\lambda})$ ,  $K^1(\lambda)$ , and  $K^2(\lambda)$  be the groups from Lemma 6.25. Then:*

- (a) *maximal extendibility holds with respect to  $W^2(\tilde{\lambda}) \triangleleft K^2(\lambda)$ , and*
- (b) *maximal extendibility holds with respect to  $W^1(\tilde{\lambda}) \triangleleft K^1(\lambda)$ , if  $h_0 \notin \ker(\lambda)$ .*

Table 6.1. Isomorphism types of  $W^1(\tilde{\lambda})$  and  $K^1(\lambda)$

	$W^1(\tilde{\lambda})$	$K^1(\lambda)$
$J_{-1} \notin \mathcal{O}_c(\hat{\lambda})$	$W(D_{l_1}) \times W(D_{l_2})$	$C_2 \times (W(B_{l_1}) \wr C_2)$ , if $l_1 = l_2$ $C_2 \times W(B_{l_1}) \times W(B_{l_2})$ , if $l_1 \neq l_2$
$J_{-1} \in \mathcal{O}_{c,1}$	$W(B_{l_1}) \times W(D_{l_2})$	$C_2 \times W(B_{l_1}) \times W(B_{l_2})$
$J_{-1} \in \mathcal{O}_{c,-1}$	$W(D_{l_1}) \times W(B_{l_2})$	$C_2 \times W(B_{l_1}) \times W(B_{l_2})$

*Proof.* Let  $Y(\hat{\lambda}) \vdash \mathcal{O}_c(\hat{\lambda})$  and  $Y'(\hat{\lambda}) \vdash \mathcal{O} \setminus \mathcal{O}_c(\hat{\lambda})$  be the partitions from Lemma 6.12. In order to prove part (a), we can assume  $Q^2(\hat{\lambda}) = \mathcal{O}$  without loss of generality. We have  $\overline{W}(\tilde{\lambda}) = \overline{W}^2(\lambda) = \mathcal{Y}_{\pm Y(\hat{\lambda})} \times \mathcal{Y}_{Y'(\hat{\lambda})}$  (see Lemma 6.13).

If  $h_0 \in \ker(\lambda)$ , then  $\mathcal{O}_c(\hat{\lambda}) \cap (\mathcal{O}_1 \cup \mathcal{O}_{-1}) = \emptyset$  by the choice of  $Q^1(\hat{\lambda})$  according to Corollary 6.22. If  $h_0 \notin \ker(\lambda)$ , Proposition 6.21 implies  $\mathcal{O}_c(\hat{\lambda}) \cap \mathcal{O}_{-1} = \emptyset$  and analogously we see  $\mathcal{O}_c(\hat{\lambda}) \cap \mathcal{O}_1 = \emptyset$ .

This implies  $\mathcal{O}_c(\hat{\lambda}) \subseteq \bigcup_{d \in \mathbb{D}_{\text{even}}} \mathcal{O}_d$ . Accordingly,  $W(\tilde{\lambda})$  is the direct product of groups  $W_d(\tilde{\lambda})$  for  $d \in \mathbb{D}$ . It suffices to consider the case where  $\mathcal{O} = \mathcal{O}_d = Q^2(\hat{\lambda})$  for some  $d \in \mathbb{D}$  and  $\mathcal{O}_c(\hat{\lambda}) \in \{\mathcal{O}, \emptyset\}$ . Additionally, we can assume that  $Y(\hat{\lambda})$  and  $Y'(\hat{\lambda})$  are partitions whose elements have all the same cardinality. If  $\mathcal{O}_c(\hat{\lambda}) = \mathcal{O}$ , then  $W^2(\lambda) \cong (C_2 \wr \mathcal{S}_k)^a$  for some positive integers  $k$  and  $a$ . Then  $K^2(\lambda) \cong (C_2 \wr \mathcal{S}_k) \wr \mathcal{S}_a$ , and hence maximal extendibility holds with respect to  $W^2(\tilde{\lambda}) \triangleleft K^2(\lambda)$ .

If  $\mathcal{O}_c(\hat{\lambda}) = \emptyset$ , then  $W^2(\lambda) = \mathcal{Y}_{Y'}$  and hence it is isomorphic to a direct product of symmetric groups. The group  $K^2(\lambda) \leq N_{\mathcal{S}_{\pm \mathcal{O}_d}}(\mathcal{Y}_Y)$  is isomorphic to  $(C\mathcal{Y}_Y) \rtimes \mathcal{S}_Y$ , where  $C := \langle \prod_{k \in y} (k, -k) \mid y \in Y \rangle \leq \mathcal{S}_{\pm \mathcal{O}_d}$ . By Lemma 4.6, maximal extendibility holds with respect to  $W_2(\tilde{\lambda}) \triangleleft K_2(\lambda)$ . This proves part (a).

For part (b), we assume  $\mathcal{O}_{c,-1}(\lambda) = \mathcal{O}$ ,  $h_0 \notin \ker(\lambda)$ , and as before  $Q^1(\hat{\lambda}) = \mathcal{O}$ . By Corollary 6.22(c), we have  $K(\lambda) = W$  and  $|W : W(\tilde{\lambda})| = 2$  (see Corollary 6.16(c)). As  $W(\lambda)/W(\tilde{\lambda})$  is cyclic, maximal extendibility holds with respect to  $W(\tilde{\lambda}) \triangleleft K(\lambda)$ .  $\square$

It remains to prove the following.

PROPOSITION 6.27. *Maximal extendibility holds with respect to  $W^1(\tilde{\lambda}) \triangleleft K^1(\lambda)$ , if  $h_0 \in \ker(\lambda)$ .*

*Proof.* Let  $\mathcal{O}_{1,i} = \{I \in \mathcal{O}_1 \mid o(\hat{\lambda}_I) = i\}$  for  $i \in \underline{4}$  and  $l_i := |\mathcal{O}_{1,i}|$ . By Lemmas 6.12 and 6.13, we have  $\overline{W}_{Q^1(\hat{\lambda})}(\tilde{\lambda}) \leq C \times \mathcal{S}_{\pm \mathcal{O}_{1,1}} \times \mathcal{S}_{\pm \mathcal{O}_{1,2}}$ , where  $C \leq W_{\mathcal{O}_{-1}}(\hat{\lambda})$  and  $C$  is then either trivial or a cyclic group of order 2. The group structures depend on  $J_{-1}$  and those groups are described in Table 6.1, where  $W(B_j)$  and  $W(D_j)$  are Coxeter groups of type  $B_j$  and  $D_j$ , respectively.

Note that in all cases  $\overline{W}_{\mathcal{O}_{1,1}}(\tilde{\lambda})$  is a Coxeter group of type  $B_{l_1}$ . Considering the structure, we observe that in all cases the statement holds according to Lemma 4.6.  $\square$

Recall that  $\mathbb{M}^{(X)} = \{\lambda \in \text{Irr}_{\text{cusp}}(L) \mid \tilde{L}_\lambda = X\}$  and  $\mathbb{M}_0 := \text{Irr}_{\text{cusp}}(L) \setminus (\mathbb{M}^{(L)} \cup \mathbb{M}^{(\tilde{L})})$  for  $X$  with  $L \leq X \leq \tilde{L}$ . For characters in  $\mathbb{M}^{(\tilde{L})} \cup \mathbb{M}_0$ , the above proves the following:

PROPOSITION 6.28. *Let  $\lambda \in \mathbb{M}^{(\tilde{L})} \cup \mathbb{M}_0$ , that is,  $\tilde{L}\tilde{L}_\lambda = \tilde{L}$ . For every  $\tilde{\lambda} \in \text{Irr}(\tilde{L} \mid \lambda)$  and  $\eta_0 \in \text{Irr}(W(\tilde{\lambda}))$ , there exists some  $K(\lambda)_{\eta_0}$ -stable  $\eta \in \text{Irr}(W(\lambda) \mid \eta_0)$ .*

*Proof.* According to Lemma 6.25, Proposition 6.27, and Lemma 6.26 imply the statement.  $\square$

**6.4 Clifford theory for  $W(\widehat{\lambda}) \triangleleft W(\lambda)$  in the case of  $\widetilde{L}_\lambda = \widehat{L}$**

We now study  $W(\lambda)$  and  $W(\widehat{\lambda})$  for characters  $\lambda \in \mathbb{M}(\widehat{L})$ , where  $\widehat{\lambda} \in \text{Irr}(\widehat{L} \mid \lambda)$  is standardized. We prove statements on the characters of  $W(\widehat{\lambda})$  and their possible extensions to  $W(\lambda)$ . The results later imply that there exists some  $E_L$ -stable  $\widetilde{N}$ -transversal in  $\text{Irr}(N \mid \mathbb{M}(\widehat{L}))$ .

In the following, we study the Clifford theory of  $W(\widehat{\lambda}) \triangleleft K(\lambda)$  for  $\lambda \in \mathbb{M}(\widehat{L})$ , where  $K(\lambda) = \overline{W}_{\lambda^{L\langle F_p \rangle}} \Big|_L$ .

LEMMA 6.29. *Assume  $|Z(\mathbf{G}^F)| = 2$  or equivalently  $q \equiv 3(4)$  and  $2 \nmid l$ . Every  $\lambda \in \text{Irr}_{\text{cusp}}(L)$  satisfies  $\widehat{L} \leq \widetilde{L}_\lambda$ . Then  $\text{Irr}_{\text{cusp}}(N \mid \mathbb{T} \cap \mathbb{M}(\widehat{L}))$  is an  $E_L$ -stable  $\widetilde{L}$ -transversal of  $\text{Irr}_{\text{cusp}}(N \mid \mathbb{M}(\widehat{L}))$ .*

*Proof.* The arguments given in Lemma 6.3 show the statement.  $\square$

According to Lemma 6.29, we can now assume  $Z(\mathbf{G}^F) = Z(\mathbf{G})$ . We do that until the end of the section.

LEMMA 6.30. *If  $|Z(\mathbf{G}^F)| = 4$  and  $\lambda \in \text{Irr}_{\text{cusp}}(L)$  with  $\widetilde{L}_\lambda = \widehat{L}$ , then  $-1 \in \mathbb{D}$  and  $\widehat{\lambda}_{-1}^{t_{l,2}} \neq \widehat{\lambda}_{-1}$ . Moreover,  $\widehat{\lambda}_{-1}^{\mathbf{n}_{e_1}(\varpi)} = \widehat{\lambda}_{-1}$ , if  $h_0 \in \ker(\lambda)$  or  $\text{type}(\Phi_{-1})$  is not of type  $D_{|J_{-1}|}$ .*

*Proof.* Recall that maximal extendibility holds with respect to  $L \triangleleft \widetilde{L}$  (see Theorem 2.17). Accordingly,  $\widetilde{L}_\lambda = \widehat{L}$  implies that  $\lambda$  is not  $t_{l,2}$ -stable for the element  $t_{l,2} \in \mathbf{T}$  from Lemma 3.13. If  $\zeta \in \mathbb{F}^\times$  with  $\zeta^{(q-1)^2} = \varpi$  and  $t_{I,2} := \mathbf{h}_I(\zeta)$  as in Lemma 6.13, then  $t_{l,2} = \prod_{I \in \mathcal{O}} t_{I,2}$ . Recall that  $\widetilde{L} = \widehat{L} \langle t_{l,2} \rangle$ . The character  $\widehat{\lambda}$  is  $t_{l,2}$ -stable, if  $\widehat{\lambda}_I$  is  $t_{I,2}$ -stable for every  $I \in \mathcal{O}$ . For  $I \in \mathcal{O} \setminus \{J_{-1}\}$ , we see  $t_{I,2} \in C_{\mathbf{L}_I}(\widehat{L}_I)$  and hence  $\widehat{\lambda}_I$  is  $t_{I,2}$ -stable. As  $\widehat{\lambda}$  is not  $t_{l,2}$ -stable,  $-1 \in \mathbb{D}$  and  $\widehat{\lambda}_{-1}$  is not  $t_{l,2}$ -stable.

In the next step, we prove  $\widehat{\lambda}_{-1}^{\mathbf{n}_{e_1}(\varpi)} = \widehat{\lambda}_{-1}$ . Since  $\widehat{\lambda}_{-1}^{t_{l,2}} \neq \widehat{\lambda}_{-1}$ , Proposition 5.3 implies that  $\widehat{\lambda}_{-1}$  is  $\gamma$ -stable, if  $\text{type}(\Phi_{-1}) = D_{|J_{-1}|}$ . We consider the other possible values of  $\text{type}(\Phi_{-1})$ . We first assume  $\text{type}(\Phi_{-1}) = A_1 \times A_1$ . Then  $L_{-1} = \text{SL}_2(q) \times \text{SL}_2(q)$ . Let  $\lambda_{-1,1}, \lambda_{-1,2} \in \text{Irr}(\text{SL}_2(q))$  such that  $\lambda \Big|_{L_{-1}} = \lambda_{-1,1} \times \lambda_{-1,2}$ . By the proof of Proposition 4.8,  $\widetilde{L}_\lambda \leq \widehat{L}$  implies that both characters  $\lambda_{-1,1}$  and  $\lambda_{-1,2}$  are not  $\text{GL}_2(q)$ -stable. Additionally, they are cuspidal. Following [B 2, Table 5.4], the characters  $\lambda_{-1,1}$  and  $\lambda_{-1,2}$  are uniquely determined up to  $\text{GL}_2(q)$ -conjugation. After applying some  $\widetilde{L}$ -conjugation, we obtain that  $\lambda_{-1,1}$  and  $\lambda_{-1,2}$  are  $\mathbf{n}_{e_1}(\varpi)$ -conjugate. As  $\widehat{L}$  induces on the  $\text{SL}_2(q)$ -factors of  $L_{-1}$  simultaneous (non-inner) diagonal automorphisms, the set  $\text{Irr}(\widehat{L}_{-1} \mid \lambda_{-1})$  contains only one character; hence,  $\widehat{\lambda}_{-1}$  is again  $\mathbf{n}_{e_1}(\varpi)$ -stable.

It remains to consider the case where  $\text{type}(\Phi_{-1}) = A_3$ . Again, the character  $\lambda_{-1}$  is not  $\widetilde{L}$ -stable. Via the isomorphism  $L_{-1} \cong \text{SL}_4(q)$ , we see that  $t_{l,2}$  induces on  $\text{SL}_4(q)$  a diagonal automorphism corresponding to a generator of  $Z(\text{SL}_4(q))$  in the sense of 2.16(b) (see also the proof of Proposition 4.8). We take any  $\chi \in \text{Irr}(\text{GL}_4(q) \mid \lambda_{-1})$ . Then  $\chi$  is cuspidal (see Lemma 2.15(c)). Using the description of cuspidal characters of general linear groups recalled in the proof of Proposition 5.1, we let  $s \in \text{GL}_4(q)$  and  $\zeta \in \mathbb{F}^\times$  such that  $\chi$  belongs to the rational Lusztig series of  $s$  and  $\zeta \in \mathbb{F}_{q^4} \setminus \mathbb{F}_{q^2}$  is an eigenvalue of  $s$ . Let  $\det : \text{GL}_4(\mathbb{F}) \rightarrow \mathbb{F}^\times$  denote the determinant and  $\det^*$  the associated linear character of  $\text{GL}_4(\mathbb{F})$  with kernel

$SL_4(\mathbb{F})$ . By the assumptions on  $\chi$ , we see that  $\chi = \chi(\det^*)^{\frac{q-1}{2}}$  and hence  $s$  and  $-s$  are conjugate. Then  $-\zeta \in \{\zeta, \zeta^q, \zeta^{q^2}, \zeta^{q^3}\}$ . Hence, using again  $o$  to denote multiplicative order,  $o(\zeta)_2 = 2(q^2 - 1)_2$ , as  $-\zeta \in \{\zeta, \zeta^q, \zeta^{q^3}\}$  would imply that  $\zeta \in \mathbb{F}_{q^2}$  or  $\zeta \in \mathbb{F}_{q^6}$ , contradicting  $\zeta \in \mathbb{F}_{q^4} \setminus \mathbb{F}_{q^2}$ . In order to compute  $\ker(\chi|_{Z(SL_4(q))})$ , we see that  $\det s = \zeta^{\frac{q^4-1}{q-1}}$  is not a square in  $\mathbb{F}_q^\times$  since  $o(\zeta)_2 = 2(q^2 - 1)_2$ . This contradicts  $h_0 \in \ker(\lambda)$ , as  $h_0$  corresponds to the central involution of  $SL_4(q)$ . Hence, there exists no cuspidal character  $\lambda_{-1}$  of  $L_{-1}$  that satisfies  $h_0 \in \ker(\lambda)$  and  $\lambda_{-1}^{t_{-1,2}} \neq \lambda_{-1}$ . This shows that  $\text{type}(\Phi_{-1}) = A_3$  is not possible. This finishes our proof.  $\square$

LEMMA 6.31. *Let  $\lambda \in \text{Irr}_{\text{cusp}}(L)$  and  $\widehat{\lambda} \in \text{Irr}(\widehat{L} | \lambda)$  such that  $h_0 \notin \ker(\lambda)$  and  $W(\widehat{\lambda}) \neq W(\lambda)$ . Then maximal extendibility holds with respect to  $W(\widehat{\lambda}) \triangleleft K(\lambda)$ .*

*Proof.* We first determine  $W(\widehat{\lambda})$ . Denote  $c_{-1} := c_{J_{-1}}$ . As the character  $\text{Irr}(\lambda|_{Z(L_{-1})})$  is not  $c_{-1}$ -stable,  $\lambda_{-1}$  is not  $c_{-1}$ -stable. This implies  $J_{-1} \notin \mathcal{O}_c(\widehat{\lambda})$ .

If  $I \in \mathcal{O}_1$ ,  $\widehat{\lambda}_I$  is a linear character. Since  $h_0 \notin \ker(\lambda)$  and hence  $h_0 \notin \ker(\widehat{\lambda}_I)$ , the order of  $\widehat{\lambda}_I$  is divisible by  $2(q-1)_2 \geq 4$ . Hence,  $\mathcal{O}_c(\widehat{\lambda}) \cap \mathcal{O}_1 = \emptyset$ .

Together with Lemma 6.14 and Corollary 6.22(a), this leads to  $\mathcal{O}_c(\widehat{\lambda}) \subseteq \bigcup_{d \in \mathbb{D}_{\text{even}}} \mathcal{O}_d$ . The structure of  $W(\widehat{\lambda})$  is given by Lemma 6.13, and we observe  $\overline{W}(\widehat{\lambda}) = W(\widehat{\lambda})$ . As in the proof of Lemma 6.26, we can apply Lemma 4.6, and we see that maximal extendibility holds with respect to  $W(\widehat{\lambda}) \triangleleft N_{\overline{W}}(W(\widehat{\lambda}))$ . Because of  $K(\lambda) \leq N_{\overline{W}}(W(\widehat{\lambda}))$ , this proves maximal extendibility with respect to  $W(\widehat{\lambda}) \triangleleft K(\lambda)$ .  $\square$

As in Lemma 6.25, we associate with  $\lambda$  subsets  $Q^1(\widehat{\lambda})$  and  $Q^2(\widehat{\lambda})$  of  $\mathcal{O}$ . Recall  $K(\lambda) = \overline{W}_{\lambda^{L(\mathbb{F}_p)}}|_L$ , whenever  $\mathbf{G}$  is not of type  $D_4$ .

LEMMA 6.32. *Let  $\lambda \in \mathbb{M}(\widehat{L}) \cap \text{Irr}(L | 1_{(h_0)})$  and  $\widehat{\lambda} \in \text{Irr}(\widehat{L} | \lambda)$  with  $W(\lambda) \neq W(\widehat{\lambda})$ . Let*

$$Q^1(\widehat{\lambda}) := \{I \in \mathcal{O}_1 \mid o(\widehat{\lambda}_I) \in \{1, 2, 4\}\} \cup \mathcal{O}_{-1} \text{ and } Q^2(\widehat{\lambda}) := \mathcal{O} \setminus Q^1(\widehat{\lambda}).$$

*Let  $\overline{W}^j(\widehat{\lambda}) := \overline{W}(\widehat{\lambda}) \cap \mathcal{S}_{\pm Q^j(\widehat{\lambda})}$ ,  $W^j(\widehat{\lambda}) := \overline{W}^j(\widehat{\lambda}) \cap W$ ,  $\widehat{L}^{(i)} := \langle \widehat{L}_I \mid I \in Q^i(\widehat{\lambda}) \rangle$ , and  $L^{(i)} := L \cap \widehat{L}^{(i)}$ , for  $i, j \in \underline{2}$ .*

(a) *Then  $K(\lambda)$  stabilizes  $Q^1(\widehat{\lambda})$  and*

$$W(\widehat{\lambda}) = W^1(\widehat{\lambda}^{(1)}) \times W^2(\widehat{\lambda}^{(2)}),$$

*where  $\widehat{\lambda}^{(i)} \in \text{Irr}(\widehat{\lambda}|_{\widehat{L}^{(i)}})$ .*

(b) *If  $x \in W(\lambda) \setminus W(\widehat{\lambda})$ , then  $x = x_1 x_2$  for some  $x_i \in W^i(\lambda^{(i)})$  ( $i \in \underline{2}$ ), where  $\lambda^{(i)} \in \text{Irr}(\lambda|_{L^{(i)}})$ .*

(c)  *$|\mathcal{O}_d \cap Q^2(\widehat{\lambda})|$  is even for every  $d \in \mathbb{D}_{\text{odd}} \setminus \{-1\}$ .*

(d) *Let  $\overline{W}^i := \overline{W} \cap \mathcal{S}_{\pm Q^i(\widehat{\lambda})}$  and  $K^i(\lambda) := (\overline{W}^i)_{(\lambda^{(i)})^{L^{(i)}(\mathbb{F}_p)}}|_{L^{(i)}}$  for  $i \in \underline{2}$ . If  $\mathcal{O}_1 \cap Q^1(\widehat{\lambda}) \neq \emptyset$ ,*

*then  $K(\lambda) \leq K^1(\lambda) \times N_{\overline{W}^2}(W^2(\widehat{\lambda}))$ .*

*Proof.* First note that  $\widehat{N}$  normalizes the groups  $\mathbf{L}_I$  ( $I \in \mathcal{O}$ ) and hence there is a well-defined action of  $\widehat{W}$  on  $\mathcal{O}$ . Now,  $Q^1(\widehat{\lambda})$  is defined using  $\widehat{\lambda}$  (and is independent of the choice of  $\widehat{\lambda} \in \text{Irr}(\widehat{L} | \lambda)$ ). Note that by this definition any element in  $\widehat{N}_\lambda$  stabilizes  $Q^1(\widehat{\lambda})$ . Without loss of generality, we can assume that  $\widehat{\lambda}$  is standardized and hence  $\overline{W}(\lambda)$  is given in Lemma 6.13.

Accordingly,  $\overline{W}(\widehat{\lambda}) = \overline{W}^1(\widehat{\lambda}) \times \overline{W}^2(\widehat{\lambda})$ . As  $\lambda \in \text{Irr}(L \mid 1_{(h_0)})$ , Corollary 6.22 implies  $\mathcal{O}_c(\widehat{\lambda}) \subseteq \bigcup_{d \in \mathbb{D}_{\text{even}}} \mathcal{O}_d$ . By the definition of  $Q^2(\widehat{\lambda})$ , we observe  $\overline{W}^2(\widehat{\lambda}) = W^2(\widehat{\lambda})$ .

According to Lemma 6.8(d),  $\overline{W}(\widehat{\lambda})$  is the direct product of the groups  $\overline{W}_d(\widehat{\lambda})$ , where  $\overline{W}_d(\widehat{\lambda}) := (\overline{V}_d)_{\widehat{\lambda}}/H_d$ . For  $\overline{W}^1(\widehat{\lambda})$  we note that

$$\overline{W}^1(\widehat{\lambda}) = \overline{W}^{1,1}(\widehat{\lambda}) \times \overline{W}^{1,2}(\widehat{\lambda}) \times \overline{W}^{1,4}(\widehat{\lambda}),$$

where  $Q^{1,j}(\lambda) := \{I \in \mathcal{O}_1 \mid o(\widehat{\lambda}_I) = j\}$  and  $\overline{W}^{1,j}(\widehat{\lambda}) := \overline{W}^1(\widehat{\lambda}) \cap \mathcal{S}_{\pm Q^{1,4}(\lambda)}$ . This proves that  $\overline{W}(\widehat{\lambda}) = \overline{W}^1(\widehat{\lambda}) \times \overline{W}^2(\widehat{\lambda})$ . By the above  $W^2(\widehat{\lambda}) = \overline{W}^2(\widehat{\lambda})$  and hence  $W(\widehat{\lambda}) = \overline{W}^1(\widehat{\lambda}) \times W^2(\widehat{\lambda}) = W^1(\widehat{\lambda}) \times W^2(\widehat{\lambda})$ . Since  $\widehat{\lambda} = \widehat{\lambda}^{(1)} \times \widehat{\lambda}^{(2)}$ , we note that  $W^1(\widehat{\lambda}^{(1)}) = W^1(\widehat{\lambda}^{(1)})$  and  $W^2(\widehat{\lambda}) = W^2(\widehat{\lambda}^{(2)})$ , proving (a).

As  $x \in K(\lambda)$  stabilizes  $Q^1(\widehat{\lambda})$  by (a), it can be written as product  $x_1 x_2$  where  $x_i \in \overline{W}^i(\lambda^{(i)})$ . Since  $\widehat{\lambda}^x = \widehat{\lambda} \mu$  for the faithful character  $\mu$  of  $\widehat{L}/L$ , it satisfies  $(\widehat{\lambda}^{(i)})^x = \widehat{\lambda}^{(i)} \mu^{(i)}$  where  $\mu^{(i)} = \mu \upharpoonright_{\widehat{L}^{(i)}}$  with  $\widehat{L}^{(i)} := \langle \widehat{L}_I \mid I \in Q^i(\lambda) \rangle$ . Hence,  $(\widehat{\lambda}^{(i)})^{x_i} = \widehat{\lambda}^{(i)} \mu^{(i)}$ .

In the following, we show that any element  $x_2 \in \overline{W}^2$  with  $(\widehat{\lambda}^{(2)})^{x_i} = \widehat{\lambda}^{(2)} \mu^{(i)}$  also satisfies  $x_2 \in W$ . This then implies the statement in part (b). Recall that  $\overline{W}_d \leq W$  for  $d \in \mathbb{D}_{\text{even}}$ . Hence, without loss of generality, we can assume that  $Q^2(\widehat{\lambda}) \subseteq \mathcal{O}_d$  for some  $d \in \mathbb{D}_{\text{odd}} \setminus \{-1\}$ .

For  $I_0 \in \mathcal{O}$  and  $\kappa \in \text{Irr}_{\text{cusp}}(\widehat{L}_{I_0})$ , we set

$$\mathcal{O}_\kappa(\widehat{\lambda}) := \{I \in \mathcal{O} \mid \widehat{\lambda}_I \text{ is } V_S\text{-conjugate to } \kappa \text{ or } \kappa^{c_{I_0}}\}.$$

Let  $\mu_I$  be defined as in Lemma 6.14. Then  $x_2(\mathcal{O}_\kappa(\widehat{\lambda})) = \mathcal{O}_{\kappa\mu_{I_0}}(\widehat{\lambda})$  (see 6.14(a)). With  $\overline{\mathcal{O}}_\kappa(\widehat{\lambda}) := \mathcal{O}_\kappa(\widehat{\lambda}) \cup \mathcal{O}_{\kappa\mu_{I_0}}(\widehat{\lambda})$ , the element  $x$  can be written as product of  $x_{\overline{\mathcal{O}}_\kappa(\lambda)} \in \mathcal{S}_{\pm \overline{\mathcal{O}}_\kappa(\lambda)}$  where  $I \in \mathcal{O}$  and  $\kappa \in \text{Irr}_{\text{cusp}}(\widehat{L}_I)$  runs over the  $\langle \mu_I \rangle \times \langle c_I \rangle$ -orbits in  $\text{Irr}_{\text{cusp}}(\widehat{L}_I)$ . To prove  $x_2 \in W$ , it is sufficient to prove  $x_{\overline{\mathcal{O}}_\kappa(\lambda)} \in W$ . Hence, we assume  $Q^2(\widehat{\lambda}) = \mathcal{O}_\kappa(\widehat{\lambda}) \cup \mathcal{O}_{\kappa\mu_{I_0}}(\widehat{\lambda})$  for some  $\kappa \in \text{Irr}(\widehat{L}_{I_0})$ .

If  $I_0 \in \mathcal{O}_1$ , we observe that  $o(\kappa) \notin \{1, 2, 4\}$  by the definition of  $Q^2(\widehat{\lambda})$ . This implies  $\kappa\mu_{I_0} \notin \{\kappa, \kappa^{c_{I_0}}\}$  and hence  $\mathcal{O}_\kappa(\widehat{\lambda}) \cap \mathcal{O}_{\kappa\mu_{I_0}}(\widehat{\lambda}) = \emptyset$ . Note that  $\overline{W}^2(\widehat{\lambda}) \leq W$ . The element  $x_2$  satisfies  $x_2(\mathcal{O}_\kappa(\widehat{\lambda})) = \mathcal{O}_{\kappa\mu_{I_0}}(\widehat{\lambda})$  as element of  $\mathcal{S}_{Q^2(\widehat{\lambda})}$ . Recall  $\widehat{\lambda}$  is standardized. Let  $I \in \mathcal{O}_\kappa(\widehat{\lambda})$  and  $I' \in \mathcal{O}_{\kappa\mu_{I_0}}(\widehat{\lambda})$ . If  $\widehat{\lambda}_I$  and  $\widehat{\lambda}_{I'}$  are  $V_S$ -conjugate,

$$x_2(\epsilon \mathcal{O}_\kappa(\widehat{\lambda})) = \epsilon \mathcal{O}_{\kappa\mu_{I_0}}(\widehat{\lambda}) \text{ and } x_2(\epsilon \mathcal{O}_{\kappa\mu_{I_0}}(\widehat{\lambda})) = \epsilon \mathcal{O}_\kappa(\widehat{\lambda}) \tag{6.3}$$

for every  $\epsilon \in \{\pm 1\}$  as element of  $\mathcal{S}_{\pm Q^2(\widehat{\lambda})}$ . Otherwise,

$$x_2(\epsilon \mathcal{O}_\kappa(\widehat{\lambda})) = -\epsilon \mathcal{O}_{\kappa\mu_{I_0}}(\widehat{\lambda}) \text{ and } x_2(\epsilon \mathcal{O}_{\kappa\mu_{I_0}}(\widehat{\lambda})) = -\epsilon \mathcal{O}_\kappa(\widehat{\lambda}) \tag{6.4}$$

for every  $\epsilon \in \{\pm 1\}$  as element of  $\mathcal{S}_{\pm Q^2(\widehat{\lambda})}$ . In both cases, we see  $x_2 \in W$ .

Assume  $I_0 \in \mathcal{O}_d$  for  $d \in \mathbb{D}_{\text{odd}} \setminus \{\pm 1\}$ . Hence,  $L_{I_0} \cong \text{GL}_d(q)$  by Lemma 4.2 and  $c_{I_0}$  acts on  $L_{I_0}$  as a graph automorphism by Lemma 4.4(c). According to Lemma 6.15, we have  $\kappa \neq \kappa^{c_{I_0}}$  and  $\overline{W}^2(\widehat{\lambda}) \leq W$ . Proposition 5.1(b) leads to  $\kappa\mu_{I_0} \notin \{\kappa, \kappa^{c_{I_0}}\}$ . We see again that  $\mathcal{O}_\kappa(\widehat{\lambda})$  and  $\mathcal{O}_{\kappa\mu_{I_0}}(\widehat{\lambda})$  are disjoint. This implies again that there exists some  $\epsilon' \in \{\pm 1\}$  such that

$$x_2(\epsilon \mathcal{O}_\kappa(\widehat{\lambda})) = \epsilon' \epsilon \mathcal{O}_{\kappa\mu_{I_0}}(\widehat{\lambda}) \text{ and } x_2(\epsilon \mathcal{O}_{\kappa\mu_{I_0}}(\widehat{\lambda})) = \epsilon' \epsilon \mathcal{O}_\kappa(\widehat{\lambda}) \tag{6.5}$$

for every  $\epsilon \in \{\pm 1\}$  as elements of  $\mathcal{S}_{\pm Q^2(\widehat{\lambda})}$ . Again  $x_2 \in W$ . Altogether this proves part (b).

The considerations above imply  $|\mathcal{O}_\kappa(\widehat{\lambda})| = |\mathcal{O}_{\kappa\mu_I}(\widehat{\lambda})|$  for every  $I \in \mathcal{O}$  and  $\kappa \in \text{Irr}_{\text{cusps}}(\widehat{L}_I)$ . If  $\mathcal{O}_\kappa(\widehat{\lambda}) \subseteq Q^2(\widehat{\lambda})$ , the sets are disjoint so that  $2 \mid |\overline{\mathcal{O}}_\kappa(\widehat{\lambda})|$ . This also applies if  $I \in \mathcal{O}_1$  and  $\kappa \in \text{Irr}_{\text{cusps}}(\widehat{L}_I)$  with  $o(\kappa) \mid 2$ . This gives part (c).

Recall  $K(\lambda) = \overline{W}_{\lambda^{L\langle F_p \rangle}} \Big|_L$ . We see that  $Q^1(\kappa) = Q^1(\lambda)$  and  $Q^2(\kappa) = Q^2(\lambda)$  for every constituent  $\kappa$  of  $\lambda^{L\langle F_p \rangle} \Big|_L$ . Accordingly, we see that  $K(\lambda) \leq \mathcal{S}_{\pm Q^1(\lambda)} \times \mathcal{S}_{\pm Q^2(\lambda)}$ .

By definition,  $\lambda^{(i)}$  is uniquely determined by  $\lambda$ . Let  $w \in K(\lambda)$ ,  $w_1 \in \mathcal{S}_{\pm Q^1(\lambda)}$ , and  $w_2 \in \mathcal{S}_{\pm Q^2(\lambda)}$  with  $w = w_1 w_2$ . As  $\lambda^{w_1 w_2}$  is some  $L\langle F_p \rangle$  conjugate of  $\lambda$ , the character  $(\lambda^{(1)})^w = (\lambda^{(1)})^{w_1}$  is an  $L^{(1)}\langle F_p \rangle$ -conjugate of  $\lambda^{(1)}$ . Hence,  $w_1 \in K^1(\lambda)$ . Analogously, we can argue for  $w_2$  and get  $w_2 \in N_{\overline{W}^2}(W^2(\widehat{\lambda}))$ , as required in (d).  $\square$

We study first the Clifford theory for  $W(\widehat{\lambda}) \triangleleft W(\lambda)$  by considering subgroups associated with  $Q^1(\widehat{\lambda})$  and  $Q^2(\widehat{\lambda})$ .

**LEMMA 6.33.** *In the situation of Lemma 6.32, maximal extendibility holds with respect to  $W^2(\widehat{\lambda}^{(2)}) \triangleleft N_{\overline{W}^2}(W(\widehat{\lambda}^{(2)}))$ .*

*Proof.* Without loss of generality, we can assume that  $\widehat{\lambda}$  is standardized. The structure of  $\overline{W}(\widehat{\lambda})$  is then given by Lemma 6.12. As in the proof of Lemma 6.26(a), the groups  $W^2(\widehat{\lambda}^{(2)})$  and  $N_{\overline{W}^2}(W(\widehat{\lambda}^{(2)}))$  satisfy Lemma 4.6 and accordingly maximal extendibility holds.  $\square$

**PROPOSITION 6.34.** *Let  $\lambda \in \text{Irr}_{\text{cusps}}(L \mid 1_{\langle h_0 \rangle})$ ,  $\widehat{\lambda} \in \text{Irr}(\widehat{L} \mid \lambda)$ , and  $\eta_0 \in \text{Irr}(W(\widehat{\lambda}))$  with  $\widetilde{L}_\lambda = \widehat{L}$ .*

- (a) *If  $Q^1(\widehat{\lambda}) = \mathcal{O}$ , then every  $\eta \in \text{Irr}(W(\lambda) \mid \eta_0)$  is  $K(\lambda)_{\eta_0}$ -stable.*
- (b) *Maximal extendibility holds with respect to  $W^1(\widehat{\lambda}^{(1)}) \triangleleft K^1(\lambda^{(1)})$ .*

*Proof.* The statement in (a) is trivial if  $W(\widehat{\lambda}) = W(\lambda)$ . Hence, we assume in the following  $W(\widehat{\lambda}) \neq W(\lambda)$  and  $Q^1(\widehat{\lambda}) = \mathcal{O}$ . According to Lemma 6.30,  $-1 \in \mathbb{D}$  and  $\widehat{\lambda}_{-1}$  is  $c_{J_{-1}}$ -stable, that is,

$$|\overline{W}_{-1}(\widehat{\lambda})| = 2.$$

In order to study those groups, we introduce more notation: For  $j \in \{1, 2, 4\}$ , let  $Q^{1,j} = \{I \in \mathcal{O}_1 \mid o(\widehat{\lambda}_I) = j\}$  and  $l_j := |Q^{1,j}|$ . Then

$$\overline{W}^{Q^{1,j}}(\widehat{\lambda}^{(1)}) = \begin{cases} \mathcal{S}_{\pm Q^{1,j}}, & \text{if } j \in \{1, 2\}, \\ \mathcal{S}_{Q^{1,4}}, & \text{if } j = 4. \end{cases}$$

Accordingly,  $\overline{W}^1(\widehat{\lambda}) = \overline{W}_{-1}(\widehat{\lambda}) \times \mathcal{S}_{\pm Q^{1,1}} \times \mathcal{S}_{\pm Q^{1,2}} \times \mathcal{S}_{Q^{1,4}}$ . Additionally,  $\overline{W}_{\widehat{\lambda}^{\widehat{L}E_L}} \Big|_{\widehat{L}}$  stabilizes the sets  $J_{-1}$  and  $Q^{1,j}$  ( $j \in \{1, 2, 4\}$ ). If  $W(\lambda) \neq W(\widehat{\lambda})$ , every  $x \in W(\lambda) \setminus W(\widehat{\lambda})$  satisfies  $x(Q^{1,1}) = Q^{1,2}$  as element of  $\mathcal{S}_{Q^1(\widehat{\lambda})}$  (see the proof of Lemma 6.32). Hence, in that case  $l_1 = l_2$ .

Following the arguments given in the proof of Lemma 6.32, the element  $x_1 \in W^1(\lambda) \setminus W^1(\widehat{\lambda})$  can be written as  $x_{-1} x_{\{1,2\}} x_4$ , where  $x_{-1} \in \langle (J_{-1}, -J_{-1}) \rangle$ ,  $x_{\{1,2\}} \in \mathcal{S}_{\pm(Q^{1,1} \cup Q^{1,2})}$  with  $x_{\{1,2\}}(Q^{1,1}) = Q^{1,2}$  as element of  $\mathcal{S}_{(Q^{1,1} \cup Q^{1,2})}$  and  $x_4 \in \langle x_4^\circ \rangle$  with

$$x_4^\circ := \prod_{i \in Q^{1,4}} (i, -i) \in \mathcal{S}_{\pm Q^{1,4}}.$$

Note that  $-1 \in \mathbb{D}$  according to Lemma 6.30 and hence  $\gamma \in E_L$ . In this notation, we have

$$K(\lambda) \leq K^1 := \langle (J_{-1}, -J_{-1}) \rangle \times ((\mathcal{S}_{\pm Q^{1,1}} \times \mathcal{S}_{\pm Q^{1,2}}) \rtimes \langle x_{\{1,2\}} \rangle) \times (\mathcal{S}_{Q^{1,4}} \rtimes \langle x_4^\circ \rangle).$$

We see  $W(\widehat{\lambda}) \cong \mathcal{S}_{\pm Q^{1,1}} \times \mathcal{S}_{\pm Q^{1,2}} \times \mathcal{S}_{Q^{1,4}}$ . Since  $K(\lambda) = \overline{W}_{\lambda^L \langle F_p \rangle} \Big|_L$ , we have

$$K(\lambda) \leq \langle (J_{-1}, -J_{-1}) \rangle \times ((\mathcal{S}_{\pm Q^{1,1}} \times \mathcal{S}_{\pm Q^{1,2}}) \rtimes \langle x_{\{1,2\}} \rangle) \times \langle x_4 \rangle \mathcal{S}_{Q^{1,4}} \leq W(\widehat{\lambda}) \langle x, c_{-1}, x_4^\circ \rangle$$

for the element  $x$  from above and with  $c_{-1} := \rho_{\mathbf{T}}(c_{J_{-1}})$ . Note  $W(\widehat{\lambda}) \langle x, c_{-1}, x_4^\circ \rangle \leq K^1 = W(\lambda) \langle c_{-1}, x_4^\circ \rangle$ . We observe  $c_{-1}, x_4^\circ \in Z(W(\lambda) \langle c_{-1}, x_4^\circ \rangle)$ . This implies that every character  $\eta \in \text{Irr}(W^1(\lambda))$  is  $K^1$ -stable. This proves part (a), and even that every character of  $\text{Irr}(W(\lambda))$  extends to  $K^1$ .

Now, by the definition of  $W^1(\widehat{\lambda}^{(1)})$  and  $K^1(\lambda^{(1)})$ , we see that in the general case the groups obtained as  $K^1(\lambda)$  coincide with  $K(\lambda)$  for a group of smaller rank where for the character  $\widehat{\lambda}^{(1)}$  part (a) can be applied. This then proves part (b). □

We consider the general case.

**PROPOSITION 6.35.** *Let  $\lambda \in \text{Irr}_{\text{cusp}}(L)$  with  $\widetilde{L}_\lambda = \widehat{L}$  and  $\eta_0 \in \text{Irr}(W(\widehat{\lambda}))$ . Then every character in  $\text{Irr}(W(\lambda) \mid \eta_0)$  is  $K(\lambda)_{\eta_0}$ -stable.*

*Proof.* Note that because of  $|\widehat{L}_\lambda : L| = 2$  it is sufficient to prove that some character in  $\text{Irr}(W(\lambda) \mid \eta_0)$  is  $K(\lambda)_{\eta_0}$ -stable. According to Lemmas 6.33 and 6.34, we can assume  $Q^1(\widehat{\lambda}) \neq \mathcal{O} \neq Q^2(\widehat{\lambda})$  for the sets  $Q^1(\widehat{\lambda})$  and  $Q^2(\widehat{\lambda})$  from Lemma 6.32.

By Lemma 6.31, we can assume  $h_0 \in \ker(\lambda)$ . The groups  $W^i(\widehat{\lambda})$  ( $i \in \mathbb{2}$ ) satisfy  $W(\widehat{\lambda}) = W^1(\widehat{\lambda}) \times W^2(\widehat{\lambda})$  (see Lemma 6.32).

If  $\mathcal{O}_1 \cap Q^1(\widehat{\lambda}) \neq \emptyset$ , then  $K(\lambda) \leq K^1(\lambda) \times N_{\overline{W}^2}(W^2(\widehat{\lambda}^{(2)}))$  (see Lemma 6.32(d)). Let  $\eta_i \in \text{Irr}(W^i(\widehat{\lambda}))$  such that  $\eta_0 = \eta_1 \times \eta_2$ . According to Lemma 6.34,  $\eta_1$  has a  $K^1(\lambda)_{\eta_1}$ -stable extension to  $W^1(\lambda)_{\eta_1}$  and maximal extendibility holds with respect to  $W^2(\widehat{\lambda}) \triangleleft N_{\overline{W}^2}(W^2(\widehat{\lambda}^{(2)}))$  according to Lemma 6.33. This proves the statement in that case.

If  $\mathcal{O}_1 \cap Q^1(\widehat{\lambda}) = \emptyset$ , then  $Q^1(\widehat{\lambda}) = \{J_{-1}\}$ . Then  $|W^1(\widehat{\lambda})| = 1$  and therefore  $W(\widehat{\lambda}) = W^2(\widehat{\lambda})$ . Then the stability statement follows by applying again Lemma 6.33. □

Together with Proposition 6.28, this leads to the following statement.

**COROLLARY 6.36.** *Let  $\lambda \in \mathbb{T}$  with  $\widetilde{L}_\lambda \neq L$ ,  $\widetilde{\lambda} \in \text{Irr}(\widetilde{L}_\lambda \mid \lambda)$ , and  $\eta_0 \in \text{Irr}(W(\widetilde{\lambda}))$ . Then there exists some  $K(\lambda)_{\eta_0}$ -stable  $\eta \in \text{Irr}(W(\lambda) \mid \eta_0)$ .*

*Proof.* For  $\lambda \in \mathbb{M}(\widetilde{L}) \cup \mathbb{M}_0$ , this is Proposition 6.28. For  $\lambda \in \text{Irr}_{\text{cusp}}(L)$  with  $\widetilde{L}_\lambda = \widehat{L}$ , the statement follows from Proposition 6.35. □

### §7. Proof of Theorem A

In the following, we explain how Corollary 6.36 about the action of  $K(\lambda)$  on  $\text{Irr}(W(\lambda))$  proves Theorem 6.1. As already sketched in the beginning of §6 based on Proposition 2.11, knowing the action of  $\widehat{K}(\lambda)$  on  $\text{Irr}(W(\lambda))$  is crucial to verify Theorem 6.1. Unless  $\mathbf{G}$  is of type  $D_4$ , the action  $\widehat{K}(\lambda)$  on  $\text{Irr}(W(\lambda))$  is given by the action of  $K(\lambda)$  (see Lemma 6.5).

Via Harish-Chandra induction, we transfer the result of Theorem 6.1 on characters of  $N$  to a weak version of Theorem A. Special considerations are needed to determine the stabilizers in  $\widetilde{GE}$  of characters  $\chi \in \text{Irr}(\mathbf{G}^F \mid (L, \text{Irr}_{\text{cusp}}(L)))$ , whenever  $L$  is not  $E(\mathbf{G}^F)$ -stable.

LEMMA 7.1. *Let  $\lambda \in \text{Irr}_{\text{cusp}}(L)$ ,  $\tilde{\lambda} \in \text{Irr}(\tilde{L}_\lambda \mid \lambda)$ , and  $\eta_0 \in \text{Irr}(W(\lambda))$ . Then some character in  $\text{Irr}(W(\lambda) \mid \eta_0)$  is  $\widehat{K}(\lambda)_{\eta_0}$ -stable.*

*Proof.* According to Corollary 6.36, there exists some  $K(\lambda)_{\eta_0}$ -stable  $\eta$  in  $\text{Irr}(W(\lambda) \mid \eta_0)$ . According to Lemma 6.5, the character  $\eta$  is  $\widehat{K}(\lambda)_{\eta_0} \cap (W \rtimes E_L^\circ)$ -stable, where  $E^\circ := \langle F_p, \gamma_0 \rangle$  and  $E_L^\circ := E^\circ \cap E_L$ . If  $\mathbf{G}$  is not of type  $D_4$  or  $E_L \leq \langle F_p, \gamma_0 \rangle$ , this is the above statement.

Accordingly, we can assume in the following that  $\mathbf{G}$  is of type  $D_4$  and  $L$  is  $\gamma_3$ -stable for the graph automorphism  $\gamma_3$  of  $D_4(\mathbb{F})$  from Notation 3.3. If  $L = T_0$ , the statement follows from [MS, Th. 3.7]. Otherwise,  $L$  is one of the other two possible  $\gamma_3$ -stable Levi subgroups. In both cases, easy calculations show that  $W(L)$  is a 2-group. According to our considerations above, we know that there is some  $K(\lambda)_{\eta_0}$ -stable  $\eta \in \text{Irr}(W(\lambda) \mid \eta_0)$ . As  $[K(\lambda) : W(\lambda)] \in \{1, 2\}$ , the character  $\eta_0$  extends to its inertia group  $K(\lambda)_{\eta_0}$ . This shows that maximal extendibility holds with respect to  $W(\tilde{\lambda}) \triangleleft K(\lambda)$ . Let  $K'(\lambda) = (\widehat{K}(\lambda)\langle F_p \rangle) \cap (W \rtimes \langle \gamma, \gamma_3 \rangle)$ . When we identify  $\overline{W}$  with  $W \rtimes \langle \gamma \rangle$ , we can see  $K(\lambda)$  as a subgroup of  $K'(\lambda)$  with index 1 or 3. Hence,  $K(\lambda)_{\eta_0}$  has index 1 or 3 in  $K'(\lambda)_{\eta_0}$ . The character  $\eta_0$  extends to  $K(\lambda)_{\eta_0}$  by the above.

Assume that  $K(\lambda)_{\eta_0}/W(\tilde{\lambda})$  is a Sylow 2-subgroup of  $K'(\lambda)/W(\tilde{\lambda})$ . Let  $K_3$  be a subgroup of  $K'(\lambda)_{\eta_0}$  with  $W(\lambda)_{\eta_0} \leq K_3$  such that  $K_3/W(\lambda)_{\eta_0}$  is a Sylow 3-subgroup of  $K'(\lambda)_{\eta_0}/W(\tilde{\lambda})$ . The character  $\eta_0$  extends to  $K_3$  as  $|W(\lambda)|$  is coprime to 3 according to [I, (11.32)]. This implies that  $\eta_0$  extends to  $K(\lambda)_{\eta_0}$ . Maximal extendibility holds with respect to  $W(\tilde{\lambda}) \triangleleft K'(\lambda)$  as well. (This can be seen via an application of [I, (11.31)].)

If  $K(\lambda)_{\eta_0}/W(\tilde{\lambda})$  is not a Sylow 2-subgroup of  $K'(\lambda)_{\eta_0}/W(\tilde{\lambda})$ , the group  $K(\lambda^{\gamma_3})_{\eta_0^{\gamma_3}} = (K(\lambda)_{\eta_0})^{\gamma_3}$  contains a Sylow 2-subgroup of  $K'(\lambda^{\gamma_3})_{\eta_0^{\gamma_3}}$  and hence by the above  $(\eta_0)^{\gamma_3}$  extends to  $K'(\lambda^{\gamma_3})_{\eta_0^{\gamma_3}}$ . Via conjugation, this implies that  $\eta_0$  extends to  $K'(\lambda)_{\eta_0}$ .  $\square$

We can now show Theorem 6.1.

*Proof of Theorem 6.1.* Recall  $\mathbb{M}^{(X)} := \{\lambda \in \text{Irr}_{\text{cusp}}(L) \mid \tilde{L} = X\}$  for the subgroups  $L \leq X \leq \tilde{L}$  and  $\mathbb{M}_0 := \text{Irr}_{\text{cusp}}(L) \setminus (\mathbb{M}^{(L)} \cup \mathbb{M}^{(\tilde{L})} \cup \mathbb{M}^{(\tilde{L})})$  (see before Lemma 6.3). As the sets  $\mathbb{M}^{(L)}$ ,  $\mathbb{M}^{(\tilde{L})}$ ,  $\mathbb{M}^{(\tilde{L})}$ , and  $\mathbb{M}_0$  are  $E_L$ -stable, it is sufficient to construct an  $\widehat{N}$ -stable  $\tilde{L}$ -transversal in  $\text{Irr}(N \mid \mathbb{M}')$  for  $\mathbb{M}' \in \{\mathbb{M}^{(L)}, \mathbb{M}^{(\tilde{L})}, \mathbb{M}^{(\tilde{L})}\}$ . Note that since every character of  $N$  is  $N$ -stable, one can equivalently also construct  $E_L$ -stable  $\tilde{N}'$ -transversals. Lemma 6.3 provides an  $\widehat{N}$ -stable  $\tilde{L}$ -transversal in  $\text{Irr}(N \mid \mathbb{M}^{(L)})$ .

Lemma 7.1 shows that for every  $\lambda \in \mathbb{M}_0 \cup \mathbb{M}^{(\tilde{L})} \cup \mathbb{M}^{(\tilde{L})}$  and every character  $\eta_0 \in \text{Irr}(W(\tilde{\lambda}))$ , there is some  $K(\tilde{\lambda})_{\eta_0}$ -stable  $\eta \in \text{Irr}(W(\lambda) \mid \eta_0)$ , where  $\tilde{\lambda} \in \text{Irr}(\tilde{L}_\lambda \mid \lambda)$ .

Assumptions (i) and (ii) of Proposition 2.11 are satisfied with  $\mathbb{T}' := \mathbb{T} \cap (\mathbb{M}_0 \cup \mathbb{M}^{(\tilde{L})} \cup \mathbb{M}^{(\tilde{L})})$  from Proposition 4.10 and the extension map  $\Lambda$  from Theorem 3.1. For every  $\lambda \in \mathbb{T}'$  and  $\eta_0 \in \text{Irr}(W(\tilde{\lambda}))$ , there exists some  $K(\lambda)_{\eta_0}$ -stable  $\eta \in \text{Irr}(W(\lambda))$ . This allows us to apply Proposition 2.12 and hence some  $\widehat{N}$ -stable  $\tilde{N}$ -transversal in  $\text{Irr}_{\text{cusp}}(N \mid \mathbb{M}_0 \cup \mathbb{M}^{(\tilde{L})} \cup \mathbb{M}^{(\tilde{L})})$  exists.  $\square$

Theorem 6.1 implies according to Theorem 2.8 that the equation  $(\tilde{G}E_L)_\chi = \tilde{G}_\chi(E_L)_\chi$  holds for every character  $\chi$  of a  $\tilde{G}$ -transversal in  $\text{Irr}(\mathbf{G}^F \mid (L, \text{Irr}_{\text{cusp}}(L)))$ . Accordingly, we have constructed an  $E_L$ -stable  $\tilde{G}$ -transversal of  $\text{Irr}(\mathbf{G}^F \mid (L, \text{Irr}_{\text{cusp}}(L)))$  (see Lemma 2.4).

COROLLARY 7.2. *Let  $\mathbf{G} = D_{l,\text{sc}}(\mathbb{F})$ , let  $F : \mathbf{G} \rightarrow \mathbf{G}$  be a standard Frobenius endomorphism, and let  $E$  be defined as in Notation 3.3 and  $\tilde{\mathcal{L}} := \mathcal{L}^{-1}(Z(\mathbf{G}))$  for the Lang map  $\mathcal{L}$  defined by  $F$  on  $\mathbf{G}$ . Let  $L$  be a standard Levi subgroup of  $\mathbf{G}^F$ , and let  $E_L$  be its stabilizer in*



$E(\mathbf{G}^F)$ . If Hypothesis 2.14 holds for every  $l' < l$ , then there exists an  $E_L$ -stable  $\tilde{G}$ -transversal in  $\text{Irr}(\mathbf{G}^F \mid (L, \text{Irr}_{\text{cusp}}(L)))$ .

*Proof.* For a given fixed Levi subgroup  $L$ , we apply Theorem 2.8 whose assumptions follow from Theorems 3.1 and 6.1. □

Condition  $A'(\infty)$  from 2.3 and equivalently Theorem A require to replace in the above statement  $E_L$  by  $E$  and study  $(\tilde{G}E)_\chi$ . Hence, we study the stabilizers of characters in  $\text{Irr}(\mathbf{G}^F \mid (L, \text{Irr}_{\text{cusp}}(L)))$  in the case where  $L$  is a standard Levi subgroup that is not  $E$ -stable.

**PROPOSITION 7.3.** *We keep  $\mathbf{G} = \text{D}_{l,\text{sc}}(\mathbb{F})$  and assume Hypothesis 2.14 holds for every  $l' < l$ . Let  $\mathbf{T}$  and  $\mathbf{L}$  as in Notation 2.5. Let  $E^\circ := \langle F_p, \gamma \rangle \leq E$  in the notation of 3.3. Assume that no  $\text{N}_{\mathbf{G}^F}(\mathbf{T})$ -conjugate of  $\mathbf{L}$  is  $E$ -stable. Let  $\chi \in \text{Irr}(\mathbf{G}^F \mid (L, \text{Irr}_{\text{cusp}}(L)))$ . Then  $\tilde{G}_\chi = \tilde{G}$  or  $(\tilde{G}E^\circ)_\chi \leq \tilde{G}_\chi(E^\circ \cap E_L)$ .*

*Proof.* Let  $N_0 := \text{N}_{\mathbf{G}^F}(\mathbf{T})$ . We consider first the possible structure of  $L$ , in particular the values of  $\mathbb{D}(L)$ . Then we give the possible values of  $\tilde{L}_\chi$  via describing  $W(\lambda)$ .

We see that  $L$  is  $F_p$ -stable. If  $L$  is  $\gamma$ -stable, then  $E(\mathbf{G}^F) = E_L$ . By our assumption  $E_L \neq E$ , we have  $\gamma \notin E_L$ . We observe that then  $-1 \notin \mathbb{D}(L)$ , as otherwise the system of simple roots  $\Delta'$ , associated with  $L$  as in Notation 3.5 is  $\gamma$ -stable, which then implies  $\gamma \in E_L$ .

If  $1 \in \mathbb{D}(L)$ , then some  $N_0$ -conjugate of  $L$  is  $\gamma$ -stable: the conjugation is given by some element  $v \in N_0 := \text{N}_{\mathbf{G}^F}(\mathbf{T}_0)$  that corresponds to some  $\sigma \in \mathcal{S}_{\pm l}$  with the following properties:  $\sigma(l) = l$ ,  $1 \in \sigma(J_1)$  and  $\sigma(\Delta') \subseteq \Delta$ . The Levi subgroup  $L$  satisfies accordingly  $1 \notin \mathbb{D}(L)$ .

Let  $W_0 = N_0/\mathbf{T}^F$ . For the proof of the statement, we consider  $\chi \in \text{Irr}(\mathbf{G}^F \mid (L, \text{Irr}_{\text{cusp}}(L)))$  with  $(\tilde{G}E)_\chi \leq \tilde{G}E_L$ . Then  $\chi^\gamma$  and  $\chi$  are  $\tilde{G}E_L$ -conjugate. For the statement, we have to show that  $\tilde{G}_\chi = \tilde{G}$ . We assume that  $\chi^\gamma$  and  $\chi$  are  $\tilde{G}E_L$ -conjugate. Let  $\lambda \in \text{Irr}_{\text{cusp}}(L)$  with  $\chi \in \text{Irr}(\mathbf{G}^F \mid (L, \lambda))$ . Then  $(L, \lambda)$  and  $(\gamma(L), \lambda^\gamma)$  are  $\tilde{G}E_L$ -conjugate, in particular  $\gamma(L)$  and  $L$  are  $N_0$ -conjugate. This shows  $\overline{W}(L) \neq W(L)$ , or equivalently  $\mathbb{D}_{\text{odd}} \neq \emptyset$ . Let  $\mathcal{O}_{\text{odd}} = \bigcup_{d \in \mathbb{D}_{\text{odd}}} \mathcal{O}_d$ ,  $\mathcal{O}_{\text{even}} = \bigcup_{d \in \mathbb{D}_{\text{even}}} \mathcal{O}_d$ , and  $I_0 \in \mathcal{O}_{\text{odd}}$ . Without loss of generality, we assume  $1 \in I_0$ . Otherwise, we replace  $L$  by some  $N_0$ -conjugate. Let  $w_0 := \prod_{i \in I_0} (i, -i) \in W_0$  and  $n_0 \in \overline{N} = \text{N}_{\mathbf{G}^F}(\mathbf{T})$  the corresponding element. Hence,  $w_0 \in \gamma N_0$ . We note that  $N$  induces on  $L$  the outer automorphisms  $W$ , while any  $w' \in \overline{W} \setminus W$  is induced by elements of  $\text{N}_{\mathbf{G}^F \langle \gamma \rangle}(L) \setminus N$ . This proves that in this case  $L$  and  $\gamma(L)$  are actually  $N_0$ -conjugate. Hence, the Harish-Chandra series satisfy  $\text{Irr}(\mathbf{G}^F \mid (L, \text{Irr}_{\text{cusp}}(L))) = \text{Irr}(\mathbf{G}^F \mid (\gamma(L), \text{Irr}_{\text{cusp}}(\gamma(L))))$ .

Let  $\lambda \in \text{Irr}_{\text{cusp}}(L)$ . Assume that  $(L, \lambda)$  and  $(L, \lambda)^{e\gamma}$  are  $\mathbf{G}^F$ -conjugate for some  $e \in \langle F_p \rangle$ . This implies that  $(L, \lambda) = (L, \lambda)^{e\bar{n}}$  for some  $\bar{n} \in \overline{N} \setminus N$ . Note that  $W(\lambda)^{\bar{n}} = W(\lambda)$ . Because of  $-1 \notin \mathbb{D}(L)$  and  $\mathbb{D}_{\text{odd}} \neq \emptyset$ , we observe  $\tilde{L}_\lambda = \tilde{L}$ , as  $\langle \hat{L}_{I_0}, t_{l,2} \rangle \leq C_{\mathbf{G}}(L)L$ .

Assume  $h_0 \in \ker(\lambda)$  and that some  $\chi \in \text{Irr}(\mathbf{G}^F \mid (L, \lambda))$  satisfies  $\tilde{G}_\chi \neq \tilde{G}$ . According to Corollary 6.22, the equation  $W(\hat{\lambda}) = W(\tilde{\lambda})$  holds, as  $\{\pm 1\} \cap \mathbb{D}(L) = \emptyset$ . If  $W(\lambda) = W(\hat{\lambda})$ , then  $\tilde{G}_\chi = \tilde{G}$ . Hence, we assume  $W(\lambda) \neq W(\hat{\lambda})$  in the following. Let  $W^{\text{odd}}(\lambda) := (W(\lambda)\mathcal{S}_{\pm \mathcal{O}_{\text{even}}}) \cap \mathcal{S}_{\pm \mathcal{O}_{\text{odd}}}$ . Without loss of generality, we can assume  $\lambda$  to be standardized, as in every  $\overline{N}$ -orbit in  $\text{Irr}_{\text{cusp}}(L)$  there is at least one standardized character (see after Definition 6.10). As  $W(\lambda) \neq W(\hat{\lambda})$  and  $\lambda$  is standardized,  $W^{\text{odd}}(\hat{\lambda}) \leq \mathcal{S}_{\mathcal{O}_{\text{odd}}}$  and  $x \in W(\lambda) \setminus W(\hat{\lambda})$  can be chosen as an involution with no fixed point in  $\mathcal{O}_{\text{odd}}$ . We note that  $\text{N}_{\mathcal{S}_{\pm \mathcal{O}_{\text{odd}}}}(W^{\text{odd}}(\lambda)) \leq W$ . This implies  $\text{N}_{\overline{W}}(W(\lambda)) \leq W$  and hence  $(L, \lambda)$  and  $(L, \lambda)^{e\gamma}$  are not  $\mathbf{G}^F$ -conjugate for any  $e \in \langle F_p \rangle$ . This proves that  $(L, \lambda)^\gamma$  is not  $\mathbf{G}^F$ -conjugate to any element of the  $E_L$ -orbit of  $(L, \lambda)$ , when  $h_0 \in \ker(\lambda)$ .

Assume  $h_0 \notin \ker(\lambda)$  and  $\nu \in \text{Irr}(\lambda|_{Z(\mathbf{G}^F)})$ . In the following, we assume  $|Z(\mathbf{G}^F)| = 4$ . Then we observe that  $E_\nu^\circ = \langle F_p \rangle = C_E(Z(\mathbf{G}^F))$  and hence  $(\tilde{G}E)_\chi \leq \tilde{G}E_\nu^\circ$  for every  $\chi \in \text{Irr}(\mathbf{G}^F | (L, \lambda))$ . If  $C_E(Z(\mathbf{G}^F)) = E_L$ , this implies  $(\tilde{G}E)_\chi \leq \tilde{G}E_L$  as required. Note that if  $2 \mid l$ , then  $C_E(Z(\mathbf{G}^F)) = E_L$ . In the following, we prove  $2 \mid |\mathcal{O}_{\text{odd}}|$  as this implies  $2 \mid l$ .

For  $I \in \mathcal{O}$ , let  $\mathbf{Z}_I$  be defined as in Lemma 4.2,  $Z_I := \mathbf{Z}_I^F$ , and  $\delta_I \in \text{Irr}(\lambda|_{Z_I})$ . Fix  $d \in \mathbb{D}_{\text{odd}}$  and  $I_d \in \mathcal{O}_d$ . For  $\kappa \in \text{Irr}(Z_{I_d})$ , we define

$$a_\kappa(\lambda) := |\{I \in \mathcal{O}_d \mid \kappa \text{ and } \delta_I \text{ are } \bar{V}_d\text{-conjugate}\}|.$$

Recall  $I_d \in \mathcal{O}_{\text{odd}}$  and  $h_0 \notin \ker(\lambda)$ . Hence,  $Z_{I_d} \cong C_{q-1}$  and  $o(\delta_{I_d})_2 = (q-1)_2$ . On the other hand, we see that

$$\sum_{\kappa} a_\kappa(\lambda) = |\mathcal{O}_d|,$$

where  $\kappa$  runs over the  $\langle c_{I_d} \rangle$ -orbits in  $\text{Irr}(Z_{I_d})$ . By the above,  $a_\kappa(\lambda) = 0$  for every  $\kappa \in \text{Irr}(Z_{I_d})$  with  $o(\kappa)_2 \neq (q-1)_2$ . If  $\lambda' \in \text{Irr}_{\text{cusp}}(L)$  is  $\bar{N}$ -conjugate to  $\lambda$ , then  $a_\kappa(\lambda) = a_\kappa(\lambda')$  for every  $\kappa \in \text{Irr}(Z_{I_d})$  according to the action of  $\bar{V}_d$  on the groups  $Z_I$  ( $I \in \mathcal{O}_d$ ). Note that  $a_\kappa(\lambda) = 0$  as  $o(\delta_I)_2 = (q-1)_2$  for every  $I \in \mathcal{O}_d$ .

Recall that we assume  $(L, \lambda)$  and  $(\gamma(L), \lambda^{\gamma e})$  are  $\mathbf{G}^F$ -conjugate. As the order of  $\gamma e$  is even, we can choose some  $F_0 \in \langle F_p \rangle$  such that  $\langle F_0 \rangle$  is a Sylow 2-subgroup of  $\langle e \rangle$ . Then the  $\bar{N}$ -orbit of  $\lambda$  is  $F_0$ -stable. Then  $F_0$  acts on the characters of  $\text{Irr}(Z_{I_d})$ , inducing an action on the set of  $\langle c_{I_d} \rangle$ -orbits in  $\text{Irr}(Z_{I_d})$ . We denote this set by  $\text{Irr}(Z_{I_d})/\langle c_{I_d} \rangle$ . If  $\kappa \in \text{Irr}(Z_{I_d})$  with  $o(\kappa)_2 = (q-1)_2$ , the  $\langle c_{I_d} \rangle$ -orbit of  $\kappa$  is not  $F_0$ -stable. Hence, the  $F_0$ -orbit in  $\text{Irr}(Z_{I_d})/\langle c_{I_d} \rangle$  containing  $\kappa$  has an even length. Since the  $\bar{N}$ -orbit of  $\lambda$  is  $F_0$ -stable, we see that  $a_\kappa(\lambda^{F_0}) = a_{\kappa^{F_0}}(\lambda)$ . Accordingly,

$$2 \mid \sum_{\kappa} a_\kappa(\lambda),$$

whenever  $\kappa$  runs over a  $\langle c_{I_d} \rangle$ -transversal in  $\{\kappa' \in \text{Irr}(Z_{I_d}) \mid o(\kappa')_2 = (q-1)_2\}$ . By the above,  $a_\kappa(\lambda) = 0$  for every  $\kappa \in \text{Irr}(Z_{I_d})$  with  $o(\kappa)_2 \neq (q-1)_2$ . Altogether, this implies  $2 \mid \sum_{\kappa} a_\kappa(\lambda) = |\mathcal{O}_d|$ , where  $\kappa \in \text{Irr}(Z_{I_d})$  runs over a  $\langle c_{I_d} \rangle$ -transversal.

As  $l = \sum_{d \in \mathbb{D}} d|\mathcal{O}_d|$  and hence  $l \equiv |\mathcal{O}_{\text{odd}}| \pmod{2}$ , the rank  $l$  is even and  $C_E(Z(\mathbf{G}^F)) = \langle F_p \rangle = E_L$ . As explained above, this leads to  $(\tilde{G}E)_\chi \leq \tilde{G}E_{\chi|_{Z(\mathbf{G}^F)}} = \tilde{G}E_L$  and hence a contradiction to the assumption on  $\chi$ .

It remains to study the case of  $|Z(\mathbf{G}^F)| = 2$ . Then  $2 \nmid l$  and  $4 \nmid (q-1)$ . Note that  $4 \nmid (q-1)$  implies  $2 \nmid |E_L|$  and hence  $|E_L|$  and  $o(\gamma) = 2$  are coprime. If  $(L, \lambda)$  and  $(\gamma(L), \lambda^\gamma)$  are  $\mathbf{G}^F E_L$ -conjugate, then the pairs are already  $\mathbf{G}^F$ -conjugate. In the following, we see that the  $\mathbf{G}^F$ -orbit of  $(L, \lambda)$  cannot be  $\gamma$ -stable. By the above, we have  $\mathbb{D}_{\text{odd}} \neq \emptyset$  and  $-1 \notin \mathbb{D}(L)$ . According to Lemma 4.3,  $\lambda$  is  $\hat{L}$ -stable, as each  $\lambda_I$  is  $\hat{L}_I$ -stable for  $I \in \mathcal{O}_{\text{odd}}$ . Hence,  $\lambda$  is  $\tilde{L}$ -stable (see also Lemma 4.3). According to Lemma 6.14, the assumption  $2 \nmid l$  implies  $W(\lambda) = W(\hat{\lambda})$ , even more  $\bar{W}(\lambda) = \bar{W}(\hat{\lambda}) \leq W$ . But this implies that  $(L, \lambda)$  and  $(L, \lambda)^\gamma$  are not  $\mathbf{G}^F$ -conjugate. Hence,  $\gamma \notin (\tilde{G}E)_\chi$  and hence  $(\tilde{G}E)_\chi \leq \tilde{G}(E_L)_\chi$ .  $\square$

A last obstacle is formed by the groups  $D_{4,sc}(q)$ . We keep the same notation.

**PROPOSITION 7.4.** *If  $\mathbf{G}^F = D_{4,sc}(q)$ , every  $\tilde{G}$ -orbit in  $\text{Irr}(\mathbf{G}^F)$  contains some  $\chi$  with  $(\tilde{G}E)_\chi = \tilde{G}_\chi E_\chi$ .*

*Proof.* Let  $\chi_0 \in \text{Irr}(\mathbf{G}^F)$  and  $E^\circ := \langle \gamma, F_p \rangle$ . Then some Sylow 2-subgroup of  $E$  is contained in  $E^\circ$ . We can assume that  $\tilde{G}E^\circ/\mathbf{G}^F$  contains a Sylow 2-subgroup of  $(\tilde{G}E^\circ)_\chi/\mathbf{G}^F$ . (Otherwise we can replace  $\chi_0$  by one of its  $E$ -conjugates.) Some  $\tilde{G}$ -conjugate  $\chi$  of  $\chi_0$  satisfies  $(\tilde{G}E^\circ)_\chi = \tilde{G}_\chi E_\chi^\circ$  according to Proposition 7.3. This proves the statement if  $(\tilde{G}E)_\chi \leq \tilde{G}E^\circ$ . Additionally,  $(\tilde{G}E)_\chi = \tilde{G}_\chi E_\chi$  holds if  $\tilde{G}_\chi = \tilde{G}$ .

Accordingly, there is some  $f \in \langle F_p \rangle$  and  $t \in \tilde{G}$  such that  $\chi$  is  $\gamma_3 ft$ -stable and  $\gamma_3 ft$  has 3-power order in  $\tilde{G}E/\mathbf{G}^F$ . If  $t \in \mathbf{G}^F$ , the equation  $(\tilde{G}E)_\chi = \tilde{G}_\chi E_\chi$  holds. Clearly,  $\tilde{G}_\chi \triangleleft (\tilde{G}E)_\chi$ . Hence,  $\tilde{G}_\chi$  is normalized by  $\gamma_3 ft$ . But via the  $\langle \gamma_3, f \rangle$ -equivariant isomorphism  $\tilde{G}/\mathbf{G}^F \cong Z(\mathbf{G}^F)$ , we see that  $\tilde{G}_\chi = \mathbf{G}^F$ , as there is no  $\gamma_3 f$ -stable subgroups of  $Z(\mathbf{G}^F)$  apart from  $\{1\}$  and  $Z(\mathbf{G}^F)$ .

The element  $\gamma_3 f$  acts on  $Z(\mathbf{G}^F)$  such that only the trivial element is fixed by  $\gamma_3 f$  and  $[\gamma_3 f, Z(\mathbf{G}^F)] = Z(\mathbf{G}^F) \setminus \{1\}$ .

Hence, some  $\tilde{G}$ -conjugate  $\chi'$  of  $\chi$  satisfies  $\gamma_3 f \in (\tilde{G}E)_{\chi'}$ . We observe that  $\gamma_3 f$  is a 3-element and hence  $o(f)$  is a power of 3. Note that  $f$  acts trivially on  $Z(\mathbf{G})$ . Since  $\chi$  satisfies  $(\tilde{G}E^\circ)_\chi = \mathbf{G}^F E_\chi^\circ$  and  $[Z(\mathbf{G}^F), F_p] = 1$ , this leads to  $(\tilde{G}E^\circ)_{\chi'} \in \{GE_\chi^\circ, \mathbf{G}^F \langle F_p, \hat{\gamma}t \rangle_{\chi'}\}$  for some  $\hat{t} \in \tilde{G}$  with  $\mathcal{L}(\hat{t}) = h_0$ . Let  $\hat{G} := \mathcal{L}^{-1}(\langle h_0 \rangle)$ . In the latter case,

$$\hat{t}^{-1} \gamma_3(\hat{t}) f^2 = \gamma_3 f (\gamma_3 f)^{\gamma \hat{t}} \in (\hat{G}E)_{\chi'}.$$

Recalling that the orders of  $\tilde{G}/(Z(\tilde{G})\mathbf{G}^F)$  and  $f$  are coprime, we get  $\hat{t}^{-1} \gamma_3(\hat{t}) \in \langle \hat{t}^{-1} \gamma_3(\hat{t}) f^2 \rangle$ , but  $\hat{t}^{-1} \gamma_3(\hat{t}) \in \tilde{G}_\chi$  and  $\hat{t}^{-1} \gamma_3(\hat{t}) \notin \mathbf{G}^F$ . This leads to a contradiction, and we see that  $(\tilde{G}E^\circ)_{\chi'} = \mathbf{G}^F E_\chi^\circ$  and hence  $(\tilde{G}E)_{\chi'} = \mathbf{G}^F \langle E_{\chi'}^\circ, \gamma_3 f \rangle$ .  $\square$

We can now deduce Theorem A from Theorem 2.8.

*Proof of Theorem A.* For a given fixed Levi subgroup  $L$  of  $\mathbf{G}^F$ , we apply Theorem 2.8 whose assumptions follow from Theorems 3.1 and 6.1. In this way, we obtain an  $E_L$ -stable  $\tilde{G}$ -transversal in  $\text{Irr}(\mathbf{G}^F \mid (L, \text{Irr}_{\text{cusp}}(L)))$  (see Lemma 7.2). If  $L$  is  $E$ -stable,  $E_L = E$  and this gives the required statement. If  $L$  has an  $E$ -stable  $\mathbf{G}^F$ -conjugate  $L'$ , then we observe that  $\text{Irr}(\mathbf{G}^F \mid (L, \text{Irr}_{\text{cusp}}(L))) = \text{Irr}(\mathbf{G}^F \mid (L', \text{Irr}_{\text{cusp}}(L')))$  and there exists an  $E$ -stable  $\tilde{G}$ -transversal in  $\text{Irr}(\mathbf{G}^F \mid (L', \text{Irr}_{\text{cusp}}(L')))$ .

It remains to consider the case where  $E_L \neq E$  and no  $\mathbf{G}^F$ -conjugate of  $L$  is  $E$ -stable. Then according to Propositions 7.3 and 7.4, every  $\chi' \in \text{Irr}(\mathbf{G}^F \mid (L, \text{Irr}_{\text{cusp}}(L)))$  has some  $\tilde{G}$ -conjugate  $\chi$  with  $(\tilde{G}E)_\chi = \tilde{G}_\chi E_\chi$ .  $\square$

**Acknowledgments.** I thank Julian Brough, Lucas Ruhstorfer, Gunter Malle, and the anonymous referee for their remarks on versions of the paper.

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